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Proper Reparametrization of Rational Ruled Surface

Jia Li (李　家), Li-Yong Shen (申立勇), and Xiao-Shan Gao (高小山)

Key Laboratory of Mathematics Mechanization, Institute of Systems Science, AMSS, Chinese Academy of Sciences Beijing 100080, China

E-mail: {lijia,xgao}@mmrc.iss.ac.cn; shenly@amss.ac.cn

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Abstract In this paper, we present a proper reparametrization algorithm for rational ruled surfaces. That is, for an improper rational parametrization of a ruled surface, we construct a proper rational parametrization for the same surface. The algorithm consists of three steps. We first reparametrize the improper rational parametrization caused by improper supports. Then the improper rational parametrization is transformed to a new one which is proper in one of the parameters. Finally, the problem is reduced to the proper reparametrization of planar rational algebraic curves.

Keywords algebraic ruled surface, algebraic curve, proper reparametrization, improper support

1 Introduction

Rational parametric curves and surfaces are one of the main tools used in geometric modeling. A key property of a set of rational parametrization is whether the rational parametrization is proper, that is, whether the values of the parameters and the image points are in a one to one correspondence. For instance

$$x = \frac{2t}{t^2+1}, \quad y = \frac{t^2-1}{t^2+1}$$

are the proper parametrization for the unit circle $x^2 + y^2 = 1$, while

$$x = \frac{2t^2}{t^4+1}, \quad y = \frac{t^4-1}{t^4+1}$$

are also a parametrization for the unit circle, but they are improper. Because for a point (x, y) on the circle, there exist two corresponding parameter values $t = \pm\sqrt{\frac{x}{1-y}}$. Improper parametrizations are undesirable because they contain redundant information which could lead to more expensive computations. So a natural question is: whether we can find a proper reparametrization for a set of improper rational parametrizations?

In the case of algebraic curves, the answer is positive. Based on Lüroth's theorem[1], various proper reparametrization algorithms for rational parametrizations of algebraic curves are developed, such as the Gröbner basis method, the characteristic set method, and the GCD method[2−5].

For algebraic surfaces, we can determine whether a surface is proper using the u-resultant[6] or the Gröbner basis[2]. However, the problem of finding a proper reparametrization for an improper rational parametrization of a surface is open in the general case[3]. There exist several partial results. In [5], a proper reparametrization algorithm was proposed for rational parametrizations which are improper in each parameter independently, that is, the proper reparametrization can be found by replacing each parameter with a rational function in itself. In [7], a proper reparametrization algorithm was proposed for rational parametrizations which are improper in only one of the parameters. In [8], a class of inherently improper parametric supports was studied.

Rational ruled surfaces are an important class of algebraic surfaces widely used in geometric modeling. Even for this simple class of surfaces, proper reparametrization algorithms do not exist. In [9, 10], the μ-basis was used to reparameterize rational ruled surface to obtain a new parametrization with lower degrees. But, the new parametrization is not necessarily proper.

In this paper, we give a solution to the proper reparametrization problem for algebraic ruled surfaces. Our algorithm works as follows. We first check if the improperness of the rational parametrization is caused by certain improper supports, and if it is, we

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will reparametrize the parametrization to get rid of this kind of improperness. Then, the improper rational reparametrization is transformed to a new one which is proper in one of the variables. Finally, by considering the improper parameter only, the rational parametrization can be treated as a rational parametrization of an algebraic curve with the proper parameter in the coefficients. Finally, we find a proper reparametrization for this curve with known methods and show that this reparametrization also provides a proper reparametrization for the ruled surface.

The rest of this paper is organized into four sections. In Section 2, we introduce the notations and preliminary results. In Section 3, we analyses the proper parametrization process of the ruled surface. In Section 4, we give the reparametrization algorithm and the example is provided. In Section 5, conclusions are given.

2 Notations and Preliminary Results

In this section, we introduce the notations and preliminary results needed in our algorithm. Let $\mathbb{Q}[s]$ be the polynomial ring over the field of rational numbers, and $\mathbb{Q}[s]^4$ the set of four-dimensional row vectors whose entries belong to $\mathbb{Q}[s]$. We consider a *rational ruled surface* in homogeneous form defined by a bi-degree $(n, 1)$ tensor product:

$$\begin{aligned}(x, y, z, w) &= \boldsymbol{P}(s,t) = \boldsymbol{P}_0(s) + \boldsymbol{P}_1(s)t \\ &= (a(s,t), b(s,t), c(s,t), d(s,t)) \end{aligned} \tag{1}$$

where $\boldsymbol{P}_i(s) = (a_i(s), b_i(s), c_i(s), d_i(s)) \in \mathbb{Q}[s]^4$, $i = 0, 1$ are called the *directrices* of $\boldsymbol{P}(s,t)$, $\gcd(a(s,t), b(s,t), c(s,t), d(s,t)) = 1$, and $d_0 + d_1 t \neq 0$. We assume that the rational parametrization (1) is nontrivial, that is, it defines a surface $f(x, y, z, w) = 0$. $\boldsymbol{P}(s,t)$ is nontrivial if and only if $\boldsymbol{P}_0(s)$ and $\boldsymbol{P}_1(s)$ are linearly independent over $\mathbb{Q}[s]$. A *reparametrization* of (1) is another rational parametrization which defines the same surface $f(x, y, z, w) = 0$.

The *inverse problem* for (1) is to find the values of the corresponding parameters s and t for a given point on the surface $f(x, y, z, w) = 0$. Equivalently, we need to solve t and s from the following equations[2]:

$$\begin{cases} (d_0(s) + d_1(s)t)x = (a_0(s) + a_1(s)t)w, \\ (d_0(s) + d_1(s)t)y = (b_0(s) + b_1(s)t)w, \\ (d_0(s) + d_1(s)t)z = (c_0(s) + c_1(s)t)w, \\ d_0(s) + d_1(s)t \neq 0. \end{cases} \tag{2}$$

The parametrization (1) is called *proper*, if for a generic point on the surface, t and s have a unique solution, that is, from (2), we have the *inverse map*:

$$t = r_1(x, y, z, w), \quad s = r_2(x, y, z, w)$$

where $r_1, r_2 \in \mathbb{Q}(x, y, z, w)$.

The parametrization (1) is called *proper for variable* t, if for a generic point on the surface, t has a unique solution, that is, from (2), we have the *inverse map* for t:

$$t = r_3(x, y, z, w)$$

where $r_3 \in \mathbb{Q}(x, y, z, w)$.

If a surface is not proper, then for a generic point on the surface, there exists a fixed number of parametric values corresponding to this point[2,6,11]. This fixed number is called the *improper index* of the parametrization (1), denoted by $IX(\boldsymbol{P})$. The improper index of (1) can be found by computing the u-resultant[6] or by computing the Gröbner basis of (2)[2].

One natural idea is to reparametrize (1) by finding a nontrivial decomposition

$$\begin{cases} a_i(s) = \bar{a}_i(h(s)), \\ b_i(s) = \bar{b}_i(h(s)), \\ c_i(s) = \bar{c}_i(h(s)), \\ d_i(s) = \bar{d}_i(h(s)), \quad i = 0, 1, \end{cases} \tag{3}$$

where $h(s) \in \mathbb{Q}(s)$. The following example shows that this is generally not possible.

Example 2.1. $\boldsymbol{P}(s,t) = (3s + (s+1)t, 2s + st, s - 1 + t, 1)$ is a parametrization of the plane $x - y - z - w = 0$. $\boldsymbol{P}(s,t)$ is not proper. With the method in [2], we could show that the improper index is two. It is clear that the polynomials $3s, s+1, 2s, s, s-1$ do not have a nontrivial decomposition like (3), so we cannot reparameterize $\boldsymbol{P}(s,t)$ with a set of new variables like $\bar{t} = t, \bar{s} = h(s)$. Similarly, we may check that $\boldsymbol{P}(s,t)$ is not proper for t (or s). So, the methods in [5, 7] cannot be used to solve this problem.

3 Proper Reparametrization

3.1 Improper Support for Ruled Surface

In this subsection, we will examine the improper parametrization for ruled surfaces caused by certain kinds of improper supports.

The *support* of $\boldsymbol{P}$ of (1) is the set of (m, n) such that $s^m t^n$ is a monomial in (1). Denote the support of $\boldsymbol{P}$ by $S(\boldsymbol{P})$. We assume that the supports of $a_0 + a_1 t, b_0 + b_1 t, c_0 + c_1 t$, and $d_0 + d_1 t$ are the same. It is clear that the supports of (1) could be arranged into the following form

$$\begin{cases} (g_1,0),(g_2,0),\ldots,(g_p,0); \\ (e_1,1),(e_2,1),\ldots,(e_q,1), \end{cases} \tag{4}$$

where $g_1 < g_2 < \cdots < g_p$ and $e_1 < e_2 < \cdots < e_q$. For a triangle S', the *normalized area* $NA(S')$ is twice the usual Euclidean area of S'. We define the *gcd of the degree gap* of (1) to be $\mathrm{gcddg}(\boldsymbol{P}) = \gcd\{g_{i+1}-g_i, e_{j+1}-e_j\colon i=1,\ldots,p-1;\ j=1,\ldots,q-1\}$.

If $\mathrm{gcddg}(\boldsymbol{P}) > 1$, we will show that $\boldsymbol{P}$ is always improper and this kind of support is called an *improper support*. We have the following proposition.

Proposition 3.1.1. *Let $\boldsymbol{P}(s,t)$ be a rational parametrization like* (1) *with support as* (4). *Then we have the following results.*

1) *If $\mathrm{gcddg}(\boldsymbol{P}) > 1$, then $\boldsymbol{P}$ is improper with improper index $\mathrm{gcddg}(\boldsymbol{P}) \cdot I$ for a positive integer I. In this case, we can reparametrize $\boldsymbol{P}$ such that the improper index for the new parametrization is I.*

2) *If $\mathrm{gcddg}(\boldsymbol{P}) = 1$, then $\boldsymbol{P}$ with randomly chosen coefficients is proper with probability one. In other words, $\boldsymbol{P}$ is improper only in a lower dimensional subspace of the coefficients space.*

Proof. First, we consider the rational parametric ruled surface like 1) with generic (indeterminant) coefficients. According to Theorem 1 of [8], the improper index of 1) is

$$IX(\boldsymbol{P}) = \gcd\{NA(S') : S' \subseteq S(\boldsymbol{P}), |S'| = 3\}.$$

The triangle in the support set $S(\boldsymbol{P})$ can only be $\{(g_{i_1},0),\ (g_{i_2},0),\ (e_j,1)\}$ or $\{(g_i,0),\ (e_{j_1},1),\ (e_{j_2},1)\}$, $i_1 < i_2$, $j_1 < j_2$. The corresponding normalized area is $g_{i_2} - g_{i_1}$ or $e_{j_2} - e_{j_1}$. Then

$$\begin{aligned} IX(\boldsymbol{P}) &= \gcd\{NA(S') : S' \subseteq S(\boldsymbol{P}), |S'| = 3\} \\ &= \gcd\{g_{i_2} - g_{i_1}, e_{j_2} - e_{j_1} :\ 1 \leqslant i_1 < i_2 \leqslant p; \\ &\quad 1 \leqslant j_1 < j_2 \leqslant q\} \\ &= \gcd\{g_{i+1} - g_i, e_{j+1} - e_j :\ 1 \leqslant i \leqslant p-1; \\ &\quad 1 \leqslant j \leqslant q-1\} \\ &= \mathrm{gcddg}(\boldsymbol{P}). \end{aligned}$$

For a rational parametrization (1) with coefficients in $\mathbb{Q}$, its improper index is $IX(\boldsymbol{P}) \cdot I = \mathrm{gcddg}(\boldsymbol{P}) \cdot I$ for a positive integer I ([8], Theorem 2).

When $\mathrm{gcddg}(\boldsymbol{P}) > 1$, we can reparametrize $\boldsymbol{P}$ such that the improper index for the new parametrization is I. We need only to reparametrize $\boldsymbol{P}$ such that the gcd of degree gap for the new parametrization is 1. If $(0,0)$ is not in the support $S(\boldsymbol{P})$. We use $s' = s$, $t' = ts^{e_1 - g_1}$ to reparametrize $\boldsymbol{P}$ as $\boldsymbol{P}'$. Let $g = \mathrm{gcddg}(\boldsymbol{P})$. Then the support for $\boldsymbol{P}'$ consists of

$$\begin{aligned} &(0,0),(k_1 g,0),\ldots,(k_{p-1} g,0), \\ &(0,1),(h_1 g,1),\ldots,(h_{q-1} g,1), \end{aligned}$$

where k_i and h_i are positive integers. Then the support of the new parametrization $\boldsymbol{P}'$ contains $(0,0)$ and $\mathrm{gcddg}(\boldsymbol{P}') = \mathrm{gcddg}(\boldsymbol{P})$. So we can reparametrize $\boldsymbol{P}$ with the new parameters $\bar{s} = s^g$, $\bar{t} = t$ and obtain the reparametrization $\overline{\boldsymbol{P}}$ of $\boldsymbol{P}$ such that $\mathrm{gcddg}(\overline{\boldsymbol{P}}) = 1$.

If $\mathrm{gcddg}(\boldsymbol{P}) = 1$, for coefficients of (1) taken from a Zariski open set in the coefficients space $\mathbb{Q}^{4|S(\boldsymbol{P})|}$, parametrization (1) is proper ([8], Theorem 3). Since the Zariski open set is the whole coefficients space minus a set with a lower dimension, we prove the proposition. $\square$

Example 3.1.1.

$$\begin{aligned} &\boldsymbol{P}(s,t) = (s + s^3 + (1+s^2)t, \\ &s + 2s^3 + (1-s^2)t, s^3 + t, s + t). \end{aligned}$$

The support of $\boldsymbol{P}$ is

$$S(\boldsymbol{P}) = \{(1,0),(3,0),(0,1),(2,1)\},$$

and $\mathrm{gcddg}(\boldsymbol{P}) = \gcd\{3-1, 2-0\} = 2$. S is an improper support, and $\boldsymbol{P}$ is improper. Note that $(0,0) \notin S(\boldsymbol{P})$, $g_1 = 1$, and $e_1 = 0$. Let $s' = s$, $t' = t/s$ be the new parameters. We obtain a reparametrization:

$$\begin{aligned} &\boldsymbol{P}'(s',t') = (1 + {s'}^2 + (1+{s'}^2)t', \\ &1 + 2{s'}^2 + (1-{s'}^2)t', {s'}^2 + t', 1 + t'). \end{aligned}$$

The support for $\boldsymbol{P}'$ is

$$S(\boldsymbol{P}') = \{(0,0),(2,0),(0,1),(2,1)\},$$

$\mathrm{gcddg}(\boldsymbol{P}') = 2$ and $(0,0) \in S(\boldsymbol{P}')$. Using $\bar{s} = {s'}^2$, $\bar{t} = t'$ to reparametrize $\boldsymbol{P}'$, we obtain a new parametrization of $\boldsymbol{P}$:

$$\begin{aligned} &\overline{\boldsymbol{P}}(\bar{s},\bar{t}) = (1 + \bar{s} + (1+\bar{s})\bar{t}, \\ &1 + 2\bar{s} + (1-\bar{s})\bar{t}, \bar{s} + \bar{t}, 1 + \bar{t}). \end{aligned}$$

We have $S(\overline{\boldsymbol{P}}) = \{(0,0),(1,0),(0,1),(1,1)\}$ and $\mathrm{gcddg}(\overline{\boldsymbol{P}}) = 1$.

The above result shows that if $\mathrm{gcddg}(\boldsymbol{P}) > 1$, we may reparametrize the parametrization such that the degree gap of the new parametrization is 1. Almost all parametric ruled surfaces whose degree gap of the parametrization is 1 are proper.

3.2 Proper Reparametrization for Parameter t

Example 2.1 shows that an improper parametrization of (1) for a ruled surface is generally not proper for the variable t. In this subsection, we will first reparametrize such an improper parametrization to make the new parametrization proper for the variable t.

Theorem 3.2.1. *Let $\boldsymbol{P}(s,t)$ be a rational ruled surface of* (1). *Then we can reparameterize $\boldsymbol{P}(s,t)$ as $\overline{\boldsymbol{P}}(\bar{s},\bar{t})$ such that $\bar{t}$ is proper in $\overline{\boldsymbol{P}}(\bar{s},\bar{t})$. More precisely, we have*

$$\overline{\boldsymbol{P}}(\bar{s},\bar{t}) = (\bar{a}_0(\bar{s}) + \bar{a}_1(\bar{s})\bar{t}, \bar{b}_0(\bar{s}) + \bar{b}_1(\bar{s})\bar{t}, \bar{d}(\bar{s})\bar{t}, \bar{d}(\bar{s})),$$

where $\bar{a}_i$, $\bar{b}_i$, $\bar{d} \in \mathbb{Q}[\bar{s}]$, $i = 0, 1$.

Proof. Since $\boldsymbol{P}(s,t) = (a_0(s)+a_1(s)t,\ b_0(s)+b_1(s)t,\ c_0(s)+c_1(s)t,\ d_0(s)+d_1(s)t)$ is a rational ruled surface, d_0 and d_1 cannot be zero simultaneously.

If $d_1 = 0$, at least one of $a_1(s)$, $b_1(s)$, $c_1(s)$ is not zero since $\boldsymbol{P}$ is a rational ruled surface. Without loss of generality, we assume $c_1(s) \neq 0$. Let

$$\bar{t} = \frac{c_0(s) + c_1(s)t}{d_0(s)}, \quad \bar{s} = s \tag{5}$$

be the new parameters. Substituting

$$t = \frac{d_0(\bar{s})\bar{t} - c_0(\bar{s})}{c_1(\bar{s})}, \quad s = \bar{s}$$

into (1), we have

$$\begin{aligned}\overline{\boldsymbol{P}}(\bar{s},\bar{t}) &= \Big(\frac{a_0(\bar{s})c_1(\bar{s}) - a_1(\bar{s})c_0(\bar{s}) + a_1(\bar{s})d_0(\bar{s})\bar{t}}{c_1(\bar{s})d_0(\bar{s})},\\ &\quad \frac{b_0(\bar{s})c_1(\bar{s}) - b_1(\bar{s})c_0(\bar{s}) + b_1(\bar{s})d_0(\bar{s})\bar{t}}{c_1(\bar{s})d_0(\bar{s})}, \bar{t}, 1\Big)\\ &= (a_0(\bar{s})c_1(\bar{s}) - a_1(\bar{s})c_0(\bar{s}) + a_1(\bar{s})d_0(\bar{s})\bar{t},\\ &\quad b_0(\bar{s})c_1(\bar{s}) - b_1(\bar{s})c_0(\bar{s}) + b_1(\bar{s})d_0(\bar{s})\bar{t},\\ &\quad c_1(\bar{s})d_0(\bar{s})\bar{t}, c_1(\bar{s})d_0(\bar{s})).\end{aligned}$$

$\overline{\boldsymbol{P}}(\bar{s},\bar{t})$ is proper in $\bar{t}$, because from $(x,y,z,w) = \overline{\boldsymbol{P}}(\bar{s},\bar{t})$, we can find the inverse map for $\bar{t}$: $\bar{t} = \frac{z}{w}$.

If $d_1 \neq 0$, let

$$t' = \frac{1}{d_0(s) + d_1(s)t}, \quad s = s' \tag{6}$$

be the new parameters. Substituting

$$s = s', \quad t = \frac{1 - d_0(s')t'}{d_1(s')t'}$$

into $\boldsymbol{P}(s,t)$, we have

$$\begin{aligned}\boldsymbol{P}'(s',t') &= \Big(\frac{(a_0(s')d_1(s') - a_1(s')d_0(s'))t' + a_1(s')}{d_1(s')},\\ &\quad \frac{(b_0(s')d_1(s') - b_1(s')d_0(s'))t' + b_1(s')}{d_1(s')},\\ &\quad \frac{(c_0(s')d_1(s') - c_1(s')d_0(s'))t' + c_1(s')}{d_1(s')}, 1\Big)\\ &= ((a_0(s')d_1(s') - a_1(s')d_0(s'))t' + a_1(s'),\\ &\quad (b_0(s')d_1(s') - b_1(s')d_0(s'))t' + b_1(s'),\\ &\quad (c_0(s')d_1(s') - c_1(s')d_0(s'))t' + c_1(s'),\\ &\quad d_1(s')),\end{aligned}$$

which becomes case one. This proves the theorem. □

Corollary 3.2.1. *$\boldsymbol{P}(s,t)$ is proper if and only if $\overline{\boldsymbol{P}}(\bar{s},\bar{t})$ is proper.*

Proof. Note that the transformations (5) and (6) are birational transformations of the same rational ruled surface. Hence we have the result. □

Example 3.2.1. Continuing from Example 2.1, for the improper parametrization

$$\boldsymbol{P}(s,t) = (3s + (s+1)t, 2s + st, s - 1 + t, 1)$$

of the plane $x - y - z - w = 0$, let $\bar{t} = s - 1 + t$, $\bar{s} = s$. Substituting $t = \bar{t} - \bar{s} + 1$, $s = \bar{s}$ into $\boldsymbol{P}$, we have the new parametrization of $\boldsymbol{P}(s,t)$

$$\overline{\boldsymbol{P}}(\bar{s},\bar{t}) = (1 + 3\bar{s} - \bar{s}^2 + (\bar{s}+1)\bar{t}, 3\bar{s} - \bar{s}^2 + \bar{s}\bar{t}, \bar{t}, 1)$$

which is proper for $\bar{t}$.

3.3 Proper Reparametrization for Parameter s

From Theorem 3.2.1, we may assume that the ruled surface has the following form in affine space:

$$\begin{cases} x = \dfrac{a_0(s) + a_1(s)t}{d(s)}, \\ y = \dfrac{b_0(s) + b_1(s)t}{d(s)}, \\ z = t, \end{cases} \tag{7}$$

where a_0, a_1, b_0, b_1, $d \in \mathbb{Q}[s]$.

Our idea is to treat (7) as a rational parametrization of a planar algebraic curve of (x,y) in the parameter s and with coefficients in $\mathbb{Q}(t)$. Then we can find a proper parametrization for this curve and show that this reparametrization also provides a proper reparametrization for the ruled surface.

Here, we need a proper reparametrization algorithm for algebraic curves. There exist several such algorithms. The method given in [5] is the simplest one and will be used in this paper.

Theorem 3.3.1[5]**.** *Let*

$$\boldsymbol{C}(s) = \Big(\frac{P_{11}(s)}{P_{12}(s)}, \frac{P_{21}(s)}{P_{22}(s)}\Big)$$

be a rational planar curve, where $P_{ij}(s) \in \mathbb{Q}[s]$, $i, j = 1, 2$. Let

$$\begin{aligned} H_1(s,\bar{s}) &= P_{11}(s)P_{12}(\bar{s}) - P_{12}(s)P_{11}(\bar{s}), \\ H_2(s,\bar{s}) &= P_{21}(s)P_{22}(\bar{s}) - P_{22}(s)P_{21}(\bar{s}), \\ H(s,\bar{s}) &= \gcd(H_1, H_2). \end{aligned}$$

If $H = c(s-\bar{s})$ for $c \in \mathbb{Q}$, then $\boldsymbol{C}(s)$ is proper. Otherwise, write H as a polynomial in $\bar{s}$:

$$H = c_d\bar{s}^d + \cdots + c_1\bar{s} + c_0, \quad c_d \neq 0,$$

where $c_i \in \mathbb{Q}[s]$, $i = 0, \ldots, d$. Then, there exist $k, l, k \neq l$ such that $\frac{c_k}{c_l} \notin \mathbb{Q}$ and $\bar{s} = \bar{s}(s) = \frac{c_k}{c_l}$ is a new parameter for the curve.

Furthermore, let

$$L_i(\bar{s}, x_i) = \mathrm{resl}(x_iP_{i2}(s) - P_{i1}(s), c_l\bar{s} - c_k, s), \quad i = 1, 2$$

be the resultant w.r.t. s. Then

$$L_i(\bar{s}, x_i) = (Q_{i2}x_i - Q_{i1})^{\deg(\bar{s},s)}, \quad i = 1, 2,$$

and

$$\overline{\boldsymbol{C}}(\bar{s}) = \Big(\frac{Q_{11}(\bar{s})}{Q_{12}(\bar{s})}, \frac{Q_{21}(\bar{s})}{Q_{22}(\bar{s})}\Big)$$

is a proper reparametrization of the curve $\boldsymbol{C}(s)$, where $\deg(\bar{s}, s) = \max\{\deg(c_k, s), \deg(c_l, s)\}$.

As a consequence of Theorem 3.3.1, we have the following result.

Theorem 3.3.2. *Consider a ruled surface (7). Let*

$$\begin{aligned} H_1(s,\bar{s}) &= d(s)(a_0(\bar{s}) + a_1(\bar{s})t) - (a_0(s) + a_1(s)t)d(\bar{s}), \\ H_2(s,\bar{s}) &= d(s)(b_0(\bar{s}) + b_1(\bar{s})t) - (b_0(s) + b_1(s)t)d(\bar{s}), \\ H(s,\bar{s}) &= \gcd(H_1, H_2). \end{aligned}$$

If $H = c(s-\bar{s})$ for $c \in \mathbb{Q}[t]$, then (7) is proper; otherwise, write H as a polynomial in $\bar{s}$:

$$H = c_d\bar{s}^d + \cdots + c_1\bar{s} + c_0, \quad c_d \neq 0,$$

where $c_i \in \mathbb{Q}[t][s]$, $i = 0, \ldots, d$. Then there exist $k, l, k \neq l$ such that $\frac{c_k}{c_l} \notin \mathbb{Q}(t)$, and a set of new parameters for the surface are

$$\bar{s} = \frac{c_k(s,t)}{c_l(s,t)}, \quad \bar{t} = t.$$

Furthermore, let

$$\begin{aligned} L_1(\bar{s}, x) &= \mathrm{resl}(G_1(s,x), c_l\bar{s} - c_k, s), \\ L_2(\bar{s}, y) &= \mathrm{resl}(G_2(s,y), c_l\bar{s} - c_k, s), \end{aligned}$$

where

$$\begin{aligned} G_1(s,x) &= xd(s) - (a_0(s) + a_1(s)t), \\ G_2(s,y) &= yd(s) - (b_0(s) + b_1(s)t). \end{aligned}$$

Then

$$\begin{aligned} L_1 &= (Q_{12}(\bar{s},t)x - Q_{11}(\bar{s},t))^{\deg(\bar{s},s)}, \\ L_2 &= (Q_{22}(\bar{s},t)y - Q_{21}(\bar{s},t))^{\deg(\bar{s},s)}, \end{aligned}$$

where $Q_{ij} \in \mathbb{Q}[\bar{s}, t]$. A proper reparametrization of (7) using the new parameters $\bar{s}, \bar{t}$ is $\big(\frac{Q_{11}(\bar{s},\bar{t})}{Q_{12}(\bar{s},\bar{t})}, \frac{Q_{21}(\bar{s},\bar{t})}{Q_{22}(\bar{s},\bar{t})}, \bar{t}\big)$.

Proof. We can consider

$$\begin{cases} x = \dfrac{a_0(s) + a_1(s)t}{d(s)} \\ y = \dfrac{b_0(s) + b_1(s)t}{d(s)} \end{cases} \tag{8}$$

as a planar curve over the base field $\mathbb{Q}(t)$. From Theorem 3.3.1,

$$\begin{cases} x = \dfrac{Q_{11}(\bar{s},\bar{t})}{Q_{12}(\bar{s},\bar{t})} \\ y = \dfrac{Q_{21}(\bar{s},\bar{t})}{Q_{22}(\bar{s},\bar{t})} \end{cases} \tag{9}$$

is a proper reparametrization of (8). Let the inversion map of (9) be $\bar{s} = r(x, y, \bar{t})$, where $r(x, y, \bar{t}) \in \mathbb{Q}(x, y, \bar{t})$. Since $z = \bar{t} = t$. The inversion map for (7) is $\bar{s} = r(x, y, z)$, $\bar{t} = z$. Therefore,

$$\Big(\frac{Q_{11}(\bar{s},\bar{t})}{Q_{12}(\bar{s},\bar{t})}, \frac{Q_{21}(\bar{s},\bar{t})}{Q_{22}(\bar{s},\bar{t})}, \bar{t}\Big) \tag{10}$$

is a proper reparametrization of (7). □

Example 3.3.1. In Example 3.2.1, we obtain the following new parametrization of the ruled surface

$$\overline{\boldsymbol{P}}(\bar{s},\bar{t}) = (1 + 3\bar{s} - \bar{s}^2 + (\bar{s}+1)\bar{t}, 3\bar{s} - \bar{s}^2 + \bar{s}\bar{t}, \bar{t}, 1)$$

which is proper for $\bar{t}$, but not proper for $\bar{s}$.

Following Theorem 3.3.2, the corresponding affine form of $\overline{\boldsymbol{P}}(\bar{s},\bar{t})$ is

$$\begin{aligned} (x, y, z) &= \overline{\boldsymbol{AP}} \\ &= (1 + 3\bar{s} - \bar{s}^2 + (\bar{s}+1)\bar{t}, 3\bar{s} - \bar{s}^2 + \bar{s}\bar{t}, \bar{t}). \end{aligned}$$

We treat the first two parameters of $\overline{\boldsymbol{AP}}$ as a rational parametrization of an algebraic curve with parameter s with coefficients in the field $\mathbb{Q}(\bar{t})$.

Using Theorem 3.3.2, we have

$$\begin{aligned} H_1(\bar{s}, s') &= (1 + 3s' - s'^2 + (s'+1)\bar{t}) - (1 + 3\bar{s} - \bar{s}^2 + (\bar{s}+1)\bar{t}), \\ H_2(\bar{s}, s') &= (3s' - s'^2 + s'\bar{t}) - (3\bar{s} - \bar{s}^2 + \bar{s}\bar{t}), \\ H &= s'^2 + (-\bar{t} - 3)s' - \bar{s}^2 + \bar{s}\bar{t} + 3\bar{s}. \end{aligned}$$

Then $c_2 = 1$, $c_1 = (-\bar{t} - 3)$, $c_0 = -\bar{s}^2 + \bar{s}\bar{t} + 3\bar{s}$. We obtain the new parameters $s' = \frac{c_0}{c_2} = 3\bar{s} - \bar{s}^2 + \bar{t}\bar{s}$, $t' = \bar{t}$

and a proper reparametrization of $\overline{\boldsymbol{AP}}$

$$(s'+1+t', s', t').$$

Corollary 3.3.1. *In Theorem* 3.3.2*, if* c_k*,* $c_l \in \mathbb{Q}[s]$*, then the parameters* s *and* t *can be separated. More precisely, we can obtain a new parameter* $\bar{s} = \frac{\psi_1(s)}{\psi_2(s)}$ *free of* t*. The new parametrization* (10) *is linear in* $\bar{t}$ *and* Q_{12} *and* Q_{22} *are free of* $\bar{t}$*.*

Proof. If c_k, $c_l \in \mathbb{Q}[s]$, then the new parameter is $\bar{s} = \frac{c_k}{c_l} \in \mathbb{Q}(s)$. So the parameters s and t can be separated. We will prove that the new parametrization is also linear in $\bar{t}$. We obtain the new parametrization by computing resultants. Note that G_1 is linear in t and x, and $c_k\bar{s} - c_l$ does not contain the parameter t. So $Q_{12}(\bar{s})x - Q_{11}(\bar{s})$ has the same degree in x and t, and hence is linear in t. More precisely, the coefficient of x in G_1 is free of t, so the leading coefficient of the resultant is free of t. It means that Q_{12} is free of t. The same is true for $Q_{22}(\bar{s})y - Q_{21}(\bar{s})$. We prove the corollary. □

In Theorem 3.3.2, we reparametrize the ruled surface in the affine space. The new parametric ruled surface is still proper when we consider it in homogenous form.

Theorem 3.3.3. *Let*

$$\boldsymbol{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t))$$

as (1) *be a rational ruled surface in homogenous form, and*

$$\boldsymbol{AP}(s,t) = \Big(\frac{a(s,t)}{d(s,t)}, \frac{b(s,t)}{d(s,t)}, \frac{c(s,t)}{d(s,t)}\Big) \tag{11}$$

a parametric surface of (1) *in affine space. Then* $\boldsymbol{P}$ *is a proper parametrization in homogenous form if and only if* $\boldsymbol{AP}$ *is a proper parametrization in affine space.*

Proof. Let the algebraic degree of the ruled surface be n, which is the degree of the implicit equation of the ruled surface, and the improper indexes of (1) and (11) be IX_1 and IX_2 respectively. To prove this theorem, we need only to prove $IX_1 = IX_2$. According to [11], we can compute the improper index of a surface by computing the number of intersections of a generic line with this surface. We use two generic planes to represent a generic line:

$$\begin{cases} u_1x + u_2y + u_3z + u_0w = 0, \\ v_1x + v_2y + v_3z + v_0w = 0. \end{cases} \tag{12}$$

In projective space, the number of solutions of s and t for

$$\begin{cases} u_1a(s,t) + u_2b(s,t) + u_3c(s,t) + u_0d(s,t) = 0, \\ v_1a(s,t) + v_2b(s,t) + v_3c(s,t) + v_0d(s,t) = 0. \end{cases} \tag{13}$$

is $nIX_1+\alpha$, where α is the number of the common zeros of $a(s,t), b(s,t), c(s,t), d(s,t)$[6].

In affine space, we consider the solutions of the polynomial equations obtained by substituting (11) into (12) with $w = 1$,

$$\begin{cases} u_1\dfrac{a(s,t)}{d(s,t)} + u_2\dfrac{b(s,t)}{d(s,t)} + u_3\dfrac{c(s,t)}{d(s,t)} + u_0 = 0, \\ v_1\dfrac{a(s,t)}{d(s,t)} + v_2\dfrac{b(s,t)}{d(s,t)} + v_3\dfrac{c(s,t)}{d(s,t)} + v_0 = 0. \end{cases} \tag{14}$$

The number of the solutions of (14) is nIX_2. And the solutions of (14) are the solutions of (13) by removing solutions of

$$\begin{cases} u_1a(s,t) + u_2b(s,t) + u_3c(s,t) = 0, \\ v_1a(s,t) + v_2b(s,t) + v_3c(s,t) = 0, \\ d(s,t) = 0. \end{cases} \tag{15}$$

Now, we prove that (15) has α solutions for s and t. Computing the resultant of two equations in (13) with respect to t. Note that these two equations are both linear in t, we have:

$$Resl = c(u,v)\prod_{i=1}^{\alpha}(s-\alpha_i)\prod_j(s-s_j),$$

where $c(u,v) \in \mathbb{Q}[u,v]$ and α_i are the s-coordinates corresponding to the common zeros of $a(s,t)$, $b(s,t)$, $c(s,t)$, $d(s,t)$ and s_j are the s-coordinates corresponding to other solutions of (13).

If (15) has more than α solutions, then there is at least one solution of (15) which does not vanish $a(s,t)$ or $b(s,t)$ or $c(s,t)$. Without loss of generality, we assume it is not the zero of $a(s,t)$. That is to say the equation system

$$\begin{cases} u_1a(s,t) + u_2b(s,t) + u_3c(s,t) = 0, \\ v_1a(s,t) + v_2b(s,t) + v_3c(s,t) = 0, \\ \eta a(s,t) - 1 = 0, \\ d(s,t) = 0. \end{cases} \tag{16}$$

has at least one solution for s, t, η. Denote

$$P_1 = u_1a(s,t) + u_2b(s,t) + u_3c(s,t),$$
$$P_2 = v_1a(s,t) + v_2b(s,t) + v_3c(s,t),$$
$$P_3 = \eta a(s,t) - 1.$$

It is easy to show that the ideal $(P_1, P_2, P_3) \subset \mathbb{Q}[u_1, u_2, u_3, v_1, v_2, v_3, s, t, \eta]$ is prime with dimension six. It is also easy to see that $d(s,t) \notin (P_1, P_2, P_3)$. By a well known result in algebraic geometry[11], (P_1, P_2, P_3, d) is of dimension less than or equal to five.

This means that there exists a nonzero polynomial R in $(P_1, P_2, P_3, d) \cap \mathbb{Q}[u_1, u_2, u_3, v_1, v_2, v_3]$. This contradicts to the fact that (12) are two generic algebraic planes. So $nIX_1 = nIX_2$, and hence $IX_1 = IX_2$. $\square$

4 Proper Reparametrization Algorithm

Base on the results proved in Section 3, we can now give the proper reparametrization algorithm.

Algorithm. *The Proper Reparametrization Algorithm*

Input: $\boldsymbol{P}(s,t) = (a_0(s) + a_1(s)t,\ b_0(s) + b_1(s)t,\ c_0(s) + c_1(s)t,\ d_0(s) + d_1(s)t)$, where a_i, b_i, c_i, $d_i \in \mathbb{Q}[s]$.

Output: If $\boldsymbol{P}$ is proper, then return "$\boldsymbol{P}$ is proper." Otherwise, return a proper reparametrization of $\boldsymbol{P}(s,t)$.

S1. Get the support of $\boldsymbol{P}$:

$$S(\boldsymbol{P}) = \{(g_1, 0), (g_2, 0), \ldots, (g_p, 0); (e_1, 1), (e_2, 1), \ldots, (e_q, 1)\},$$

where $g_1 < g_2 < \cdots < g_p$ and $e_1 < e_2 < \cdots < e_q$. Compute $\mathrm{gcddg}(\boldsymbol{P})$. If $\mathrm{gcddg}(\boldsymbol{P}) = 1$, then goto S3. If $(0,0) \in S(\boldsymbol{P})$, then goto S2.

Let $t' = ts^{e_1 - g_1}$ and substitute $t = \frac{t'}{s^{e_1 - g_1}}$ into $\boldsymbol{P}(s,t)$. We obtain a new reparametrization $\boldsymbol{P}'(s,t')$.

Let $\boldsymbol{P}(s,t) = \boldsymbol{P}'(s,t')$.

S2. Let $s' = s^{\mathrm{gcddg}(\boldsymbol{P})}$ and substitute $s = s'^{\frac{1}{\mathrm{gcddg}(\boldsymbol{P})}}$ into $\boldsymbol{P}(s,t)$. We obtain a new reparametrization $\boldsymbol{P}'(s',t)$.

Set $\boldsymbol{P}(s,t) = \boldsymbol{P}'(s',t)$ by set s' to be s.

S3. If $d_1 = 0$, goto S4.

Let $t' = \frac{1}{d_0(s) + d_1(s)t}$ and substitute $t = \frac{1 - d_0(s)t'}{d_1(s)t'}$ into $\boldsymbol{P}(s,t)$. We obtain a new reparametrization $\boldsymbol{P}'(s,t')$. Set $\boldsymbol{P}(s,t) = \boldsymbol{P}'(s,t')$ by set t' to be t.

S4. Without loss of generality, we may assume $c_1(s) \neq 0$. Let $\bar{t} = \frac{c_0(s) + c_1(s)t}{d(s)}$ and substitute $t = \frac{d(s)\bar{t} - c_0(s)}{c_1(s)}$ into $\boldsymbol{P}$. We obtain a new reparametrization:

$$\begin{cases} x = \dfrac{\bar{a}_0(s) + \bar{a}_1(s)\bar{t}}{\bar{d}(s)}, \\ y = \dfrac{\bar{b}_0(s) + \bar{b}_1(s)\bar{t}}{\bar{d}(s)}, \\ z = \bar{t}, \end{cases}$$

where $\bar{a}_0$, $\bar{a}_1$, $\bar{b}_0$, $\bar{b}_1$, $\bar{d} \in \mathbb{Q}[s]$.

S5. Compute

$$\begin{aligned} H_1(s,\bar{s}) &= \bar{d}(s)(\bar{a}_0(\bar{s}) + \bar{a}_1(\bar{s})\bar{t}) - (\bar{a}_0(s) + \bar{a}_1(s)\bar{t})\bar{d}(\bar{s}), \\ H_2(s,\bar{s}) &= \bar{d}(s)(\bar{b}_0(\bar{s}) + \bar{b}_1(\bar{s})\bar{t}) - (\bar{b}_0(s) + \bar{b}_1(s)\bar{t})\bar{d}(\bar{s}), \\ H(s,\bar{s}) &= \gcd(H_1, H_2). \end{aligned}$$

If $H = c(s - \bar{s})$ for $c \in \mathbb{Q}[\bar{t}]$, return "$\boldsymbol{P}(s)$ is proper."

S6. Write H as a polynomial in $\bar{s}$:

$$H = c_d \bar{s}^d + \cdots + c_1 \bar{s} + c_0, \quad c_d \neq 0,$$

where $c_i \in Q[\bar{t}][s]$, $i = 0, \ldots, d$. There exists a $k \neq d$ such that $\frac{c_k}{c_d} \notin Q(\bar{t})$.

S7. Let $D = \max\{\deg(c_d, s), \deg(c_k, s)\}$. Compute

$$\begin{aligned} L_1(\bar{s}, x) &= \mathrm{resl}(G_1(s,x), c_d\bar{s} - c_k, s) \\ &= (Q_{12}(\bar{s})x - Q_{11}(\bar{s}))^D, \\ L_2(\bar{s}, y) &= \mathrm{resl}(G_2(s,y), c_d\bar{s} - c_k, s) \\ &= (Q_{22}(\bar{s})y - Q_{21}(\bar{s}))^D, \end{aligned}$$

where $G_1(s,x) = xd(s) - (a_0(s) + a_1(s)\bar{t})$, $G_2(s,y) = yd(s) - (b_0(s) + b_1(s)\bar{t})$.

S8. Let

$$\begin{aligned} &(Q_1(\bar{s},\bar{t}), Q_2(\bar{s},\bar{t}), Q_3(\bar{s},\bar{t}), Q_4(\bar{s},\bar{t})) \\ &= (Q_{11}Q_{22}, Q_{21}Q_{12}, Q_{12}Q_{22}\bar{t}, Q_{12}Q_{22}) \end{aligned}$$

and

$$F = \gcd(Q_1, Q_2, Q_3, Q_4).$$

Return

$$\left(\frac{Q_1(\bar{s},\bar{t})}{F}, \frac{Q_2(\bar{s},\bar{t})}{F}, \frac{Q_3(\bar{s},\bar{t})}{F}, \frac{Q_4(\bar{s},\bar{t})}{F}\right).$$

It is easy to see that the algorithm works for rational parametrizations of the affine form $\boldsymbol{AP}(s,t) = (r_1(s,t), r_2(s,t), r_3(s,t))$ where $r_i \in \mathbb{Q}(s,t)$ and one of r_i is linear in s or t.

Computationally, the most difficult step of the algorithm is S7, where we need to compute two resultants of polynomials with degrees not higher than the degrees of a_i, b_i, c_i, d_i in s. Therefore, the algorithm is efficient for rational parametrizations (1) with moderately high degrees.

Note that the algorithm still outputs a proper reparametrization if we remove Steps S1 and S2. By adding them, the efficiency of the algorithm can be enhanced in two aspects. First, these steps may reduce the degree of s and hence reduce computation costs in later steps. Second, by Proposition 3.1.1, almost all reparametrizations obtained by Steps S1 and S2 are proper. As a consequence, Step S7 is not needed.

Also note that if the given rational parametrization is from $\mathbb{Q}[s,t]$, then the algorithm will work over $\mathbb{Q}$ and the proper reparametrization also has coefficients in $\mathbb{Q}$.

We use the following example to illustrate the algorithm.

Example 4.1. $\boldsymbol{P} = ((2s^3 + 2s^2 - 2s)t - s^4 + s^2 - 2s + 1,\ st + s - 1,\ st - s^2, 1)$.

S1. The support of $\boldsymbol{P}$ is

$$S(\boldsymbol{P}) = \{(0,0), (1,0), (2.0), (4,0), (1,1), (2,1), (3,1)\}.$$

We have $\mathrm{gcddg}(\boldsymbol{P}) = 1$, goto S3.

S3. $d_1 = 0$, goto S4.

S4. $c_1(s) = s \neq 0$. Let $\bar{t} = \frac{c_0(s)+c_1(s)t}{d(s)} = st - s^2$ and substitute $t = \frac{\bar{t}+s^2}{s}$ into $\boldsymbol{P}$. We obtain a new parametrization:

$$\begin{cases} x = -s^2 - 2s + 1 + s^4 + 2\bar{t}s - 2\bar{t} + 2\bar{t}s^2 + 2s^3, \\ y = s - 1 + \bar{t} + s^2, \\ z = \bar{t}. \end{cases}$$

S5. Compute

$$\begin{aligned} H_1(s,\bar{s}) = &(-s^2 - 2s + 1 + s^4 + 2\bar{t}s - 2\bar{t} + 2\bar{t}s^2 + 2s^3) \\ &- (-\bar{s}^2 - 2\bar{s} + 1 + \bar{s}^4 + 2\bar{t}\bar{s} \\ &- 2\bar{t} + 2\bar{t}\bar{s}^2 + 2\bar{s}^3), \\ H_2(s,\bar{s}) = &(s - 1 + \bar{t} + s^2) - (\bar{s} - 1 + \bar{t} + \bar{s}^2), \\ H(s,\bar{s}) = &\gcd(H_1, H_2) = -\bar{s}^2 - \bar{s} + s + s^2. \end{aligned}$$

S6. $\boldsymbol{P}(s)$ is not proper. Write H as a polynomial in $\bar{s}$:

$$H = c_2\bar{s}^2 + c_1\bar{s} + c_0$$

where $c_2 = -1$, $c_1 = -1$, $c_0 = s + s^2$. Note that $\frac{c_0}{c_2} = s + s^2 \notin Q(\bar{t})$.

S7. $D = 2$. Compute

$$\begin{aligned} L_1(\bar{s},x) = &\,\mathrm{resl}(x - (-s^2 - 2s + 1 + s^4 \\ &+ 2\bar{t}s - 2\bar{t} + 2\bar{t}s^2 + 2s^3), \\ &- \bar{s} - s - s^2, s) \\ = &\,(x - 1 + 2\bar{t} - 2\bar{s} + 2\bar{t}\bar{s} - \bar{s}^2)^2, \\ L_2(\bar{s},y) = &\,\mathrm{resl}(y - (s - 1 + \bar{t} + s^2), -\bar{s} - s - s^2, s) \\ = &\,(-y - 1 + \bar{t} - \bar{s})^2, \end{aligned}$$

S8. Return the following proper reparametrization:

$$(-2(1+\bar{s})\bar{t} + 1 + 2\bar{s} + \bar{s}^2, \bar{t} - \bar{s} - 1, \bar{t}, 1).$$

The implicit equation for the parametrization is $x = y^2 - z^2$.

5 Conclusion

In this paper, we give a proper reparametrization algorithm for rational ruled surfaces. In general, we cannot guarantee that the new parametrization is still linear in one of the variables. It is an interesting problem to find a proper reparametrization for a rational ruled surface which is still of (1).

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Jia Li is a Ph.D. candidate in the Institute of Systems Science, Chinese Academy of Sciences. Her research interests include: symbolic computation, computer aided geometry design and algebraic geometry.

Li-Yong Shen is a post-doctoral researcher in the Institute of Systems Science, Chinese Academy of Sciences. His research interests include: symbolic computation, computer aided geometry design and trustworthy computing.

Xiao-Shan Gao is a professor of the Institute of Systems Science, Chinese Academy of Sciences. His research interests include: automated reasoning, symbolic computation, intelligent CAD, CAGD, and robotics. He has published over one hundred research papers, two monographs and edited four books or conference proceedings. His webpage is http://www.mmrc.iss.ac.cn/∼xgao.