Sparse Differential Resultant

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ABSTRACT

In this paper, the concept of sparse differential resultant for a differentially essential system of differential polynomials is introduced and its properties are proved. In particular, a degree bound for the sparse differential resultant is given. Based on the degree bound, an algorithm to compute the sparse differential resultant is proposed, which is single exponential in terms of the order, the number of variables, and the size of the differentially essential system.

Categories and Subject Descriptors
I.1.2 [Computing Methodologies]: Symbolic and Algebraic Manipulation - Algorithms for differential equations

General Terms
Algorithms, Theory

Keywords
Sparse differential resultant, differentially essential system, Chow form, degree bound, single exponential algorithm.

1. INTRODUCTION

The resultant, which gives conditions for a system of polynomial equations to have common solutions, is a basic concept in algebraic geometry and a powerful tool in elimination theory [2, 8, 16, 6, 19, 9, 24, 28]. The sparse resultant was introduced by Gelfand, Kapranov, and Zelevinsky as a generalization of the usual resultant [13]. Basic properties for the sparse resultant were given by Sturmfels and co-authors [23, 28, 29]. A Sylvester style matrix based method to compute sparse resultants was first given by Canny and Emiris [2, 8, 16, 6, 19, 24, 28]. The sparse resultant was introduced and its properties are proved. In particular, we give a degree bound for the sparse differential resultant, which also leads to a degree bound for the differential resultant. Based on the degree bound, we give an algorithm to compute the sparse differential resultant. The complexity of the algorithm in the worst case is single exponential of the form \(O(n^{3.376}(s+1)^{O(n)}(m+1)^{O(n^2)})\), where \(s, m, n, \) and \(l\) are the order, the degree, the number of variables, and the size of the differentially essential system respectively. The sparseness is reflected in the quantity \(l\).

In principle, the sparse differential resultant can be computed with any differential elimination method, and in particular with the change of order algorithms given by Boulier-Lemaire-Maza [1] and Golubitsky-Kondratieva-Ovchinnikov [14]. The differentially essential system already forms a triangular set when considering their constant coefficients as leading variables, and the sparse differential resultant is the first element of the characteristic set of the prime ideal generated by the differentially essential system under a different special ranking. Therefore, the change of order strategy proposed in [1, 14] can be used. In our case, due to the special structure of the differentially essential system, we can give specific bounds for the order and degree needed to compute the resultant, which allows us to reduce the problem to linear algebra directly and give explicit complexity bounds.

As preparations for the main results of the paper, we prove...
several properties about the degrees of the elimination ideal and the generalized Chow form in the algebraic case, which are also interesting themselves.

The rest of the paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we define the sparse differential resultant and give its properties. And in Section 4, we present an algorithm to compute the sparse differential resultant. In Section 5, we conclude the paper by proposing several problems for future research.

2. DEGREE OF ELIMINATION IDEAL AND GENERALIZED CHOW FORM

In this section, we will prove several properties about the degrees of elimination ideals and generalized Chow forms in the algebraic case, which will be used later in the paper. These properties are also interesting themselves.

2.1 Degree of elimination ideal

Let \( P \) be a polynomial in \( K[X] \) where \( X = \{x_1, \ldots, x_n\} \). We use \( \deg(P) \) to denote the total degree of \( P \). Let \( I \) be a prime algebraic ideal in \( K[X] \) with dimension \( d \). We use \( \deg(I) \) to denote the degree of \( I \), which is defined to be the number of the zero dimensional prime ideal \( (I, L_1, \ldots, L_d) \), where \( \sum_{i=1}^{n} u_{ij} x_i (i = 1, \ldots, d) \) are \( d \) generic primes [17]. That is,

\[
\deg(I) = |V(I, L_1, \ldots, L_d)|.
\]

Clearly, \( \deg(I) = \deg(I, L_1, \ldots, L_d) \) for \( i = 1, \ldots, d \). \( \deg(I) \) is also equal to the maximal number of intersection points of \( V(I) \) with \( d \) hyperplanes under the condition that the number of these points is finite [18]. That is,

\[
\deg(I) = \max\{|V(I) \cap H_1 \cap \cdots \cap H_d| : H_i \text{ are affine hyperplanes with } |V(I) \cap H_1 \cap \cdots \cap H_d| < \infty \}
\]

The relation between the degree of an ideal and that of its elimination ideal is given by the following result.

Theorem 2.1 Let \( I \) be a prime ideal in \( K[X] \) and \( J_k = I \cap K[x_1, \ldots, x_k] \) for any \( 1 \leq k \leq n \). Then \( \deg(J_k) \leq \deg(I) \).

Proof: Suppose \( \dim(I) = d \) and \( \dim(J_k) = d_k \). Two cases are considered:

1. \( J_k \cap K(u_0) \cap \cdots \cap K(u_d) = J \).

2. \( J_k \cap K(u_0, \ldots, u_d) = 0 \)-dimensional prime ideal.

To prove 1), it suffices to show that whenever \( f \) is in the left ideal, \( f \) belongs to \( J \). Without loss of generality, suppose \( f \in K(u_0) \cap \cdots \cap K(u_d) \). Then there exist \( h_i, q_i \in K(u_0) \cap \cdots \cap K(u_d) \) such that \( f = \sum_i h_i q_i \). Substituting \( u_{ij} = -\sum_{j=1}^{k} u_{ij} x_j \) into the above equality, we have \( f = \sum_i h_i q_i \in J_k \) and \( f = 0 \mod(P_1, \ldots, P_d) \). So, \( f \in J \).

To prove 2), suppose \( (\xi_1, \ldots, \xi_n) \) is a generic point of \( I \). Denote \( U_0 = \{u_{10}, \ldots, u_{0d}\} \). Then \( J_k = (I, P_1, \ldots, P_d) \subseteq K(u_0, u_1) \cap \cdots \cap K(u_0, u_d) \) is a prime ideal of dimension \( d \) with a generic point \( (\xi_1, \ldots, \xi_n, -\sum_{j=1}^{k} u_{ij} \xi_j, \ldots, -\sum_{j=1}^{k} u_{ij} \xi_j) \).

Since \( d_1 = d \), there exist \( e \) elements in \( \{\xi_1, \ldots, \xi_n\} \) algebraically independent over \( K \). So by [16, p.168-169], \( J_k \cap K(u_0, u_1, \ldots, u_d) = 0 \) and \( \deg(I) = \deg(J_k) \).

Case (b): \( d_1 < d \). Let \( \mathcal{L}_i = u_{i0} + u_{i1} x_1 + \cdots + u_{id} x_d \) (\( i = 1, \ldots, d - d_1 \)). By [17, Theorem 1, p. 54], \( J = (I, L_1, \ldots, L_{d-d_1}) \subseteq K(u_i) \) is a prime ideal of dimension \( d \) and \( \deg(J) = \deg(I) \), where \( u_i = \{u_{ij} : i = 1, \ldots, d - d_1, j = 0, \ldots, n\} \). Of course, \( J_k \subseteq J \). Since both \( J_k \) and \( J \) are prime ideals and \( \dim(I) = d_1 \), it suffices to prove that \( \dim(J_k) = d_1 \).

Let \( J_0 = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{d-d_1}) \subseteq K(u_0, u_1, \ldots, u_d) \) with \( u_0 = \{u_{i0}, \ldots, u_{id-d_1}\} \). Suppose \( \{x_1, \ldots, x_{d-d_1}\} \) is a parameter set of \( I_0 \). Similarly to the procedure of proving 2) in case (a), we can show that \( J_0 \cap K(u_i) \cap \cdots \cap K(u_{d-d_1}) = 0 \), and \( J_0 \cap K(u_0, \ldots, u_{d-d_1}) = 0 \) follows. So \( \dim(J_0) = d_1 \).

Since \( \dim(J_0) = \dim(J_k) \), by case (a), we have \( \deg(J_k) \leq \deg(J) = \deg(I) \). And due to the fact that \( \deg(J_k) = \deg(I_k) \), \( \deg(I_k) \leq \deg(I) \) follows.

In this article, we will use the following result.

Lemma 2.2 [22, Proposition 1] Let \( F_1, \ldots, F_m \in K[X] \) be polynomials generating an ideal \( I \) of dimension \( r \). Suppose \( \deg(F_1) \geq \cdots \geq \deg(F_m) \) and \( d := \prod_{i=1}^{m} \deg(F_i) \). Then \( \deg(I) \leq D \).

2.2 Degree of algebraic generalized Chow form

Let \( I \) be a prime ideal in \( K[X] \) with dimension \( d \),

\[
P_i = u_{i0} + \sum_{1 \leq \alpha_1 + \cdots + \alpha_n \leq n} u_{i,\alpha_1} \cdots u_{i,\alpha_n} x_\alpha (i = 0, \ldots, d)
\]

generic polynomials of degree \( m_i \) and \( u_i \) the vector of coefficients of \( P_i \). Philippon [24] proved that

\[
(I, P_0, \ldots, P_d) \subseteq K[u_0, \ldots, u_d] = (Chow(I))
\]

is a prime principal ideal and \( G(u_0, \ldots, u_d) \) is defined to be the generalized Chow form of \( I \), denoted by \( G(I) \).

In this section, we will give the degree of the generalized Chow form in terms of the degrees of \( I \) and that of \( P_i \) by proving Theorem 2.4.

At first, we will give another description of the degree for a prime ideal. In (3), when \( P_i \) become generic primes

\[
\mathcal{L}_i = v_{i0} + \sum_{j=1}^{n} v_{ij} x_j (i = 0, 1, \ldots, d),
\]

generated by the set of coefficients of \( \mathcal{L}_i \). A basic property of Chow forms is that [17] for each \( i \) between 0 and \( d \),

\[
\deg(I) = \deg_{v_i} (Chow(I)).
\]

In the following lemma, we will give the degree of an ideal intersected by a generic primal. To prove the lemma, we apply the following Bezout inequality (see [15] or [18]): Let \( V, W \) be affine algebraic varieties. Then

\[
\deg(V \cap W) \leq \deg(V) \cdot \deg(W).
\]
**Lemma 2.3** Let $I$ be a prime ideal in $K[X]$ with $\text{dim}(I) = d > 0$ and $P$ a generic polynomial. Then $\deg(I, P) = \deg(P) \cdot \deg(I)$. 

**Proof:** Firstly, we prove the lemma holds for $d = 1$. Let $v$ be the vector of coefficients of $P$, $m = \deg(P)$, and $J = (I, P) \subset K(v)[X]$. Then by [17, p. 110], $J$ is a prime algebraic ideal of dimension zero. Let $L_0$ be a generic prime with $u_0$ the vector of coefficients. By (4), $(J, L_0) \cap K(v)[u_0] = (\text{Chow}(J))$. Here, we choose Chow($J$) to be an irreducible polynomial in $K[v, u_0]$. From (5), we have $\deg(J) = \deg(u_0) \text{Chow}(J)$.

Let $M = (I, L_0) \subset K(u_0)[X]$. Then $M$ is a prime ideal of dimension zero with $\deg(M) = \deg(I)$. And $(M, P) \cap K(u_0)[v] = (G(M))$ where $G(M) \in K[v, u_0]$ is irreducible. Clearly, $G(M) = c \cdot \text{Chow}(J)$ for some $c \in K^*$ and $G(M)$ can be factored as 

$$G(M) = A(u_0) \prod_{\tau = 1}^{\deg(I)} P(\xi_\tau),$$

where $\xi_\tau$ are all the elements of $V(M)$ and $A(u_0)$ is an extraneous factor lying in $K[u_0]$. Now, specialize $P$ to $L^n_0$, where $L_0 = u_{10} + \sum_{i=1}^{m} u_{ix_i}$ is a generic prime. Then we have $\deg(M) = A(u_0) \sum_{i=1}^{\deg(I)} L^n_i (\xi_i)$ and $\deg(G(M) u_0) = \deg(J)$. Since Chow($I$) = $B(u_0) \prod_{\tau = 1}^{\deg(I)} L^n_i(\xi_i)$ for some $B \in K[u_0]$, $G(M)$ is irreducible. And by Bézout inequality (6), $\deg(I, P) \leq \deg(I) \cdot \deg(P)$, so $\deg(I, P) = \deg(G(M) \cdot \deg(P)$). 

For the case $d > 1$, let $L_0, \ldots, L_{d-1}$ be generic primes, then $I_1 = (I, L_1, \ldots, L_{d-1})$ is a prime ideal of dimension one and $\deg(I_1) = \deg(I)$. By the case $d = 1$, $\deg(I, P) = \deg(I_1) \cdot \deg(P)$. So $\deg(I, P) = \deg(I, P, L_1, \ldots, L_{d-1}) = \deg(I_1) \cdot \deg(P) = \deg(I_1) \cdot \deg(P) = \deg(G(M) \cdot \deg(P)$).

The following result generalizes Lemma 1.8 in [24].

**Theorem 2.4** Let $G(u_0, \ldots, u_d)$ be the generalized Chow form of a prime ideal $I$ of dimension $d$ w.r.t. $F_0, \ldots, F_d$. Then $G$ is of degree $\deg(I) \prod_{\tau = 1}^{\deg(I)} \deg(F_\tau)$ in each set $u_i$.

**Proof:** It suffices to prove the result for $i = 0$. 

If $d = 0$, then $G(u_0) = \prod_{\tau = 1}^{\deg(I)} P_0(\xi_\tau)$, where $\xi_\tau \in V(I)$. Clearly, $\deg(G, u_0) = \deg(I)$. 

We consider the case $d > 0$. Let $J_0 = (I, F_1, \ldots, F_d) \subset K[u_0, u_1, \ldots, u_d, X]$ and $J = (J_0) \subset K(u_0, \ldots, u_d, x_1, \ldots, x_n)$. Then $J$ is a prime ideal of dimension zero and by Lemma 2.3, $\deg(J) = \deg(I) \prod_{\tau = 1}^{\deg(I)} \deg(F_\tau)$. We claim that $\deg(G(u_0, \ldots, u_d)$ is also the generalized Chow form of $J$, hence $G(u_0, \ldots, u_d) = \deg(I) \prod_{\tau = 1}^{\deg(I)} \deg(F_\tau)$. Since $G(u_0, \ldots, u_d)$ is the generalized Chow form of $I$, we have $(J, F_0, \ldots, F_d) \cap K[u_0, \ldots, u_d] = (G(u_0, \ldots, u_d)) = (J_0, F_0) \cap K[u_0, \ldots, u_d]$. Let $G_1(u_0, \ldots, u_d) \subset K[u_0, \ldots, u_d]$ be the generalized Chow form of $I$ and irreducible. Then $(J, F_0) \cap K[u_0, \ldots, u_d] = (G_1)$. So $G \in (G_1)$. But $G, G_1$ are irreducible polynomials in $K[u_0, \ldots, u_d]$ and $G = c \cdot G_1$ for some $c \in K^*$ and $G$ is the generalized Chow form of $I$. 

**3. SPARSE DIFFERENTIAL RESULTANT**

In this section, we define the sparse differential resultant and prove its basic properties.

**3.1 Definition of sparse differential resultant**

Let $F$ be an ordinary differential field and $F(Y)$ the ring of differential polynomials in the differential indeterminates $Y = (y_1, \ldots, y_n)$. For any element $e \in F(Y)$, we use $e(k)$ to represent the $k$-th derivative of $e$ and $e^{[k]}$ to denote the set $\{e(i) : i = 0, \ldots, k\}$. Details about differential algebra can be found in [20, 26].

The following theorem presents an important property on differential specialization, which will be used later.

**Theorem 3.1** [12, Theorem 2.14] Let $\{u_1, \ldots, u_r\}$ be a set of differential indeterminates, and $P(I, Y) \in F(I, Y)$ a differential polynomial in the differential indeterminates $U = (u_1, \ldots, u_r)$ and $Y = (y_1, \ldots, y_n)$. Let $u^0 = (u_1^0, \ldots, u_r^0)$, where $u_i^0$ are some differential extension field of $F$. If $P(I, Y^0)$ ($i = 1, \ldots, m$) are differentially dependent over $F(U)$, then for any specialization $U$ to $Y^0$ in $F$, $P(I, Y^0)$ ($i = 1, \ldots, m$) are differentially dependent over $F$.

To define the sparse differential resultant, consider $n + 1$ differential polynomials with differential indeterminates as coefficients

$$P_i = u_{i0} + \sum_{k=1}^{l_i} u_{ik} M_{ik} (i = 0, \ldots, n) \quad (7)$$

where $M_{ik} = (Y^{[i]} i^\alpha)_{\alpha k}$ is a monomial in $(y_1, \ldots, y_r, y_{1}^{[i]}, \ldots, y_{1}^{[i]}, \ldots, y_{n}^{[i]})$ with exponent vector $\alpha k$ and $|\alpha k| \geq 1$. The set of exponent vectors $S_{l_i} = \{0, \alpha k : k = 1, \ldots, l_i\}$ is called the support of $P_i$, where $0$ is the exponent vector for the constant term. The number $|S_{l_i}| = l_i + 1$ is called the size of $P_i$. Note that $s_i$ is the order of $P_i$ and an exponent vector of $P_i$ contains $n(s_i + 1)$ elements.

Denote $u = \{u_{ik} : i = 0, \ldots, n; k = 1, \ldots, l_i\}$. Let $\eta_1, \ldots, \eta_n$ be $n$ elements which are differentially independent over $Q(u)$ and denote $\eta = (\eta_1, \ldots, \eta_n)$, where $Q$ is the field of rational numbers. Let

$$\zeta_i = -\sum_{k=1}^{l_i} u_{ik}(\eta^{[i]} i^\alpha)_{\alpha k} (i = 0, \ldots, n). \quad (8)$$

Denote the differential transcendence degree by $d.\text{tr.deg}$. Then, we have

**Lemma 3.2** $d.\text{tr.deg} Q(u) \langle \zeta_0, \ldots, \zeta_n \rangle / Q(u) = n$ if and only if there exist $n$ monomials $M_{ik}(i = 1, \ldots, n)$ in (7) such that $r_i \neq r_j$ for $i \neq j$ and $M_{ik}(\eta) = (\eta^{[i]} i^\alpha)_{\alpha k}$ are differentially independent over $Q(u)$.

**Proof:** “$\Leftarrow$” Without loss of generality, we assume $r_i = i (i = 1, \ldots, n)$ and $M_{ik}(\eta) (i = 1, \ldots, n)$ are differentially independent. It suffices to prove that $\zeta_1, \ldots, \zeta_n$ are differentially independent over $Q(u)$. Suppose the contrary, i.e., $\zeta_1, \ldots, \zeta_n$ are differentially dependent. Now specialize $u_{ij}$ to $-\delta_{ik}$. By Theorem 3.1 and (8), $M_{ik}(\eta) (i = 1, \ldots, n)$ are differentially dependent, which is a contradiction.

“$\Rightarrow$” Suppose the contrary, i.e., $M_{ik}(\eta) (i = 1, \ldots, n)$ are differentially dependent for any $n$ different $r_i$ and $k_i = 1, \ldots, l_i$. Since each $\zeta_i$ is a linear combination of $M_{ik}(\eta) (k_i = 1, \ldots, l_i)$, $\zeta_1, \ldots, \zeta_n$ are differentially dependent, contradicting that $d.\text{tr.deg} Q(u) \langle \zeta_0, \ldots, \zeta_n \rangle / Q(u) = n$. 


Definition 3.3 A set of differential polynomials of form (7) satisfying the condition in Lemma 3.2 is called a differentially essential system.

A differential polynomial $f$ of form (7) is called quasi-generic [12] if for each $1 \leq i \leq n$, $f$ contains at least one monomial in $F\{y_i\} \setminus F$. Clearly, $n + 1$ quasi-generic differential polynomials form a differentially essential system.

Now let $\mathcal{P}_0, \ldots, \mathcal{P}_n$ be the differential ideal generated by $\mathcal{P}_i$ in $Q(u)\{y_0, u_{00}, \ldots, u_n\}$. Then it is a prime differential ideal with a generic point $(\eta_1, \ldots, \eta_n, \zeta_0, \ldots, \zeta_n)$ and of dimension $n$. Clearly, $I = [\mathcal{P}_0, \ldots, \mathcal{P}_n] \cap Q(t)\{u_{00}, \ldots, u_n\}$ is a prime differential ideal with a generic point $(\zeta_0, \ldots, \zeta_n)$. As a consequence of Lemma 3.2, we have

Corollary 3.4 $I$ is of codimension one if and only if $\{\mathcal{P}_0, \ldots, \mathcal{P}_n\}$ is a differentially essential system.

Now suppose $\{\mathcal{P}_0, \ldots, \mathcal{P}_n\}$ is a differentially essential system. Since $I$ is of codimension one, then by [26, line 14, p. 45], there exists an irreducible differential polynomial $R(u_0, u_{00}, \ldots, u_n) \in Q(u)\{u_{00}, \ldots, u_n\}$ such that
\[
[\mathcal{P}_0, \ldots, \mathcal{P}_n] \cap Q(u)\{u_{00}, \ldots, u_n\} = \text{sat}(R)
\]
where sat($R$) is the saturation ideal of $R$. More explicitly, sat($R$) is the whole set of differential polynomials having zero pseudo-remainder w.r.t. $R$ under any ranking endowed on $u_{00}, \ldots, u_n$. And by clearing denominators when necessary, we suppose $R \in Q(u)\{u_0, u_{00}, \ldots, u_n\}$ is irreducible and also denoted by $R(u_0, u_{00}, \ldots, u_n)$. Let $u_i = (u_0, u_1, \ldots, u_l_i)$ be the vector of coefficients of $F_i$ and denote $R(u_0, \ldots, u_n) = R(u_0, over dots, u_n).$ Now we give the definition of sparse differential resultant as follows:

Definition 3.5 $R(u_0, \ldots, u_n) \in Q(u_0, \ldots, u_n)$ in (9) is defined to be the sparse differential resultant of the differentially essential system $\mathcal{P}_0, \ldots, \mathcal{P}_n$.

Example 3.6 For $n = 2$, let $\mathcal{P}_0 = u_{00} + u_{01}y_1 + y_2, \mathcal{P}_1 = u_{10} + u_{01}y_1 + y_2,$ and $\mathcal{P}_2 = u_{20} + u_{21}y_1 + y_2$. Using differential elimination algorithms [5], we can show that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ form a differentially essential system and their sparse differential resultant is $R = -u_{10}u_{10}u_{11} - u_{00}u_{21}u_{10} + u_{01}u_{10}u_{22}u_{00} - u_{11}u_{10}u_{22}u_{01}$. The following properties can be proved easily.

1. When all $\mathcal{P}_i$ become generic differential polynomials of the form $\mathcal{P}_i = u_i + \sum_{1 \leq i \leq m_i} u_{i, a}(y^{[\bar{l}_i]})^a$, the sparse differential resultant is the differential resultant defined in [12].

2. $R$ is the vanishing polynomial of $(\zeta_0, \ldots, \zeta_n)$ with minimal order in each $u_{00}$. Since $R \in Q(u)\{u_0, \ldots, u_n\}$ is irreducible, ord($R, u_i$) = ord($R, u_{00}$).

3. Suppose ord($R, u_i$) = $h_i \geq 0$ and denote $\sigma = \sum_{i=1}^{n} h_i$. Given a vector $(q_0, \ldots, q_n) \in \mathbb{N}^{n+1}$ with $\sum_{i=0}^{n} q_i = q$, if $q < \sigma$, then there is no polynomial $P$ in sat($R$) with ord($P, u_i$) = $q_i$. And $R$ is the unique irreducible polynomial in sat($R$) with total order $q = \sigma$ up to some $\alpha \in \mathbb{Q}$. This property will be used in our algorithm to search for the sparse differential resultant.

Remark 3.7 It is not easy to define the sparse differential resultant as the algebraic sparse resultant of $P^{(k)}$ considered as polynomials in $y_{i}^{(j)}$. The reason is that it is difficult to check whether the supports of $P^{(k)}$ and $P^{(l)}$ satisfy the conditions for the existence of the algebraic sparse resultant [29]. Furthermore, the coefficients of $P^{(k)}$ are not generic.

3.2 Properties of sparse differential resultant

Following Kolchin [21], we introduce the concept of differentially homogenous polynomials.

Definition 3.8 A differential polynomial $p \in F\{y_0, \ldots, y_n\}$ is called differentially homogenous of degree $m$ if for a new differential indeterminate $\lambda$, we have $p(\lambda y_0, \lambda y_1, \ldots, \lambda y_n) = \lambda^m p(y_0, y_1, \ldots, y_n)$.

The differential analog of Euler’s theorem related to homogenous polynomials is valid.

Theorem 3.9 [21] $f \in F\{y_0, y_1, \ldots, y_n\}$ is differentially homogenous of degree $m$ if and only if
\[
\sum_{j=0}^{n} \sum_{k \in \mathbb{N}} \binom{k+r}{r} y_j^{(k)} \frac{\partial f(y_0, \ldots, y_n)}{\partial y_j^{(k+r)}} = \begin{cases} m f & r = 0 \\ 0 & r \neq 0 \end{cases}
\]

Sparse differential resultants have the following property.

Theorem 3.10 The sparse differential resultant is differentially homogenous in each $u_i$ which is the coefficient set of $P_i$.

Proof: Similar to the proof of [12, Theorem 4.16], we can show that $R$ satisfies the conditions of Theorem 3.9 for each $u_i$. The proof is omitted due to the page limit.

Continue from Example 3.6. In this example, $R$ is differentially homogenous of degree 2 in $u_0$, of degree 1 in $u_1$ and of degree 2 in $u_2$ respectively.

In the following, we prove formulas for sparse differential resultants, which are similar to the Poisson type formulas for Chow forms and algebraic resultants [23]. Denote ord($R, u_i$) by $h_i$ ($i = 0, \ldots, n$). We have the following theorem.

Theorem 3.11 Let $R(u_0, \ldots, u_n)$ be the sparse differential resultant of $\mathcal{P}_0, \ldots, \mathcal{P}_n$. Let $\deg(R, u_{00}) = t_0$. Then there exist $\xi_{k}$ for $\tau = 1, \ldots, t_0$ and $k = 1, \ldots, l_0$ such that
\[
R = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{k}^{(h_k)}),
\]
where $A$ is a polynomial in $F[u_{00}, \ldots, u_{n+1}^{(h_n)} \setminus u_{00}^{(h_0)}]$. The proof: Consider $R$ as a polynomial in $u_{00}$ with coefficients in $Q_0 = Q(u_{00}^{(h_0)} \setminus u_{00}^{(h_0)}).$ Then, in an algebraic extension field of $Q_0$, we have
\[
R = A \prod_{\tau=1}^{t_0} (u_{00}^{(h_0)} - \xi_{k}^{(h_k)}),
\]
where $t_0 = \deg(R, u_{00}^{(h_0)}).$ Note that $\xi_{k}$ is an algebraic root of $R(u_{00}^{(h_0)}) = 0$ and a derivative for $\xi_{k}$ can be naturally defined.
to make $\mathcal{F}(\tau)$ a differential field. From $R(u; \varphi, \ldots, \varphi_n) = 0$, if we differentiate this equality w.r.t. $\varphi_{0k}$, then we have

$$\frac{\partial R}{\partial \varphi_{0k}} + \frac{\partial R}{\partial \varphi_{00}} (-\eta^{[k]})^{\alpha_0k} = 0 \tag{11}$$

where $\frac{\partial R}{\partial \varphi_{0k}}$ and $\frac{\partial R}{\partial \varphi_{00}}$ are obtained by substituting $u_0$ by $\varphi_i$ and $\frac{\partial R}{\partial \varphi_{0k}}$ respectively.

Now multiply equation (11) by $u_{0k}$ and for $k$ from 1 to $l_0$ add all of the equations obtained together, we get

$$\frac{\partial R}{\partial \varphi_{00}} \varphi_0 + \sum_{k=1}^{l_0} u_{0k} \frac{\partial R}{\partial \varphi_{0k}} = 0 \tag{12}$$

Thus, the polynomial $G_1 = u_{00} \frac{\partial R}{\partial \varphi_{00}} + \sum_{k=1}^{l_0} u_{0k} \frac{\partial R}{\partial \varphi_{0k}}$ vanishes at $(u_{00}, \ldots, u_{0n}) = (\varphi_0, \ldots, \varphi_n).$ Since $\text{ord}(G_1) \leq \text{deg}(G_1) = \text{deg}(R)$, there exists some $a \in \mathcal{F}$ such that $G_1 = aR.$ Setting $u_{0(k)} = \varphi_i$ in both sides of $G_1$, we have $u_{0(k)}R_{0} + \sum_{k=1}^{l_0} u_{0k} R_{0k} = 0$, where $R_{0k} = \frac{\partial R}{\partial \varphi_{0k}} u_{0(k)} = \varphi_i.$ Since $R$ is irreducible as an algebraic polynomial in $u_{0(k)}$, $R_{0} \neq 0$. Denote $\xi_k = R_{0k}/R_{0}$. Thus, $u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_k = 0$ under the condition $u_{0(k)} = \varphi_i.$ Consequently, $\varphi_i = -\left(\sum_{k=1}^{l_0} u_{0k} \xi_k\right)^{(b)}$ and (10) follows. $\square$

If $\mathcal{F}_0$ contains the linear terms $y_i$ ($i = 1, \ldots, n$), then the above result can be strengthened as follows.

**Theorem 3.12** Suppose $\mathcal{F}_0$ has the form

$$\mathcal{F}_0 = u_{00} + \sum_{i=1}^{n} \sum_{k=0}^{l_i} u_{0k}^{(y_i^{[s]})} \alpha_{0i}. \tag{13}$$

Then there exist $\xi_{\tau}$ ($\tau = 1, \ldots, t_0, k = 1, \ldots, n$) such that

$$R = A \prod_{r=1}^{t_0} \left( u_{00} + \sum_{i=1}^{n} u_{0r} \xi_{\tau} + \sum_{k=0}^{l_i} u_{0k} (\xi_{\tau}^{[s]})^{\alpha_{0i}} \right)^{(b)}$$

$$= A \prod_{r=1}^{t_0} \mathcal{F}_0(\xi_{\tau})^{(b)}, \quad \text{where } \xi_{\tau} = (\xi_{\tau_1}, \ldots, \xi_{\tau_n}).$$

Moreover, $\xi_{\tau}$ ($\tau = 1, \ldots, t_0$) lies on $\mathcal{F}_1, \ldots, \mathcal{F}_n.$

**Proof:** For the first part, from Theorem 3.11, it remains to show that for $i = n + 1$ to $l_0$, $\tau_i = (\xi_{\tau_i}^{[s]})^{\alpha_{0i}}$. From equation (11), we have $\eta_i = \frac{\partial R}{\partial u_{0k}} \frac{\partial R}{\partial \varphi_{00}}$ and $\left(\xi_{\tau_i}^{[s]}\right)^{\alpha_{0i}} = \frac{\partial R}{\partial u_{0k}} \frac{\partial R}{\partial \varphi_{00}}$. If $(\xi_{\tau_i}^{[s]})^{\alpha_{0i}} = \prod_{j=1}^{n} \prod_{k=0}^{l_j} \left( \frac{\partial R}{\partial u_{0j}} \frac{\partial R}{\partial \varphi_{00}} \right)^{(k)} \left(\alpha_{0j}\right)^{(k)}$, then

$$\frac{\partial R}{\partial u_{0k}} \frac{\partial R}{\partial \varphi_{00}} = \frac{\partial R}{\partial u_{00}} \frac{\partial R}{\partial \varphi_{00}}.$$ 

It follows that

$$\prod_{j=1}^{n} \prod_{k=0}^{l_j} \left( \frac{\partial R}{\partial u_{0j}} \frac{\partial R}{\partial \varphi_{00}} \right)^{(k)} \left(\alpha_{0j}\right)^{(k)} - \frac{\partial R}{\partial u_{00}} \frac{\partial R}{\partial \varphi_{00}}$$

vanishes at $(u_{00}, \ldots, u_{0n}) = (\varphi_0, \ldots, \varphi_n).$ Since there exists some $a \in \mathcal{N}$, such that $G_i = \left(\frac{\partial R}{\partial u_{0k}} \frac{\partial R}{\partial \varphi_{00}} \right)^{(n)} \prod_{j=1}^{n} \prod_{k=0}^{l_j} \left( \frac{\partial R}{\partial u_{0j}} \frac{\partial R}{\partial \varphi_{00}} \right)^{(k)} \left(\alpha_{0j}\right)^{(k)} \left(\alpha_{0k}\right)^{(k)} - \frac{\partial R}{\partial u_{00}} \frac{\partial R}{\partial \varphi_{00}}$ is a polynomial in $Q\{u_0, \ldots, u_n\}$, $G_i \in \text{sat}(R)$. Now substituting $u_{00} = \varphi_i$ for $h \geq 0$ into $G_i$, we obtain that $\xi_{\tau_i} = \prod_{j=1}^{n} \prod_{k=0}^{l_j} \left( (\xi_{\tau_j})^{(k)} \right)^{(\alpha_{0j})} = (\xi_{\tau_i}^{[s]})^{\alpha_{0i}}$.

The proof of the second assertion is based on generalized differential Chow form introduced in [12] and is omitted. $\square$

As in algebra, the sparse differential resultant gives a necessary condition for a system of differential polynomials to have common solutions, as shown by the following theorem.

**Theorem 3.13** Let $\mathcal{P}_0, \ldots, \mathcal{P}_n$ be a differentially essential system of the form (7) and $R(u_0, \ldots, u_n)$ be their sparse differential resultant. Denote $\text{ord}(R, u_i) = h_i$ and $S_R = \frac{\partial R}{\partial u_{0k}}$. Suppose that when $u_i$ ($i = 0, \ldots, n$) are specialized to sets $v_i$ which are elements in an extension field of $\mathcal{F}$, $\mathcal{P}_i$ are specialized to $\mathcal{F}_i$ ($i = 0, \ldots, n$). If $\mathcal{F}_i = 0$ ($i = 0, \ldots, n$) have a common solution, then $R(v_0, \ldots, v_n)$ should be zero.

From equation (11), it is clear that the polynomial $\frac{\partial R}{\partial u_{0k}} = \frac{\partial R}{\partial u_{00}} \frac{\partial R}{\partial v_{0k}}$ vanishes at $\xi$. So (14) follows. $\square$

Again, if $\mathcal{F}_0$ contains the linear terms $y_i$ ($i = 1, \ldots, n$), then the above result can be strengthened as follows.

**Corollary 3.14** Suppose $\mathcal{F}_0$ has the form (13). If $R(v_0, \ldots, v_n) = 0$ and $S_R(v_0, \ldots, v_n) \neq 0$, then $\mathcal{F}_i = 0$ have a common solution.

**Proof:** From the above theorem, we know that for $k$ from 1 to $n$,

$$A_k = \frac{\partial R}{\partial u_{0k}} + \frac{\partial R}{\partial u_{00}} (-\eta_k) \in [\mathcal{F}_0, \ldots, \mathcal{F}_n].$$

Clearly, $A_k$ is linear in $y_k$. Suppose the differential remainder of $\mathcal{F}_i$ w.r.t. $A_1, \ldots, A_n$ in order to eliminate $y_1, \ldots, y_n$ is $g_i$, then $S_{\mathcal{F}_i} \equiv g_i$ mod $[A_1, \ldots, A_n]$ for $a \in \mathcal{N}$. Thus, $g_i \in [\mathcal{F}_0, \ldots, \mathcal{F}_n] \cap Q\{u_0, \ldots, u_n\} = \text{sat}(R)$. So we have $S_{\mathcal{F}_i} \equiv 0$ mod $[A_1, \ldots, A_n]$ for some $b \in \mathcal{N}$. Now specialize $u_k$ to $v_i$, for $i = 0, \ldots, n$, then we have

$$S_{\mathcal{F}_i}(v_0, \ldots, v_n) \cdot S_R(v_0, \ldots, v_n) \equiv 0 \mod [A_1, \ldots, A_n]. \tag{15}$$

Let $\xi_k = \frac{\partial R}{\partial u_{0k}} (v_0, \ldots, v_n) / S_R(v_0, \ldots, v_n)$ $(k = 1, \ldots, n)$, and denote $\xi = (\xi_1, \ldots, \xi_n)$. Then from equation (15), $\mathcal{F}_i(\xi)$ is a common solution of $\mathcal{F}_0, \ldots, \mathcal{F}_n$. $\square$

**4. AN ALGORITHM TO COMPUTE SPARSE DIFFERENTIAL RESULTANT**

In this section, we give an algorithm to compute the sparse differential resultant with single exponential complexity.
4.1 Degree bounds for sparse differential results

In this section, we give an upper bound for the degree and order of the sparse differential resultant, which will be crucial to our algorithm to compute the sparse resultant.

Theorem 4.1. Let $P_0, \ldots, P_n$ be a differentially essential system of form (7) with $\text{ord}(P_i) = s_i$ and $\deg(P_i, Y) = m_i$. Let $R(u_0, \ldots, u_n)$ be the sparse differential resultant of $P_i (i = 0, \ldots, n)$. Suppose $\text{ord}(R, u_i) = h_i$ for each $i$. We have

1) $h_i \leq s - s_i$ for $i = 1, \ldots, n$ where $s = \sum_{i=0}^{n} s_i$.

2) $R$ can be written as a linear combination of $P_i$ and their derivatives up to order $h_i$. Precisely,

$$R(u_0, \ldots, u_n) = \sum_{i=0}^{n} h_i G_{ik} P_{ik}^{(h_i)} \quad (16)$$

for some $G_{ik} \in \mathbb{Q}[u_0^{[h]}, \ldots, u_n^{[h]}, Y^{[h]}]$ where $h = \max\{h_i + 1\}$.

3) $\deg(R) \leq \prod_{i=0}^{n} (m_i + 1) h_i + 1 \leq (m + 1)^{n + s - s_i + 1}$, where $m = \max\{m_i\}$.

Proof: 1) Let $\theta_\eta = -\sum_{1 \leq |\alpha| \leq m_i} u_{\alpha} (Y^{[\alpha_1]})^\alpha (i = 0, \ldots, n)$ where $\eta = (\eta_1, \ldots, \eta_n)$ is the generic point of the zero differential ideal $[0]$, and $W_{\theta_\eta} = u_{\alpha 0} + \sum_{1 \leq |\alpha| \leq m_i} u_{\alpha} (Y^{[\alpha_1]})^\alpha$ is a generic polynomial of order $s_i$ and degree $m_i$. Then from the property of differential resultant ([12, Theorem 1.3.]), we know the minimal polynomial of $(\theta_\eta, \ldots, \theta_n)$ is of order $s - s_i$ in each $u_{\alpha 0}$. Now specialize all the $u_{\alpha 0}$ such that $\theta_i$ is specialized to the corresponding $\zeta_i$. By the procedures in the proof of Theorem 3.1, we can obtain a nonzero differential polynomial vanishing at $(\zeta_0, \ldots, \zeta_n)$ with order not greater than $s - s_i$ in each variable $u_{\alpha 0}$. Since $R$ is the minimal polynomial of $\langle \zeta_0, \ldots, \zeta_n \rangle$, $\text{ord}(R, u_{\alpha 0}) = \text{ord}(R, u_i) \leq s - s_i$.

2) Substituting $u_{\alpha 0}$ by $P_i - \sum_{1 \leq |\alpha| \leq m_i} u_{\alpha} (Y^{[\alpha_1]})^\alpha$ in the polynomial $R(u_0, \ldots, u_n)$ for $i = 0, \ldots, n$, we get

$$R(u_0, \ldots, u_n) = R(u_0, P_0 - \sum_{1 \leq |\alpha| \leq m_i} u_{\alpha} (Y^{[\alpha_1]})^\alpha, \ldots, P_n - \sum_{1 \leq |\alpha| \leq m_i} u_{\alpha} (Y^{[\alpha_1]})^\alpha)$$

$$= \sum_{i=0}^{n} h_i G_{ik} P_{ik}^{(h_i)} + T(u, Y)$$

for $G_{ik} \in \mathbb{Q}[u_0^{[h]}, \ldots, u_n^{[h]}, Y^{[h]}]$ and $T(u, Y)$ is a polynomial of $u_{\alpha 0}$, $Y$. Since $[P_0, \ldots, P_n] \cap \mathbb{Q}(u) \{Y\}$ is $[0]$, $T = 0$ and 2) is proved. Moreover, $(\mathbb{Q}[u_0^{[h]}, \ldots, P_n] \cap \mathbb{Q}[u_0^{[h]}, \ldots, u_n^{[h]}] = (R(u_0, \ldots, u_n).$

3) Let $J_0 = (P_0^{[h]}, \ldots, P_n^{[h]} \cap \mathbb{Q}[u_0^{[h]}, \ldots, u_n^{[h]}, Y])$ where $Y$ is the $y_i$ and their derivatives appearing in $P_i$, $P_i^{[h]}$. By Lemma 2.2, $\deg(J_0) \leq \prod_{i=0}^{n} (m_i + 1) h_i + 1$ and $(R) = J_0 \cap \mathbb{Q}[u_0^{[h]}, \ldots, u_n^{[h]}]$ is the eliminant ideal of $J_0$. Thus, by Theorem 2.1,

$$\deg(R) \leq \deg(J_0) \leq \prod_{i=0}^{n} (m_i + 1) h_i + 1 \quad (17)$$

Together with 1), 3) is proved.

The following theorem gives an upper bound for degrees of differential resultants, the proof of which is not valid for sparse differential resultants. In the following result, when we estimate the degree of $R$, only the degrees of $P_i$ in $Y$ are considered, while in Theorem 4.1, the degrees of $P_i$ in both $Y$ and $u_{\alpha}$ are considered.

Theorem 4.2. Let $F_i (i = 0, \ldots, n)$ be generic differential polynomials in $Y = \{y_1, \ldots, y_s\}$ with order $s_i$, degree $m_i = \deg(P_i, Y)$, and $s = \sum_{i=0}^{n} s_i$. Let $R(u_0, \ldots, u_n)$ be the differential resultant of $F_0, \ldots, F_n$. Then we have $\deg(R, u_i) \leq \prod_{i=0}^{n} m_i + s_i + 1$ for each $k = 0, \ldots, n$.

Proof: Without loss of generality, we consider $k = 0$.

[Insert proof details here]
Corollary 4.4 Let \( f_0, \ldots, f_n \in \mathbb{F}\{y_1, \ldots, y_n\} \) have no common solutions with \( \text{ord}(f_i) = r_i, s = \sum r_i, \) and \( \text{deg}(f_i) \leq m. \) If the sparse differential resultant of \( f_0, \ldots, f_n \) is nonzero, then there exist \( H_{ij} \in \mathbb{F}\{y_1, \ldots, y_n\} \) such that \( \sum_{i=0}^n \sum_{j=0}^n H_{ij} f_i^{(j)} = 1 \) and \( \text{deg}(H_{ij} f_i^{(j)}) \leq (m+1)n+2 \).

Proof: The hypothesis implies that \( \mathbb{P}(f_i) \) form a differentially essential system. Clearly, \( R(u_0, \ldots, u_n) \) has the property stated in Theorem 4.3, where \( u_i \) are coefficients of \( \mathbb{P}(f_i) \). The result follows directly from Theorem 4.3 by specializing \( u_i \) to the coefficients of \( f_i \).

Now, we give an algorithm \textsf{SDResultant} to compute sparse differential resultants. The algorithm works adaptively by searching \( R \) with an order vector \((h_0, \ldots, h_n) \in \mathbb{N}^{n+1} \) with \( h_i \leq s - s_i \) by Theorem 4.1. Denote \( o = \sum h_i \). We start with \( o = 0. \) And for this \( o, \) choose one vector \((h_0, \ldots, h_n) \) at a time. For this \((h_0, \ldots, h_n) \), we search for \( R \) from degree \( D = 1. \) If we cannot find an \( R \) with such a degree, then we repeat the procedure with degree \( D + 1 \) until \( D \geq \sum_{i=0}^n (m_i + 1). \) In that case, we choose another \((h_0, \ldots, h_n) \) with \( \sum h_i = o. \) But if for all \((h_0, \ldots, h_n) \) with \( h_i \leq s - s_i \) and \( \sum h_i = o, \) we cannot find an \( R \) with degree \( o + 1. \) In this way, we need only to handle problems with the real size and need not go to the upper bound in most cases.

Theorem 4.5 Algorithm \textsf{SDResultant} computes sparse differential resultants with at most \( O(n^{3.376}(s+1)^{O(n)}}) (m+1)O(n^2) \) \( \mathbb{Q} \)-arithmetic operations.

Proof: In each loop of Step 3, the complexity of the algorithm is clearly dominated by Step 3.1.2., where we need to solve a system of linear equations \( \mathcal{P} = 0 \) over \( \mathbb{Q} \) in \( c_0 \) and \( c_{ij}. \) It is easy to show that \( |c_0| = \mathcal{O}((D+L-1) \sum_{i=0}^n (m_i + 1)), \) and \( |c_{ij}| = \mathcal{O}(m+1)^D \sum_{i=0}^n \sum_{j=0}^n (n+1) \sum_{k=0}^n (k+1)), \) where \( L \) is the number of linear equations with \( \mathcal{P} = 0. \) Then \( \mathcal{P} = 0 \) is a linear equation system with

Algorithm 1 — \textsf{SDResultant}(\( \mathbb{F}_0, \ldots, \mathbb{F}_n \))

**Input:** A differentially essential system \( \mathbb{F}_0, \ldots, \mathbb{F}_n. \)

**Output:** The sparse differential resultant of \( \mathbb{F}_0, \ldots, \mathbb{F}_n. \)

1. For \( i = 0, \ldots, n, \) set \( s_i \) as \( \text{ord}(\mathbb{F}_i), m_i = \text{deg}(\mathbb{F}_i), \) \( \mathbb{F}_i = \text{coeff}(\mathbb{F}_i) \) and \( |\mathbb{F}_i| = l_i + 1. \)
2. Set \( R = 0, o = 0, s = \sum_{i=0}^n s_i, m = \max\{m_i\}. \)
3. While \( R = 0 \) do
   3.1. For each vector \((h_0, \ldots, h_n) \in \mathbb{N}^{n+1} \) with \( \sum_{i=0}^n h_i = o \) and \( l_i \leq s - s_i \) do
      3.1.1. \( U = \sum_{i=0}^n \mathbb{F}_i[h_i], \) \( h_{max} = (h_1, s_1) \), \( D = 1. \)
      3.1.2. While \( R = 0 \) and \( D \leq \sum_{i=0}^n (m_i + 1)h_i^{+1} \) do
         3.1.2.1. Set \( R_0 \) to be a homogenous GPol of degree \( D \) in \( U. \)
         3.1.2.2. Set \( c_0 = \text{coeff}(R_0, U). \)
         3.1.2.3. Set \( H_{ij}(i = 0, \ldots, n; j = 0, \ldots, h_i) \) to be GPol of degree \( m_i + 1 \) in \( Y^{h_i} \U. \)
         3.1.2.4. Set \( c_{ij} = \text{coeff}(H_{ij}, Y^{h_i} \U). \)
         3.1.2.5. Set \( \mathcal{P} \) to be the set of coefficients of \( R_0(u_0, \ldots, u_n) = \sum_{i=0}^n \sum_{j=0}^n H_{ij} f_i^{(j)} \) as an algebraic polynomial in \( \mathbb{Y}^{h_i} \U. \)
      3.1.2.6. Solve the linear equation \( \mathcal{P} = 0 \) in variables \( c_0 \) and \( c_{ij}. \)
      3.1.2.7. If \( c_0 \) has a nonzero solution, then substitute it into \( R_0 \) to get \( R \) and go to Step 4., else \( R = 0. \)
   3.1.2.8. \( D = D + 1. \)
3.2. \( o = o + 1. \)
4. Return \( R. \)

/* \textsf{GPol} stands for generic ordinary polynomial. */

/* \( \text{coeff}(P, V) \) returns the set of coefficients of \( P \) as an ordinary polynomial in variables \( V. \) */

\[ N = \binom{(D+L-1)}{L-1} + \sum_{i=0}^n (h_i + 1)(m+1)^{D-m_i-1+L+n(h_i+1)} \] variables and \( M = \binom{(m+1)D+L+n(h_i+1)+1}{L+n(h_i+1)+1} \) equations. To solve it, we need at most \( (\max(M, N))^2 \) \( \mathbb{Q} \)-arithmetic operations over \( \mathbb{Q}. \) In the above inequalities, we assume that \( (m+1)^{n+s+2} + n(s+1) \) use the fact that \( l \geq (n+1)^2, \) where \( l = \sum_{i=0}^n (l_i + 1). \) Our complexity assumes an \( O(1) \)-complexity cost for all field operations over \( \mathbb{Q}. \) Thus, the complexity follows.

Remark 4.6 Algorithm \textsf{SDResultant} can be improved by using a better search strategy. If \( D \) is not big enough, instead of checking \( D + 1, \) we can check \( 2D. \) Repeating this procedure, we may find a \( k \) such that \( k \leq \text{deg}(R) \leq 2k+1. \) We then bisecting the interval \([2^k, 2^{k+1}] \) again to find the proper degree for \( R. \) This will lead to a better complexity, which is still single exponential.
5. CONCLUSION AND PROBLEM
In this paper, the sparse differential resultant is defined and its basic properties are proved. In particular, degree bounds for the sparse differential resultant and the usual differential resultant are given. Based on these degree bounds, we propose a single exponential algorithm to compute the sparse differential resultant.

In the algebraic case, there exists a necessary and sufficient condition for the existence of sparse resultants in terms of the supports [29]. It is interesting to find such a condition for sparse differential resultants.

It is useful to represent the sparse resultant as the quotient of two determinants, as done in [7] in the algebraic case. In the differential case, we do not have such formulas, even in the simplest case of the resultant for two generic differential polynomials in one variable. The treatment in [4] is not complete. For instance, let \( f, g \) be two generic differential polynomials in one variable \( g \) with order one and degree two. Then, the differential resultant for \( f, g \) defined in [4] is zero, because all elements in the first column of the matrix \( M(\delta, n, m) \) in [4, p.543] are zero. Furthermore, it is not easy to fix the problem.

The degree of the algebraic sparse resultant is equal to the mixed volume of certain polytopes generated by the supports of the polynomials [23] or [13, p.255]. A similar degree bound is desirable for the sparse differential resultant.

There exist very efficient algorithms to compute the algebraic sparse resultants ([10, 11]). How to apply the principles behind these algorithms to compute sparse differential resultants is an important problem.

6. REFERENCES