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An algorithm for solving partial differential parametric systems[☆]

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Abstract

For a partial differential parametric system

$$P_1 = 0, \dots, P_r = 0, \quad Q_1 \neq 0, \dots, Q_s \neq 0,$$

where P_i, Q_j are differential polynomials in $K\{u_1, \dots, u_t, y_1, \dots, y_n\}$ and u_k are parameters, an algorithm to solve the parametric system is presented in this paper. The algorithm finds not just the values of the parameters u_k such that the system has solutions, but also all solutions of this system.

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1. Introduction

Let K be a differential field of characteristic 0 with a finite number of differential operators, and $K\{u_1, \dots, u_t, y_1, \dots, y_n\}$ (or $K\{U, Y\}$ for short) be the differential polynomial ring with parameters $u_1, \dots, u_t$ and indeterminates $y_1, \dots, y_n$. By a parametric system, we mean

$$P_1 = 0, \dots, P_r = 0, \quad Q_1 \neq 0, \dots, Q_s \neq 0, \tag{1.1}$$

where P_i, Q_j are differential polynomials in $K\{U, Y\}$, $(i = 1, \dots, r; j = 1, \dots, s)$.

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For the parametric system, two questions need to be answered: (1) for what value of the parameters does, the system (1.1) has solutions; (2) how to find all solutions of (1.1).

Sit [13] studied linear algebraic parametric systems. Weispfenning [14] investigated the general algebraic parametric system by introducing the *Comprehensive Groebner Basis*. Gao and Chou in [3] gave a solution to questions (1) and (2) for algebraic case based on Wu–Ritt's zero decomposition algorithm [17] and Sit's notation of covers [13]. Furthermore, they studied the parametric system for ordinary differential case in [4]. This paper will extend these results to partial differential case.

The idea of constructing a cover of differential zeros of a partial differential system is as follows. Firstly, we use Wu–Ritt's differential zero decomposition algorithm [15] to decompose the system into a disjuction union of some special differential quasi-algebraic variety. Applying Rosenfeld's lemma [11] on the relation of differential zeros and algebraic zeros of a special system of differential polynomials and Seidenberg's technique [12], we then give a projection algorithm. Finally, we get an algorithm to solve the partial differential system, whence we obtain a differential zero structure theorem for (1.1) of the form $(S_1, ASC_1), \ldots, (S_l, ASC_l)$, where S_i are some differential quasi-algebraic variety in the parameters U in a universal differential field L of K and ASC_i are ascending sets in the indeterminates Y, such that for each $\xi \in S_i$, when replacing the U by ξ, the ASC_i has solutions which are also solutions of (1.1). Furthermore, all solutions can be given in this way.

The *differential projection algorithm* presented in this paper is also based on some new results in theory of differential algebra, such as Morrison [8] and Li [7]. The background of its applications comes from control theory and quantifier elimination, see [1–4,9,10].

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries and Wu–Ritt's differential zero decomposition theorem. Section 3 gives a projection algorithm to eliminate differential indeterminates for a partial differential parametric system. In Section 4, we give an algorithm for solving the parametric system and two practical examples.

2. Preliminaries

2.1. Concept and notation

Let the ground field K be a differential field of characteristic 0 with a finite number of *differential operators* $\delta_1, \ldots, \delta_m$; Θ a free commutative monoid generated by $\delta_1, \ldots, \delta_m$ with multiplicative identity $\varepsilon = \delta_1^0 \cdots \delta_m^0$. An element of Θ is called a *derivative operator*. The *order* of a derivative operator $\theta = \delta_1^{i_1} \cdots \delta_m^{i_m}$ equals $i_1 + \cdots + i_m$ and is denoted by $ord(\theta)$. A derivative operator is *proper* if its order is positive.

In order to introduce a partial ordering between two differential polynomials, we first define below an ordering among the derivative operators of Θ. Let $\theta_1, \theta_2 \in \Theta$, $\theta_1 = \delta_1^{i_1} \cdots \delta_m^{i_m}$, $\theta_2 = \delta_1^{j_1} \cdots \delta_m^{j_m}$. θ_1 is said to be *higher than* θ_2, denoted by $\theta_1 > \theta_2$, if either $ord(\theta_1) > ord(\theta_2)$ or $ord(\theta_1) = ord(\theta_2)$ and there exists a positive integer k,

$(1 \leqslant k \leqslant m)$, such that

$$i_s = j_s \quad (\text{for } s > k), \quad \text{and} \quad i_k < j_k.$$

Let $y_1, \ldots, y_n$ be differential indeterminates over K. For each y_k, the symbol θy_k denotes the partial derivative of y_k with respect to θ, which is again an indeterminate over K. As a matter of convention, εy_k equals y_k. Let

$$\Theta(Y) = \{\theta y_k | 1 \leqslant k \leqslant n, \theta \in \Theta\}.$$

An element θy_k of $\Theta(Y)$ is called a *derivative* of y_k; and k, the *class* of θy_k.

Among these elements of $\Theta(Y)$, we define an ordering as follows. Let $\theta_i y_i$, $\theta_j y_j \in \Theta(Y)$, and $\theta_i y_i = \delta_1^{i_1} \cdots \delta_m^{i_m} y_i$, $\theta_j y_j = \delta_1^{j_1} \cdots \delta_m^{j_m} y_j$. We say $\theta_i y_i$ is *higher than* $\theta_j y_j$, denoted by $\theta_i y_i > \theta_j y_j$, if either $i > j$ or $i = j$ and $\theta_1 > \theta_2$.

The differential polynomial ring $K\{y_1, \ldots, y_n\}$ is the usual commutative polynomial ring generated by $\Theta(Y)$ over K, an element P of which is called a *differential polynomial* (abbreviated as: *d-pol*). If we ignore the differential structure, then $K\{y_1, \ldots, y_n\}$ is an algebraic polynomial ring which is referred to as the *underlying ring*. The notation $Elm(P)$ denotes the algebraic polynomial corresponding to *d-pol* P.

A subset J of $K\{y_1, \ldots, y_n\}$ is called a *differential ideal* if for all $P \in K\{y_1, \ldots, y_n\}$, $Q_1, Q_2 \in J$, $\delta \in \Theta$, we have $PQ_1 \in J$, $Q_1 + Q_2 \in J$ and $\delta Q_1 \in J$. For a set of *d-pols* DP, we use the notation $[DP]$ to denote a differential ideal generated by DP.

For $P \in K\{y_1, \ldots, y_n\} \backslash K$, the *class* of P is defined to be the greatest k such that θy_k occurs in P and is denoted by $cls(P)$. Assume that $cls(P) = k$, the highest derivative in y_k appearing in P is θ, then θy_k is called the *Lead* of P and is denoted by $ld(P)$. The *order* of P is defined to be $ord(\theta)$ and denoted by $ord(P)$. The *degree* of P is the degree of P in θy_k and denoted by $deg(P)$. The leading coefficient of P with respect to θy_k is called the *initial* of P and denoted by $ini(P)$. The formal partial derivative of P with respect to θy_k is called the *separant* of P and denoted by $sep(P)$. A power product of $ini(P)$ and $sep(P)$ is called an *IS-product* of P.

Suppose that $P, Q \in K\{y_1, \ldots, y_n\} \setminus K$, and $ld(P) = \theta_i y_i, ld(Q) = \theta_j y_j$. P is said to be *higher than* Q, denoted by $P > Q$ (or $Q < P$), if either $ld(P) > ld(Q)$ or $ld(P) = ld(Q) = u$ and $deg_u P > deg_u Q$. In addition, we convent that for all $P \in K\{y_1, \ldots, y_n\} \setminus K, Q \in K$, $P > Q$. If neither $P > Q$ nor $Q > P$, then we say P and Q are of the same rank, written as $rank(P) = rank(Q)$.

Let L be a universal differential field (see [6]) of K. For a subset $DP \subseteq K\{y_1, \ldots, y_n\}$, a differential zero of DP is understood to be an n-tuple $(z_1, \ldots, z_n) \in L^n$ such that every P in DP becomes zero after θy_k is replaced by θz_k, for all θy_k occurring in DP. The set of differential zeros of DP is denoted by *d-zero*(DP). When DP is regarded as a subset of the underlying ring, an algebraic zero of DP we mean a solution of $DP = 0$, and the set of algebraic zeros of DP is denoted by *zero*(DP). It is clear that *d-zero*$(DP) \subseteq$ *zero*(DP), and the converse fails in general. For $DP, DQ \subset K\{y_1, \ldots, y_n\}$, *d-zero*$(DP/DQ)$ and *zero*(DP/DQ) denote, respectively, the sets of all common *d-zeros* and zeros (in L) of the *d-pols* in DP which do not annihilate any *d-pol* in DQ.

2.2. Wu–Ritt's differential zero decomposition theorem

For $P \in K\{y_1,\ldots,y_n\} \setminus K, Q \in K\{y_1,\ldots,y_n\} \setminus \{0\}$, Q is said to be *reduced* with respect to P if

(1) no proper derivative of $ld(P)$ occurs in Q; or
(2) if $ld(P)$ occurs in Q, then $deg_{ld(P)} Q < deg_{ld(P)} P$.

A *differential ascending set* (abbreviated as: *d-asc set*) is either a single non-zero element in K or a finite sequence of non-zero *d-pols*, none of which is in $K : A_1,\ldots,A_r$, such that $A_1 < \cdots < A_r$ with each A_i is reduced with respect to A_j for $i > j$. By an *IS-product* J of a *d-asc set ASC*, we mean a product of the IS-product of all *d-pols* in *ASC*.

A non-zero *d-pol Q* is said to be *reduced* with respect to a *d-asc set ASC* if it is reduced with respect to every *d-pol P* in *ASC*.

Lemma 2.1. *Let ASC be a d-asc set*: $A_1,\ldots,A_r$, *Q any d-pol. There exist non-negative integers* $s_i, t_i, (i=1,\ldots,r)$, *such that when a suitable linear combination of some d-pols and their derivatives among ASC are subtracted from*

$$ini(P_1)^{i_1}\cdots ini(P_r)^{i_r} sep(P_1)^{t_1}\cdots sep(P_r)^{t_r} Q \tag{2.1}$$

the remainder, *R is reduced with respect to ASC.*

In [15], the above R is called the *differential remainder* of G with respect to *ASC*, denoted by $d\text{-}rem(Q, ASC)$. Furthermore, (2.1) can be written simply as

$$J * Q \equiv R \quad d\text{-mod}(ASC), \tag{2.2}$$

where J is an *IS-product* of *ASC*. Actually, we may give the concrete expression of the differential remainder R as follows.

$$R = J * Q - \sum_{i,\theta\in\Theta} C_{i,\theta}\theta A_i, \tag{2.3}$$

where $C_{i,\theta} \in K\{y_1,\ldots,y_n\}$.

A passive system is a special system of differential polynomials, which was studied and developed by Wu [15]. The reader is referred to Wu [15] or Li [7] for a general theory for passive *d-asc sets* and precise definitions of terms not defined in this section.

Theorem 2.2 (Wu–Ritt's Differential Zero Decomposition Theorem (weak form) (Wu [15])) *Let DP be a differential polynomial system*, *there exists an algorithm for decomposing DP as the following form*

$$d\text{-}zero(DP) = \bigcup_k d\text{-}zero(ASC_k/J_k),$$

where ASC_k *are some passive d-asc sets*, J_k *is an IS-product of* ASC_k.

Note that Wu–Ritt's differential zero decomposition algorithm needs only arithmetic and differential operations, and no factorization is needed.

3. A differential projection algorithm

A *differential quasi-algebraic variety* is defined to be $D = \cup_{i=1}^{k}(d\text{-}zero(DP/DQ))$, where DP and DQ are two *d-pol* sets in $K\{U,Y\}$. $U=\{u_1,\dots,u_t\}$ is called a set of parameter and $Y=\{y_1,\dots,y_n\}$ a set of indeterminates. For given two *d-pol* sets DP and DQ in $K\{U,Y\}$, we define the *Projection* with the $y_n,\dots,y_1$ as follows:

$$Proj_{y_1,\dots,y_n} d\text{-}zero(DP/DQ) = \{e \in L^t \mid \exists a \in L^n, \text{s.t.}(e,a) \in d\text{-}zero(DP/DQ)\}.$$

If $t=0$, we define $Proj_{y_1,\dots,y_n} d\text{-}zero(DP/DQ) = True$ if $d\text{-}zero(DP/DQ) \neq \emptyset$, and $False$ otherwise.

We firstly study the case occurring a differential indeterminate y in the parametric system.

Lemma 3.1. *Let $ASC \subset K\{U,y\}$ be a d-asc set, $Q \in K\{U,y\}$, the differential remainder $R = d\text{-}rem(Q,P)$. Then*

$$Proj_{y_n} d\text{-}zero(ASC/J * Q) = Proj_{y_n} d\text{-}zero(ASC/J * R),$$

in which J is an IS-product of ASC.

Proof. The previous expression (2.3) can be written as

$$J * Q = \sum_{i,\theta \in \Theta} C_{i,\theta}\theta A_i + R. \tag{3.1}$$

From (3.1), we have that those differential zeros of ASC which do not annihilate J, $R \neq 0$ will lead to $Q \neq 0$ and vice versa, i.e., Q and R have the same differential zeros under the given hypothesis. Then $d\text{-}zero(ASC/J*Q) = d\text{-}zero(ASC/J*R)$. Furthermore,

$$Proj_y d\text{-}zero(ASC/J * Q) = Proj_y d\text{-}zero(ASC/J * R).$$

The proof is completed. □

In what follows, Algorithm 3.2 gives the method of eliminating a differential indeterminate y, i.e., computing $Proj_y d\text{-}zero(ASC/J*Q)$, where $ASC \subset K\{U,y\}, Q \in K\{U,y\}$.

Algorithm 3.2.

Input: A passive *d-asc set* $ASC = \{B_1,\dots,B_r,A_1,\dots,A_s\}$, where $B_i \in K\{U\}$, $A_j \in K\{U,y\} \setminus K\{U\}$, and a *d-pol* $Q \in K\{U,y\}$.

Output: $Proj_y d\text{-}zero(ASC/J * Q)$, where J is an *IS*-product of ASC.

Step 1: Let $Z = d\text{-}zero(ASC/J * Q)$, $R = d\text{-}rem(Q, ASC)$. By Lemma 3.1, we have

$$Proj_y Z = Proj_y d\text{-}zero(ASC/J * R). \tag{3.2}$$

Step 2: If ASC does not involve differential indeterminate y, i.e. $s=0$, then we split it into two cases.

Step 2a: If $R\in K\{U\}$, then

$$Proj_y Z = d\text{-}zero(ASC/J * R).$$

Step 2b: If $R\in K\{U,y\}\setminus K\{U\}$, write R as a polynomial in y and its derivatives (say, $\theta_j y$), i.e., $R=\sum_{i=0}^{k_0} C_i * Q_i$, where $C_i\in K\{U\}$, every Q_i is a monomial in y and $\theta_j y$. From (3.2), we thus have

$$Proj_y Z = \bigcup_{i=0}^{k_0} d\text{-}zero(ASC/J * C_i).$$

Step 3: If ASC involves y, i.e., $s>0$. Let $ASC' = ASC - \{A_s\} = \{B_1,\ldots,B_r,A_1,\ldots,A_{s-1}\}$. We distinguish three cases below.

Step 3a: If R does not involve y or $ld(R) < ld(A_s)$, then

$$Proj_y Z = Proj_y d\text{-}zero(ASC'/J * R).$$

Step 3b: If $ld(R) = ld(A_s) = v$, let $d = deg_v(A_s)$, $R_0 = Rem(R^d, P)$, we have

$$Proj_y Z = Proj_y d\text{-}zero(ASC'/J * R_0).$$

Step 3c: If $ld(R) > ld(A_s)$, write R as a polynomial of $ld(R)$, $R=\sum_{i=0}^{k_1} C_i * (ld(R))^i$, we have

$$Proj_y Z = \bigcup_{i=1}^{k_1} Proj_y(d\text{-}zero(ASC/J * R * C_i)).$$

For $i=1,\ldots,k_1$, let $R_1 = R * C_i$, repeat this step until $ld(R_1)\leqslant ld(A_s)$. Furthermore, by Step 3a or Step 3b, we can eliminate A_s and obtain

$$Proj_y Z = \bigcup_{i=1}^{k_2} Proj_y(d\text{-}zero(ASC'/JR_{i0})).$$

Step 3d: By now, we have eliminated A_s. Let $J = J' * J_s$, where J' and J_s are the *IS-product* of ASC' and A_s, respectively. Let $R_i' = d\text{-}rem(J_s * R_{i0}, ASC')$. Thus we have

$$Proj_y Z = \bigcup_{i=1}^{k_3} Proj_y(d\text{-}zero(ASC'/J'R_i')).$$

For $i=1,\ldots,k_3$, repeat Steps 3a–c to eliminate $A_{s-1},\ldots,A_1$ in succession. Finally, we get

$$Proj_y Z = \bigcup_{i=1}^{k_4} Proj_y(d\text{-}zero(B_1,\ldots,B_r/J_b * P_i)), \tag{3.3}$$

where J_b is an *IS-product* of $\{B_1,\ldots,B_r\}$, $P_i\in K\{U,y\}$.

Step 3e: By Step 2, we get

$$Proj_{y}Z=\bigcup_{i=1}^{l} d\text{-}zero(B_1,\ldots,B_r/J_b*Q_i), \tag{3.4}$$

where $Q_i\in K\{U\}$.

Note: $\{B_1,\ldots,B_r\}$ in (3.4) is still a passive *d-asc set* which includes only indeterminates U (i.e., $u_1,\ldots,u_t$).

Algorithm 3.3 computes a projection of a differential quasi-algebraic variety, we refer to it as a *differential projection algorithm*, which computes $Proj_{y_1,\ldots,y_n}d\text{-}zero(DP/DQ)$ where $DP,DQ\subset K\{u_1,\ldots,u_t,y_1,\ldots,y_n\}$.

Algorithm 3.3. *Input*: Two *d-pol* sets $DP=\{P_1,\ldots,P_r\}$, $DQ=\{Q_1,\ldots,Q_s\}$ in $K\{U,Y\}$ (i.e., $K\{u_1,\ldots,u_t,y_1,\ldots,y_n\}$).

Output: $Proj_{y_1,\ldots,y_n}d\text{-}zero(DP/DQ)$.

Step 1: Let $Q=\Pi_{i=1}^{s}Q_i$. By Theorem 2.2, under the differential indeterminates order $u_1<\cdots<u_s<y_1<\cdots<y_n$, we have

$$d\text{-}zero(DP/DQ)=\bigcup_{i=1}^{k_1}(d\text{-}zero(ASC_i/J_i*Q)). \tag{3.5}$$

For $i=1,\ldots,k_1$, do Steps 2–3.

Step 2: By Algorithm 3.2, we may eliminate y_n from ASC_i and obtain the result

$$\begin{aligned}&Proj_{y_n}d\text{-}zero(ASC_i/J_i*Q)\\ &\quad=\bigcup_{j=1}^{k_2}(d\text{-}zero(\{B_{j1},\ldots,B_{jl_j},A_{j1},\ldots,A_{jq_j}\}/J'_j*R_j)),\end{aligned} \tag{3.6}$$

where J'_j is an *IS-product* of $\{B_{j1},\ldots,B_{jl_j},A_{j1},\ldots,A_{jq_j}\}$ and $A_k,R_j,J'_j\in K\{U,y_1,\ldots,y_{n-1}\}$, $B_k\in K\{U\}$.

Step 3: Note that $B_{j1},\ldots,B_{jl_j},A_{j1},\ldots,A_{jq_j}$ is still a passive *d-asc set* which does not include y_n. Repeat Step 2, $y_{n-1},\ldots,y_1$ can be eliminated in succession. Then we get

$$Proj_{y_1,\ldots,y_n}d\text{-}zero(ASC_i/J_i*Q)=\bigcup_{j=1}^{k_3}(d\text{-}zero(\{B_{j1},\ldots,B_{jl_j}/\overline{J_j}*\overline{R_j}\})),$$

where $B_{jk},\overline{J_j},\overline{R_j}\in K\{U\}$, $\overline{J_j}$ is an *IS-product* of $\{B_{j1},\ldots,B_{jl_j}\}$.

Step 4: By the elimination of Steps 2–3 and (3.5), we have

$$d\text{-}zero(DP/DQ)=\bigcup_{i=1}^{k_5}(d\text{-}zero(\{B_{i1},\ldots,B_{il_i}/\overline{J_i}*\overline{R_i}\})),$$

in which $B_i,\overline{J_i},\overline{R_i}\in K\{U\}$, $\overline{J_i}$ is an *IS-product* of $\{B_{i1},\ldots,B_{il_i}\}$.

Algorithm 3.3 is an immediate consequence of Theorem 2.2 and Algorithm 3.2. In order to prove Algorithm 3.2, we first give a lemma.

Lemma 3.4. *Let $A \subset K\{y_1,\dots,y_n\}$ be a passive d-asc set and Q a d-pol which is reduced with respect to A. Then the system $A=0$, $J*Q\neq 0$ has an algebraic solution iff it has a differential solution.*

Proof. If the system $A=0$, $J*Q\neq 0$ has a differential solution, it obviously has an algebraic solution.

The proof of the other direction involves some concepts and notations appeared in [11] (also see [7,8]). Suppose that $A=0$, $J*Q\neq 0$ has an algebraic solution. Then no power of $J*Q$ is in the ideal $[A]$. If $(J*Q)^r\in[A]$, in which r is a positive integer, then $Q^r\in[A]:J^r$. By Li's proof [7], a passive *d-asc set* is a coherent autoreduced set. (A *d-asc set* is called an autoreduced set in [11]).

Since Q is reduced with respect to A, it is also partially reduced with respect to A. Furthermore, Q^r is partially reduced with respect to A. By Rosenfeld Lemma, we have $Q^r\in(A):J^r$. Hence, $J^r*Q^r\in(A)$, whence $J*Q\in\ <A>$. That is to say, that all algebraic solutions of the system $A=0$, $J*Q\neq 0$ annihilate $J*Q$, a contradiction. Using Zorn's Lemma [5], we see that $[A]$ is contained in a differential prime ideal, which furnishes a differential solution to $A=0$, $J*Q\neq 0$. This proof is complete. □

In what follows we shall prove the projection of a differential system is the same with that of corresponding algebraic system under the conditions: ASC being a passive *d-asc set* and Q reduced with respect to ASC.

Without loss of generality, suppose that $ASC\subset K\{U,y\}$, $Q\in K\{U,y\}$, and $\theta_1 y,\dots,\theta_n y$ are all derivatives appeared in ASC and Q. Let

$$V_1 = Proj_y d\text{-}zero(ASC/J*Q),$$

J an *IS-product* of ASC.

Consider the corresponding algebraic system $Elm(ASC)=0$, $Elm(J*Q)\neq 0$. Let

$$V_2 = Proj_{\theta_1 y,\dots,\theta_n y} zero(Elm(ASC)/Elm(J*Q)).$$

According to the fact that a projection of a quasi-algebraic variety is a quasi-algebraic variety [16], V_2 can be written as

$$V_2 = \bigcup_{i=1}^{l} zero(Elm(ASC_i)/Elm(J_i*Q_i)),$$

where $ASC_i\subset K\{U\}$, $Q\in K\{U\}$.

Corresponding to their differential form, let

$$\bigcup_{i=1}^{l} d\text{-}zero((ASC_i)/(J_i*Q_i)) = V_3.$$

We have the following result.

Theorem 3.5. *Under the notations above, $V_1=V_3$.*

Proof. Write $Z=d\text{-}zero(ASC/J*Q)$, $\bar{Z}=zero(Elm(ASC)/Elm(J*Q))$.

$\forall a \in V_1$, $\exists y$, such that $(a, y) \in Z$. We denote a zero of $\bar{Z}$ as (Elm(a), Elm(y)), where $Elm(y) = \{\theta_1 y, \ldots, \theta_n y\}$. We get $Elm(a) \in V_2$. Furthermore, $a \in V_3$, thus $V_1 \subseteq V_3$.

On the contrary, $\forall a \in V_3$, i.e., $Elm(a) \in V_2$. We can extend $Elm(a)$ to a zero of $\bar{Z}$, i.e. $(Elm(a), \theta_1 y, \ldots, \theta_n y) \in \bar{Z}$. By Rosenfeld Lemma [11], $\exists y$, such that $(a, y) \in d$-$zero(ASC)/(J * Q)$. Thus $a \in V_1$, i.e. $V_3 \subseteq V_1$. This proof is finished. □

Proof of Algorithm 3.2. In Step 2a, if $R \in K\{U\}$, the ASC and R are free of y, so (3.1) holds.

Steps 2b and 3 are pure algebraic projections, some techniques of which come from Gao and Chou [3] and Seidenberg [12]. By Theorem 3.5, Algorithm 3.2 gives a computation of differential projection.

4. Computing a cover of zeros of a partial differential system

We now continue to consider the parametric system

$$P_1 = 0, \ldots, P_r = 0, \quad Q_1 \neq 0, \ldots, Q_s \neq 0, \tag{4.1}$$

where P_i, Q_j are in $K\{U, Y\}$. Let $DP = \{P_1, \ldots, P_r\}, DQ = \{Q_1, \ldots, Q_s\}$. Following Sit [13] and Gao and Chou [4], we have

Definition 4.1. A solution function of (4.1) is a pair (S, ASC), where S is a differential quasi-algebraic variety in L^t and ASC is a *d-asc set* in $K\{U\}\{Y\} - K\{U\}$, such that

(a) for each $u \in S$, let ASC', DQ' be obtained from ASC, DQ by replacing the u by u', then d-$zero(ASC'/\{J'\} \cup DQ')$ is a non-empty differential quasi-algebraic variety in L^n.

(b) for each $y' \in d$-$zero(ASC'/\{J'\} \cup DQ')$, $(u', y') \in d$-$zero(DP/DQ)$. We call also (u', y') a solution of (S, ASC).

Definition 4.2. A cover of (4.1) is a set of solution functions of (4.1)

$$\{(S_1, ASC_1), \ldots, (S_l, ASC_l)\},$$

such that each $(u', y') \in d$-$zero(DP/DQ)$ is a solution of some (S_i, ASC_i).

Theorem 4.3. *There is an algorithm to find a cover for the parameter system* (4.1).

Algorithm 4.4 below gives a constructive proof of Theorem 4.3. Furthermore, let

$$C = \{(S_1, ASC_1), \ldots, (S_l, ASC_l)\}$$

be a cover of (4.1). Then we have

$$Proj_{y_1,\ldots,y_n} d\text{-}zero(DP/DQ) = \bigcup_{i=1}^{l} S_i.$$

Algorithm 4.4. *Input*: Two *d-pol* sets $DP = \{P_1, \ldots, P_r\}, DQ = \{Q_1, \ldots, Q_s\}$ in $K\{U, Y\}$.

Output: A cover for $d\text{-}zero(DP/DQ)$.

Step 1: Identical to Step 1 of Algorithm 3.3, we have

$$d\text{-}zero(DP/DQ)=\bigcup_{i=1}^{k_1}(d\text{-}zero(ASC_i/J_i*Q)).$$

For $i=1,\ldots,k_1$, do Steps 2–3.

Step 2: By Algorithm 3.3, we have

$$Proj_{y_1,\ldots,y_n}d\text{-}zero(ASC_i/J_i*Q)=\bigcup_{j=1}^{k_2}d\text{-}zero(B_{j1},\ldots,B_{jl_j}/J_j*Q'_j). \tag{4.2}$$

Step 3: In (4.2), let $S_i=\cup_{j=1}^{k_2}d\text{-}zero(B_{j1},\ldots,B_{jl_j}/J'_j*Q'_j)$, continue to eliminate $u_t,\ldots,u_1$ by using Algorithm 3.3, and we get $D_i=Proj_{u_1,\ldots,u_t}S_i$.

If $D_i=True$, then (S_i,ASC_i) is a solution function of $d\text{-}zero(DP/DQ)$, otherwise, $d\text{-}zero(ASC_i/J_i*Q)=\emptyset$, discard it.

Step 4: By now, all solution functions obtained from Step 3:

$$(S_1,ASC_1),\ldots,(S_l,ASC_l)$$

furnish a cover for $d\text{-}zero(DP/DQ)$.

Example 1 comes from Pommaret and Quadrat [10], which computes a tree of integrability conditions and each leaf of which represents a formal solution of the system. We here compute a cover of its differential zeros.

Example 1. Given a partial differential parametric system:

$$DP\begin{cases}\frac{\partial^4 y}{\partial x_1^2\partial x_2^2}-a*\frac{\partial y}{\partial x_2}=0,\\ \frac{\partial^3 y}{\partial x_1\partial x_2^2}=0,\end{cases}$$

where y is a differential indeterminate, and a as a differential parametric indeterminate. Compute a cover of $d\text{-}zero(DP)$.

Let $a<y$, according to Theorem 2.2, we have

$$d\text{-}zeroDP=d\text{-}zero\left(\frac{1}{a},a*\frac{\partial y}{\partial x_2}\right)\cup d\text{-}zero\left(a,\frac{\partial^3 y}{\partial x_1\partial x_2^2}\right).$$

Using Algorithm 4.4, we get a cover of $d\text{-}zero(DP)$:

$$\left(\left(\frac{1}{a}\right),a*\frac{\partial y}{\partial x_2}\right),\left((a),\frac{\partial^3 y}{\partial x_1\partial x_2^2}\right).$$

Example 2. Compute a cover of differential zeros of the partial differential parametric system below in parameter y_1.

$$P_1=0,\qquad P_2=0,\qquad Q\neq 0, \tag{4.3}$$

in which $P_1 = (\partial y_1/\partial x_1) + 2y_2 + y_1^2; P_2 = -(\partial y_1/\partial x_2) + 4y_2 * y_1^2 - 4y_1 * (\partial y_2/\partial x_1) + 2(\partial^2 y_2/\partial x_1^2) + 8y_2^2; Q = (\partial y_1/\partial x_1) * (\partial y_2/\partial x_2) + (\partial y_1/\partial x_2)^2$. ($P_1, P_2$ occurs in [15]).

Let the ordering between y_1 and y_2 be $y_2 > y_1$. By Wu–Ritt's differential zero decomposition theorem, we have

$$d\text{-}zero(\{P_1, P_2\}/Q) = d\text{-}zero(\{H, P_1\}/Q),$$

where $H = (\partial y_1/\partial x_2) + (\partial^3 y_1/\partial x_1^3) - 6y_1^2 * (\partial y_1/\partial x_1)$.

According to Algorithm 3.2, we compute the remainder of Q with respect to *d-asc set* H, P_1 and get

$$R = 2\left(\frac{\partial y_1}{\partial x_2}\right)^2 - \frac{\partial y_1}{\partial x_1} * \frac{\partial^2 y_1}{\partial x_2 * \partial x_1} - 2\,\frac{\partial y_1}{\partial x_1} * y_1 * \frac{\partial y_1}{\partial x_2},$$

whence,

$$Proj_{y_2} d\text{-}zero(\{P_1, P_2\}/Q) = Proj_{y_2} = d\text{-}zero(\{H, P_1\}/R) = d\text{-}zero(H/R).$$

Furthermore, $Proj_{y_1} d\text{-}zero(H/R) \neq \emptyset$. Therefore, a cover of differential zeros of (4.3) is:

$$((H/R), \{H, P_1\}).$$

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