Planar Generalized Stewart Platforms and Their Direct Kinematics

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Abstract. In this paper, we introduce the concept of *planar generalized Stewart platform* (GSP) consisting of two rigid bodies connected with three constraints between three pairs of geometric primitives in the two rigid bodies respectively. This problem can be treated as a special but important class of geometric constraint solving problems. We show that there exist sixteen forms of planar GSPs. We also obtain the closed-form solutions of the direct kinematics for the planar GSPs. For a class of GSPs with two distance and one angular constraints, we may give pure geometric solutions based on ruler and compass constructions.

Keywords: Planar generalized Stewart platform, geometric constraint solving, direct kinematics, closed-form solution.

1 Introduction

The Stewart platform, originated from the mechanism designed by Stewart for flight simulation [22] and the mechanism designed by Gough for tire test [10], is a spatial parallel manipulator consisting of two rigid bodies: a moving platform, or simply a platform, and a base. The position and orientation (pose) of the base are fixed. The base and platform are connected with six extensible legs. For a set of given lengths of the six legs, the pose of the platform could generally be determined. The Stewart platform has been studied extensively in the past twenty years and has many applications. Comparing to serial mechanisms, the main advantage of the Stewart platform is its inherent stiffness and high load/weight ratio. For more information on the platform, please consult [2, 4, 13, 15, 18, 19]. A large portion of the work on Stewart platform is focused on the *direct kinematics*[13, 15, 18, 19].

On the other hand, *geometric constraint solving* is the central topic in much of the current work of developing intelligent CAD systems [5, 11, 12, 14, 20]. It

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also has applications in molecular modelling, linkage design, computer vision and computer aided instruction. Geometric constraint solving algorithms accept the declarative description of geometric diagrams or engineering drawings as the input and output a drawing procedure. In [6, 7], as a special class of geometric constraint problems, we introduce the spatial generalized Stewart platform (GSP) consisting of two rigid bodies connected with six distance and/or angular constraints between six pairs of points, lines and/or planes on the base and the moving platform respectively, which could be considered as the most general form of parallel manipulators with six DOFs in certain sense. We prove that there exist 3850 possible forms of GSPs which could provide more practical six DOFs parallel manipulators. The original Stewart platform is one of the GSPs in [6], where the six constraints are distance constraints between points.

While a majority of the work on Stewart platform focuses on the spatial case, several people also considered the planar Stewart platform which consists a moving platform and a base connected with three extensible legs. In [21], Pennock and Kanssner proved that the the upper bound of the number of solutions for the direct kinematics of the planar Stewart platform is six. Gosselin and Merlet developed robust solving schemes and established sharper bounds for special planar Stewart platforms [9]. Other interesting work on the planar Stewart platform could be found in [1, 3, 16, 17].

In this paper, we introduce the *planar generalized Stewart platform* which could be considered as the most general form of planar parallel manipulators with three DOFs in certain sense. A planar *GSP* consists of a base and a moving platform connected with three distance or/and angular constraints between three pairs of points and/or lines on the base and platform respectively. We show that there exist sixteen forms of planar GSPs. The planar Stewart platform considered in previous work such as [21, 9] is a planar GSP where the three constraints are three distance constraints among three pairs of points.

The direct kinematics is to solve an algebraic equation system. The characteristic set method is a convenient and powerful tool to deal with such equations[25]. Using the characteristic set method, we could reduce the solving of an equation system into the solving of equations in triangular form and hence the solving of univariate equations. It should be noticed that these univariate polynomial equations are in “cascade” form, that is, the coefficients of an equation involve the roots of the previous equations. These equations in triangular form are called *closed-form solutions* in this paper. We show that closed-form solutions to the direct kinematics of all planar GSPs could be found with the characteristic set method [25]. With these closed-form solutions, upper bounds for the number of solutions of the direct kinematics in the general cases can also be given. For a class of GSPs involving an angular constraint, we provide a solution to the direct kinematics based on ruler and compass constructions.

The rest of the paper is organized as follows. In Section 2, we define the planar GSP. In Section 3, we give the solutions to direct kinematics for the planar GSPs. In Section 4, conclusions are given. The results presented in this paper were reported in the un-published technical report [8].

2 Geometric Constraint Solving and Generalized Stewart Platform

In this section, we will introduce the generalized Stewart platform as a special class of geometric constraint problems.

2.1 A General Method of Geometric Constraint Solving

We consider two types of *geometric primitives*: points and lines in the two dimensional Euclidean plane and two types of *geometric constraints*: the distance constraint between point/point, point/line and the angular constraint between line/line. A *geometric constraint problem* is to find all the possible solutions of a set of geometric primitives satisfying a set of geometric constraints.

In [7], we proposed a geometric constraint solving method. As shown in Figure 1, to solve a geometric constraint problem, we first use the C-tree decomposition algorithm to reduce the problem to general construction sequences, and then reduce the solving of general construction sequences to the solving of basic merge patterns, which are the smallest problems we have to solve in order to solve the original problem.

Let $\mathcal{B}$ and $\mathcal{U}$ be two sets of geometric primitives. A *basic merging pattern* is to determine the position of $\mathcal{U}$ assuming that the position of $\mathcal{B}$ are known and there exists a set of geometric constraints among geometric primitives in $\mathcal{B}$ and $\mathcal{U}$. We further assume that a basic merge pattern $(\mathcal{B},\mathcal{U})$ has the following properties.

1. $\mathcal{B}$ and $\mathcal{B}\cup\mathcal{U}$ are rigid bodies. Here, by a rigid body, we mean a structurally well-constrained problem [7].
2. There is no subset V of $\mathcal{U}$ such that $\mathcal{B}\cup V$ is a rigid body.

As shown in Figure 1, there are three classes of basic merge patterns. The type of explicit constructions means to construct one geometric primitive, that is, $\mathcal{U}$

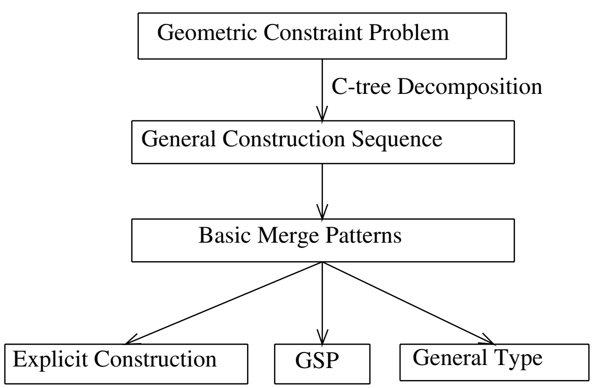

Fig. 1. Solving a constraint problem

consists of one geometric primitive. Explicit constructions are generally easy to solve. The next easy case is the general Stewart platform (GSP), where both $\mathcal{B}$ and $\mathcal{U}$ are rigid bodies and the problem is to determine the relative position of two rigid bodies according to three constraints. In the general case, $\mathcal{U}$ is not a rigid body and we need to determine the position of $\mathcal{U}$ using the constraints between primitives in $\mathcal{B}$ and $\mathcal{U}$ and constraints between primitives inside $\mathcal{U}$. In this paper, we will give closed-form solutions to the 2D GSPs.

2.2 Planar Generalized Stewart Platform

A rigid body in the plane has three DOFs. Therefore, to determine its position and orientation, we need three geometric constraints. This leads to the following definition.

Definition 1. *A planar* generalized Stewart platform *consists of two rigid bodies connected with three geometric constraints. One of the rigid bodies called* base *is fixed and the other rigid body called* platform *is movable. The position and orientation of the platform are determined by the values of the three constraints.*

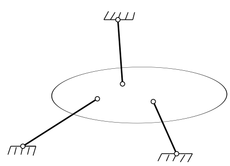

Fig. 2. Planar GSP

The planar GSP can be divided into two classes:

DDA. The GSP has two distance and one angular constraints.
DDD. The GSP has three distance constraints.

We cannot have more than one angular constraints due to the fact that a rigid body in the plane has one rotational DOF and the rotational DOF can generally be determined by one angular constraint.

Proposition 1. *If we assume that the geometric primitives in the base and platform are distinct, there are 6 types of* **DDA** *planar GSPs and 10 types of* **DDD** *planar GSPs. Totally, there are 16 types of planar GSPs.*

Proof. Let $d_i(a_i)$ be the number of possible ways to assign i distance(angular) constraints between the platform and the base. There is one type of angular constraint: line/line. For the point/line constraint, we need to consider two cases: line/point and point/line meaning that the line is on the platform and the base respectively. So we need only to consider three types of distance constraints: point/point, point/line and line/point.

The number of possible ways to select m objects from n types of objects is C_{m+n-1}^{m}. Then the number of possible types of GSPs with j distance constraints and i angular constraints is:

$$a_i d_j = C_{i+1-1}^{i} \cdot C_{j+3-1}^{j} = C_{i}^{i} \cdot C_{j+2}^{j} = C_{j+2}^{j}.$$

Then the number of DDA GSPs is: $d_2 = C_{2+2}^{2} = 6$ and the number of DDD GSPs is $d_3 = C_{3+2}^{3} = 10$. ∎

3 Closed-Form Solutions to the Direct Kinematics of Planar GSPs

The *direct kinematics* of a GSP $(\mathcal{B},\mathcal{U})$ is to find the position and direction of $\mathcal{U}$ relative to $\mathcal{B}$ assuming that the position and direction of $\mathcal{B}$ is fixed and the values for the three constraints between $\mathcal{B}$ and $\mathcal{U}$ are given.

3.1 The Characteristic Set Method

In what follows, we will use Ritt-Wu's characteristic set method [25, 23] to find the closed-form solutions of the direct kinematics of a GSP. Let V be a set of parameters, $x_i, i = 1, \ldots, p$ the variables to be determined, and PS $= 0$ a set of polynomial equations in V and the x_i. The method could be used to find a set of equations in *triangular form*, that is, an equation system

$$\mathrm{CS} = \begin{cases} f_1(V, x_1) = I_1(V)x_1^{d_1} + R_1(V, x_1) \\ f_2(V, x_1, x_2) = I_2(V, x_1)x_2^{d_2} + R_2(V, x_1, x_2) \\ \vdots \\ f_p(V, x_1, \ldots, x_p) = I_p(V, x_1, \ldots, x_{p-1})x_p^{d_p} + R_p(V, x_1, \ldots, x_p) \end{cases} \tag{1}$$

where $deg_{x_i} R_i(V, x_1, \ldots, x_i) < d_i (i = 1, \ldots, p)$. Variable x_i is called the *leading variable* of f_i. I_i is called the *initial* of f_i. For a set of values of the parameters V, we may solve x_i with the univariate equation $f_p(V, x_1, \ldots, x_i) = 0$ recursively under the condition $I_i \neq 0$. These univariate equations could be solved with either numerical methods or symbolic methods such as methods of real root isolation. It is clear that in order to solve a set of equations in triangular form, we need only to solve univariate equations.

For a set of polynomials PS and a polynomial D, let Zero(PS $/D$) be the set of solutions for all $P \in$ PS which are not solutions of $D = 0$. With Ritt-Wu's characteristic set method, we may decompose the solution set Zero(PS) as the union of the zero sets of several triangular sets:

$$\mathrm{Zero}(\mathrm{PS}) = \cup_{i=1}^{m} \mathrm{Zero}(\mathcal{A}_i / J_i) \tag{2}$$

where each $\mathcal{A}_i$ is a triangular set and J_i is the product of the initials of the polynomials in $\mathcal{A}_i$. In this paper, when we say that the *closed-form solutions* of an equations system PS $= 0$ are given, we mean that we have reduced the PS $= 0$ to the solutions of triangular sets.

3.2 The DDA Planar GSPs

For planar **DDA** GSPs, we may solve the direct kinematic problem in two steps. First, we impose an angular constraint to determine the rotational DOF of the platform. Then we impose the distance constraints without breaking the angular constraint imposed previously. In this way, we generate a solution to the direct kinematic problem based on the ruler and compass construction.

1. Imposing Angular Constraint

Let $\mathcal{B}$ and $\mathcal{U}$ be the base and the platform of the GSP. After an angular constraint is imposed between $\mathcal{B}$ and $\mathcal{U}$, we need only to find a rotational matrix $\mathbf{R}$ such that $\mathbf{R}\mathcal{U}$ satisfies the angular constraint. We need only to consider angular constraints between two unit vectors on $\mathcal{B}$ and $\mathcal{U}$ respectively. Let s_1 be a unit vector on the base and s_2 a unit vector on the platform. Without loss of generality, we may further assume that $s_1 = s_2$. Let $\mathbf{R} = (r_{ij})_{2\times 2}$ be the rotational matrix. The angular constraint is imposed as follows. We assume that the platform is at some known place at the beginning. After imposing the angular constraint, the platform moves to the correct position by a rotation represented by the rotational matrix $\mathbf{R}$. So the angular constraint can be represented by

$$\cos(\angle(s_1, \mathbf{R}s_2)) = d.$$

Let $s_2 = s_1 = (l_1, m_1)$ where $l_1^2 + m_1^2 = 1$. We can obtain the following equation system:

$$\begin{cases} \mathbf{R^T R} = \mathbf{I} \\ det(\mathbf{R}) = 1 \\ s_1 \cdot \mathbf{R}s_2 = d \\ l_1^2 + m_1^2 = 1 \end{cases} \tag{3}$$

Applying Ritt-Wu's characteristic set method [24, 25] to equations (3) under the variable order $r_{11} > r_{22} > r_{12} > r_{21} > d > l_1 > m_1$, we have

$$\text{Zero}((3)) = \text{Zero}(\text{CS})$$

where CS is given below.

$$\text{CS} = \begin{cases} l_1^2 + m_1^2 = 1 \\ {r_{21}}^2 - 1 + d^2 = 0 \\ r_{12} + r_{21} = 0 \\ r_{22} - r_{11} = 0 \\ r_{11} - d = 0. \end{cases} \tag{4}$$

Proposition 2. *After imposing an angular constraint, the number of real solutions for the direction of the platform is at most two and this bound can be reached. Furthermore, the equations CS in triangular form provide closed-form solutions to the problem.*

Proof. Since equation system (4) consists of one quadratic equation and three linear equations in the variables $r_{i,j}$, the direct kinematics problem has at most two solutions. Furthermore, the problem has two real solutions if and only if $1 - d^2 > 0$ which is possible since $d = \cos(\angle(s_1, \mathbf{R}s_2))$. ∎

2. Imposing Distance Constraints

As mentioned in Section 2, there exist three kinds of distance constraints:

DPP: the distance constraint between two points,
DLP: the distance constraint between a line on the platform and a point on the base, and
DPL: the distance constraint between a point on the platform and a line on the base.

Definition 2. *For each distance constraint, say C =***DPL***, the locus of the corresponding geometric element e on the platform under the angular constraint and this distance constraint is called the* locus induced by this constraint*, and is denoted by $\mathcal{L}_C$ or $\mathcal{L}_{DPL}$.*

Proposition 3. *Let D be a distance constraint between a geometric element e on the platform and a geometric element on the base. If the direction of the platform is fixed, then the locus of e, that is $\mathcal{L}_D$, could be a circle or two lines.*

Proof. We use $\mathrm{DIS}(e_1, e_2)$ to denote the distance between a geometric element e_1 on the platform and a geometric element e_2 on the base. The loci induced by the three distance constraints can be determined as follows.

$\mathcal{L}_{DPP}$**.** For constraint $\mathrm{DIS}(p_1, p_2) = d$, the locus of point p_1 is a circle with center p_2 and radius d.
$\mathcal{L}_{DLP}$**.** For constraint $\mathrm{DIS}(l, p) = d$, if we only consider the distance constraint, then line l could be all the tangent lines of a circle with center p and radius d. If we further assume that the direction of line l is fixed, then the locus of l is two lines l_1 and l_2 which are parallel to the line l and with distance d to p.
$\mathcal{L}_{DPL}$**.** For constraint $\mathrm{DIS}(p, l) = d$, the locus of point p is two lines l_1 and l_2 which are parallel to the line l and with distance d to l.

In Figure 3, the circle in diagram (a) is the locus of **DPP**; the bold line l tangent to the circle in diagram (b) represents the line on the platform and the lines l_1 coincident to l and l_2 parallel to l is the locus of **DLP**; and two thin lines parallel to line l in diagram (c) is the locus of **DPL**. ∎

Proposition 4. *Let D be a distance constraint between a geometric element e on the platform and a geometric element on the base. If the direction of the platform is fixed, then the locus of any given point on e is $\mathcal{L}_D$.*

Proof. If e is a point, D must be either **DPP** or **DPL**. In this case, the statement is obviously valid. Otherwise, e is a line. From the above discussion, we know

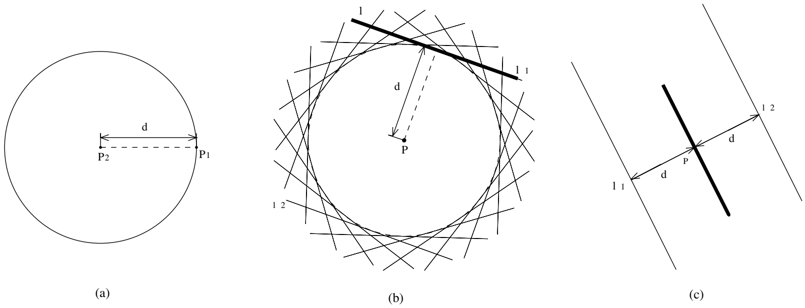

Fig. 3. Loci of distance constraints

that the collection of points on e is $\mathcal{L}_D$. Hence $\mathcal{L}_D$ could be considered as the locus for a given point on e. ∎

After the angular constraint is imposed, the direction of the platform is fixed. To find the position of the platform, we need only to find the position of a point on the platform.

Algorithm 1. *The input includes two distance constraints $D_i, i = 1, 2$ between geometric elements on the platform and the base. We further assume that the directions of the platform and hence the directions of the lines on the platform are fixed. The output is a new position for the platform such that the two distance constraints are satisfied.*

1. Determine the equations $E_i(x, y) = 0, i = 1, 2$ for the loci $\mathcal{L}_{D_i}$ as shown in Proposition 3.
2. Let D_i be a constraint between a geometric element e_i on the platform and f_i on the base. If e_i is a point, let $p_i = e_i$. Otherwise e_i is a line. Select an arbitrary fixed point on e_i as p_i. Let $p_i = (x_i, y_i)$.
3. By Proposition 4, after imposing the distance constraint D_i, point p_i is on the locus $\mathcal{L}_{D_i}$. Furthermore, since the direction of the platform is fixed, when imposing the constraint D_2, point p_1 must also be on the locus $\mathcal{L}'_2$ which is the translation of $\mathcal{L}_{D_2}$ at the direction $p_1 - p_2$. Then after imposing the two distance constraints, the new position p'_1 for point p_1 must be the intersection of two equations:
$$\begin{aligned} &E_1(x, y) = 0, \\ &E_2(x - x_1 + x_2, y - y_1 + y_2) = 0. \end{aligned} \tag{5}$$
4. By Proposition 3, (5) are equations for lines or circles. Then we need only to find the intersections of pairs of lines and circles, which are very easy to be solved. We generally could have two or four solutions.
5. Move the platform along the translation vector $t = p'_1 - p_1$, it will satisfy the two distance constraints.

So for the DDA case, we have the following conclusions.

1. To impose the angular constraint, we usually have two solutions.
2. To impose the two distance constraints, the problem is reduced to the intersection of a pair of lines/a circle which has four real solutions; a pair of lines/a pair of lines which has four real solutions; circle/circle which has two real solutions.

As a consequence, we have proved the following result.

Theorem 2. *We generally could have four or eight real solutions for a* **DDA** *problem depending on the types of the constraints imposed on it. Furthermore, these solutions can be obtained by rotating the platform and taking intersections between line/line, line/circle, or circle/circle.*

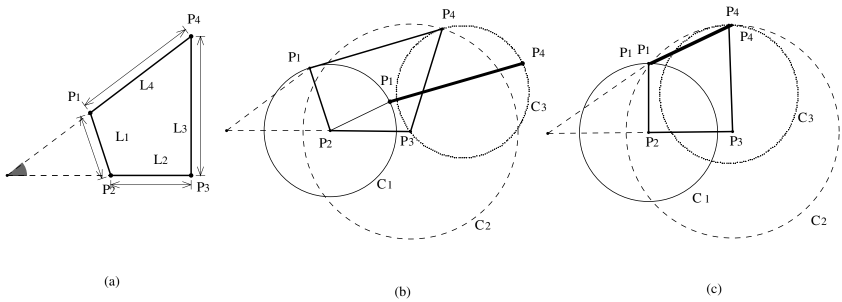

Fig. 4. A **DDA** geometric constraint problem and its geometric solution

Note that the solution given above is pure geometric. Let us illustrate this with the example in Figure 4. We may consider this as a **DDA** GSP by considering $p_1p_4l_4$ as the platform and $p_2p_3l_2$ as the base. We may solve this problem as follows.

1. Rotate line p_1p_4 so that the angle between line p_1p_4 and line p_2p_3 is the given angle.
2. Let c_1 be the circle with p_2 as center and $|p_2p_1|$ as the radius, c_2 the circle with p_3 as center and $|p_3p_4|$ as the radius, and c_3 the translation of c_1 along vector $p_4 - p_1$. The correct position for p_4 is the intersection of c_2 and c_3. Denote this intersection as p'_4.
3. The position for p'_1 is $p'_4 + p_1 - p_4$. The problem could have either one or two solutions as shown in (c) and (b) of Figure 4.

3.3 The DDD GSPs

A problem is called ruler and compass constructible, or *RC-constructible*, if the coordinates of its geometric elements can be found by solving univariate linear or quadratic equations.

Theorem 3. *As mentioned in section 3.2, there exist three kinds of distance constraints:* **DPP**, **DLP** *and* **DPL**. *The* **DDD** *GSPs can be divided into ten different sub-cases shown below. We use a new notation to represent these ten cases. For instance,* **PPP-LLL** *represents the GSP where the three distance constraints are between three points on the platform and three lines on the base respectively.*

1. *Direct kinematics for* **PPP-LLL** *and* **LLL-PPP** *can be reduced to the solving of one quadratic and three linear equations in the general case. Hence these GSPs are RC-constructible. Considering the fact that the distance constraint between a point and a line has two forms ($|pl| = \pm d$), each of* **PPP-LLL** *and* **LLL-PPP** *has at most 8 solutions.*
2. *Direct kinematics for* **PPP-LLP**, **LLP-PPP**, **LPP-PLL** *and* **LLP-PPL** *can be reduced to the solving of one quartic and three linear equations in the general case. We use the method in [5] to decide that the problems are not RC-constructible. Each of* **PPP-LLP** *and* **LLP-PPP** *has at most 16 solutions. Each of* **LPP-PLL** *and* **LLP-PPL** *has at most 32 solutions.*
3. *Direct kinematics for* **PPP-LPP**, **LPP-PPP**, **LPP-PLP** *and* **PPP-PPP** *can be reduced to the solving of one equation of degree six and three linear equations in the general case. The polynomials of degree six in these cases are irreducible. Then it is obvious that the problems are not RC-constructible. Each of* **PPP-LPP** *and* **LPP-PPP** *has at most 12 solutions.* **LPP-PLP** *has at most 24 solutions and* **PPP-PPP** *has at most six solutions.*

Proof. Let us consider the case **LPP-PLP**, which is to impose three distance constrains: **DPP**, **DPL** and **DLP** simultaneously. It is obvious that we can always get three non collinear points on the base and on the platform, respectively. If the primitive involved is a line, we can take a point on it.

Let the three points on the base be B_1, B_2 and B_3. Assuming that B_1 is the origin of the fixed coordinate system on the base, B_1B_2 the x-axis. The coordinates of three points on the base are $B_1 = (0, 0)$, $B_2 = (b_1, 0)$ and $B_3 = (b_2, b_3)$. Let the three points on the platform be D_1, D_2 and D_3. Assuming that point p is the origin of the moving coordinate system on the platform. The coordinate of point p in the fixed coordinate system is $p = (x_3, x_4)$, and point p is the foot of perpendicular line from point D_3 to line D_1D_2. Let $\angle(B_1B_2, D_1D_2) = \theta$, $x_1 = \cos\theta, x_2 = \sin\theta$. The moving coordinates of the three points on the platform are $D_1 = (-h_1, 0)$, $D_2 = (h_2, 0)$, $D_3 = (0, h_3)$, where h_1, h_2, h_3 are three nonnegative parameters. D_1D_2 is the x-axis of the moving coordinate system. The coordinates of D_1, D_2, D_3 in the fixed coordinate system are $D_{11} = (-h_1x_1+x_3, -h_1x_2+x_4)$, $D_{22} = (h_2x_1+x_3, h_2x_2+x_4)$ and $D_{33} = (-h_3x_2+x_3, h_3 x_1 + x_4)$.

Let the parametric equation of the line l on the base be $p = B_3 + u_1s_1$, where $s_1 = (l_1, m_1)$ and $|s_1| = 1$. Let the parametric equation of line l_0 on the platform in the moving coordinate system be $P = D_2 + u_2s_2$ where $s_2 = (l_2, m_2)$ and $|s_2| = 1$. Then the parametric equation of line l_0 in the fixed coordinate system is $p = D_{22} + u_2s_{22}$, where $|s_{22}| = 1$ and $s_{22} = (l_2x_1 - m_2x_2, l_2x_2 + m_2x_1)$.

Let the three constraints be $|B_1D_{11}| = t$, $|B_2l_0| = t_1$ and $|D_{33}l| = t_2$, we have

$$\begin{array}{l} x_1^2 + x_2^2 - 1 = 0 \\ (-h_1x_1 + x_3)^2 + (-h_1x_2 + x_4)^2 - t^2 = 0 \\ (l_2x_2 + m_2x_1)(h_2x_1 + x_3 - b_1) - (l_2x_1 - m_2x_2)(h_2x_2 + x_4) - d_1 = 0 \\ m_1(-h_3x_2 + x_3 - b_2) - l_1(h_3x_1 + x_4 - b_3) - d_2 = 0 \end{array} \tag{6}$$

where $d_1 = \pm t_1$ and $d_2 = \pm t_2$.

Equation system (6) can be reduced to the following triangular form with Ritt-Wu's characteristic set method under the variable order $x_1 < x_2 < x_3 < x_4$.

$$\begin{array}{l} z_{41}\mathbf{x_1}^6 + z_{42}x_1^5 + z_{43}x_1^4 + z_{44}x_1^3 + z_{45}x_1^2 + z_{46}x_1 + z_{47} = 0 \\ (z_{31}x_1^2 + z_{32}x_1 + z_{33})\mathbf{x_2} + z_{34}x_1^3 + z_{35}x_1^2 + z_{36}x_1 + z_{37} = 0 \\ ((-m_1m_2 - l_1l_2)x_2 + (-l_1m_2 + m_1l_2)x_1)\mathbf{x_3} + m_1h_3x_2^2m_2 + ((-m_1h_3l_2 \\ +l_1h_3m_2)x_1 + l_1l_2b_1 - l_1b_3m_2 - d_2m_2 + m_1b_2m_2)x_2 - l_1h_3x_1^2l_2 + (d_2l_2 \\ +l_1b_1m_2 - m_1b_2l_2 + l_1b_3l_2)x_1 - l_1m_2h_2 - l_1d_1 = 0 \\ -l_1\mathbf{x_4} - m_1h_3x_2 + m_1x_3 - l_1h_3x_1 - m_1b_2 + l_1b_3 + d_2 = 0 \end{array} \tag{7}$$

where z_{ij} are the polynomials in the parameters l_i, m_j, and h_k, which may be found in the technical report [8]. The equations in (7) give the solution to the GSP in the generic case and hence the platform has at most six solutions. Considering the fact that $d_1 = \pm t_1$ and $d_2 = \pm t_2$, the problem could have twenty four solutions. For the other nine planar DDD GSPs, the proofs are quite similar. Details could be found in the technical report [8]. ∎

Example 1. The problem in Figure 5 can be reduced into merging two rigid bodies $p_1p_2p_3p_4$ and $p_5p_6p_7p_8$. We take $p_5p_6p_7p_8$ as the the base object and $p_1p_2p_3p_4$ the dependent object. The constraints are $|l_1p_4| = 0$, $|l_2p_3| = 0$ and $|p_5l_3| = 0$, which is an **LPP-PLL** GSP. Let $p_7 = (0, 0)$. The parametric equations for lines l_1, l_2 are $p = (0, 0) + u_1(0, 1)$ and $p = (0, 0) + u_2(1, 0)$. Let point p_3 be the origin of the moving coordinate system. Then $p_3 = (x_3, x_4)$. Let $|p_6p_7| = b_2$, $|p_5p_6| = b_3$ and $|p_3p_4| = h_3$. Thus the coordinates for points p_4 and p_5 are

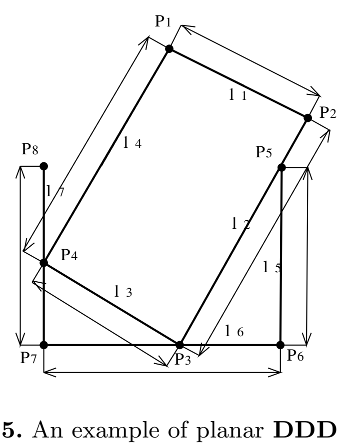

Fig. 5. An example of planar **DDD** GSP

$p_4 = (-x_2h_3 + x_3, x_1h_3 + x_4)$ and $p_5 = (b_2, b_3)$. The parametric equation of line l_3 is $p = (x_3, x_4) + u_3(x_1, x_2)$. The equation system is

$$\begin{cases} x_1^2 + x_2^2 - 1 = 0 \\ |x_2(b_2 - x_3) - x_1(b_3 - x_4)| = 0 \\ |-h_3x_2 + x_3| = 0 \\ |x_4| = 0 \end{cases} \tag{8}$$

Applying Ritt-Wu's characteristic set method to equation system (8) under the variable order $x_3 > x_2 > x_1 > b_2 > b_3 > h_3$, we obtain the following decomposition:

$$\mathrm{Zero}((8)) = \cup_{i=1}^{6} \mathrm{Zero}(\mathrm{CS}_i/J_i)$$

where CS_i and J_i are given below.

$\mathrm{CS}_1 = [b_2\mathbf{x_3} + h_3^2\mathbf{x_1}^2 - \mathbf{x_1}b_3h_3 - h_3^2, b_2\mathbf{x_2} + h_3\mathbf{x_1}^2 - \mathbf{x_1}b_3 - h_3, h_3^2\mathbf{x_1}^4 - 2b_3h_3\mathbf{x_1}^3 + (b_2^2 + b_3^2 - 2h_3^2)\mathbf{x_1}^2 + 2\mathbf{x_1}b_3h_3 + h_3^2 - b_2^2]$, $J_1 = b_2h_3$.
$\mathrm{CS}_2 = [\mathbf{x_3}, -\mathbf{x_2}b_2 + \mathbf{x_1}b_3, b_2^2\mathbf{x_1}^2 - b_2^2 + \mathbf{x_1}^2b_3^2, h_3]$,$J_2 = b_2$.
$\mathrm{CS}_3 = [\mathbf{x_3}, \mathbf{x_2} - 1, \mathbf{x_1}, b_2, h_3]$, $J_3 = 1$.
$\mathrm{CS}_4 = [\mathbf{x_3}, \mathbf{x_2} + 1, \mathbf{x_1}, b_2, h_3]$,$J_4 = 1$.
$\mathrm{CS}_5 = [\mathbf{x_3}, \mathbf{x_1}^2 + \mathbf{x_2}^2 - 1, b_2, b_3, h_3]$, $J_5 = 1$.
$\mathrm{CS}_6 = [h_3\mathbf{x_2} - \mathbf{x_3}, h_3\mathbf{x_2}^2 + \mathbf{x_1}b_3, h_3\mathbf{x_1}^2 - \mathbf{x_1}b_3 - h_3, b_2]$, $J_6 = h_3$.

With the above zero decomposition, the solutions of (8) are reduced to the solutions of $\mathrm{CS}_i = 0, i = 1, \ldots, 6$.

From the structure of these triangular sets, we could solve equation (8) as follows.

1. If $h_3 \neq 0, b_2 \neq 0$, we will use $\mathrm{CS}_1 = 0$ to find the solutions.
2. If $h_3 \neq 0, b_2 = 0$, we will use $\mathrm{CS}_6 = 0$ to find the solutions.
3. If $h_3 = 0, b_2 = 0, b_3 = 0$, we will use $\mathrm{CS}_5 = 0$ to find the solutions.
4. If $h_3 = 0, b_2 = 0, b_3 \neq 0$, we will use $\mathrm{CS}_3 = 0, \mathrm{CS}_4 = 0$ to find the solutions.
5. If $h_3 = 0, b_2 \neq 0$, we will use $\mathrm{CS}_2 = 0$ to find the solutions.

If we take $b_2 = \frac{1}{2}$, $b_3 = 0$ and $h_3 = 1$, we obtain four real solutions from $\mathrm{CS}_1 = 0$, which are $(\frac{\sqrt{3}}{2}, 0, 0, 0)$, $(-\frac{\sqrt{3}}{2}, 0, 0, 0)$, $(1, 0, 0, 0)$ and $(-1, 0, 0, 0)$. So the problem has four real solutions at most.

4 Conclusions

A generalization of the planar Stewart platform is introduced by considering all possible geometric constraints between three pairs of geometric primitives on the base and the platform respectively. This gives 16 types of planar GSPs. The purpose of introducing these new types of planar Stewart platforms is to find new and better parallel mechanisms. We give closed-form solutions to the direct kinematics of these GSPs. For the six GSPs with two distance constraints and one angular constraint, we are able to give a pure geometric solution based on ruler and compass constructions.

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