Sparse Differential Resultant*

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ABSTRACT

In this paper, the concept of sparse differential resultant for a differentially essential system of differential polynomials is introduced and its properties are proved. In particular, a degree bound for the sparse differential resultant is given. Based on the degree bound, an algorithm to compute the sparse differential resultant is proposed, which is single exponential in terms of the order, the number of variables, and the size of the differentially essential system.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation - Algorithms for differential equations

General Terms

Algorithms, Theory

Keywords

Sparse differential resultant, differentially essential system, Chow form, degree bound, single exponential algorithm.

1. INTRODUCTION

The resultant, which gives conditions for a system of polynomial equations to have common solutions, is a basic concept in algebraic geometry and a powerful tool in elimination theory [2, 8, 16, 6, 19, 9, 24, 28]. The sparse resultant was introduced by Gelfand, Kapranov, and Zelevinsky as a generalization of the usual resultant [13]. Basic properties for the sparse resultant were given by Sturmfels and co-authors [23, 28, 29]. A Sylvester style matrix based method to compute sparse resultants was first given by Canny and Emiris [3, 10]. A determinant representation for the sparse resultant was given by D'Andrea [7].

The differential resultant for two nonlinear differential polynomials in one variable was studied by Ritt in [25, p.47]. General differential resultants were defined by Carrà Ferro using Macaulay's definition of algebraic resultants [4]. But, the treatment in [4] is not complete. For instance, the differential resultant for two generic differential polynomials with degrees greater than one is always zero if using the definition in [4]. Differential resultants for linear ordinary differential polynomials were studied by Rueda and Sendra in [27]. In [12], a rigorous definition for the differential resultant of $n+1$ generic differential polynomials in n variables was presented.

A generic differential polynomial with order o and degree d contains an exponential number of differential monomials in terms of o and d. Since most of the differential polynomials encountered in practice do not contain all of these monomials, it is useful to define the sparse differential resultant which can be considered as the differential analog for the algebraic sparse resultant [7, 10, 13, 28].

In this paper, the concept of sparse differential resultant for a differentially essential system consisting of $n+1$ differential polynomials in n differential variables is introduced and its properties similar to that of the Sylvester resultant are proved. In particular, we give a degree bound for the sparse differential resultant, which also leads to a degree bound for the differential resultant. Based on the degree bound, we give an algorithm to compute the sparse differential resultant. The complexity of the algorithm in the worst case is single exponential of the form $O(n^{3.376}(s+1)^{O(n)}(m+1)^{O(ns^2l)})$, where s, m, n, and l are the order, the degree, the number of variables, and the size of the differentially essential system respectively. The sparseness is reflected in the quantity l.

In principle, the sparse differential resultant can be computed with any differential elimination method, and in particular with the change of order algorithms given by Boulier-Lemaire-Maza [1] and Golubitsky-Kondratieva-Ovchinnikov [14]. The differentially essential system already forms a triangular set when considering their constant coefficients as leading variables, and the sparse differential resultant is the first element of the characteristic set of the prime ideal generated by the differentially essential system under a different special ranking. Therefore, the change of order strategy proposed in [1, 14] can be used. In our case, due to the special structure of the differentially essential system, we can give specific bounds for the order and degree needed to compute the resultant, which allows us to reduce the problem to linear algebra directly and give explicit complexity bounds.

As preparations for the main results of the paper, we prove

* Partially supported by a National Key Basic Research Project of China (2011CB302400) and by a grant from NSFC (60821002).

ISSAC'11, June 8–11, 2011, San Jose, California, USA.

several properties about the degrees of the elimination ideal and the generalized Chow form in the algebraic case, which are also interesting themselves.

The rest of the paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we define the sparse differential resultant and give its properties. And in Section 4, we present an algorithm to compute the sparse differential resultant. In Section 5, we conclude the paper by proposing several problems for future research.

2. DEGREE OF ELIMINATION IDEAL AND GENERALIZED CHOW FORM

In this section, we will prove several properties about the degrees of elimination ideals and generalized Chow forms in the algebraic case, which will be used later in the paper. These properties are also interesting themselves.

2.1 Degree of elimination ideal

Let P be a polynomial in $K[\mathbb{X}]$ where $\mathbb{X} = \{x_1, \ldots, x_n\}$. We use $\deg(P)$ to denote the total degree of P. Let $\mathcal{I}$ be a prime algebraic ideal in $K[\mathbb{X}]$ with dimension d. We use $\deg(\mathcal{I})$ to denote the *degree* of $\mathcal{I}$, which is defined to be the number of solutions of the zero dimensional prime ideal $(\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_d)$, where $\mathbb{L}_i = u_{i0} + \sum_{j=1}^n u_{ij}x_j$ $(i = 1, \ldots, d)$ are d generic primes [17]. That is,

$$\deg(\mathcal{I}) = |\mathbb{V}(\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_d)|. \tag{1}$$

Clearly, $\deg(\mathcal{I}) = \deg(\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_i)$ for $i = 1, \ldots, d$. $\deg(\mathcal{I})$ is also equal to the maximal number of intersection points of $\mathbb{V}(\mathcal{I})$ with d hyperplanes under the condition that the number of these points is finite [18]. That is,

$$\begin{aligned}\deg(\mathcal{I}) = \max\{|\mathbb{V}(\mathcal{I}) \cap H_1 \cap \cdots \cap H_d| :\ & H_i \text{ are affine} \\ & \text{hyperplanes with } |\mathbb{V}(\mathcal{I}) \cap H_1 \cap \cdots \cap H_d| < \infty\}\end{aligned} \tag{2}$$

The relation between the degree of an ideal and that of its elimination ideal is give by the following result.

Theorem 2.1 *Let $\mathcal{I}$ be a prime ideal in $K[\mathbb{X}]$ and $\mathcal{I}_k = \mathcal{I} \cap K[x_1, \ldots, x_k]$ for any $1 \le k \le n$. Then $\deg(\mathcal{I}_k) \le \deg(\mathcal{I})$.*

Proof: Suppose $\dim(\mathcal{I}) = d$ and $\dim(\mathcal{I}_k) = d_1$. Two cases are considered:

Case (a): $d_1 = d$. Let $\mathbb{P}_i = u_{i0} + u_{i1}x_1 + \cdots + u_{ik}x_k$ $(i = 1, \ldots, d)$. Denote $\mathbf{u} = \{u_{ij} : i = 1, \ldots, d; j = 0, \ldots, k\}$. Then by [17, Theorem 1, p. 54], $\mathcal{J} = (\mathcal{I}_k, \mathbb{P}_1, \ldots, \mathbb{P}_d)$ is a prime ideal of dimension zero in $K(\mathbf{u})[x_1, \ldots, x_k]$ and has the same degree as $\mathcal{I}_k$. We claim that

1) $(\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d) \cap K(\mathbf{u})[x_1, \ldots, x_k] = \mathcal{J}$.

2) $(\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d)$ is a 0-dimensional prime ideal.

To prove 1), it suffices to show that whenever f is in the left ideal, f belongs to $\mathcal{J}$. Without loss of generality, suppose $f \in K[\mathbf{u}][x_1, \ldots, x_k]$. Then there exist $h_l, q_i \in K[\mathbf{u}][\mathbb{X}]$ and $g_l \in \mathcal{I}$ such that $f = \sum_l h_l g_l + \sum_{i=1}^d q_i \mathbb{P}_i$. Substituting $u_{i0} = -\sum_{j=1}^k u_{ik}x_k$ into the above equality, we get $\bar{f} = \sum_l \bar{h}_l g_l \in \mathcal{I}_k$ and $f \equiv \bar{f} \,\mathrm{mod}(\mathbb{P}_1, \ldots, \mathbb{P}_d)$. So, $f \in \mathcal{J}$.

To prove 2), suppose $(\xi_1, \ldots, \xi_n)$ is a generic point of $\mathcal{I}$. Denote $U_0 = \{u_{10}, \ldots, u_{d0}\}$. Then $\mathcal{J}_0 = (\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d) \subseteq K(\mathbf{u}\backslash U_0)[\mathbb{X}, U_0]$ is a prime ideal of dimension d with a generic point $(\xi_1, \ldots, \xi_n, -\sum_{j=1}^k u_{1j}\xi_j, \ \ldots, -\sum_{j=1}^k u_{dj}\xi_j)$. Since $d_1 = d$, there exist d elements in $\{\xi_1, \ldots, \xi_k\}$ algebraically independent over K. So by [16, p.168-169], $\mathcal{J}_0 \cap K(\mathbf{u}\backslash U_0)[U_0] = (0)$ and 2) follows.

Since $\mathcal{J}$ and $(\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d)$ are zero dimensional ideals, by [30, Proposition 9, p.7], $\deg(\mathcal{J}) \le \deg(\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d)$. So by (2), $\deg(\mathcal{I}) \ge |\mathbb{V}(\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d)| \ge \deg(\mathcal{J}) = \deg(\mathcal{I}_k)$.

Case (b): $d_1 < d$. Let $\mathbb{L}_i = u_{i0} + u_{i1}x_1 + \cdots + u_{in}x_n$ $(i = 1, \ldots, d - d_1)$. By [17, Theorem 1, p. 54], $\mathcal{J} = (\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_{d-d_1}) \subseteq K(\mathbf{u})[\mathbb{X}]$ is a prime ideal of dimension d_1 and $\deg(\mathcal{J}) = \deg(\mathcal{I})$, where $\mathbf{u} = \{u_{ij} : i = 1, \ldots, d - d_1; j = 0, \ldots, n\}$. Let $\mathcal{J}_k = \mathcal{J} \cap K(\mathbf{u})[x_1, \ldots, x_k]$. We claim that $\mathcal{J}_k = (\mathcal{I}_k)$ in $K(\mathbf{u})[x_1, \ldots, x_k]$. Of course, $\mathcal{J}_k \supseteq (\mathcal{I}_k)$. Since both $\mathcal{J}_k$ and $(\mathcal{I}_k)$ are prime ideals and $\dim((\mathcal{I}_k)) = d_1$, it suffices to prove that $\dim(\mathcal{J}_k) = d_1$.

Let $\mathcal{J}_0 = (\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_{d-d_1}) \subseteq K(\mathbf{u}\backslash U_0)[\mathbb{X}, U_0]$ with $U_0 = \{u_{10}, \ldots, u_{d-d_1,0}\}$. Suppose $\{x_1, \ldots, x_{d_1}\}$ is a parametric set of $\mathcal{I}_k$. Similarly to the procedure of proving 2) in case (a), we can show that $\mathcal{J}_0 \cap K(\mathbf{u}\backslash U_0)[x_1, \ldots, x_{d_1}, U_0] = (0)$, and $\mathcal{J}_k \cap K(\mathbf{u})[x_1, \ldots, x_{d_1}] = (0)$ follows. So $\dim(\mathcal{J}_k) = d_1$.

Since $\dim(\mathcal{J}_k) = \dim(\mathcal{J})$, by case (a), we have $\deg(\mathcal{J}_k) \le \deg(\mathcal{J}) = \deg(\mathcal{I})$. And due to the fact that $\deg(\mathcal{J}_k) = \deg((\mathcal{I}_k)) = \deg(\mathcal{I}_k)$, $\deg(\mathcal{I}_k) \le \deg(\mathcal{I})$ follows. □

In this article, we will use the following result.

Lemma 2.2 *[22, Proposition 1] Let $F_1, \ldots, F_m \in K[\mathbb{X}]$ be polynomials generating an ideal $\mathcal{I}$ of dimension r. Suppose $\deg(F_1) \ge \cdots \ge \deg(F_m)$ and let $D := \prod_{i=1}^{n-r} \deg(F_i)$. Then $\deg(\mathcal{I}) \le D$.*

2.2 Degree of algebraic generalized Chow form

Let $\mathcal{I}$ be a prime ideal in $K[\mathbb{X}]$ with dimension d,

$$\mathbb{P}_i = u_{i0} + \sum_{1 \le \alpha_1 + \cdots + \alpha_n \le m_i} u_{i,\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ (i = 0, \ldots, d)$$

generic polynomials of degree m_i, and $\mathbf{u}_i$ the vector of coefficients of $\mathbb{P}_i$. Philippon [24] proved that

$$(\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d) \cap K[\mathbf{u}_0, \ldots, \mathbf{u}_d] = (G(\mathbf{u}_0, \ldots, \mathbf{u}_d)) \tag{3}$$

is a prime principal ideal and $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is defined to be the *generalized Chow form* of $\mathcal{I}$, denoted by $G(\mathcal{I})$.

In this section, we will give the degree of the generalized Chow form in terms of the degrees of $\mathcal{I}$ and that of $\mathbb{P}_i$ by proving Theorem 2.4.

At first, we will give another description of the degree for a prime ideal. In (3), when $\mathbb{P}_i$ become generic primes

$$\mathbb{L}_i = v_{i0} + \sum_{j=1}^n v_{ij}x_j (i = 0, 1, \ldots, d),$$

the generalized Chow form becomes the usual *Chow form*, denoted by $\mathrm{Chow}(\mathcal{I})$. That is

$$(\mathcal{I}, \mathbb{L}_0, \ldots, \mathbb{L}_d) \cap K[\mathbf{v}_0, \ldots, \mathbf{v}_d] = (\mathrm{Chow}(\mathcal{I})) \tag{4}$$

where $\mathbf{v}_i$ is the set of coefficients of $\mathbb{L}_i$. A basic property of Chow forms is that [17] for each i between 0 and d,

$$\deg(\mathcal{I}) = \deg_{\mathbf{v}_i} \mathrm{Chow}(\mathcal{I}). \tag{5}$$

In the following lemma, we will give the degree of an ideal intersected by a generic primal. To prove the lemma, we apply the following Bezout inequality (see [15] or [18]): Let V, W be affine algebraic varieties. Then

$$\deg(V \cap W) \le \deg(V) \cdot \deg(W). \tag{6}$$

Lemma 2.3 *Let $\mathcal{I}$ be a prime ideal in $K[\mathbb{X}]$ with $dim(\mathcal{I}) = d > 0$ and P a generic polynomial. Then* $\deg(\mathcal{I}, P) = \deg(P) \cdot \deg(\mathcal{I})$.

Proof: Firstly, we prove the lemma holds for $d = 1$. Let $\mathbf{v}$ be the vector of coefficients of P, $m = \deg(P)$, and $\mathcal{J} = (\mathcal{I}, P) \subset K(\mathbf{v})[\mathbb{X}]$. Then by [17, p. 110], $\mathcal{J}$ is a prime algebraic ideal of dimension zero. Let $\mathbb{L}_0$ be a generic prime with $\mathbf{u}_0$ the vector of coefficients. By (4), $(\mathcal{J}, \mathbb{L}_0) \cap K(\mathbf{v})[\mathbf{u}_0] = (\mathrm{Chow}(\mathcal{J}))$. Here, we choose $\mathrm{Chow}(\mathcal{J})$ to be an irreducible polynomial in $K[\mathbf{v}, \mathbf{u}_0]$. From (5), we have $\deg(\mathcal{J}) = \deg_{\mathbf{u}_0}\mathrm{Chow}(\mathcal{J})$.

Let $\mathcal{M} = (\mathcal{I}, \mathbb{L}_0) \subset K(\mathbf{u}_0)[\mathbb{X}]$. Then $\mathcal{M}$ is a prime ideal of dimension zero with $\deg(\mathcal{M}) = \deg(\mathcal{I})$. And $(\mathcal{M}, P) \cap K(\mathbf{u}_0)[\mathbf{v}] = (G(\mathcal{M}))$ where $G(\mathcal{M}) \in K[\mathbf{v}, \mathbf{u}_0]$ is irreducible. Clearly, $G(\mathcal{M}) = c \cdot \mathrm{Chow}(\mathcal{J})$ for some $c \in K^*$ and $G(\mathcal{M})$ can be factored as

$$G(\mathcal{M}) = A(\mathbf{u}_0) \prod_{\tau=1}^{\deg(\mathcal{I})} P(\xi_\tau),$$

where ξ_τ are all the elements of $\mathbb{V}(\mathcal{M})$ and $A(\mathbf{u}_0)$ is an extraneous factor lying in $K[\mathbf{u}_0]$. Now, specialize P to $\mathbb{L}_1^m$ where $\mathbb{L}_1 = u_{10} + \sum_{i=1}^n u_{1i}x_i$ is a generic prime. Then we have $\overline{G(\mathcal{M})} = A(\mathbf{u}_0)\prod_{\tau=1}^{\deg(\mathcal{I})} \mathbb{L}_1^m(\xi_\tau)$ and $\deg(\overline{G(\mathcal{M})}, \mathbf{u}_0) = \deg(\mathcal{J})$. Since $\mathrm{Chow}(\mathcal{I}) = B(\mathbf{u}_0)\prod_{\tau=1}^{\deg(\mathcal{I})} \mathbb{L}_1(\xi_\tau)$ for some $B \in K[\mathbf{u}_0]$ is irreducible and $\overline{G(\mathcal{M})} \in K[\mathbf{u}_0, \mathbf{u}_1]$, there exists $g \in K[\mathbf{u}_0]^*$ such that $\overline{G(\mathcal{M})} = g \cdot (\mathrm{Chow}(\mathcal{I}))^m$. So, $\deg(\overline{G(\mathcal{M})}, \mathbf{u}_0) \geq m \cdot \deg(\mathrm{Chow}(\mathcal{I}), \mathbf{u}_0) = m \cdot \deg(\mathcal{I})$. And by Bézout inequality (6), $\deg(\mathcal{I}, P) \leq \deg(\mathcal{I}) \cdot \deg(P)$, so $\deg(\mathcal{I}, P) = \deg(\mathcal{I}) \cdot \deg(P)$.

For the case $d > 1$, let $\mathbb{L}_1, \ldots, \mathbb{L}_{d-1}$ be generic primes, then $\mathcal{I}_1 = (\mathcal{I}, \mathbb{L}_1, \ldots, \mathbb{L}_{d-1})$ is a prime ideal of dimension one and $\deg(\mathcal{I}_1) = \deg(\mathcal{I})$. By the case $d = 1$, $\deg(\mathcal{I}_1, P) = \deg(\mathcal{I}_1) \cdot \deg(P)$. So $\deg(\mathcal{I}, P) = \deg(\mathcal{I}, P, \mathbb{L}_1, \ldots, \mathbb{L}_{d-1}) = \deg(\mathcal{I}_1, P) = \deg(\mathcal{I}_1) \cdot \deg(P) = \deg(\mathcal{I}) \cdot \deg(P)$. ☐

The following result generalizes Lemma 1.8 in [24].

Theorem 2.4 *Let $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ be the generalized Chow form of a prime ideal $\mathcal{I}$ of dimension d w.r.t. $\mathbb{P}_0, \ldots, \mathbb{P}_d$. Then G is of degree $\deg(\mathcal{I}) \prod_{j \neq i} \deg(\mathbb{P}_j)$ in each set $\mathbf{u}_i$.*

Proof: It suffices to prove the result for $i = 0$.

If $d = 0$, then $G(\mathbf{u}_0) = \prod_{\tau=1}^{\deg(\mathcal{I})} \mathbb{P}_0(\xi_\tau)$, where $\xi_\tau \in \mathbb{V}(\mathcal{I})$. Clearly, $\deg(G, \mathbf{u}_0) = \deg(\mathcal{I})$.

We consider the case $d > 0$. Let $\mathcal{J}_0 = (\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d) \subset K[\mathbf{u}_1, \ldots, \mathbf{u}_d, \mathbb{X}]$ and $\mathcal{J} = (\mathcal{J}_0) \subset K(\mathbf{u}_1, \ldots, \mathbf{u}_d)[x_1, \ldots, x_n]$. Then $\mathcal{J}$ is a prime ideal of dimension zero and by Lemma 2.3, $\deg(\mathcal{J}) = \deg(\mathcal{I}) \prod_{i=1}^d \deg(\mathbb{P}_i)$. We claim that $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is also the generalized Chow form of $\mathcal{J}$, hence $\deg(G, \mathbf{u}_0) = \deg(\mathcal{J}) = \deg(\mathcal{I}) \prod_{i=1}^d \deg(\mathbb{P}_i)$. Since $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is the generalized Chow form of $\mathcal{I}$, we have $(\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d) \cap K[\mathbf{u}_0, \ldots, \mathbf{u}_d] = (G(\mathbf{u}_0, \ldots, \mathbf{u}_d)) = (\mathcal{J}_0, \mathbb{P}_0) \cap K[\mathbf{u}_0, \ldots, \mathbf{u}_d]$. Let $G_1(\mathbf{u}_0, \ldots, \mathbf{u}_d) \in K[\mathbf{u}_0, \ldots, \mathbf{u}_d]$ be the generalized Chow form of $\mathcal{J}$ and irreducible. Then $(\mathcal{J}, \mathbb{P}_0) \cap K(\mathbf{u}_1, \ldots, \mathbf{u}_d)[\mathbf{u}_0] = (G_1)$. So $G \in (G_1)$. But G, G_1 are irreducible polynomials in $K[\mathbf{u}_0, \ldots, \mathbf{u}_d]$, so $G = c \cdot G_1$ for some $c \in K^*$ and G is the generalized Chow form of $\mathcal{J}$. ☐

3. SPARSE DIFFERENTIAL RESULTANT

In this section, we define the sparse differential resultant and prove its basic properties.

3.1 Definition of sparse differential resultant

Let $\mathcal{F}$ be an ordinary differential field and $\mathcal{F}\{\mathbb{Y}\}$ the ring of differential polynomials in the differential indeterminates $\mathbb{Y} = \{y_1, \ldots, y_n\}$. For any element $e \in \mathcal{F}\{\mathbb{Y}\}$, we use $e^{(k)} = \delta^k e$ to represent the k-th derivative of e and $e^{[k]}$ to denote the set $\{e^{(i)} : i = 0, \ldots, k\}$. Details about differential algebra can be found in [20, 26].

The following theorem presents an important property on differential specialization, which will be used later.

Theorem 3.1 *[12, Theorem 2.14] Let $\{u_1, \ldots, u_r\}$ be a set of differential indeterminates, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$ $(i = 1, \ldots, m)$ differential polynomials in the differential indeterminates $\mathbb{U} = (u_1, \ldots, u_r)$ and $\mathbb{Y} = (y_1, \ldots, y_n)$. Let $\mathbb{Y}^0 = (y_1^0, y_2^0, \ldots, y_n^0)$, where y_i^0 are in some differential extension field of $\mathcal{F}$. If $P_i(\mathbb{U}, \mathbb{Y}^0)$ $(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\mathbb{U}^0$ in $\mathcal{F}$, $P_i(\mathbb{U}^0, \mathbb{Y}^0)$ $(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}$.*

To define the sparse differential resultant, consider $n + 1$ differential polynomials with differential indeterminates as coefficients

$$\mathbb{P}_i = u_{i0} + \sum_{k=1}^{l_i} u_{ik} M_{ik} \; (i = 0, \ldots, n) \tag{7}$$

where $M_{ik} = (\mathbb{Y}^{[s_i]})^{\alpha_{ik}}$ is a monomial in $\{y_1, \ldots, y_n, \ldots, y_1^{(s_i)}, \ldots, y_n^{(s_i)}\}$ with exponent vector α_{ik} and $|\alpha_{ik}| \geq 1$. The set of exponent vectors $\mathbb{S}_i = \{\bar{0}, \alpha_{ik} : k = 1, \ldots, l_i\}$ is called the *support* of $\mathbb{P}_i$, where $\bar{0}$ is the exponent vector for the constant term. The number $|\mathbb{S}_i| = l_i + 1$ is called the *size* of $\mathbb{P}_i$. Note that s_i is the order of $\mathbb{P}_i$ and an exponent vector of $\mathbb{P}_i$ contains $n(s_i + 1)$ elements.

Denote $\mathbf{u} = \{u_{ik} : i = 0, \ldots, n; k = 1, \ldots, l_i\}$. Let $\eta_1, \ldots, \eta_n$ be n elements which are differentially independent over $\mathbb{Q}\langle\mathbf{u}\rangle$ and denote $\eta = (\eta_1, \ldots, \eta_n)$, where $\mathbb{Q}$ is the field of rational numbers. Let

$$\zeta_i = -\sum_{k=1}^{l_i} u_{ik} (\eta^{[s_i]})^{\alpha_{ik}} \; (i = 0, \ldots, n). \tag{8}$$

Denote the differential transcendence degree by d.tr.deg. Then, we have

Lemma 3.2 d.tr.deg $\mathbb{Q}\langle\mathbf{u}\rangle\langle\zeta_0, \ldots, \zeta_n\rangle/\mathbb{Q}\langle\mathbf{u}\rangle = n$ *if and only if there exist n monomials $M_{r_i k_i}$ $(i = 1, \ldots, n)$ in (7) such that $r_i \neq r_j$ for $i \neq j$ and $M_{r_i k_i}(\eta) = (\eta^{[s_{r_i}]})^{\alpha_{r_i k_i}}$ are differentially independent over $\mathbb{Q}\langle\mathbf{u}\rangle$.*

Proof: " $\Leftarrow$ " Without loss of generality, we assume $r_i = i \, (i = 1, \ldots, n)$ and $M_{ik_i}(\eta) \, (i = 1, \ldots, n)$ are differentially independent. It suffices to prove that $\zeta_1, \ldots, \zeta_n$ are differentially independent over $\mathbb{Q}\langle\mathbf{u}\rangle$. Suppose the contrary, i.e. $\zeta_1, \ldots, \zeta_n$ are differentially dependent. Now specialize u_{ij} to $-\delta_{ik_i}$. By Theorem 3.1 and (8), $M_{ik_i}(\eta) \, (i = 1, \ldots, n)$ are differentially dependent, which is a contradiction.

" $\Rightarrow$ " Suppose the contrary, i.e., $M_{r_i k_i}(\eta) \, (i = 1, \ldots, n)$ are differentially dependent for any n different r_i and $k_i = 1, \ldots, l_{r_i}$. Since each ζ_{r_i} is a linear combination of $M_{r_i k_i}(\eta)$ $(k_i = 1, \ldots, l_{r_i})$, $\zeta_{r_1}, \ldots, \zeta_{r_n}$ are differentially dependent, contradicting that d.tr.deg $\mathbb{Q}\langle\mathbf{u}\rangle\langle\zeta_0, \ldots, \zeta_n\rangle/\mathbb{Q}\langle\mathbf{u}\rangle = n$. ☐

Definition 3.3 *A set of differential polynomials of form (7) satisfying the condition in Lemma 3.2 is called a* differentially essential system.

A differential polynomial f of form (7) is called *quasi-generic* [12] if for each $1 \leq i \leq n$, f contains at least one monomial in $\mathcal{F}\{y_i\} \setminus \mathcal{F}$. Clearly, $n+1$ quasi-generic differential polynomials form a differentially essential system.

Now let $[\mathbb{P}_0, \ldots, \mathbb{P}_n]$ be the differential ideal generated by $\mathbb{P}_i$ in $\mathbb{Q}\langle\mathbf{u}\rangle\{\mathbb{Y}, u_{00}, \ldots, u_{n0}\}$. Then it is a prime differential ideal with a generic point $(\eta_1, \ldots, \eta_n, \zeta_0, \ldots, \zeta_n)$ and of dimension n. Clearly, $\mathcal{I} = [\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbb{Q}\langle\mathbf{u}\rangle\{u_{00}, \ldots, u_{n0}\}$ is a prime differential ideal with a generic point $(\zeta_0, \ldots, \zeta_n)$. As a consequence of Lemma 3.2, we have

Corollary 3.4 *$\mathcal{I}$ is of codimension one if and only if $\{\mathbb{P}_0, \ldots, \mathbb{P}_n\}$ is a differentially essential system.*

Now suppose $\{\mathbb{P}_0, \ldots, \mathbb{P}_n\}$ is a differentially essential system. Since $\mathcal{I}$ is of codimension one, then by [26, line 14, p. 45], there exists an irreducible differential polynomial $R(\mathbf{u}; u_{00}, \ldots, u_{n0}) \in \mathbb{Q}\langle\mathbf{u}\rangle\{u_{00}, \ldots, u_{n0}\}$ such that

$$[\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbb{Q}\langle\mathbf{u}\rangle\{u_{00}, \ldots, u_{n0}\} = \mathrm{sat}(R) \tag{9}$$

where $\mathrm{sat}(R)$ is the saturation ideal of R. More explicitly, $\mathrm{sat}(R)$ is the whole set of differential polynomials having zero pseudo-remainders w.r.t. R under any ranking endowed on $u_{00}, \ldots, u_{n0}$. And by clearing denominators when necessary, we suppose $R \in \mathbb{Q}\{\mathbf{u}; u_{00}, \ldots, u_{n0}\}$ is irreducible and also denoted by $R(\mathbf{u}; u_{00}, \ldots, u_{n0})$. Let $\mathbf{u}_i = (u_{i0}, u_{i1}, \ldots, u_{il_i})$ be the vector of coefficients of $\mathbb{P}_i$ and denote $R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = R(\mathbf{u}; u_{00}, \ldots, u_{n0})$. Now we give the definition of sparse differential resultant as follows:

Definition 3.5 *$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) \in \mathbb{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ in (9) is defined to be the* sparse differential resultant *of the differentially essential system $\mathbb{P}_0, \ldots, \mathbb{P}_n$.*

Example 3.6 *For $n = 2$, let $\mathbb{P}_0 = u_{00}+u_{01}y_1y_2$, $\mathbb{P}_1 = u_{10}+u_{01}y_1'y_2'$, and $\mathbb{P}_2 = u_{20}+u_{21}y_1'y_2$. Using differential elimination algorithms [5], we can show that $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ form a differentially essential system and their sparse differential resultant is $R = -u_{11}u_{20}^2u_{01}^2 - u_{01}u_{00}u_{21}^2u_{10} + u_{01}u_{11}u_{20}u_{21}u_{00}' - u_{11}u_{20}u_{00}u_{21}u_{01}'$.*

The following properties can be proved easily.

1. When all $\mathbb{P}_i$ become generic differential polynomials of the form $\mathbb{P}_i = u_{i0} + \sum\limits_{1\leq|\alpha|\leq m_i} u_{i,\alpha}(\mathbb{Y}^{[s_i]})^\alpha$, the sparse differential resultant is the differential resultant defined in [12].

2. R is the vanishing polynomial of $(\zeta_0, \ldots, \zeta_n)$ with minimal order in each u_{i0}. Since $R \in \mathbb{Q}\{\mathbf{u}; u_{00}, \ldots, u_{n0}\}$ is irreducible, $\mathrm{ord}(R, \mathbf{u}_i) = \mathrm{ord}(R, u_{i0})$.

3. Suppose $\mathrm{ord}(R, \mathbf{u}_i) = h_i \geq 0$ and denote $o = \sum_{i=0}^n h_i$. Given a vector $(q_0, \ldots, q_n) \in \mathbb{N}^{n+1}$ with $\sum_{i=0}^n q_i = q$, if $q < o$, then there is no polynomial P in $\mathrm{sat}(R)$ with $\mathrm{ord}(P, \mathbf{u}_i) = q_i$. And R is the unique irreducible polynomial in $\mathrm{sat}(R)$ with total order $q = o$ up to some $a \in \mathbb{Q}$. This property will be used in our algorithm to search for the sparse differential resultant.

Remark 3.7 *It is not easy to define the sparse differential resultant as the algebraic sparse resultant of $\mathbb{P}_i^{(k)}$ considered as polynomials in $y_i^{(j)}$. The reason is that it is difficult to check whether the supports of $\mathbb{P}_i$ and $\mathbb{P}_i^{(k)}$ satisfy the conditions for the existence of the algebraic sparse resultant [29]. Furthermore, the coefficients of $\mathbb{P}_i^{(k)}$ are not generic.*

3.2 Properties of sparse differential resultant

Following Kolchin [21], we introduce the concept of differentially homogenous polynomials.

Definition 3.8 *A differential polynomial $p \in \mathcal{F}\{y_0, \ldots, y_n\}$ is called differentially homogenous of degree m if for a new differential indeterminate λ, we have $p(\lambda y_0, \lambda y_1 \ldots, \lambda y_n) = \lambda^m p(y_0, y_1, \ldots, y_n)$.*

The differential analog of Euler's theorem related to homogenous polynomials is valid.

Theorem 3.9 *[21] $f \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is differentially homogenous of degree m if and only if*

$$\sum_{j=0}^n \sum_{k\in\mathbb{N}} \binom{k+r}{r} y_j^{(k)} \frac{\partial f(y_0, \ldots, y_n)}{\partial y_j^{(k+r)}} = \begin{cases} mf & r = 0 \\ 0 & r \neq 0 \end{cases}$$

Sparse differential resultants have the following property.

Theorem 3.10 *The sparse differential resultant is differentially homogenous in each $\mathbf{u}_i$ which is the coefficient set of $\mathbb{P}_i$.*

Proof: Similar to the proof of [12, Theorem 4.16], we can show that R satisfies the conditions of Theorem 3.9 for each $\mathbf{u}_i$. The proof is omitted due to the page limit. □

Continue from Example 3.6. In this example, R is differentially homogenous of degree 2 in $\mathbf{u}_0$, of degree 1 in $\mathbf{u}_1$ and of degree 2 in $\mathbf{u}_2$ respectively.

In the following, we prove formulas for sparse differential resultants, which are similar to the Poisson type formulas for Chow forms and algebraic resultants [23]. Denote $\mathrm{ord}(R, \mathbf{u}_i)$ by h_i $(i = 0, \ldots, n)$. We have the following theorem.

Theorem 3.11 *Let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the sparse differential resultant of $\mathbb{P}_0, \ldots, \mathbb{P}_n$. Let $\deg(R, u_{00}^{(h_0)}) = t_0$. Then there exist $\xi_{\tau k}$ for $\tau = 1, \ldots, t_0$ and $k = 1, \ldots, l_0$ such that*

$$R = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k}\xi_{\tau k})^{(h_0)}, \tag{10}$$

where A is a polynomial in $\mathcal{F}[\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]} \backslash u_{00}^{(h_0)}]$.

Proof: Now consider R as a polynomial in $u_{00}^{(h_0)}$ with coefficients in $\mathbb{Q}_0 = \mathbb{Q}(\cup_{l=0}^n \mathbf{u}_l^{[h_l]} \setminus \{u_{00}^{(h_0)}\})$. Then, in an algebraic extension field of $\mathbb{Q}_0$, we have

$$R = A \prod_{\tau=1}^{t_0} (u_{00}^{(h_0)} - z_\tau)$$

where $t_0 = \deg(R, u_{00}^{(h_0)})$. Note that z_τ is an algebraic root of $R(u_{00}^{(h_0)}) = 0$ and a derivative for z_τ can be naturally defined to make $\mathcal{F}\langle z_\tau\rangle$ a differential field. From $R(\mathbf{u};\zeta_0,\ldots,\zeta_n)=0$, if we differentiate this equality w.r.t. $u_{0k}^{(h_0)}$, then we have

$$\overline{\frac{\partial R}{\partial u_{0k}^{(h_0)}}}+\frac{\partial R}{\partial \zeta_0^{(h_0)}}(-\eta^{[s_0]})^{\alpha_{0k}}=0 \tag{11}$$

where $\overline{\frac{\partial R}{\partial u_{0k}^{(h_0)}}}$ and $\frac{\partial R}{\partial \zeta_0^{(h_0)}}$ are obtained by substituting u_{i0} by ζ_i in $\frac{\partial R}{\partial u_{0k}^{(h_0)}}$ and $\frac{\partial R}{\partial u_{00}^{(h_0)}}$ respectively.

Now multiply equation (11) by u_{0k} and for k from 1 to l_0 add all of the equations obtained together, then we get

$$\frac{\partial R}{\partial \zeta_0^{(h_0)}}\zeta_0+\sum_{k=1}^{l_0}u_{0k}\overline{\frac{\partial R}{\partial u_{0k}^{(h_0)}}}=0 \tag{12}$$

Thus, the polynomial $G_1=u_{00}\frac{\partial R}{\partial u_{00}^{(h_0)}}+\sum_{k=1}^{l_0}u_{0k}\frac{\partial R}{\partial u_{0k}^{(h_0)}}$ vanishes at $(u_{00},\ldots,u_{n0})=(\zeta_0,\ldots,\zeta_n)$. Since $\mathrm{ord}(G_1)\le\mathrm{ord}(R)$ and $\deg(G_1)=\deg(R)$, there exists some $a\in\mathcal{F}$ such that $G_1=aR$. Setting $u_{00}^{(h_0)}=z_\tau$ in both sides of G_1, we have $u_{00}R_{\tau0}+\sum_{k=1}^{l_0}u_{0k}R_{\tau k}=0$, where $R_{\tau k}=\frac{\partial R}{\partial u_{0k}^{(h_0)}}\big|_{u_{00}^{(h_0)}=z_\tau}$. Since R is irreducible as an algebraic polynomial in $u_{00}^{(h_0)}$, $R_{\tau0}\neq0$. Denote $\xi_{\tau k}=R_{\tau k}/R_{\tau0}$. Thus, $u_{00}+\sum_{k=1}^{l_0}u_{0k}\xi_{\tau k}=0$ under the condition $u_{00}^{(h_0)}=z_\tau$. Consequently, $z_\tau=-(\sum\limits_{k=1}^{l_0}u_{0k}\xi_{\tau k})^{(h_0)}$ and (10) follows. □

If $\mathbb{P}_0$ contains the linear terms $y_i\,(i=1,\ldots,n)$, then the above result can be strengthened as follows.

Theorem 3.12 *Suppose $\mathbb{P}_0$ has the form*

$$\mathbb{P}_0=u_{00}+\sum_{i=1}^n u_{0i}y_i+\sum_{i=n+1}^{l_0}u_{0i}(\mathbb{Y}^{[s_0]})^{\alpha_{0i}}. \tag{13}$$

Then there exist $\xi_{\tau k}$ $(\tau=1,\ldots,t_0;k=1,\ldots,n)$ such that

$$\begin{aligned}R&=A\prod_{\tau=1}^{t_0}\Big(u_{00}+\sum_{i=1}^n u_{0i}\xi_{\tau i}+\sum_{i=n+1}^{l_0}u_{0i}\big(\xi_\tau^{[s_0]}\big)^{\alpha_{0i}}\Big)^{(h_0)}\\&=A\prod_{\tau=1}^{t_0}\mathbb{P}_0(\xi_\tau)^{(h_0)},\quad \textit{where } \xi_\tau=(\xi_{\tau1},\ldots,\xi_{\tau n}).\end{aligned}$$

Moreover, $\xi_\tau\,(\tau=1,\ldots,t_0)$ lies on $\mathbb{P}_1,\ldots,\mathbb{P}_n$.

Proof: For the first part, from Theorem 3.11, it remains to show that for $i=n+1$ to l_0, $\xi_{\tau i}=\big(\xi_\tau^{[s_0]}\big)^{\alpha_{0i}}$. From equation (11), we have $\eta_j=\overline{\frac{\partial R}{\partial u_{0j}^{(h_0)}}}\Big/\frac{\partial R}{\partial \zeta_0^{(h_0)}}$ and $(\eta^{[s_0]})^{\alpha_{0i}}=\overline{\frac{\partial R}{\partial u_{0i}^{(h_0)}}}\Big/\frac{\partial R}{\partial \zeta_0^{(h_0)}}$. If $(\mathbb{Y}^{[s_0]})^{\alpha_{0i}}=\prod_{j=1}^n\prod_{k=0}^{s_0}\big(y_j^{(k)}\big)^{(\alpha_{0i})_{jk}}$, then

$$\prod_{j=1}^n\prod_{k=0}^{s_0}\Big(\Big(\overline{\frac{\partial R}{\partial u_{0j}^{(h_0)}}}\Big/\frac{\partial R}{\partial \zeta_0^{(h_0)}}\Big)^{(k)}\Big)^{(\alpha_{0i})_{jk}}=\overline{\frac{\partial R}{\partial u_{0i}^{(h_0)}}}\Big/\frac{\partial R}{\partial \zeta_0^{(h_0)}}.$$

It follows that

$$\prod_{j=1}^n\prod_{k=0}^{s_0}\Big(\Big(\frac{\partial R}{\partial u_{0j}^{(h_0)}}\Big/\frac{\partial R}{\partial u_{00}^{(h_0)}}\Big)^{(k)}\Big)^{(\alpha_{0i})_{jk}}-\frac{\partial R}{\partial u_{0i}^{(h_0)}}\Big/\frac{\partial R}{\partial u_{00}^{(h_0)}}$$

vanishes at $(u_{00},\ldots,u_{n0})=(\zeta_0,\ldots,\zeta_n)$. Since there exists some $a\in\mathbb{N}$, such that $G_i=$

$$\Big(\frac{\partial R}{\partial u_{00}^{(h_0)}}\Big)^a\Big(\prod_{j=1}^n\prod_{k=0}^{s_0}\Big(\Big(\frac{\partial R}{\partial u_{0j}^{(h_0)}}\Big/\frac{\partial R}{\partial u_{00}^{(h_0)}}\Big)^{(k)}\Big)^{(\alpha_{0i})_{jk}}-\frac{\partial R}{\partial u_{0i}^{(h_0)}}\Big/\frac{\partial R}{\partial u_{00}^{(h_0)}}\Big)$$

is a polynomial in $\mathbb{Q}\{\mathbf{u}_0,\ldots,\mathbf{u}_n\}$, $G_i\in\mathrm{sat}(R)$. Now substituting $u_{00}^{(h_0+h)}=z_\tau^{(h)}$ for $h\ge0$ into G_i, we obtain that $\xi_{\tau i}=\prod_{j=1}^n\prod_{k=0}^{s_0}\big((\xi_{\tau j})^{(k)}\big)^{(\alpha_{0i})_{jk}}=(\xi_\tau^{[s_0]})^{\alpha_{0i}}$.

The proof of the second assertion is based on generalized differential Chow form introduced in [12] and is omitted. □

As in algebra, the sparse differential resultant gives a necessary condition for a system of differential polynomials to have common solutions, as shown by the following theorem.

Theorem 3.13 *Let $\mathbb{P}_0,\ldots,\mathbb{P}_n$ be a differentially essential system of the form (7) and $R(\mathbf{u}_0,\ldots,\mathbf{u}_n)$ be their sparse differential resultant. Denote $ord(R,\mathbf{u}_i)=h_i$ and $S_R=\frac{\partial R}{\partial u_{00}^{(h_0)}}$. Suppose that when $\mathbf{u}_i\,(i=0,\ldots,n)$ are specialized to sets $\mathbf{v}_i$ which are elements in an extension field of $\mathcal{F}$, $\mathbb{P}_i$ are specialized to $\overline{\mathbb{P}}_i\,(i=0,\ldots,n)$. If $\overline{\mathbb{P}}_i=0(i=0,\ldots,n)$ have a common solution, then $R(\mathbf{v}_0,\ldots,\mathbf{v}_n)=0$. Moreover, if $S_R(\mathbf{v}_0,\ldots,\mathbf{v}_n)\neq0$, in the case that $\overline{\mathbb{P}}_i=0(i=0,\ldots,n)$ have a common solution ξ, then for each k, we have*

$$\big((\xi)^{[s_0]}\big)^{\alpha_{0k}}=\frac{\partial R}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0,\ldots,\mathbf{v}_n)/S_R(\mathbf{v}_0,\ldots,\mathbf{v}_n). \tag{14}$$

Proof: Since $R(\mathbf{u}_0,\ldots,\mathbf{u}_n)\in[\mathbb{P}_0,\ldots,\mathbb{P}_n]$, $R(\mathbf{v}_0,\ldots,\mathbf{v}_n)\in[\overline{\mathbb{P}}_0,\ldots,\overline{\mathbb{P}}_n]$. So if $\overline{\mathbb{P}}_i=0\,(i=0,\ldots,n)$ have a common solution, then $R(\mathbf{v}_0,\ldots,\mathbf{v}_n)$ should be zero.

From equation (11), it is clear that the polynomial $\frac{\partial R}{\partial u_{0k}^{(h_0)}}+S_R\cdot(-\mathbb{Y}^{[s_0]})^{\alpha_{0k}}\in[\mathbb{P}_0,\ldots,\mathbb{P}_n]$. Thus, if ξ is a common solution of $\overline{\mathbb{P}}_i=0$, then the polynomial $\frac{\partial R}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0,\ldots,\mathbf{v}_n)+S_R(\mathbf{v}_0,\ldots,\mathbf{v}_n)\cdot(-\mathbb{Y}^{[s_0]})^{\alpha_{0k}}$ vanishes at ξ. So (14) follows. □

Again, if $\mathbb{P}_0$ contains the linear terms $y_i\,(i=1,\ldots,n)$, then the above result can be strengthened as follows.

Corollary 3.14 *Suppose $\mathbb{P}_0$ has the form (13). If $R(\mathbf{v}_0,\ldots,\mathbf{v}_n)=0$ and $S_R(\mathbf{v}_0,\ldots,\mathbf{v}_n)\neq0$, then $\overline{\mathbb{P}}_i=0$ have a common solution.*

Proof: From the proof of the above theorem, we know that for k from 1 to n,

$$A_k=\frac{\partial R}{\partial u_{0k}^{(h_0)}}+\frac{\partial R}{\partial u_{00}^{(h_0)}}(-y_k)\in[\mathbb{P}_0,\ldots,\mathbb{P}_n].$$

Clearly, A_k is linear in y_k. Suppose the differential remainder of $\mathbb{P}_i$ w.r.t. $A_1,\ldots,A_n$ in order to eliminate $y_1,\ldots,y_n$ is g_i, then $S_R^a\mathbb{P}_i\equiv g_i,\mathrm{mod}\,[A_1,\ldots,A_n]$ for $a\in\mathbb{N}$. Thus, $g_i\in[\mathbb{P}_0,\ldots,\mathbb{P}_n]\cap\mathbb{Q}\langle\mathbf{u}\rangle\{u_{00},\ldots,u_{n0}\}=\mathrm{sat}(R)$. So we have $S_R^b\mathbb{P}_i\equiv0\,\mathrm{mod}\,[A_1,\ldots,A_n,R]$ for some $b\in\mathbb{N}$. Now specialize $\mathbf{u}_i$ to $\mathbf{v}_i$ for $i=0,\ldots,n$, then we have

$$S_R^b(\mathbf{v}_0,\ldots,\mathbf{v}_n)\cdot\overline{\mathbb{P}}_i\equiv0\,\mathrm{mod}\,[\overline{A}_1,\ldots,\overline{A}_n]. \tag{15}$$

Let $\xi_k=\frac{\partial R}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0,\ldots,\mathbf{v}_n)/S_R(\mathbf{v}_0,\ldots,\mathbf{v}_n)\,(k=1,\ldots,n)$, and denote $\xi=(\xi_1,\ldots,\xi_n)$. Then from equation (15), $\overline{\mathbb{P}}_i(\xi)=0$. So, ξ is a common solution of $\overline{\mathbb{P}}_0,\ldots,\overline{\mathbb{P}}_n$. □

4. AN ALGORITHM TO COMPUTE SPARSE DIFFERENTIAL RESULTANT

In this section, we give an algorithm to compute the sparse differential resultant with single exponential complexity.

4.1 Degree bounds for sparse differential resultants

In this section, we give an upper bound for the degree and order of the sparse differential resultant, which will be crucial to our algorithm to compute the sparse resultant.

Theorem 4.1 *Let $\mathbb{P}_0, \ldots, \mathbb{P}_n$ be a differentially essential system of form (7) with ord$(\mathbb{P}_i) = s_i$ and $\deg(\mathbb{P}_i, \mathbb{Y}) = m_i$. Let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the sparse differential resultant of $\mathbb{P}_i$ $(i = 0, \ldots, n)$. Suppose ord$(R, \mathbf{u}_i) = h_i$ for each i. We have*

1) *$h_i \le s - s_i$ for $i = 0, \ldots, n$ where $s = \sum_{i=0}^n s_i$.*

2) *R can be written as a linear combination of $\mathbb{P}_i$ and their derivatives up to order h_i. Precisely,*

$$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{k=0}^{h_i} G_{ik} \mathbb{P}_i^{(k)} \tag{16}$$

for some $G_{ik} \in \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]}, \mathbb{Y}^{[h]}]$ where $h = max_i\{h_i + s_i\}$.

3) *$\deg(R) \le \prod_{i=0}^n (m_i + 1)^{h_i+1} \le (m+1)^{ns+n+1}$, where $m = max_i\{m_i\}$.*

Proof: 1) Let $\theta_i = -\sum_{1 \le |\alpha| \le m_i} u_{i\alpha}(\eta^{[s_i]})^\alpha$ $(i = 0, \ldots, n)$ where $\eta = (\eta_1, \ldots, \eta_n)$ is the generic point of the zero differential ideal [0], and $\mathbb{W}_i = u_{i0} + \sum_{1 \le |\alpha| \le m_i} u_{i\alpha}(\mathbb{Y}^{[s_i]})^\alpha$ is a generic polynomial of order s_i and degree m_i. Then from the property of differential resultants ([12, Theorem 1.3.]), we know the minimal polynomial of $(\theta_0, \ldots, \theta_n)$ is of order $s - s_i$ in each u_{i0}. Now specialize all the $u_{i\alpha}$ such that θ_i are specialized to the corresponding ζ_i. By the procedures in the proof of Theorem 3.1, we can obtain a nonzero differential polynomial vanishing at $(\zeta_0, \ldots, \zeta_n)$ with order not greater than $s - s_i$ in each variable u_{i0}. Since R is the minimal polynomial of $(\zeta_0, \ldots, \zeta_n)$, $\mathrm{ord}(R, \mathbf{u}_i) = \mathrm{ord}(R, u_{i0}) \le s - s_i$.

2) Substituting u_{i0} by $\mathbb{P}_i - \sum_{k=1}^{l_i} u_{ik}(\mathbb{Y}^{[s_i]})^{\alpha_{ik}}$ in the polynomial $R(\mathbf{u}; u_{00}, \ldots, u_{n0})$ for $i = 0, \ldots, n$, we get

$$\begin{aligned} &R(\mathbf{u}; u_{00}, \ldots, u_{n0}) \\ &= R(\mathbf{u}; \mathbb{P}_0 - \sum_{k=1}^{l_0} u_{0k}(\mathbb{Y}^{[s_0]})^{\alpha_{0k}}, \ldots, \mathbb{P}_n - \sum_{k=1}^{l_n} u_{nk}(\mathbb{Y}^{[s_n]})^{\alpha_{nk}}) \\ &= \textstyle\sum_{i=0}^n \sum_{k=0}^{h_i} G_{ik}\mathbb{P}_i^{(k)} + T(\mathbf{u}, \mathbb{Y}) \end{aligned}$$

for $G_{ik} \in \mathbb{Q}\{\cup_{i=0}^n \mathbf{u}_i, \mathbb{Y}\}$ and $T = R(\mathbf{u}; -\sum_{k=1}^{l_0} u_{0k}(\mathbb{Y}^{[s_0]})^{\alpha_{0k}}, \ldots, -\sum_{k=1}^{l_n} u_{nk}(\mathbb{Y}^{[s_n]})^{\alpha_{nk}}) \in [\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbb{Q}\langle\mathbf{u}\rangle\{\mathbb{Y}\}$. Since $[\mathbb{P}_0, \ldots, \mathbb{P}_n] \cap \mathbb{Q}\langle\mathbf{u}\rangle\{\mathbb{Y}\} = [0]$, $T = 0$ and 2) is proved. Moreover, $(\mathbb{P}_0^{[h_0]}, \ldots, \mathbb{P}_n^{[h_n]}) \cap \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]}] = (R(\mathbf{u}_0, \ldots, \mathbf{u}_n))$.

3) Let $\mathcal{J}_0 = (\mathbb{P}_0^{[h_0]}, \ldots, \mathbb{P}_n^{[h_n]}) \subset \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]}, \widetilde{\mathbb{Y}}]$ where $\widetilde{\mathbb{Y}}$ are the y_i and their derivatives appearing in $\mathbb{P}_0^{[h_0]}, \ldots, \mathbb{P}_n^{[h_n]}$. By Lemma 2.2, $\deg(\mathcal{J}_0) \le \prod_{i=0}^n \prod_{j=0}^{h_i} \deg(\mathbb{P}_i, \mathbb{Y} \cup \mathbf{u}_i) = \prod_{i=0}^n (m_i + 1)^{h_i+1}$ and $(R) = \mathcal{J}_0 \cap \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]}]$ is the elimination ideal of $\mathcal{J}_0$. Thus, by Theorem 2.1,

$$\deg(R) \le \deg(\mathcal{J}_0) \le \prod_{i=0}^n (m_i + 1)^{h_i+1}. \tag{17}$$

Together with 1), 3) is proved. □

The following theorem gives an upper bound for degrees of differential resultants, the proof of which is not valid for sparse differential resultants. In the following result, when we estimate the degree of R, only the degrees of $\mathbb{P}_i$ in $\mathbb{Y}$ are considered, while in Theorem 4.1, the degrees of $\mathbb{P}_i$ in both $\mathbb{Y}$ and u_{ik} are considered.

Theorem 4.2 *Let F_i $(i = 0, \ldots, n)$ be generic differential polynomials in $\mathbb{Y} = \{y_1, \ldots, y_n\}$ with order s_i, degree $m_i = \deg(\mathbb{P}_i, \mathbb{Y})$, and $s = \sum_{i=0}^n s_i$. Let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the differential resultant of $F_0, \ldots, F_n$. Then we have $\deg(R, \mathbf{u}_k) \le \frac{s-s_k+1}{m_k} \prod_{i=0}^n m_i^{s-s_i+1}$ for each $k = 0, \ldots, n$.*

Proof: Without loss of generality, we consider $k = 0$.

By [12, Theorem 6.8], $\mathrm{ord}(R, \mathbf{u}_i) = s - s_i$ for each i and $R \in (F_0^{[s-s_0]}, \ldots, F_n^{[s-s_n]}) \subset \mathbb{Q}[\mathbb{Y}^{[s]}, \mathbf{u}_0^{[s-s_0]}, \ldots, \mathbf{u}_n^{[s-s_n]}]$. Let $\mathcal{I}^a = (F_1^{[s-s_1]}, \ldots, F_n^{[s-s_n]}) \subset \mathbb{Q}(\widetilde{\mathbf{u}})[\mathbb{Y}^{[s]}]$, where $\widetilde{\mathbf{u}} = \cup_{i=1}^n \mathbf{u}_i^{[s-s_i]}$. Clearly, $\mathcal{I}^a$ is a prime ideal of dimension $s - s_0$.

Let $\mathbb{P}_0, \ldots, \mathbb{P}_{s-s_0}$ be independent generic polynomials of degree m_0 in $\mathbb{Y}^{[s]}$ with $\mathbf{v}_i$ coefficients of $\mathbb{P}_i$. Denote $\widetilde{\mathbf{v}} = \cup_{i=0}^{s-s_0} \mathbf{v}_i \backslash \{v_{i0}\}$ where v_{i0} is the constant term of $\mathbb{P}_i$.

Suppose η is a generic point of $\mathcal{I}^a$. Let $\zeta_i = -\mathbb{P}_i(\eta) + v_{i0}$ and $\overline{\zeta_i} = -F_0^{(i)}(\eta) + u_{00}^{(i)}$ $(i = 0, \ldots, s - s_0)$. Clearly, ζ_i and $\overline{\zeta_i}$ are free of v_{i0} and $u_{00}^{(i)}$ respectively. Let $G(\mathbf{v}_0, \ldots, \mathbf{v}_{s-s_0}) = G(\widetilde{\mathbf{v}}; v_{00}, \ldots, v_{s-s_0,0}) \in \mathbb{Q}[\widetilde{\mathbf{u}}; \mathbf{v}_0, \ldots, \mathbf{v}_{s-s_0}]$ be the generalized Chow form of $\mathcal{I}^a$. Then $G(\widetilde{\mathbf{v}}; v_{00}, \ldots, v_{s-s_0,0})$ is the vanishing polynomial of $(\zeta_0, \ldots, \zeta_{s-s_0})$ over $\mathbb{Q}(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}})$. Now specialize $\mathbf{v}_i$ to the corresponding coefficients of $F_0^{(i)}$. Then ζ_i are specialized to $\overline{\zeta_i}$. By [16, p.168-169], there exists a nonzero polynomial $H(\mathbf{u}_0^{[s-s_0]} \backslash u_{00}^{[s-s_0]}; u_{00}, \ldots, u_{00}^{(s-s_0)}) \in \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \ldots, \mathbf{u}_n^{[s-s_n]}]$ such that

1) $H(\mathbf{u}_0^{[s-s_0]} \backslash u_{00}^{[s-s_0]}; \overline{\zeta_0}, \ldots, \overline{\zeta_{s-s_0}}) = 0$ and

2) $\deg(H) \le \deg(G)$.

So $H \in (F_0^{[s-s_0]}, \ldots, F_n^{[s-s_n]}) \cap \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \ldots, \mathbf{u}_n^{[s-s_n]}] = (R)$. Thus, $\deg(R, \mathbf{u}_0^{[s-s_0]}) \le \deg(H, \mathbf{u}_0^{[s-s_0]}) \le \deg(G(\mathbf{v}_0, \ldots, \mathbf{v}_{s-s_0}))$. By Theorem 2.4, $\deg(G, \mathbf{v}_i) = \deg(\mathcal{I}^a) m_0^{s-s_0}$ for each i. Since $\mathcal{I}^a$ is generated by $(F_1^{[s-s_1]}, \ldots, F_n^{[s-s_n]})$ in $\mathbb{Q}(\widetilde{\mathbf{u}})[\mathbb{Y}^{[s]}]$, $\deg(\mathcal{I}^a) \le \prod_{i=1}^n m_i^{s-s_i+1}$ by Lemma 2.2. So, $\deg(R, \mathbf{u}_0) \le \frac{s-s_0+1}{m_0} \prod_{i=0}^n m_i^{s-s_i+1}$. □

4.2 Algorithm

If a polynomial R is the linear combination of some known polynomials $F_i (i = 1, \ldots, s)$, that is $R = \sum_{i=1}^s H_i F_i$, then a general idea to estimate the computational complexity of R is to estimate the upper bounds of the degrees of R and $H_i F_i$ and to use linear algebra to find the coefficients of R.

For sparse differential resultant, we already gave its degree in Theorem 4.1. Now we will give the degrees of the expressions in the linear combination.

Theorem 4.3 *Let $\mathbb{P}_0, \ldots, \mathbb{P}_n$ be a differentially essential system with order s_i and degree m_i respectively. Denote $s = \sum_{i=0}^n s_i$, $m = max_{i=0}^n\{m_i\}$. Let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the sparse differential resultant of $\mathbb{P}_0, \ldots, \mathbb{P}_n$ with ord$(R, \mathbf{u}_i) = h_i$ for each i. Then we have $\deg(G_{ik}\mathbb{P}_i^{(k)}) \le (m+1)\deg(R) \le (m+1)^{ns+n+2}$ in formula (16).*

Proof: By Theorem 4.1 and its proof, R can be written as $R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^n \sum_{k=0}^{h_i} G_{ik}\mathbb{P}_i^{(k)}$.

To estimate the degree of $G_{ik}\mathbb{P}_i^{(k)}$, we need only to consider every monomial $M(\mathbf{u}; u_{00}, \ldots, u_{n0})$ in $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$. Consider one monomial $M = \mathbf{u}^\gamma \prod_{i=0}^n \prod_{k=0}^{h_i} (u_{i0}^{(k)})^{d_{ik}}$ with $|\gamma| = d$ and $d + \sum_{i=0}^n \sum_{k=0}^{h_i} d_{ik} \le \deg(R)$, where $\mathbf{u}^\gamma$ represents a monomial in $\mathbf{u}$ and their derivatives with exponent vector γ. Using the substitution in the proof of Theorem 4.1, we have

$$M = \mathbf{u}^\gamma \prod_{i=0}^n \prod_{k=0}^{h_i} \left(\left(\mathbb{P}_i - \sum_{j=1}^{l_i} u_{ij} (\mathbb{Y}^{[s_i]})^{\alpha_{ij}} \right)^{(k)} \right)^{d_{ik}}.$$

When expanded, every term has total degree bounded by $d + \sum_{i=0}^n \sum_{k=0}^{h_i} (m_i+1) d_{ik}$ in $\mathbf{u}_0^{[h_0]}, \ldots, \mathbf{u}_n^{[h_n]}$ and $\mathbb{Y}^{[h]}$ with $h = \max\{h_i + s_i\}$. Since $d + \sum_{i=0}^n \sum_{k=0}^{h_i} (m_i+1) d_{ik} \le (m+1)(d + \sum_{i=0}^n \sum_{k=0}^{h_i} d_{ik}) \le (m+1)\deg(R)$, applying Theorem 4.1, the theorem is proved. □

For a given system $f_0, \ldots, f_n \in \mathcal{F}\{y_1, \ldots, y_n\}$, let $\mathbf{v}_i$ be the set of coefficients of f_i and $\mathbb{P}(f_i)$ the differential polynomial of the form (7) with the same support as f_i. When $\mathbb{P}(f_i)$ form a differentially essential system, let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be their sparse differential resultant. Then $R(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ is defined to be the *sparse differential resultant of* f_i. The following result gives an effective differential Nullstellensatz under certain conditions.

Corollary 4.4 *Let $f_0, \ldots, f_n \in \mathcal{F}\{y_1, \ldots, y_n\}$ have no common solutions with ord$(f_i) = s_i, s = \sum_{i=0}^n s_i$, and $\deg(f_i) \le m$. If the sparse differential resultant of $f_0, \ldots, f_n$ is nonzero, then there exist $H_{ij} \in \mathcal{F}\{y_1, \ldots, y_n\}$ s.t. $\sum_{i=0}^n \sum_{j=0}^{s-s_i} H_{ij} f_i^{(j)} = 1$ and $\deg(H_{ij} f_i^{(j)}) \le (m+1)^{ns+n+2}$.*

Proof: The hypothesis implies that $\mathbb{P}(f_i)$ form a differentially essential system. Clearly, $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ has the property stated in Theorem 4.3, where $\mathbf{u}_i$ are coefficients of $\mathbb{P}(f_i)$. The result follows directly from Theorem 4.3 by specializing $\mathbf{u}_i$ to the coefficients of f_i. □

Now, we give an algorithm **SDResultant** to compute sparse differential resultants. The algorithm works adaptively by searching R with an order vector $(h_0, \ldots, h_n) \in \mathbb{N}^{n+1}$ with $h_i \le s - s_i$ by Theorem 4.1. Denote $o = \sum_{i=0}^n h_i$. We start with $o = 0$. And for this o, choose one vector $(h_0, \ldots, h_n)$ at a time. For this $(h_0, \ldots, h_n)$, we search for R from degree $D = 1$. If we cannot find an R with such a degree, then we repeat the procedure with degree $D+1$ until $D > \prod_{i=0}^n (m_i+1)^{h_i+1}$. In that case, we choose another $(h_0, \ldots, h_n)$ with $\sum_{i=0}^n h_i = o$. But if for all $(h_0, \ldots, h_n)$ with $h_i \le s - s_i$ and $\sum_{i=0}^n h_i = o$, R cannot be found, then we repeat the procedure with $o + 1$. In this way, we need only to handle problems with the real size and need not go to the upper bound in most cases.

Theorem 4.5 *Algorithm* **SDResultant** *computes sparse differential resultants with at most $O(n^{3.376}(s+1)^{O(n)}(m+1)^{O(nls^2)})$ $\mathbb{Q}$-arithmetic operations.*

Proof: In each loop of Step 3, the complexity of the algorithm is clearly dominated by Step 3.1.2., where we need to solve a system of linear equations $\mathcal{P} = 0$ over $\mathbb{Q}$ in $\mathbf{c}_0$ and $\mathbf{c}_{ij}$. It is easy to show that $|\mathbf{c}_0| = \binom{D+L-1}{L-1}$ and $|\mathbf{c}_{ij}| = \binom{(m+1)D-m_i-1+L+n(h+1)}{L+n(h+1)}$, where $L = \sum_{i=0}^n (h_i + 1)(l_i+1)$. Then $\mathcal{P} = 0$ is a linear equation system with

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Algorithm 1 — SDResultant$(\mathbb{P}_0, \ldots, \mathbb{P}_n)$

Input: A differentially essential system $\mathbb{P}_0, \ldots, \mathbb{P}_n$.
Output: The sparse differential resultant of $\mathbb{P}_0, \ldots, \mathbb{P}_n$.

1. For $i = 0, \ldots, n$, set $s_i = \mathrm{ord}(\mathbb{P}_i)$, $m_i = \deg(\mathbb{P}_i, \mathbb{Y})$, $\mathbf{u}_i = \mathrm{coeff}(\mathbb{P}_i)$ and $|\mathbf{u}_i| = l_i + 1$.
2. Set $R = 0$, $o = 0$, $s = \sum_{i=0}^n s_i$, $m = \max_i\{m_i\}$.
3. While $R = 0$ do
   3.1. For each vector $(h_0, \ldots, h_n) \in \mathbb{N}^{n+1}$ with $\sum_{i=0}^n h_i = o$ and $h_i \le s - s_i$ do
      3.1.1. $U = \cup_{i=0}^n \mathbf{u}_i^{[h_i]}$, $h = \max_i\{h_i + s_i\}$, $D = 1$.
      3.1.2. While $R = 0$ and $D \le \prod_{i=0}^n (m_i+1)^{h_i+1}$ do
         3.1.2.1. Set R_0 to be a homogenous GPol of degree D in U.
         3.1.2.2. Set $\mathbf{c}_0 = \mathrm{coeff}(R_0, U)$.
         3.1.2.3. Set $H_{ij} (i = 0, \ldots, n; j = 0, \ldots, h_i)$ to be GPols of degree $(m+1)D - m_i - 1$ in $\mathbb{Y}^{[h]}, U$.
         3.1.2.4. Set $\mathbf{c}_{ij} = \mathrm{coeff}(H_{ij}, \mathbb{Y}^{[h]} \cup U)$.
         3.1.2.5. Set $\mathcal{P}$ to be the set of coefficients of $R_0(\mathbf{u}_0, \ldots, \mathbf{u}_n) - \sum_{i=0}^n \sum_{j=0}^{h_i} H_{ij} \mathbb{P}_i^{(j)}$ as an algebraic polynomial in $\mathbb{Y}^{[h]}, U$.
         3.1.2.6. Solve the linear equation $\mathcal{P} = 0$ in variables $\mathbf{c}_0$ and $\mathbf{c}_{ij}$.
         3.1.2.7. If $\mathbf{c}_0$ has a nonzero solution, then substitute it into R_0 to get R and go to Step 4., else $R = 0$.
         3.1.2.8. D:=D+1.
   3.2. o:=o+1.
4. Return R.

/*/ GPol stands for generic ordinary polynomial.

/*/ coeff(P, V) returns the set of coefficients of P as an ordinary polynomial in variables V.

$N = \binom{D+L-1}{L-1} + \sum_{i=0}^n (h_i+1)\binom{(m+1)D-m_i-1+L+n(h+1)}{L+n(h+1)}$ variables and $M = \binom{(m+1)D+L+n(h+1)}{L+n(h+1)}$ equations. To solve it, we need at most $(\max\{M, N\})^\omega$ arithmetic operations over $\mathbb{Q}$, where ω is the matrix multiplication exponent and the currently best known ω is 2.376.

The iteration in Step 3.1.2. may go through 1 to $d_i = \prod_{i=0}^n (m_i+1)^{h_i+1} \le (m+1)^{ns+n+1}$, and the iteration in Step 3.1. at most will repeat $\prod_{i=0}^n (s - s_i + 1) \le (s+1)^{n+1}$ times. And by Theorem 4.1, Step 3 may loop from $o = 0$ to ns. The whole algorithm needs at most

$$\begin{aligned} &\textstyle\sum_{o=0}^{ns} \sum_{\substack{h_i \le s-s_i \\ \sum_i h_i = o}} \sum_{D=1}^{d_i} \left(\max\{M, N\}\right)^{2.376} \\ &\le O(n^{3.376}(s+1)^{O(n)}(m+1)^{O(nls^2)}) \end{aligned}$$

arithmetic operations over $\mathbb{Q}$. In the above inequalities, we assume that $(m+1)^{ns+n+2} \ge ls + n(s+1)$ and use the fact that $l \ge (n+1)^2$, where $l = \sum_{i=0}^n (l_i+1)$. Our complexity assumes an $O(1)$-complexity cost for all field operations over $\mathbb{Q}$. Thus, the complexity follows. □

Remark 4.6 *Algorithm* **SDResultant** *can be improved by using a better search strategy. If D is not big enough, instead of checking $D+1$, we can check $2D$. Repeating this procedure, we may find a k such that $2^k \le \deg(R) \le 2^{k+1}$. We then bisecting the interval $[2^k, q2^{k+1}]$ again to find the proper degree for R. This will lead to a better complexity, which is still single exponential.*

5. CONCLUSION AND PROBLEM

In this paper, the sparse differential resultant is defined and its basic properties are proved. In particular, degree bounds for the sparse differential resultant and the usual differential resultant are given. Based on these degree bounds, we propose a single exponential algorithm to compute the sparse differential resultant.

In the algebraic case, there exists a necessary and sufficient condition for the existence of sparse resultants in terms of the supports [29]. It is interesting to find such a condition for sparse differential resultants.

It is useful to represent the sparse resultant as the quotient of two determinants, as done in [7] in the algebraic case. In the differential case, we do not have such formulas, even in the simplest case of the resultant for two generic differential polynomials in one variable. The treatment in [4] is not complete. For instance, let f, g be two generic differential polynomials in one variable y with order one and degree two. Then, the differential resultant for f, g defined in [4] is zero, because all elements in the first column of the matrix $M(\delta, n, m)$ in [4, p.543] are zero. Furthermore, it is not easy to fix the problem.

The degree of the algebraic sparse resultant is equal to the mixed volume of certain polytopes generated by the supports of the polynomials [23] or [13, p.255]. A similar degree bound is desirable for the sparse differential resultant.

There exist very efficient algorithms to compute the algebraic sparse resultants ([10, 11]). How to apply the principles behind these algorithms to compute sparse differential resultants is an important problem.

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