Generalized Stewart–Gough Platforms and Their Direct Kinematics

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***Abstract*—In this paper, we introduce the *generalized Stewart–Gough platform* (GSP) consisting of two rigid bodies connected with six distance and/or angular constraints between six pairs of points, lines, and/or planes in the base and the moving platform, respectively. We prove that there exist 3850 possible forms of GSPs. We give the upper bounds for the number of solutions of the direct kinematics for all the GSPs. We also obtain closed-form solutions and the best upper bounds of real solutions of the direct kinematics for a class of 1120 GSPs.**

***Index Terms*—Closed-form solution, decoupled mechanisms, direct kinematics, generalized Stewart–Gough platform (GSP), parallel manipulator.**

I. Introduction

THE Stewart–Gough platform (SP), originated from the mechanism designed by Stewart for flight simulation [23] and the mechanism designed by Gough for tire testing [12], is a parallel manipulator consisting of two rigid bodies: a moving platform, or simply a platform, and a base. The position and orientation (pose) of the base are fixed. The base and platform are connected with six extensible legs. For a set of given values for the lengths of the six legs, the pose of the platform could be generally determined. The SP has been studied extensively in the past 20 years and has many applications. Compared with serial mechanisms, the main advantages of the SP are its inherent stiffness and high load/weight ratio. For recent surveys, please consult [5] and [16].

Many variants of the SP were introduced for different purposes. Most of these variants are special forms of the SP. In [7], Faugere and Lazard gave a classification of all special forms of the SP. In [2], Baron *et al.* studied all the possibilities of using three joints to connect the legs and the platforms. The SP uses distances between points as the driving parameters. New parallel manipulators of six degrees of freedom (DOFs) based on distances between points and planes were proposed by Artigue *et al.* [1] and Dafaoui *et al.* [4]. Parallel manipulators based on distances between points and lines are proposed by Hunt [13] and Pritschow *et al.* [20]. Bonev *et al.* and Zhang added extra sensors to the SP in order to find a unique position of the platform or to do calibration [3], [27].

In this paper, we introduce the *generalized Gough–Stewart platform* (GSP), which could be considered as the most general form of parallel manipulators with six DOFs in a certain sense. A GSP consists of two rigid bodies connected with six distance or/and angular constraints between six pairs of points, lines, and/or planes in the base and platform, respectively. The original SP is one of the GSP, where the six constraints are distance constraints between points. The SPs introduced in [1], [4], [13], and [20] are all special cases of the GSP, where the six constraints are distance constraints between points/planes or points/lines. We prove that there exist 3850 possible forms of GSPs. We also show how to realize these designs by using different combinations of revolute, prismatic, cylindrical, spherical, and planar joints. The purpose of introducing these new GSPs is to find new and more practical parallel mechanisms for various purposes.

A large portion of the work on SP is focused on the *direct kinematics*: for a given set of lengths of the legs, determine the pose of the platform. This problem is still not solved completely. For the SP, Lazard [7], [15], Mourrain [18], [19], Raghavan [21], and Ronga *et al.* [22] proved that the number of complex solutions of the direct kinematics is at most 40, or infinite. Dietmaier showed that the SP could have 40 real solutions [6]. On the other hand, Wen and Liang [24] and Zhang and Song [26] gave the closed-form solutions for the SP with planar base and platforms. For the general case, Husty derived a set of six polynomial equations which lead to an equation of degree 40 [14].

In this paper, we give the upper bounds for the numbers of solutions of the direct kinematics for all 3850 GSPs by borrowing techniques from Lazard [15] and Mourrain [19]. One interesting fact is that the direct kinematics for many GPSs are much easier than that of the SP. We identify a class of 35 GPSs which could have at most 20 solutions. We show that a class of 1220 GSPs is decoupled, in that we can first determine the direction and then the position of the platform. We also obtain the closed-form solutions and the optimized upper bounds for the numbers of real solutions of these GSPs.

One specific reason that leads us to introduce the GSP is that the direct kinematic problem for the original SP is considered a very difficult task [5], while for some of the GSPs, the direct kinematic problem is much easier. The difficulty in solving the direct kinematic problem is considered to be a major obstacle in using the SP in many applications. These new GSPs might provide new parallel manipulators which have the stiffness and

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lightness of the SP, and with an easy-to-solve direct kinematic problem.

The rest of the paper is organized as follows. In Section II, we define the GSP. In Section III, we introduce the techniques used to determine the number of solutions. In Section IV, we give the equations for the distance and angular constraints. In Section V, we give the direct kinematic analysis. In Section VI, we give the closed-form solutions to a class of GSPs. In Section VII, we show how to realize the GSPs. In Section VIII, conclusions are given.

II. Generalized Stewart–Gough Platform

We consider three types of *geometric primitives*: points, planes, and lines in the three-dimensional (3-D) Euclidean space, and two types of *geometric constraints*: the distance constraints between point/point, point/line, point/plane, line/line, and the angular constraints between line/line, line/plane, and plane/plane.

For a constraint c, the *valence* of c is the number of scalar equations required to define c. Besides some special cases, all constraints mentioned above have valence one. One of the special cases is "two planes form an angle of zero," which needs two equations to describe. These special cases could be omitted, since they only occur in rare or special cases. Therefore, we assume that every constraint has valency one.

A rigid body in the space has six DOFs. Therefore, to determine its position and orientation, we need six geometric constraints. This leads to the following definition.

Definition 2.1: A GSP consists of two rigid bodies connected with six geometric constraints. One of the rigid bodies, called *base*, is fixed, and the other rigid body, called *platform*, is movable. The pose of the platform is determined by the values of the six constraints.

The GSP can be divided into four classes.

- 3D3A: The GSP has three distance and three angular constraints.
- 4D2A: The GSP has four distance and two angular constraints.
- 5D1A: The GSP has five distance and one angular constraints.
- 6D: The GSP has six distance constraints.

We cannot have more than three angular constraints, due to the fact that a rigid body in the space has three rotational DOFs, and these rotational DOFs can generally be determined by three angular constraints. A fourth angular constraint will conflict with the other angular constraints.

Fig. 1 is one of the simplest GSPs, where l_1, l_2 are two lines on the platform and l_3, l_4 are two lines on the base. Lines l_1, l_2 and l_3, l_4 are connected with six geometric constraints represented by lines marked with an "a" or a "d" representing an angular constraint or a distance constraint, respectively. Therefore, this is a 3D3A GSP. It is proved that this platform has at most four solutions [10]. Note that this is a special case of GSP. In the general case, there should exist six lines in the base and the platform, respectively, and the six constraints are between six pairs of different lines.

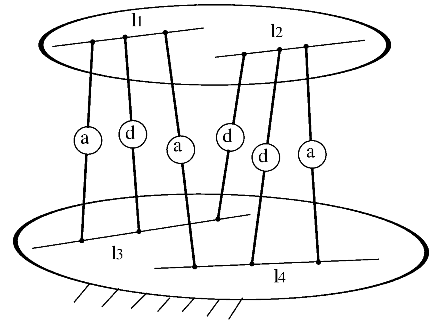

Fig. 1. Special 3D3A GSP.

Proposition 2.2: If we assume that the geometric primitives in the base and platform are distinct, there are 1120 types of 3D3A GSPs, 1260 types of 4D2A GSPs, 1008 types of 5D1A GSPs, and 462 types of 6D GSPs. Totally, there are 3850 types of GSPs.

Proof: Let d_i (a_i) be the number of possible ways to assign i distance (angular) constraints between the platform and the base. There are three types of angular constraints: line/line, line/plane, plane/plane. For the line/plane constraint, we need to consider two cases: line/plane and plane/line, meaning that the line is on the platform and the base, respectively. Hence, we need to consider four types of angular constraints. Similarly, we need to consider six types of distance constraints. Then the number of possible types of jDiA GSPs is

$$a_i d_j = C_{i+4-1}^{i} \cdot C_{j+6-1}^{j} = C_{i+3}^{i} \cdot C_{j+5}^{j}.$$

By simple computations, the possible types of 3D3A, 4D2A, 5D1A, and 6D are $a_3 d_3 = 1120$, $a_2 d_4 = 1260$, $a_1 d_5 = 1008$, and $d_6 = 462$, respectively.

III. The Rigid-Motion Varieties of Lazard and Mourrain

In order to find the maximal number of solutions to the direct kinematics, we need the rigid-motion varieties introduced by Lazard and Mourrain. A rigid motion in space can be described by (R, T), where R is a 3×3 matrix representing a rotation, and T is a vector in $\mathbb{R}^3$ representing a translation. R satisfies the following equations: $R^T R = R R^T = I$ and $\det(R) = 1$.

Lazard introduced a representation of rigid motions as an algebraic variety in $\mathbb{R}^{15}$ [15]. The coordinates are the elements of the matrix R, and vectors T and $U = R^T T$. The defining equations of this variety are

$$R^T R = R R^T = I, \quad \det(R) = 1, \quad U = R^T T. \qquad (1)$$

We call this variety the *Lazard variety* and the variables *Lazard's coordinate system*. By computing the Hilbert function of this ideal, it is easy to see that the variety is of dimension 6 and of degree 20 [15]. The rotation R and the translation T corresponding to the usual description of rigid motions are the projection of this variety onto the space of the elements of the matrix R and vector T.

Mourrain defined the set of rigid motions in the space $\mathbb{R}^3$ as an algebraic variety in $\mathbb{R}^{16}$. Let $R_j = (r_{1,j}, r_{2,j}, r_{3,j})^T$ be the

jth column of R, $R_i^* = (r_{i,1}, r_{i,2}, r_{i,3})^T$ the ith row of R, and $T = (t_1, t_2, t_3)$. These vectors satisfy

$$\det([R_1, R_2, R_3]) = 1, \quad [R_i, R_j] = [R_i^*, R_j^*] = \delta_{i,j} \quad (2)$$

where $[X, Y]$ represents the internal product of X and Y, $\delta_{i,j} = 1$ for $i = j$, and $\delta_{i,j} = 0$ for $i \neq j$. Introducing new variables $U = (u_1, u_2, u_3)$, and ρ which satisfy the following equations:

$$U = R^T T, \quad \rho = [T, T] \quad (3)$$

the set of rigid motions is described as a variety in $\mathbb{R}^{16}$ defined by the above equations. The variables are $r_{i,j}$, t_i, u_i, and ρ. A variable z of homogenization is introduced in order to work in the projection space $\mathbb{P}^{16}$. Setting $z = 0$ in the ideal generated by the previous equations, we obtain the part of the variety at infinity. In order to eliminate the component of the variety at infinity in $\mathbb{P}^{16}$, Mourrain introduced the new variables

$$(i, j|k, 4) = r_{i,k} \cdot t_j - r_{j,k} \cdot t_i. \quad (4)$$

Now the variables are $r_{i,j}$, t_i, $(i, j|k, 4)$, u_i, and $\rho (1 \leq i, j, k \leq 3)$. Equations (2)–(4) define a variety in $\mathbb{R}^{25}$. We call it *Mourrain variety* and the variables *Mourrain's coordinate system*. This variety is of dimension 6 and of degree 40 [19]. Mourrain proved that the rotation R and translation T corresponding to the usual description of rigid motions are the projection of this variety in $\mathbb{P}^{16}$ [18].

IV. Distance Constraints and Angular Constraints

In this section, we will give the equations for these constraints in Lazard's and Mourrain's coordinate systems. We use **P**, **L**, and **H** to represent a point, a line, and a plane, respectively. Then there exist six kinds of distance constraints in the GSPs: **DPP**, **DPL**, **DLP**, **DPH**, **DHP**, and **DLL**. For instance, **DPL** represents a distance constraint between a point on the platform and a line on the base. There exist four kinds of angular constraints in the GSPs: **ALL**, **ALH**, **AHL**, and **AHH**. For instance, **AHL** represents an angular constraint between a plane on the platform and a line on the base.

DPP. This is the usual distance constraint in the SP between two points. Suppose that a point Y is fixed on the platform, and a corresponding point X is fixed on the base. After performing a rotation R and a translation T to the platform, the image of point Y is $RY + T$. The distance between X and the image of Y under the rigid motion (R, T) is given by the norm of $RY + T - X$

$$d^2 = [RY + T - X, RY + T - X]$$

where $[X, Y]$ represents $X^T Y$. In Lazard's coordinate system, we have

$$\begin{aligned} d^2 &= [RY + T - X, RY + T - X] \\ &= [Y, Y] + [T, T] + [X, X] + 2[T, R^T Y] \\ &\quad - 2[X, T] - 2[RY, X] \\ &= [Y, Y] + [T, T] + [X, X] + 2[Y, U] \\ &\quad - 2[X, T] - 2[RY, X]. \end{aligned} \quad (5)$$

This is a quadratic equation in the variables R, T, and U. In Mourrain's coordinate system, we have

$$\begin{aligned} d^2 &= [RY + T - X, RY + T - X] \\ &= [Y, Y] + \rho + [X, X] + 2[Y, U] \\ &\quad - 2[X, T] - 2[RY, X]. \end{aligned} \quad (6)$$

This is a linear equation in the variables R, T, U, and ρ

DPL. A point Y is fixed on the platform. A line L is fixed on the base. Let A be a point on L, and u a unit vector parallel to L. Then under the rigid motion (R, T), the distance between the image of Y and L is the norm of $(RY + T - A) \times u$. The distance constraint can be represented by

$$d^2 = [(RY + T - A) \times u, (RY + T - A) \times u]. \quad (7)$$

It is clear that the above expression is independent of the choices of A and u. This is a quadratic equation, both in the variables R, T, and U, and variables R, T, U, and ρ.

DLP. A line L is fixed on the platform, and a point X is fixed on the base. Let B be a point on L, and v a unit vector parallel to L. Then under the rigid motion (R, T), the distance between the image of L and X is the norm of $(RB + T - X) \times Rv$. The distance constraint can be represented by

$$d^2 = [(RB + T - X) \times Rv, (RB + T - X) \times Rv].$$

Since $[V \times W, V \times W] = [V, V][W, W] - [V, W]^2$, we have

$$\begin{aligned} d^2 &= [(RB + T - X) \times Rv, (RB + T - X) \times Rv] \\ &= ([B, B] + [T, T] + [X, X] + 2[B, U] \\ &\quad -2[RB, X] - 2[T, X])[v, v] \\ &\quad - ([B, v] + [U, v] - [X, Rv])^2. \end{aligned} \quad (8)$$

This is a quadratic equation, both in the variables R, T, and U, and variables R, T, U, and ρ.

DPH. A point Y is a fixed on the platform, and π a plane on the base. Let v be the unit normal vector of π. The equation of π can be represented in the form $v^T X = D$, where D is the distance from the origin point to π. The distance constraint can be represented by

$$d = \pm \left(v^T \cdot (RY + T) - D\right). \quad (9)$$

This constraint can be represented by two linear equations or a quadratic equation, both in the variables R, T, and U, and variables R, T, U, and ρ.

DHP. A plane π is fixed on the platform. A point X is fixed on the base. Let B be a point on π and v the unit normal vector of π. The equations of π and the image of π under the rigid motion (R, T) are $v^T (Y - B) = 0$ and $(Rv)^T (Y - RB - T) = 0$, respectively. The distance constraint can be represented by

$$\begin{aligned} d &= \pm \left((Rv)^T \cdot (X - RB - T)\right) \\ &= \pm \left((Rv)^T \cdot X - v^T \cdot B - v^T \cdot U\right). \end{aligned}$$

This constraint can be represented by two linear equations or a quadratic equation, both in the variables R, T, and U, and variables R, T, U, and ρ.

DLL. The last distance constraint is between two lines. Let L_1 be a line on the platform, and L_2 a line on the base. Let M_i be a point on L_i, and u_i a unit vector parallel to L_i. The distance constraint between L_2 and the image of L_1 under a rigid motion (R, T) can be represented by

$$d^2 \cdot [Rv_1 \times v_2, Rv_1 \times v_2] = ((RM_1 + T - M_2) \cdot (Rv_1 \times v_2))^2 .$$

This is a quartic equation, both in the variables R, T, and U, and variables R, T, U, and ρ.

ALL. Let L_1 be a line on the platform, and L_2 a line on the base. Let u_i be a unit vector parallel to L_i. The angular constraint between L_2 and the image of L_1 under a rigid motion (R, T) can be represented by

$$(Ru_1)^T \cdot u_2 = \cos\alpha$$

where α is the angle formed by the two lines. This is a linear equation in R.

ALH. Let L be a line on the platform, and H a plane on the base. Let u_1 and u_2 be the unit normal vectors of L and H. The angular constraint between H and the image of L under a rigid motion (R, T) can be represented by

$$(Ru_1)^T \cdot u_2 = \cos\left(\frac{\pi}{2} - \alpha\right)$$

where α is the angle formed by L and H. This is a linear equation in R.

AHL. Let H be a plane on the platform, and L a line on the base. Let u_1 and u_2 be the unit normal vectors of H and L. The angular constraint between L and the image of H under a rigid motion (R, T) can be represented by

$$(Ru_1)^T \cdot u_2 = \cos\left(\frac{\pi}{2} - \alpha\right).$$

This is a linear equation in R.

AHH. Let H_1 be a plane on the platform, and H_2 a plane on the base. Let u_i be the unit normal vector of H_i. The angular constraint between H_2 and the image of H_1 under a rigid motion (R, T) can be represented by

$$(Ru_1)^T \cdot u_2 = -\cos\alpha.$$

This is a linear equation in R.

If the equation for a constraint is of degree s, we say that the constraint is of degree s. In summary, we have the following result.

Propositon 4.1: Angular constraints are of degree one. **DPL**, **DLP**, **DPH**, and **DHP** constraints are of degree two. **DLL** constraints are of degree four. A **DPP** constraint is of degree one in Mourrain's variable system, and is of degree two in Lazard's variable system.

V. Direct Kinematics of GSP

The *direct kinematic problem* of a GSP is to find the pose of its platform, supposing that the pose of the base is known and the values for the six constraints are given. In this section, we will give upper bounds for the numbers of solutions for the direct kinematics of the GSPs.

We need the following well-known result [8].

Theorem 5.1 (Bezout's Theorem): Let $f_i = 0, i = 1, \ldots, m$ be polynomial equations in n variables, such that the variety defined by $f_i = 0, i = 1, \ldots, k, (k < m)$ is of degree d, and the variety V defined by $f_i = 0, i = 1, \ldots, m$ has a finite number of solutions. Then V contains at most $d \prod_{j=k+1}^{m} \deg(f_i)$ solutions, where $\deg(f_i)$ is the (total) degree of f_i in the n variables.

Theorem 5.2: For a GSP with m constraints of degree two and s constraints of degree four in Mourrain's coordinate system, if the direct kinematic problem has a finite number of complex solutions, then the number of solutions is at most $40 \cdot 2^m \cdot 4^s$.

Proof: We illustrate the proof with a special case. Consider a GSP with five **DPP** distance constraints and one **DPL** constraint. Five points $Y_1, \ldots, Y_5$ on the platform are imposed with the **DPP** constraints with five points $X_1, \ldots, X_5$ on the base, respectively. Point Y_6 on the platform is imposed a **DPL** constraint with a line L on the base. Let X_6 be a point on L, and v the unit normal vector of L. Following Mourrain's treatment [19], we need to find the common solutions of (2)–(4), and the following equations in $\mathbb{R}^{25}$:

$$\begin{aligned} &[RY_i + T - X_i, RY_i + T - X_i] = d_i^2, \quad (i = 1, \ldots, 5) \\ &[(RY_6 + T - X_6) \times v, (RY_6 + T - X_6) \times v] = d_6^2. \end{aligned} \tag{10}$$

By (6), the first five equations are linear in the variables R, T, U, and ρ. By (7), the last equation is of degree two. According to Section III, as a variety in $\mathbb{R}^{25}$, the Mourrain variety is of degree 40. By Bezout's theorem, if the direct kinematic problem has a finite number of complex solutions, the problem has at most $40 \cdot 2 = 80$ solutions. The general case can be proved similarly. Note that constraints of degree one do not affect the number of solutions.

Theorem 5.3: For a GSP with m constraints of degree two other than the **DPP** constraint and s constraints of degree four. Suppose that the direct kinematic problem has a finite number of complex solutions. If there exist **DPP** constraints in the GSP, then the number of solutions is at most $40 \cdot 2^m \cdot 4^s$. Otherwise, the number of solutions is at most $20 \cdot 2^m \cdot 4^s$.

Proof: We still use the special case in the proof of *Theorem 5.2* to illustrate the case that there exist **DPP** constraints. In this example, $m = 1$, $s = 0$. Following Lazard's treatment [23], we need to find the common solutions of (1) and the following equations in $\mathbb{R}^{15}$:

$$\begin{aligned} &[RY_i + T - X_i, RY_i + T - X_i] = d_i^2, \quad (i = 1, \ldots, 5) \\ &[(RY_6 + T - X_6) \times v, (RY_6 + T - X_6) \times v] = d_6^2. \end{aligned} \tag{11}$$

We may set $Y_1 = X_1 = (0, 0, 0)$. The first equation in (11) becomes $[T, T] = d_1^2$. Then for $i =$2, 3, 4, 5, we have

$$\begin{aligned} d_i^2 &= [RY_i + T - X_i, RY_i + T - X_i] \\ &= [Y_i, Y_i] + [T, T] + [X_i, X_i] + 2[Y_i, U] \\ &\quad - 2[X_i, T] - 2[RY_i, X_i] \\ &= [Y_i, Y_i] + d_1^2 + [X_i, X_i] + 2[Y_i, U] \\ &\quad - 2[X_i, T] - 2[RY_i, X_i]. \end{aligned}$$

Then these equations are linear in R, T, and U. The last one in (11) is of degree two. According to Section III, as a variety in $\mathbb{R}^{15}$, the Lazard variety is of degree 20. By Bezout's theorem, the problem has at most $20 \cdot 2 \cdot 2^s \cdot 2^k = 80$ solutions. If there exist no **DPP** constraints, the result is a direct application of the Bezout theorem and the fact that the Lazard variety is of degree 20.

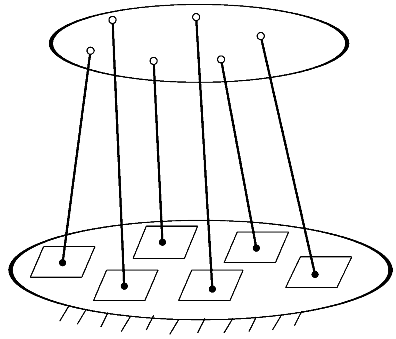

Fig. 2. 6D GSP driven by point/plane distances.

In general, *Theorems 5.2* and *5.3* give the same bound, except that the **DPP** distance constraint is not imposed. In that case, the equation system does not contain equation $[T,T] = l_1$ in Lazard's coordinate system, and the bounds given by Lazard's method is half of the bound given by Mourrain's method. We will use this property to give a class of GSPs whose maximal number of solutions is 20. This bound is half of that of the original SP. As a consequence, the direct kinematic problem is easier.

Constraint **DPH** can be represented by two linear equations (9) which represent the fact that the point is in different sides of the plane. Since in a real mechanism, we know on which side of the base the platform is located, we need only one of the linear equations. We call solutions obtained in this way *feasible solutions*. The GSP in Fig. 2 is a GSP with six **DPH** distance constraints. The solid dots in the base mean that the legs are perpendicular to the corresponding planes, and the hollow dots in the platform represent sphere joints. Since the **DPH** constraint is of degree two, by *Theorem 5.3*, this GSP could have at most $20 \cdot 2^6 = 1280$ solutions. But, we will show that it has at most 20 feasible solutions.

Theorem 5.4: If we assume that the geometric primitives in the base and platform are distinct, there are 35 types of GSPs using **DPH** distance constraints and angular constraints. The direct kinematic problem for these 35 GSPs has at most 20 feasible solutions, or an infinite number of solutions.

Proof: Use the notations introduced in *Proposition 2.2*. Let a_i, $i = 1, 2, 3$ be the number of possible ways to assign i angular constraints between the platform and the base. Since there is only one type of distance constraint, the number of possible GSPs mentioned in the theorem is $1 + a_1 + a_2 + a_3 = 1 + C_4^1 + C_5^2 + C_6^3 = 35$. Since the equation for the feasible solutions of constraint **DPH** is linear, and angular constraints are all linear, by Bezout's theorem, the maximal number of feasible solutions, if finite, is no more than the degree of Lazard's variety, which is 20.

VI. Closed-Form Solutions to the Direct Kinematics of 3D3A GSPs

For 3D3A GSPs, we may solve the direct kinematic problem in two steps. First, we impose three angular constraints to determine the rotational DOFs of the platform. It is clear that imposing distance constraints will not break the angular constraints imposed previously. Then, we may impose three distance constraints to determine the position of the platform. In other words, the 3D3A GSPs are *decoupled* with respect to the angular and distance constraints. This allows us to find the closed-form solutions to all 1120 GSPs.

In what follows, we will use Wu-Ritt's characteristic set method [17], [25] to find the closed-form solutions. Let U be a set of parameters, x_i, $i = 1, \ldots, p$ the variables to be determined, and $PS = 0$ a set of polynomial equations in U and the x_i. The method could be used to find a set of equations in *triangular form*, that is, an equation system like

$$f_1(U, x_1) = 0, f_2(U, x_1, x_2) = 0, \ldots, f_p(U, x_1, \ldots, x_p) = 0$$

which could be used to solve x_i for all values of U, except a set with lower dimension than that of U. In other words, the triangular set provides solutions in the generic case. Variable x_i is called the *leading variable* of f_i.

A. Imposing Three Angular Constraints

According to Section IV, the expressions of angular constraints only involve unit vectors parallel to the corresponding lines or perpendicular to the corresponding planes in the platform or the base. So we need only to consider angular constraints between two unit vectors. Let $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ be unit vectors on the base, and $\mathbf{s}_4$, $\mathbf{s}_5$, and $\mathbf{s}_6$ unit vectors on the platform. Assuming that the rotational matrix is $\mathbf{R} = (r_{ij})_{3\times 3}$ and the angular constraints are $\cos(\angle(\mathbf{R}\mathbf{s}_1, \mathbf{s}_4)) = d_1$, $\cos(\angle(\mathbf{R}\mathbf{s}_2, \mathbf{s}_5)) = d_2$, and $\cos(\angle(\mathbf{R}\mathbf{s}_3, \mathbf{s}_6)) = d_3$. Since $\mathbf{s}_i$ are fixed on the platform and the base, we may assume without loss of generality that $\mathbf{s_1} = (0, 0, 1)$, $\mathbf{s_2} = (0, m_0, n_0)$, $\mathbf{s_3} = (l_1, m_1, n_1)$, $\mathbf{s_4} = (0, 0, 1)^T$, $\mathbf{s_5} = (0, m_2, n_2)^T$, and $\mathbf{s_6} = (l_3, m_3, n_3)^T$. We obtain the following equation system:

$$\begin{cases} \mathbf{R^T R} = \mathbf{I}, \det(\mathbf{R}) = 1 \\ \mathbf{s_1} \cdot \mathbf{R s_4} = d_1, \mathbf{s_2} \cdot \mathbf{R s_5} = d_2, \mathbf{s_3} \cdot \mathbf{R s_6} = d_3 \end{cases} \tag{12}$$

where the l_k, m_s, n_t, and d_l are parameters, and the $z_{i,j}$ are the variables to be solved.

Equation system (12) can be reduced to the following triangular form with Wu-Ritt's characteristic set method [17], [25] under the variable order $r_{32} < r_{31} < r_{23} < r_{22} < r_{21} < r_{13} < r_{12} < r_{11} < r_{33}$:

$$\begin{cases} h_1 = z_{10}\mathbf{r}_{32}^8 + z_{11}\mathbf{r}_{32}^7 + z_{32}\mathbf{r}_{32}^6 + z_{13}\mathbf{r}_{32}^5 + z_{14}\mathbf{r}_{32}^4 \\ \quad +z_{15}\mathbf{r}_{32}^3 + z_{16}\mathbf{r}_{32}^2 + z_{17}\mathbf{r}_{32} + z_{18} = 0 \\ h_2 = z_2\mathbf{r}_{31} + z_{24}r_{32}^3 + z_{25}r_{32}^2 + z_{26}r_{32} + z_{27} = 0 \\ h_3 = z_{30}\mathbf{r}_{23} + z_{31} \\ h_4 = m_0 m_2 \mathbf{r}_{22} + q_2 r_{23} + q_3 r_{32} + d_1 n_0 n_2 - d_2 = 0 \\ h_5 = z_5\mathbf{r}_{21} - d_1 r_{22} r_{23} r_{31} - r_{31} r_{32} + r_{23}^2 r_{31} r_{32} = 0 \\ h_6 = z_6\mathbf{r}_{13} + r_{31} r_{23}^2 + \left(d_1^2 - 1\right) r_{31} = 0 \\ h_7 = r_{31}\mathbf{r}_{12} - d_1 r_{32} r_{22} + \left(r_{32}^2 - 1\right) r_{23} = 0 \\ h_8 = \mathbf{r}_{11} + r_{32} r_{23} - r_{22} d_1 = 0 \\ h_9 = \mathbf{r}_{33} = d_1 \end{cases} \tag{13}$$

where the bold variable in each equation is the leading variable, $z_2 = z_{20}r_{32}^3 + z_{21}r_{32}^2 + z_{22}r_{32} + z_{23}$, $q_2 = m_0 n_2$, $q_3 = n_0 m_2$, $z_5 = (d_1^2 - 1)r_{22} - r_{23}r_{32}d_1$, $z_6 = (d_1^2 - 1)r_{22} - r_{23}r_{32}d_1$, and all the $z_{1,j} (j = 0, \ldots, 8)$, $z_{2,j} (j = 0, \ldots, 7)$, and $z_{3,j} (j =$

TABLE I
EIGHT REAL SOLUTIONS TO ROTATION MATRIX R

r_{12}	r_{13}	r_{21}	r_{23}	r_{31}	r_{32}
$\frac{1+\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$
$\frac{1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$
$\frac{-1-\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$
$\frac{-1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$
$\frac{1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$
$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$
$\frac{-1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$
$\frac{-1+\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$

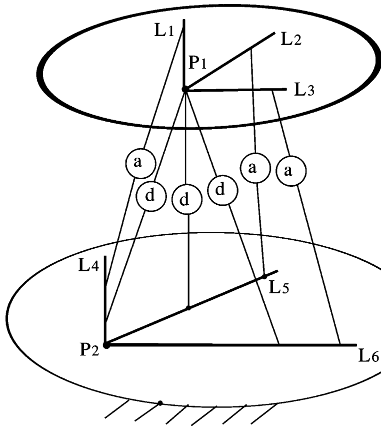

Fig. 3. Example of 3D3A GSP.

$1, 2)$ are polynomials in parameters l_k, m_s, n_t, and d_l, which may be found in [11].

Proposition 6.1: After imposing three angular constraints, the number of real solutions for the direction of the platform is at most eight, and this bound cannot be improved. Furthermore, (13) provides closed-form solutions to the problem in the generic case.

Proof: Lazard showed that the rotational matrix satisfying (2) defines a variety of dimension three and degree eight in $\mathbb{R}^9$ [15]. By Bezout's theorem, the number of directions for the platform is at most eight or infinity when three angular constraints are imposed, since the equations for angular constraints are linear. From *Example 6.2* below, the problem could indeed have eight real solutions. So the bound can not be improved. According to the basic theory of the characteristic set method [17], [25], (13) provides a solution for a given set of values of the parameters when these values do not vanish the coefficients of the leading variables. Let $J = z_{10}z_2z_{30}m_0m_2z_5z_6r_{31}$ be the product of these coefficients. Since J is a polynomial in the parameters only, the values vanishing J clearly consist of a set with lower dimension than that of the parameters.

Example 6.2: Fig. 3 represents a 3A3D GSP, where lines connecting the platform and the base labeled by "a" or "d" represent angular or distance constraints, respectively, and $L_i (i = 1, \ldots, 6)$ are different lines. Let $L_1L_2L_3$ be on the platform and $L_4L_5L_6$ on the base. We further assume that L_1, L_2, and L_3 are perpendicular to each other; L_4, L_5, and L_6 are also perpendicular to each other.

The angular constraints are $\cos(\angle(L_1, L_4)) = d_1$, $\cos(\angle(L_2, L_5)) = d_2$, and $\cos(\angle(L_3, L_6)) = d_3$. If taking $\mathbf{s_1} = (0, 0, 1)$, $\mathbf{s_2} = (0, 1, 0)$, $\mathbf{s_3} = (1, 0, 0)$, $\mathbf{s_4} = (0, 0, 1)^T$, $\mathbf{s_5} = (0, 1, 0)^T$, and $\mathbf{s_6} = (1, 0, 0)^T$, we can get $r_{11} = d_3$, $r_{22} = d_2$, and $r_{33} = d_1$ from (12). The equation system (12) can be reduced to the following triangular form with Wu-Ritt's characteristic set method [17], [25] under the variable order $r_{32} < r_{31} < r_{23} < r_{21} < r_{13} < r_{12}$:

$$\begin{cases} \mathbf{r}_{32}^4 - \left(d_3^2 - d_1^2 - d_2^2 + 1\right)\mathbf{r}_{32}^2 + d_3^2 - 2d_3d_2d_1 + d_2^2d_1^2 = 0 \\ \left(d_1r_{32}^2 - d_3d_2 + d_2^2d_1\right)\mathbf{r}_{31}^2 + \left(-d_3d_2 + d_1d_3^2\right)r_{32}^2 \\ \quad + d_2d_1^2d_3 - d_1d_3^2 + d_3d_2 - d_2^2d_1 = 0 \\ r_{32}\mathbf{r}_{23} + d_3 - d_2d_1 = 0 \\ \left(d_1r_{32}^2 - d_3d_2 + d_2^2d_1\right)r_{31}\mathbf{r}_{21} \\ \quad + \left(d_2d_1 - d_3d_1^2 + d_3^2d_2d_1 - d_2^2d_3\right)r_{32} = 0 \\ r_{31}\mathbf{r}_{13} - d_1d_3 + d_2 = 0 \\ \left(-d_1r_{32}^3 + \left(d_3d_2 - d_2^2d_1\right)r_{32}\right)r_{31}\mathbf{r}_{12} \\ \quad + \left(-d_3^2d_2 - d_1^3d_3 + d_2d_1^2 + d_1d_3^3\right)r_{32}^2 + 2d_2d_1^2d_3^2 \\ \qquad - d_1d_3^3 + d_3^2d_2 + d_3^2d_1^2 - 2d_2^2d_1d_3 - d_2^2d_1^3d_3 = 0 \end{cases} \tag{14}$$

Let $\angle(L_1, L_4)) = \pi/3$, $\angle(L_2, L_5) = \pi/3$, and $\angle(L_3, L_6) = \pi/3$. Then $r_{33} = d_1 = 1/2$, $r_{22} = d_2 = 1/2$, and $r_{33} = d_3 = 1/2$. We obtain eight sets of real solutions to the rotation matrix R, which are listed in Table I.

B. Imposing Three Distance Constraints

Definition 6.3: As mentioned in Section IV, there exist six kinds of distance constraints: **DPP**, **DPL**, **DPH**, **DLP**, **DHP**, and **DLL**. For each them, say $C = \mathbf{DPL}$, the locus of the corresponding geometric element e on the platform, that is the set of all possible points on e, under the three angular constraints, and this distance constraint is called the *locus induced by this constraint*, and is denoted by $\mathcal{L}_C$ or $\mathcal{L}_{\mathbf{DPL}}$.

Proposition 6.4: Let D be a distance constraint between a geometric element e on the platform and a geometric element on the base. If the direction of the platform is fixed, then the locus of e, that is $\mathcal{L}_D$, could be a (feasible) plane, a sphere, or a cylinder.

Proof: We use $\mathrm{DIS}(e_1, e_2)$ to denote the distance between two geometric elements e_1, e_2. The loci induced by the six distance constraints can be determined as follows.

$\mathcal{L}_{\mathbf{DPP}}$. For constraint $\mathrm{DIS}(p_1, p_2) = d$, the locus of p_1 is a sphere with center p_2 and radius d.

$\mathcal{L}_{\mathbf{DPH}}$. For constraint $\mathrm{DIS}(p, h) = d$, the locus of the p is two planes h_1 and h_2, which are parallel to the plane h and with distance d to h. Similar to *Theorem 5.4*, we may consider feasible solutions of the platform by treating one of the planes as feasible.

$\mathcal{L}_{\mathbf{DPL}}$. For constraint $\mathrm{DIS}(p, l) = d$, the locus of p is a cylinder with axis l and radius d.

Please note that in the three cases mentioned above, the loci have nothing to do with the three angular constraints. In the following cases, the angular constraints are necessary to obtain the loci.

$\mathcal{L}_{\mathbf{DLP}}$. For constraint $\mathrm{DIS}(l, p) = d$, if we only consider the distance constraint, then line l could be all the tangent lines of a sphere with center p and radius d. If we further assume that the direction of line l is fixed, then the locus of l is a cylinder with axis l_1 and radius d, where l_1 is the line passing through p and parallel to l.

$\mathcal{L}_{\mathbf{DHP}}$. Let the constraint be $\mathrm{DIS}(h, p) = d$. Since the direction of h is fixed, h has two solutions: the two planes which are parallel to h and with distance d to p. Similar to *Theorem 5.4*, we may consider feasible solutions of the platform by treating one of the planes as feasible.

$\mathcal{L}_{\mathbf{DLL}}$. Let the constraint be $\mathrm{DIS}(l_1, l_2) = d$. We need consider two cases. (1) The two lines l_1 and l_2 are not parallel to each other. Since the direction of line l_1 is fixed, the locus of l_1 is two planes which are parallel to l_1 and l_2, and with distance d to l_2. (2) If l_1 and l_2 are parallel, the locus of l_1 is the cylinder c.

Proposition 6.5: Let D be a distance constraint between a geometric element e on the platform and a geometric element on the base. If the direction of the platform is fixed, then the locus of any given point on e is $\mathcal{L}_D$.

Proof: If e is a point, D must be one of **DPP**, **DPL**, or **DPH**. In this case, the statement is obviously valid. Otherwise, e is either a line or a plane. From the above discussion, we know that the collection of points on e is $\mathcal{L}_D$. Hence, $\mathcal{L}_D$ could be considered as the locus for a given point on e.

After the three angular constraints are imposed, the direction of the platform is fixed. To find the pose of the platform, we need only to find the position of a point on the platform.

Algorithm 6.6: The input includes three distance constraints D_i, $i = 1, 2, 3$ between geometric elements on the platform and the base. We further assume that the directions of the platform, and hence, the directions of the lines and planes in D_i are known. The output is a new position for the platform such that the three distance constraints are satisfied.

1) Determine the equations $E_i(x, y, z) = 0$, $i = 1, 2, 3$ for the loci $\mathcal{L}_{D_i}$ as shown in *Proposition 6.4*.
2) Let D_i be a constraint between a geometric element e_i on the platform and f_i on the base. If e_i is a point, let $p_i = e_i$. Otherwise, e_i is either a line or a plane. Select an arbitrary fixed point on e_i as p_i. Let $p_i = (x_i, y_i, z_i)$.
3) By *Proposition 6.5*, after imposing the distance constraint D_i, point p_i is on the locus $\mathcal{L}_{D_i}$. Furthermore, since the direction of the platform is fixed, when imposing each constraint D_i $(i = 2, 3)$, point p_1 must also be on the locus $\mathcal{L}'_i$ $(i = 2, 3)$ which is the translation of $\mathcal{L}_{D_i}$ at the direction $p_1 - p_i$. Then after imposing the three distance constraints, the position p'_1 for point p_1 must be the intersection of three surfaces

$$\begin{aligned} &E_1(x, y, z) = 0 \\ &E_2(x + x_1 - x_2, y + y_1 - y_2, z + z_1 - z_2) = 0 \\ &E_3(x + x_1 - x_2, y + y_1 - y_2, z + z_1 - z_2) = 0. \end{aligned} \tag{15}$$

4) Solving (15) as shown in *Proposition 6.7*, we find the new position p'_1 for point p_1.
5) Move the platform along the translation vector $p_1p'_1 = p'_1 - p_1$, and it will satisfy the three distance constraints.

Proposition 6.7: Use $\mathbf{P}$, $\mathbf{S}$, and $\mathbf{C}$ to denote a plane, a sphere and a cylinder. We use a combination of these three letters to denote the intersection of three such surfaces. For example, **PPP** means to find the intersection point of three planes. We have the following upper bound for the solutions of their intersections if the number of solutions is finite. Furthermore, these bounds are the best in terms of finding real solutions.

1) Case **PPP**. This is the intersection of three planes. Hence, it has at most one solution.
2) Cases **PPC**, **PPS**, **SSS**, or **PSS**. They have two solutions at most. The equation system for each of these cases can be reduced to a triangular set consisting of two linear equations and a quadratic equation.
3) Cases **PCC**, **SSC**, or **PSC**. They have four solutions at most. The equation system for each of these cases can be reduced to a triangular set consisting of two linear equations and one quartic equation.
4) Cases **CCC** or **SCC**. They have eight solutions at most. The equation system for each of these cases can be reduced to a triangular set consisting of two linear equations and one equation of degree eight.

Proof: Let us consider the case **PSC**, which is to find the intersection of a plane, a sphere, and a cylinder. By Bezout's theorem, it has at most four solutions. Without loss of generality, we may assume that the equations for the plane, sphere, and cylinder are as follows

$$\begin{aligned} d_1x + d_2y + d_3z + d_4 &= 0 \\ (x - x_0)^2 + y^2 + z^2 - r_1^2 &= 0 \\ x^2 + y^2 - r_2^2 &= 0 \end{aligned}$$

where x_0, d_i, and r_i are the parameters, and x, y, and z are the variables to be solved. The above equation system can be reduced into the following triangular form under the variable order $z < y < x$:

$$\begin{cases} c_1\mathbf{z^4} + c_2\mathbf{z^3} + c_3\mathbf{z^2} + c_4\mathbf{z} + c_5 = 0 \\ c_6\mathbf{y} - d_1z^2 + c_7z + c_8 = 0 \\ 2x_0\mathbf{x} - z^2 - x_0^2 + r_1^2 - r_2^2 = 0 \end{cases} \tag{16}$$

where $c_1 = d_1^2 + d_2^2$; $c_2 = 4d_1x_0d_3$; $c_3 = 4x_0^2d_3^2 + 2r_2^2d_2^2 + 2d_1^2r_2^2 + 4d_4d_1x_0 + 2d_1^2x_0^2 - 2r_1^2d_2^2 + 2x_0^2d_2^2 - 2d_1^2r_1^2$; $c_4 = 4x_0d_3d_1r_2^2 + 4d_1x_0^3d_3 + 8x_0^2d_3d_4 - 4d_1r_1^2x_0d_3$; $c_5 = d_1^2x_0^4 + r_1^4d_2^2 + d_1^2r_1^4 + d_1^2r_2^4 + r_2^4d_2^2 - 2x_0^2r_1^2d_2^2 + x_0^4d_2^2 - 2x_0^2r_2^2d_2^2 - 2d_1^2x_0^2r_1^2 + 4d_1x_0^3d_4 + 2d_1^2x_0^2r_2^2 + 4x_0d_4d_1r_2^2 - 4d_1r_1^2x_0d_4 - 2r_1^2r_2^2d_2^2 + 4x_0^2d_4^2 - 2d_1^2r_1^2r_2^2$; $c_6 = 2x_0d_2$; $c_7 = 2x_0d_3$; $c_8 =$

$d_1x_0^2 + d_1r_2^2 - d_1r_1^2 + 2x_0d_4$. Equations (16) could be used to give the solutions for all parametric values except those vanishing $J = c_1c_6x_0 = (d_1^2 + d_2^2)(2x_0d_2)x_0$. Parametric values vanishing J must satisfy $x_0 = 0$ or $d_2 = 0$. The corresponding solutions can be obtained easily with the same method. Please refer to [11] for other cases. We still need to show that the above equation system could have four real solutions. We will give a geometric proof of this fact. The intersection of a plane and a sphere is a circle c. We may assume that the intersection of the plane and the cylinder is an eclipse and choose the position of c properly so that it has four intersections with the eclipse, and hence, with the cylinder. The fact that the upper bound given in this proposition is the best for cases **PPP**, **PPC**, **PPS**, **SSS**, **PSS**, **PCC**, **SSC**, **PSC**, and **SCC** could be proved similarly as case **PSC**. The idea is first to find the intersection of two surfaces, which is a line, a circle, or two circles. Then intersect the line or circles with the third surface. *Example 6.8* shows that case **CCC** could have eight real solutions.

Example 6.8: Continue from *Example 6.2*. Now we impose three distance constraints to the GSP in Fig. 3. Here we still use the same notations to denote the lines and points on the base and the platform. We assume that the three angular constraints have been imposed.

Let the constraints be $|P_1L_4| = r_1$, $|P_1L_5| = r_2$, and $|P_1L_6| = r_3$. Without loss of generality, we may assume that $P_2 = (0,0,0)$. Assuming that the coordinates of point P_1 become $P_1^* = (x,y,z)$ after imposing the constraints. By *Algorithm 6.6*, P_1^* should be the intersection of three cylinders and satisfy the following equation system:

$$\begin{aligned} x^2 + y^2 - r_1^2 &= 0 \\ x^2 + z^2 - r_2^2 &= 0 \\ y^2 + z^2 - r_3^2 &= 0. \end{aligned} \tag{17}$$

The above equation can be easily reduced to the following triangular form:

$$\begin{cases} 2x^2 + r_2^2 - r_3^2 - r_1^2 = 0 \\ 2y^2 + r_3^2 - r_1^2 - r_2^2 = 0 \\ 2z^2 + r_1^2 - r_3^2 - r_2^2 = 0. \end{cases} \tag{18}$$

It is obvious that above equation system has eight complex solutions. Furthermore, if $r_1^2 < r_2^2 + r_3^2$, $r_2^2 < r_1^2 + r_3^2$, and $r_3^2 < r_1^2 + r_2^2$, the equation system has eight real solutions.

C. Number of Solutions for the 3D3A GSPs

As a direct consequence of *Propositions 6.1* and *6.7*, we have the following result.

Theorem 6.9: We generally could have eight solutions when imposing three angular constraints. We generally could have one, two, four, or eight solutions when imposing three distance constraints. Therefore, a 3D3A GSP generally could have 2^k, $k = 3, \ldots, 6$ solutions depending on the types of the constraints imposed on it. These bounds are the best in terms of finding real solutions.

Table II gives a classification of the 1120 3D3A GSPs according to the three distance constraints in it and their maximal number of solutions. This classification is possible, due to the fact that the angular constraints do not affect the maximal number of solutions. In the table, N_1 is the maximal number of solutions; the second column is the types of the three intersection surfaces; the other columns give all the possible combinations of the three distance constraints. For instance, **PH** represents the distance constraint between a point on the platform and a plane on the base. By *Proposition 2.2*, there exist $d_3 = C_8^3 = 56$ cases to assign three distance constraints between the base and the platform.

TABLE II
MAXIMAL NUMBER OF SOLUTIONS FOR 3D3A GSPs

N_1	Type	Combination of the three distance constraints								
8	PPP	PH PH PH	PH PH HP	PH PH LL	PH HP HP	PH HP LL	PH LL LL	HP HP HP	HP HP LL	HP LL LL
		LL LL LL								
16	PPC	PH PH PL	PH PH LP	PH HP PL	PH HP LP	PH LL PL	PH LL LP	HP HP PL	HP HP LP	HP LL PL
		HP LL LP	LL LL PL	LL LL LP						
16	PPS	PH PH PP	PH HP PP	PH LL PP	HP HP PP	HP LL PP	LL LL PP			
16	SSS	PP PP PP								
16	PSS	PP PP PP	HP PP PP	LL PP PP						
32	PCC	PH PL PL	PH PL LP	PH LP LP	HP PL PL	HP PL LP	HP LP LP	LL PL PL	LL PL LP	LL LP LP
32	SSC	PP PP PL	PP PP LP							
32	PSC	PH PP PL	PH PP LP	HP PP PL	HP PP LP	LL PP PL	LL PP LP			
64	CCC	PL PL PL	PL PL LP	PL LP LP	LL LP LP					
64	SCC	PP PL PL	PP PL LP	PP LP LP						

VII. REALIZATION OF THE GSPs

In this section, we will show how to realize the GSPs, which could be useful for people using GSPs in various fields.

Before giving the realization of these constraints, we will show that there exist many possible variants in these realizations. One variant is that each constraint could be constructed with different combinations of joints such as revolute, prismatic, cylindrical, spherical, and planar joints. For instance, the point/plane distance constraint can be constructed by a spherical and a planar joint, as shown in Fig. 4, or by a spherical and two prismatic joints, as shown in Fig. 5. The arrows in these figures show that the driver is the distance between the point and the plane.

The second variant is that there exist different arrangements for the driver. For a given constraint, the position of the driver joint can be arranged in different places. For instance, Fig. 6 shows a realization of a point/line distance constraint, where the

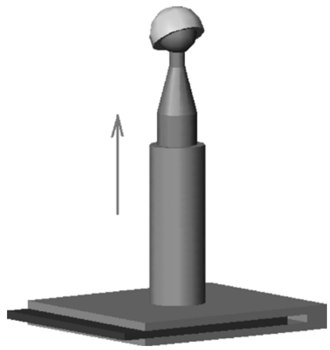

Fig. 4. P/H distance constraint.

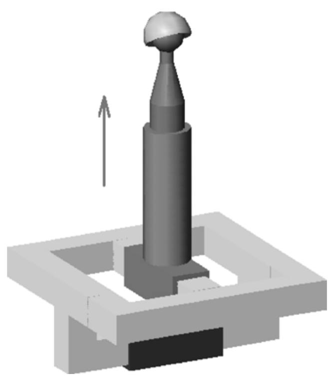

Fig. 5. P/H distance constraint.

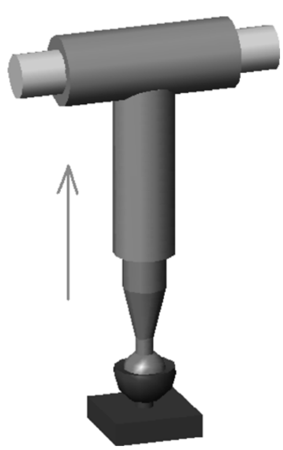

Fig. 6. Inside driver.

driver joint is placed inside the corresponding point and line. Drivers of this kind are called *inside drivers*. Fig. 7 shows a realization of the same constraint, where the driver joint is placed outside the point and line. Drivers of this kind are called *outside drivers*. Both of the mechanisms are realizations of the same constraint, but their kinematic problems are different. In this paper, we only consider the inside driver joint case.

According to Section II, we consider seven types of constraints. If we could design mechanisms that realize these seven constraints, then all the 3850 kinds of GSPs can be realized. In what follows, we will show how to realize all seven kinds of constraints. In the corresponding figures, the driver joints are shown by the arrows. As we mentioned before, each figure is only one kind of possible realization.

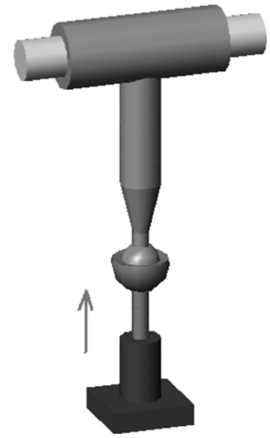

Fig. 7. Outside driver.

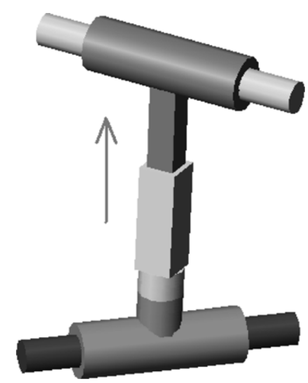

Fig. 8. L/L distance constraint.

1) The point/point distance constraint is the classical case which can be realized with two spherical joints.
2) Fig. 6 is a realization for the point/line distance constraint, which consists of a spherical joint and a cylindrical joint. The driver is a prismatic joint between the spherical joint and the cylindrical joint.
3) Fig. 4 is a realization for the point/plane distance constraint, which consists of a spherical joint and a planar joint. The driver is a prismatic joint between the spherical joint and the planar joint.
4) Fig. 8 is a realization for the line/line distance constraint which uses two cylindrical joints. Between the two joints is a revolute joint and a driver prismatic joint.
5) Fig. 9 is a realization for the line/line angular constraint, which uses two cylindrical joints. Between the two joints is a prismatic joint and a driver revolute joint.
6) Fig. 10 is a realization for the line/plane angular constraint, which consists of a cylindrical joint and a planar joint. The driver is a revolute joint between the cylindrical joint and the planar joint.
7) Fig. 11 is a realization for the plane/plane angular constraint, which uses two planar joints. The driver is a revolute joint between the two planar joints.

A realization for the 3D3A GSP in Fig. 1 is given in Fig. 12, where the distance and angular constraints between line l_1 and line l_3 are combined into one chain. So in this chain, both the

Fig. 9. L/L angular constraint.

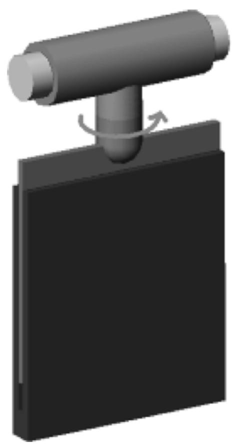

Fig. 10. L/H angular constraint.

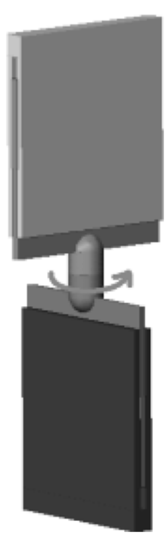

Fig. 11. H/H angular constraint.

revolute and prismatic joints are driver joints. The same happens line l_2 and line l_4.

A realization for the 6D GSP in Fig. 2 is given in Fig. 13, where the six constraints are all point/plane distances. From the figure, we can see that all six spherical joints are in the same plane. This is different from that given in [4]. Also, the mechanism in [4] uses outside drivers.

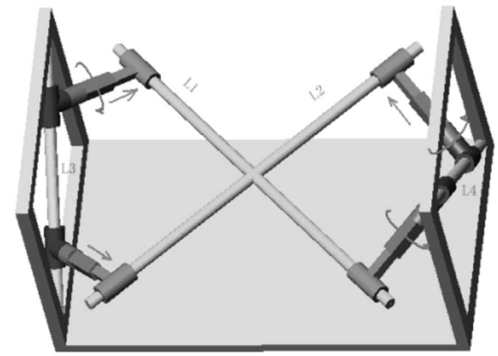

Fig. 12. Realization of the GSP in Fig. 1.

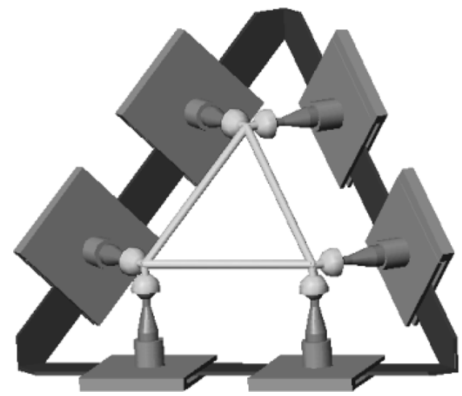

Fig. 13. Realization of the GSP in Fig. 2.

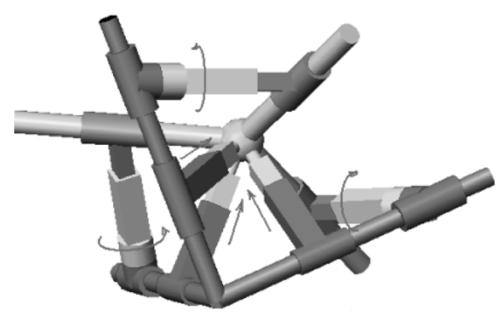

Fig. 14. Realization of the GSP in Fig. 3.

A realization for the 3D3A GSP in Fig. 3 is given in Fig. 14, where the three distance constraints are between points and lines, and the three angular constraints are between three pairs of lines.

VIII. Conclusion

A generalization of the SP is introduced by considering all possible geometric constraints between six pairs of geometric primitives on the base and the platform, respectively. This gives 3850 types of GSPs with the original SP as one of the cases. The purpose of introducing these new types of SPs is to find new and better parallel mechanisms.

We also give an upper bound for the number of solutions of the direct kinematics for each GSP. One related problem is to find the maximal number of real solutions. We show that the upper bounds given for the 1120 types of 3D3A GSPs are also the best bounds for the GSPs to have real solutions. Closed-form solutions to 3D3A general SPs are also given. It is proved that the original SP could have 40 real solutions [6]. It is interesting to see how to extend the technique used in [6] to other GSPs.

Instead of giving the upper bound for the number of solutions to the direct kinematics of the GSPs, a possible improvement is to give all the possible numbers of solutions, as done in [9] for a similar problem. Besides the direct kinematics, we may also study the workspace, the singularity, and the dynamic properties of the GSPs in order to fully understand the GSPs.

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References

[1] F. Artigue, M. Y. Amirat, and J. Pontnau, "Isoelastic behavior of parallel robots," *Robotica*, vol. 7, pp. 323–325, 1989.
[2] L. Baron and J. Angles, "The direct kinematics of parallel manipulators under joint-sensor redundancy," *IEEE Trans. Robot. Autom.*, vol. 16, pp. 12–19, Feb. 2000.
[3] I. A. Bonev, J. Ryu, S. G. Kim, and S. K. Lee, "A closed-form solution to the direct kinematics of nearly general parallel manipulators with optimally located three linear extra sensors," *IEEE Trans. Robot. Autom.*, vol. 17, pp. 148–156, Apr. 2001.
[4] E. M. Dafaoui, Y. Amirat, F. Pontnau, and C. Francois, "Analysis and design of a six-DOF parallel manipulator: Modeling, singular configurations and workspace," *IEEE Trans. Robot. Autom.*, vol. 14, pp. 78–91, Feb. 1998.
[5] B. Dasgupta and T. S. Mruthyunjaya, "The Stewart platform manipulator: A review," *Mech. Machine Theory*, vol. 35, pp. 15–40, 2000.
[6] P. Dietmaier, "The Stewart–Gough platform of general geometry can have 40 real postures," in *Advances in Robot Kinematics: Analysis and Control*, J. Lenarcic and M. L. Hust, Eds. Norwell, MA: Kluwer, 1998, pp. 7–16.
[7] J. C. Faugere and D. Lazard, "Combinatorial classes of parallel manipulators," *Mech. Machine Theory*, vol. 30, no. 6, pp. 765–776, 1995.
[8] W. Fulton, *Intersection Theory*. Berlin, Germany: Springer-Verlag, 1984.
[9] X. S. Gao, X. R. Hou, J. Tang, and H. Cheng, "Complete solution classification for the perspective-three-point problem," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 25, no. 8, pp. 930–943, Aug. 2003.
[10] X. S. Gao and G. Zhang, "Geometric constraint solving via c-tree decomposition," in *Proceedings of ACM Solid Modeling Conference*. Seattle, WA: ACM Press, 2003, pp. 45–55.
[11] X. S. Gao, D. Lei, Q. Liao, and G. Zhang. (2003) Generalized Stewart platforms and their direct kinematics. *MM Preprints* [Online], vol (22), pp. 64–85. Available: http://www.mmrc.iss.ac.cn/xgao/publ.html
[12] V. E. Gough, "Automobile stability, control, and tyre performance," Automobile Div., Inst. Mech. Eng., 1956.
[13] K. H. Hunt, "Structural kinematics of in parallel actuated robot arms," *J. Mechan., Transmission, Autom. Design*, vol. 105, pp. 705–712, 1983.
[14] M. L. Husty, "An algorithm for solving the direct kinematics of general Stewart–Gough platforms," *Mech. Machine Theory*, vol. 31, no. 4, pp. 365–379, 1996.
[15] D. Lazard, "Generalized Stewart platform: How to compute with rigid motions?," in *Proc. IMACS ACA*, Lille, France, Jun. 1993, pp. 85–88.
[16] J. P. Merlet, *Parallel Robots*. Dordrecht, The Netherlands: Kluwer, 2000.
[17] B. Mishra, *Algorithmic Algebra*. Berlin, Germany: Springer, 1993, pp. 297–381.
[18] B. Mourrain, "The 40 generic positions of a parallel robot," in *Proc. ISSAC*, M. Bronstein, Ed., 1993, pp. 173–182.
[19] ——, "Enumeration problems in geometry, robotics and vision," in *Algorithms in Algebraic Geometry and Applications*, L. Gonzalez and T. Recio, Eds. Cambridge, MA: Birkhäuser, 1996, vol. 143, pp. 285–306.
[20] G. Pritschow, C. Eppler, and W.-D. Lehner, "Highly dynamic drives for parallel kinematic machines with constant arm length," in *Proc. 1st Int. Colloq., Collaborative Res. Center*, vol. 562, Braunschweig, Germany, 2002, pp. 199–211.
[21] M. Raghavan, "The Stewart platform of general geometry has 40 configurations," *ASME J. Mech. Design*, vol. 115, pp. 277–282, 1993.
[22] F. Ronga and T. Vust, "Stewart platforms without computer?," in *Proc. Int. Conf. Real Analytic and Algebraic Geom.*. New York: Walter de Gruyter, 1995, pp. 196–212.
[23] D. Stewart, "A platform with six degrees of freedom," *Proc. Inst. Mech. Eng.*, vol. 180, no. 1, pp. 371–386, 1965.
[24] F. Wen and C. G. Liang, "Displacement analysis of the 6-6 Stewart platform mechanism," *Mech. Machine Theory*, vol. 29, no. 4, pp. 547–557, 1994.
[25] W. T. Wu, *Basic Principles of Mechanical Theorem Proving in Geometries*. Berlin, Germany: Springer, 1995.
[26] C. Zhang and S. M. Song, "Forward position analysis of nearly general Stewart platforms," *ASME J. Mech. Des.*, vol. 116, pp. 54–60, 1994.
[27] H. Zhang, "Self-calibration of parallel mechanisms: A case study on Stewart platform," *IEEE Trans. Robot. Autom.*, vol. 13, no. 3, pp. 387–397, Jun. 1997.

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