Matrix Formulae of Differential Resultant for First Order Generic Ordinary Differential Polynomials

Zhi-Yong Zhang, Chun-Ming Yuan and Xiao-Shan Gao

Abstract In this paper, a matrix representation for the differential resultant of two generic ordinary differential polynomials f_1 and f_2 in the differential indeterminate y with order one and arbitrary degree is given. That is, a nonsingular matrix is constructed such that its determinant contains the differential resultant as a factor. Furthermore, the algebraic sparse resultant of f_1, f_2, δf_1 and δf_2 treated as polynomials in y, y', y'' is shown to be a nonzero multiple of the differential resultant of f_1 and f_2. Although very special, this seems to be the first matrix representation for a class of nonlinear generic differential polynomials.

Keywords Matrix formula · Differential resultant · Sparse resultant · Macaulay resultant

1 Introduction

Multivariate resultant, which gives a necessary condition for a set of $n + 1$ polynomials in n variables to have common solutions, is an important tool in elimination theory. One of the major issues in the resultant theory is to give a matrix representation for the resultant, which allows fast computation of the resultant using existing methods of determinant computation. By a matrix representation of the resultant, we mean a nonsingular square matrix whose determinant contains the resultant as a factor. There exist stronger forms of matrix representations. For instance, in the case of two univariate polynomials in one variable, there exist matrix formulae named after Sylvester and Bézout, whose determinants equal the resultant. Unfortunately, such

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determinant formulae do not generally exist for multivariate resultants. Macaulay showed that the multivariate resultant can be represented as a ratio of two determinants of certain Macaulay matrixes [16]. D'Andrea established a similar result for the multivariate sparse resultant [8] based on the pioneering work on sparse resultant [2, 3, 11, 20]. This paper will study matrix representations for differential resultants.

Using the analog between ordinary differential operators and univariate polynomials, the differential resultant for two linear ordinary differential operators was studied by Berkovich and Tsirulik [1] using Sylvester style matrices. The subresultant theory was first studied by Chardin [6] for two differential operators and then by Li [15] and Hong [12] for the more general Ore polynomials.

For nonlinear differential polynomials, the differential resultant is more difficult to define and study. The differential resultant for two nonlinear differential polynomials in one variable was defined by Ritt in [17, p. 47]. In [23, p. 46], Zwillinger proposed to define the differential resultant of two differential polynomials as the determinant of a matrix following the idea of algebraic multivariate resultant, but did not give details. General differential resultants were defined by Carrà Ferro using Macaulay's definition of algebraic resultants [5]. But, the treatment in [5] is not complete, as will be shown in Sect. 2.2 of this paper. In [22], Yang et al. used the idea of algebraic Dixon resultant to compute the differential resultant. Although very efficient, this approach is not complete and does not provide a matrix representation for the differential resultant. Differential resultants for linear ordinary differential polynomials were studied by Rueda-Sendra [18]. In [19], Rueda gave a matrix representation for a generic sparse linear system. In [9], the first rigorous definition for the differential resultant of $n+1$ differential polynomials in n differential indeterminates was given and its properties were proved. In [13, 14], the sparse resultant for differential Laurent polynomials was defined and a single exponential time algorithm to compute the sparse resultant was given. Note that an ideal approach is used in [9, 13, 14], and whether the multivariate differential resultant admits a matrix representation is left as an open issue.

In this paper, based on the idea of algebraic sparse resultants and Macaulay resultants, a matrix representation for the differential resultant of two generic ordinary differential polynomials f_1, f_2 in the differential indeterminate y with order one and arbitrary degree is given. The constructed square matrix has entries equal to the coefficients of f_1, f_2, their derivatives, or zero, whose determinant is a nonzero multiple of the differential resultant. Furthermore, we prove that the sparse resultant of $f_1, f_2, \delta f_1$ and δf_2 treated as polynomials in y, y', y'' is not zero and contains the differential resultant of f_1 and f_2 as a factor. Although very special, this seems to be the first matrix representation for a class of nonlinear generic differential polynomials.

The rest of the paper is organized as follows. In Sect. 2, the method of Carrà Ferro is briefly introduced and the differential resultant is defined following [9]. In Sect. 3, a matrix representation for the differential resultant of two differential polynomials with order one and arbitrary degree is given. In Sect. 4, it is shown that the differential resultant can be computed as a factor of a special algebraic sparse resultant. In Sect. 5, the conclusion is presented and a conjecture is proposed.

2 Preliminaries

To motivate what we do, we first briefly recall Carrà Ferro's definition for differential resultant and then give a counter example to show the incompleteness of Carrà Ferro's method when dealing with nonlinear generic differential polynomials. Finally, definition for differential resultant given in [9] is introduced.

2.1 A Sketch of Carrà Ferro's Definition

Let K be an ordinary differential field of characteristic zero with δ as a derivation operator. $K\{y\} = K[\delta^n y, n \in \mathbb{N}]$ is the differential ring of the differential polynomials in the differential indeterminate y with coefficients in K. Let p_1 (respectively p_2) be a differential polynomial of order m and degree d_1 (respectively of order n and degree d_2) in $K\{y\}$. According to Carrà Ferro [5], the differential resultant of p_1 and p_2, denoted by $\delta\mathrm{R}(p_1, p_2)$, is defined to be the Macaulay's algebraic resultant of $m+n+2$ differential polynomials

$$\mathscr{P}(p_1, p_2) = \{\delta^n p_1, \delta^{n-1} p_1, \dots, p_1, \delta^m p_2, \delta^{m-1} p_2, \dots, p_2\}$$

in the polynomial ring $S_{m+n} = K[y, \delta y, \dots, \delta^{m+n} y]$ in $m+n+1$ variables.

Specifically, let

$$D = 1 + (n+1)(d_1 - 1) + (m+1)(d_2 - 1), \quad L = \binom{m+n+1+D}{m+n+1}.$$

Let $y_i = \delta^i y$ for all $i = 0, 1, \dots, m+n$. For each $a = (a_0, \dots, a_{m+n}) \in \mathbb{N}^{m+n+1}$, $Y^a = y_0^{a_0} \cdots y_{m+n}^{a_{m+n}}$ is a power product in S_{m+n}. M^D_{m+n+1} stands for the set of all power products in S_{m+n} of degree less than or equal to D. Obviously, the cardinality of M^D_{m+n+1} equals L. In a similar way it is possible to define $M^{D-d_1}_{m+n+1}$ which has $L_1 = \binom{m+n+1+D-d_1}{m+n+1}$ monomials, and $M^{D-d_2}_{m+n+1}$ which has $L_2 = \binom{m+n+1+D-d_2}{m+n+1}$ monomials. The monomials in M^D_{m+n+1}, $M^{D-d_1}_{m+n+1}$ and $M^{D-d_2}_{m+n+1}$ are totally ordered using first the degree and then the lexicographic order derived from $y_0 < y_1 < \cdots < y_{m+n}$.

Definition 1 The $(((n+1)L_1 + (m+1)L_2) \times L)$-matrix

$$M(\delta, n, m) = M(\delta^n p_1, \dots, \delta p_1, p_1, \delta^m p_2, \dots, \delta p_2, p_2),$$

is defined in the following way: for each i such that $(j-1)L_1 < i \le jL_1$ the coefficients of the polynomial $Y^a \delta^{n+1-j} p_1$ are the entries of the ith row for each $Y^a \in M^{D-d_1}_{m+n+1}$ and each $j = 1, \dots, n+1$, while for each i such that

$$(n+1)L_1 + (j-n-2)L_2 < i \le (n+1)L_1 + (j-n-1)L_2$$

the coefficients of the polynomial $Y^a \delta^{m+n+2-j} p_2$ are the entries of the ith row for each $Y^a \in M_{m+n+1}^{D-d_2}$ and each $j = n+2, \ldots, m+n+2$, that are written with respect to the power products in M_{m+n+1}^{D} in decreasing order.

Definition 2 The differential resultant of p_1 and p_2 is defined to be

$$\gcd(\det(P) : P \text{ is an } (L \times L)\text{-submatrix of } M(\delta, n, m)).$$

2.2 A Counter Example and Definition of Differential Resultant

In this subsection, we use Carrà Ferro's method [5] to construct matrix formula of two nonlinear generic ordinary differential polynomials with order one and degree two,

$$\begin{aligned} g_1 &= a_0 y_1^2 + a_1 y_1 y + a_2 y^2 + a_3 y_1 + a_4 y + a_5, \\ g_2 &= b_0 y_1^2 + b_1 y_1 y + b_2 y^2 + b_3 y_1 + b_4 y + b_5, \end{aligned} \tag{1}$$

where, hereinafter, $y_1 = \delta y$ and a_i, b_i with $i = 0, \ldots, 5$ are generic differential indeterminates.

For differential polynomials in (1), we have $d_1 = d_2 = 2, m = n = 1$, $D = 5$, $L = 56$, and $L_1 = L_2 = 20$. The set of column monomials is

$$M_3^5 = \{y_2^5, y_2^4 B_3^1, y_2^3 B_3^2, y_2^2 B_3^3, y_2 B_3^4, B_3^5\},$$

where, and throughout the paper, $B = \{1, y, y_1, y_2\}$ and B_i^j denotes all monomials of total degree less than or equal to j in the first i elements of B. For example, $B_2^2 = \{1, y, y^2\}$ and $B_3^2 = \{1, y, y_1, y^2, yy_1, y_1^2\}$. Note that the monomials of M_3^3 are $M_3^3 = \{y_2^3, y_2^2 B_3^1, y_2 B_3^2, B_3^3\} = B_4^3$.

According to Definition 1, $M(\delta, 1, 1)$ is an 80×56 matrix

$$\begin{array}{c} \begin{array}{cccccccc} y_2^5 & y_2^4 y_1 & \ldots & y_2^3 & \ldots & y & 1 \end{array} \\ \left(\begin{array}{ccccccc} 0 & d_1 a_0 & \ldots & \delta a_5 & \ldots & 0 & 0 \\ & & & \cdots\cdots & & & \\ 0 & 0 & \ldots & 0 & \ldots & \delta a_4 & \delta a_5 \\ 0 & 0 & \ldots & a_0 & \ldots & 0 & 0 \\ & & & \cdots\cdots & & & \\ 0 & 0 & \ldots & 0 & \ldots & a_4 & a_5 \\ 0 & d_2 b_0 & \ldots & \delta b_5 & \ldots & 0 & 0 \\ & & & \cdots\cdots & & & \\ 0 & 0 & \ldots & 0 & \ldots & \delta b_4 & \delta b_5 \\ 0 & 0 & \ldots & b_0 & \ldots & 0 & 0 \\ & & & \cdots\cdots & & & \\ 0 & 0 & \ldots & 0 & \ldots & b_4 & b_5 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} B_3^3 \delta g_1 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} B_3^3 g_1 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} B_3^3 \delta g_2 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} B_3^3 g_2 \end{array} \end{array}$$

Obviously, the entries of the first column are all zero in $M(\delta, 1, 1)$, since the monomial y_2^5 never appears in any row polynomial $Y * f$, where the monomial $Y \in M_3^3$ and $f \in \{\delta g_1, g_1, \delta g_2, g_2\}$. Consequently, the differential resultant of g_1 and g_2 is identically zero according to this definition.

Actually, the differential resultant is defined using an ideal approach for two generic differential polynomials in one differential indeterminate in [17] and $n+1$ generic differential polynomials in n differential indeterminates in [9]. f is said to be a *generic differential polynomial* in differential indeterminates $\mathbb{Y} = \{y_1, \ldots, y_n\}$ with order s and degree m if f contains all the monomials of degree up to m in $y_1, \ldots, y_n$ and their derivatives up to order s. Furthermore, the coefficients of f are also differential indeterminates. For instance, g_1 and g_2 in (1) are two generic differential polynomials.

Theorem 3 [9] *Let $p_0, p_1, \ldots, p_n$ be generic differential polynomials with order s_i and coefficient sets $\mathbf{u}_i$ respectively. Then $[p_0, p_1, \ldots, p_n]$ is a prime differential ideal in $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \ldots, \mathbf{u}_n\}$. And*

$$[p_0, p_1, \ldots, p_n] \cap \mathbb{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\} = \mathrm{sat}(\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n)) \tag{2}$$

is a prime differential ideal of codimension one, where $\mathbf{R}$ is defined to be the differential sparse resultant of $p_0, p_1, \ldots, p_n$, which has the following properties

(a) $\mathbf{R}(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ is an irreducible polynomial and differentially homogeneous in each $\mathbf{u}_i$.
(b) $\mathbf{R}(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ is of order $h_i = s - s_i$ in $\mathbf{u}_i$ $(i = 0, \ldots, n)$ with $s = \sum_{l=0}^{n} s_l$.
(c) The differential resultant can be written as a linear combination of p_i and the derivatives of p_i up to the order $s - s_i$. Precisely, we have

$$\mathbf{R}(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij} p_i^{(j)}$$

where $h_{ij} \in \mathbb{Q}[\mathbb{Y}, \mathbb{Y}^{(1)} \ldots, \mathbb{Y}^{(s)}, \mathbf{u}_0^{(s-s_0)}, \ldots, \mathbf{u}_n^{(s-s_n)}]$.

3 Matrix Formula for Differential Polynomials

In this section, we will give a matrix representation for the following generic differential polynomials in y

$$\begin{aligned} f_1 &= a_{y_1^{d_1}} y_1^{d_1} + a_{y_1^{d_1-1} y} y_1^{d_1-1} y + \cdots + a_0, \\ f_2 &= b_{y_1^{d_2}} y_1^{d_2} + b_{y_1^{d_2-1} y} y_1^{d_2-1} y + \cdots + b_0, \end{aligned} \tag{3}$$

where $y_1 = \delta y$, $1 \le d_1 \le d_2$, and $a_{y_1^{d_1}}, \ldots, a_0, b_{y_1^{d_2}}, \ldots, b_0$ are differential indeterminates.

3.1 Matrix Construction

In this subsection, we show that when choosing a proper column monomial set, a square matrix can be constructed following Macaulay's idea [16].

By (c) of Theorem 3, the differential resultant for f_1, f_2 can be written as a linear combination of $f_1, f_2, \delta f_1$ and δf_2 which are treated as polynomials in variables $y, y_1, y_2 = \delta y_1$. So, we will try to construct a matrix representation for the differential resultant of f_1 and f_2 from these four polynomials.

From Sect. 2, it is easy to see that the main problem in Carrà Ferro's definition is that the matrix $M(\delta, n, m)$ contains too many columns. Or equivalently, the monomial set M^D_{m+n+1} used to represent the columns is too large.

Consider the monomial set

$$\mathscr{E} = B_3^D \cup y_2 B_3^{D-1} \tag{4}$$

with $D = 2d_1 + 2d_2 - 3$. We will show that if using $\mathscr{E}$ as the column monomial set, a nonsingular square matrix can be constructed.

Define the *main monomial* of polynomials $p_1 = \delta f_1, p_2 = \delta f_2, p_3 = f_1, p_4 = f_2$ to be

$$\mathrm{mm}(p_1) = y_2\, y_1^{d_1-1}, \quad \mathrm{mm}(p_2) = y_1^{d_2}, \quad \mathrm{mm}(p_3) = y^{d_1}, \quad \mathrm{mm}(p_4) = 1. \tag{5}$$

Then, we can divide $\mathscr{E}$ into four mutually disjoint sets $\mathscr{E} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3 \cup \mathscr{S}_4$, where

$$\begin{aligned}
\mathscr{S}_1 &= \{Y^\alpha \in \mathscr{E} : \mathrm{mm}(p_1) \text{ divides } Y^\alpha\},\\
\mathscr{S}_2 &= \{Y^\alpha \in \mathscr{E} : \mathrm{mm}(p_1) \text{ does not divide } Y^\alpha \text{ but } \mathrm{mm}(p_2) \text{ does}\},\\
\mathscr{S}_3 &= \{Y^\alpha \in \mathscr{E} : \mathrm{mm}(p_1), \mathrm{mm}(p_2) \text{ do not divide } Y^\alpha \text{ but } \mathrm{mm}(p_3) \text{ does}\},\\
\mathscr{S}_4 &= \{Y^\alpha \in \mathscr{E} : \mathrm{mm}(p_1), \mathrm{mm}(p_2), \mathrm{mm}(p_3) \text{ do not divide } Y^\alpha\}.
\end{aligned} \tag{6}$$

As a consequence, we can write down a system of equations:

$$\begin{aligned}
Y^\alpha/\mathrm{mm}(p_1) * p_1 &= 0, \quad \text{for } Y^\alpha \in \mathscr{S}_1,\\
Y^\alpha/\mathrm{mm}(p_2) * p_2 &= 0, \quad \text{for } Y^\alpha \in \mathscr{S}_2,\\
Y^\alpha/\mathrm{mm}(p_3) * p_3 &= 0, \quad \text{for } Y^\alpha \in \mathscr{S}_3,\\
Y^\alpha/\mathrm{mm}(p_4) * p_4 &= 0, \quad \text{for } Y^\alpha \in \mathscr{S}_4.
\end{aligned} \tag{7}$$

Observe that the total number of equations is the number of elements in $\mathscr{E}$ and denoted by $N = (D+1)^2$.

Regarding the monomials in (7) as unknowns, we obtain a system of N linear equations in these monomial unknowns. Denote the coefficient matrix of the system of linear equations (7) by D_{d_1,d_2} whose entries are zero or the coefficients of f_i and $\delta f_i, i = 1, 2$.

Note that the main monomials of the polynomials are not the maximal monomials in the sense of Macaulay [16], so the monomials on the left hand side of (7) may not be contained in $\mathscr{E}$. Next, we prove that this does not occur for our main monomials.

Lemma 1 *The coefficient matrix D_{d_1,d_2} of system (7) is square.*

Proof The coefficient matrix of system (7) has $N = |\mathscr{E}|$ rows. In order to prove the lemma, it suffices to demonstrate that, for each $Y^\alpha \in \mathscr{S}_i, i = 1, \ldots, 4$, all monomials in $[Y^\alpha/\mathrm{mm}(p_i)] * p_i$ are contained in $\mathscr{E}$. Recall that $\mathscr{E} = B_3^D \cup y_2 B_3^{D-1}$. Then by (6), one has

$$\begin{aligned}
\mathscr{S}_1 &= B_3^{D-d_1} * \mathrm{mm}(p_1) = B_3^{D-d_1} * \mathrm{mm}(\delta f_1),\\
\mathscr{S}_2 &= B_3^{D-d_2} * \mathrm{mm}(p_2) = B_3^{D-d_2} * \mathrm{mm}(\delta f_2),\\
\mathscr{S}_3 &= T_1 * \mathrm{mm}(p_3) = T_1 * \mathrm{mm}(f_1),\\
\mathscr{S}_4 &= T_2 * \mathrm{mm}(p_4) = T_2 * \mathrm{mm}(f_2),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
T_1 &= \left\{ \left(\bigcup_{i=0}^{d_2-1} y_1^i B_2^{D-d_1-i} \right) \bigcup \left(y_2 \bigcup_{i=0}^{d_1-2} y_1^i B_2^{D-d_1-1-i} \right) \right\},\\
T_2 &= \left\{ \left(\bigcup_{i=0}^{d_2-1} y_1^i B_2^{d_1-1} \right) \bigcup \left(y_2 \bigcup_{i=0}^{d_1-2} y_1^i B_2^{d_1-1} \right) \right\}.
\end{aligned} \tag{9}$$

Note that the representation of $\mathscr{S}_2$ in (8) is obtained with the help of the condition $d_1 \le d_2$.

Hence, Eq. (7) become

$$\begin{aligned}
B_3^{D-d_1} * \delta f_1 &= 0,\\
B_3^{D-d_2} * \delta f_2 &= 0,\\
T_1 * f_1 &= 0,\\
T_2 * f_2 &= 0.
\end{aligned} \tag{10}$$

Since the monomial set of δf_1 is $B_3^{d_1} \cup y_2 * B_3^{d_1-1}$, the monomial set of $B_3^{D-d_1} * \delta f_1$ is $B_3^{D-d_1} * (B_3^{d_1} \cup y_2 * B_3^{d_1-1}) = B_3^D \cup y_2 B_3^{D-1} = \mathscr{E}$. So monomials in the first set of equations in (10) are in $\mathscr{E}$. Since the monomial set of f_1 is $B_3^{d_1} = \cup_{l=0}^{d_1} y_1^l B_2^{d_1-l}$, the monomial set of $T_1 * f_1$ is $T_{11} \cup y_2 T_{12}$, where $T_{11} = \bigcup_{k=0}^{d_1+d_2-1} y_1^k B_2^{D-k}$ and $T_{12} = \bigcup_{k=0}^{2d_1-2} y_1^k B_2^{D-k-1}$. Since $d_1 \ge 1$ and $d_2 \ge 1$, we have $d_1 + d_2 - 1 \le$

$D = 2d_1 + 2d_2 - 3$ and hence $T_{11} \subset B_3^D$. Since $d_1 \geq 1$ and $d_2 \geq 1$, we have $2d_1 - 2 \leq D - 1 = 2d_1 + 2d_2 - 4$ and hence $T_{12} \subset B_3^{D-1}$. As a consequence, $T_{11} \cup y_2 T_{12} \subset \mathscr{E}$ and the monomials in the third set of equations in (10) are in $\mathscr{E}$. Other cases can be proved similarly. Thus all monomials in the left hand side of (10) are in $\mathscr{E}$. This proves the lemma. □

It is worthy to say that, due to the decrease of the number of monomials in $\mathscr{E}$ compared with the method by Carrà Ferro, the size of the matrix D_{d_1,d_2} decreases significantly. Precisely, when we choose $m = n = 1$, the size of the matrix D_{d_1,d_2} is $(D+1)^2 \times (D+1)^2$ and one can see that the size of the matrix given by Carrà Ferro is $2\binom{3+D-d_2}{3} + 2\binom{3+D-d_1}{3} \times \binom{3+D}{3}$ where $D = 2d_1 + 2d_2 - 3$. One can show that $2\binom{3+D-d_2}{3} + 2\binom{3+D-d_1}{3} \geq \binom{3+D}{3} \geq (D+1)^2$ for $d_1, d_2 \geq 1$ and the last equality holds only for $d_1 = d_2 = 1$.

3.2 Matrix Representation for Differential Resultant

In this section, we show that $\det(D_{d_1,d_2})$ is not identically equal to zero and contains the differential resultant as a factor.

Lemma 2 $\det(D_{d_1,d_2})$ *is not identically equal to zero.*

Proof It suffices to show that there exists a unique monomial in the sense that it is different from all other monomials in the expansion of $\det(D_{d_1,d_2})$.

The coefficients of the main monomials in δf_1, δf_2, f_1 and f_2 are respectively

$$\begin{array}{lll} \delta f_1 : a_{y_1^{d_1}} & \text{the coefficient of } \mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}, & \\ \delta f_2 : \delta b_{y_1^{d_2}} + b_{y_1^{d_2-1} y} & \text{the coefficient of } \mathrm{mm}(\delta f_2) = y_1^{d_2}, & \\ f_1 \ : \ a_{y^{d_1}} & \text{the coefficient of } \mathrm{mm}(f_1) = y^{d_1}, & \\ f_2 \ : \ b_0 & \text{the coefficient of } \mathrm{mm}(f_2) = 1. & \end{array} \tag{11}$$

We will show that the monomial $(a_{y_1^{d_1}})^{n_1} (\delta b_{y_1^{d_2}})^{n_2} (a_{y^{d_1}})^{n_3} (b_0)^{n_4}$ is a unique one by the following four steps, where n_i is the number of elements in $\mathscr{S}_i$ with $i = 1, \ldots, 4$. From (10), $n_1 = |B_3^{D-d_1}|$, $n_2 = |B_3^{D-d_2}|$, $n_3 = |T_1|$, $n_4 = |T_2|$.

Step 1. Observe that, in δf_1, $a_{y^{d_1}}$ only occurs in the coefficient of $y_1 y^{d_1-1}$ with the form $a_{y^{d_1}} + \delta a_{y_1 y^{d_1-1}}$. Furthermore, $\delta a_{y_1 y^{d_1-1}}$ only occurs in this term given by δf_1 and no other places of D_{d_1,d_2}. So using the transformation

$$\delta a_{y_1 y^{d_1-1}} = c_{y_1 y^{d_1-1}} - d_1 a_{y^{d_1}}, \quad \text{with other coefficients unchanged,} \tag{12}$$

where $c_{y_1 y^{d_1-1}}$ is a new differential indeterminate, D_{d_1,d_2} is transformed to a new matrix which is singular if and only if the original one is singular. Still denote the matrix by D_{d_1,d_2}.

From (10), for a monomial $M \in T_1$, $a_{y^{d_1}}$ is the coefficient of the monomial My^{d_1} in each polynomial $T_1 * f_1$ and hence in each corresponding row of D_{d_1,d_2}. Then $a_{y^{d_1}}$ is in different rows and columns of D_{d_1,d_2}, and this gives the factor $(a_{y^{d_1}})^{n_3}$. Delete those rows and columns of D_{d_1,d_2} containing $a_{y^{d_1}}$ and denote the remaining matrix by $D^{(1)}_{d_1,d_2}$. From (10), the columns deleted are represented by monomials $y^{d_1}T_1$. So, $D^{(1)}_{d_1,d_2}$ is still a square matrix.

Step 2. Let $M \in B_3^{D-d_1}$. The term $a_{y_1^{d_1}}$ occurs in $M * \delta f_1$ as the coefficient of the monomial $y_2 y_1^{d_1-1}M$, or equivalently it occurs in the columns represented by $y_2 y_1^{d_1-1}M$. This gives the factor $(a_{y_1^{d_1}})^{n_1}$. It is easy to check that $a_{y_1^{d_1}}$ does not occur in other places of $D^{(1)}_{d_1,d_2}$. From the definition for T_1 (9), the columns deleted in case 1 correspond those columns represented by monomials of the form $y_2^{k_2} y_1^{k_1} y^{d_1}$ where either $k_2 = 0$ and $k_1 < d_2$ or $k_2 = 1$ and $k_1 < d_1 - 1$. Then $\{y^{d_1}T_1\} \cap \{y_2 y_1^{d_1-1} B_3^{D-d_1}\} = \emptyset$, or equivalently those columns of D_{d_1,d_2} containing $a_{y_1^{d_1}}$ are still in $D^{(1)}_{d_1,d_2}$. Similar to case 1, one can delete those rows and columns of $D^{(1)}_{d_1,d_2}$ containing $a_{y_1^{d_1}}$ and denote the remaining matrix by $D^{(2)}_{d_1,d_2}$ which is still a square matrix. From (10), the columns deleted are represented by monomials $y_2 y_1^{d_1-1} B_3^{D-d_1}$.

Step 3. At the moment, $D^{(2)}_{d_1,d_2}$ only contains coefficients of f_2 and δf_2. Observe that b_0 only occurs in the rows corresponding to $T_2 * f_2$, where T_2 is defined in (9). Note that δb_0 instead of b_0 occurs in δf_2. Since $\{y^{d_1}T_1\} \cap T_2 = \emptyset$ and $\{y_2 y_1^{d_1-1} B_3^{D-d_1}\} \cap T_2 = \emptyset$, the columns of D_{d_1,d_2} containing b_0, represented by T_2, are not deleted in case 1 and case 2. Then, we have the factor $(b_0)^{n_4}$. Similarly, delete those rows and columns of $D^{(2)}_{d_1,d_2}$ containing b_0 and denote the remaining matrix by $D^{(3)}_{d_1,d_2}$ which is still a square matrix. From (10), the columns deleted are represented by monomials T_2.

Step 4. From (10), the rows of $D^{(3)}_{d_1,d_2}$ are from coefficients of $B_3^{D-d_2} * \delta f_2$. The term $\delta b_{y_1^{d_2}}$ is in the coefficient of the monomial $M * y_1^{d_2}$ in $M * \delta f_2$ for $M \in B_3^{D-d_2}$, and $\delta b_{y_1^{d_2}}$ does not occur in other places of $M * \delta f_2$. Furthermore, since $\{y^{d_1}T_1\} \cap \{y_1^{d_2} B_3^{D-d_2}\} = \emptyset$, $\{y_2 y_1^{d_1-1} B_3^{D-d_1}\} \cap \{y_1^{d_2} B_3^{D-d_2}\} = \emptyset$, and $T_2 \cap \{y_1^{d_2} B_3^{D-d_2}\}\} = \emptyset$, the columns containing the term $\delta b_{y_1^{d_2}}$ are not deleted in the first three cases. Then, we have the factor $(\delta b_{y_1^{d_2}})^{n_2}$.

Following the above procedures step by step, the coefficients of choosing main monomials of the polynomials f_1, f_2, δf_1 and δf_2 occur in each row and each column of D_{d_1,d_2} and only once, and the monomial $\left(a_{y_1^{d_1}}\right)^{n_1}\left(\delta b_{y_1^{d_2}}\right)^{n_2}\left(a_{y^{d_1}}\right)^{n_3}\left(b_0\right)^{n_4}$ is a unique one in the expansion of the determinant of D_{d_1,d_2}. So the lemma follows. □

Note that the selection of main monomials in above algorithm is not unique, thus there may exist other ways to construct matrix formula for system (3).

Corollary 4 *Following the above notations, for any* $Y^\alpha \in \mathscr{S}_i$*, if all monomials of* $[Y^\alpha/\mathrm{mm}(p_j)] * p_j$ *are contained in* $\mathscr{E}$ $(j \neq i)$*, then the rearranged matrix, which is obtained by replacing the row polynomials* $[Y^\alpha/\mathrm{mm}(p_i)] * p_i$ *by* $[Y^\alpha/\mathrm{mm}(p_j)] * p_j$*, is not identically equal to zero.*

Corollary 4 follows from the fact that the proof of Lemma 2 is independent of the number of elements in $\mathscr{S}_i$ as long as the main monomials are the same.

The relation between $\det(D_{d_1,d_2})$ and differential resultant of f_1 and f_2, denoted by $\mathbf{R}$, is stated as the following theorem.

Theorem 5 $\det(D_{d_1,d_2})$ *is a nonzero multiple of* $\mathbf{R}$.

Proof From Lemma 2, $\det(D_{d_1,d_2})$ is nonzero. In the matrix D_{d_1,d_2}, multiply a column monomial $M \neq 1$ in $\mathscr{E}$ to the corresponding column and add the result to the constant column corresponding to the monomial 1. Then the constant column becomes $Y^\alpha * p_i$ with $p_1 = \delta f_1, p_2 = \delta f_2, p_3 = f_1, p_4 = f_2$ and $Y^\alpha \in \mathscr{S}_i/\mathrm{mm}(p_i), i = 1, \ldots, 4$. Since a determinant is multilinear on the columns, expanding the matrix by the constant column, we obtain

$$\det(D_{d_1,d_2}) = h_1 f_1 + h_2 \delta f_1 + h_3 f_2 + h_4 \delta f_2, \qquad (13)$$

where h_j are differential polynomials. From (2), $\det(D_{d_1,d_2}) \in \mathrm{sat}(\mathbf{R})$. On the other hand, from Theorem 3, $\mathbf{R}$ is irreducible and the order of $\mathbf{R}$ about the coefficients of f_1, f_2 is one. Therefore, $\mathbf{R}$ must divide $\det(D_{d_1,d_2})$. □

From Theorem 5, we can easily deduce a degree bound $N = 4(d_1 + d_2 - 1)^2$ for the differential resultant of f_1 and f_2. The main advantage to represent the differential resultant as a factor of the determinant of a matrix is that we can use fast algorithms of matrix computation to compute the differential resultant as did in the algebraic case [4].

Suppose that $\det(D_{d_1,d_2})$ is expanded as a polynomial. Then the differential resultant can be found by the following result.

Corollary 6 *Suppose* $\det(D_{d_1,d_2}) = \prod_{i=1}^{s} P_i^{e_i}$ *is an irreducible factorization of* $\det(D_{d_1,d_2})$ *in* $\mathbb{Q}[C_{f_1}, C_{f_2}]$*, where* $C_{f_i}, i = 1, 2$ *are the sets of coefficients of* f_i*. Then there exists a unique factor, say* P_1*, which is in* $[f_1, f_2]$ *and is the differential resultant of* f_1 *and* f_2*.*

Proof From (c) of Theorems 3 and 5, $\mathbf{R} \in [f_1, f_2]$ and is an irreducible factor of $\det(D_{d_1,d_2})$. Suppose $\det(D_{d_1,d_2})$ contains another factor, say P_2, which is also in $[f_1, f_2]$. Then $P_2 \in \mathrm{sat}(\mathbf{R})$ by (2). Since $\mathbf{R}$ is irreducible with order one and P_2 is of order no more than one, P_2 must equal $\mathbf{R}$, which contradicts to the hypothesis. □

3.3 Example (1) Revisited

In this section, we apply the method just proposed to construct a matrix representation for the differential resultant of the system (1).

Following the method given in the proceeding section, for system (1), we have $D = 2d_1 + 2d_2 - 3 = 5$ and select the main monomials of $\delta g_1, \delta g_2, g_1, g_2$ are $y_2 y_1, y_1^2, y^2, 1$, respectively. Then $\mathscr{E} = y_2 B_3^4 \cup B_3^5$ is divided into the following four disjoint sets

$$
\begin{aligned}
\mathscr{S}_1 &= y_2 y_1 B_3^3,\\
\mathscr{S}_2 &= y_1^2 B_3^3,\\
\mathscr{S}_3 &= y^2 \left[B_2^3 \cup y_1 B_2^2 \cup y_2 B_2^2 \right],\\
\mathscr{S}_4 &= B_2^1 \cup y_1 B_2^1 \cup y_2 B_2^1.
\end{aligned} \tag{14}
$$

Using (7) and regarding the monomials in $\mathscr{E}$ as variables, we obtain the matrix $D_{2,2}$, which is a 36×36 square matrix in the following form.

$$
\begin{array}{c}
\begin{array}{ccccccc}
y_2 y_1^4 & y_2 y_1^3 y & \dots & y_2 y_1^2 y^2 & \dots & y & 1
\end{array}\\
\left(\begin{array}{ccccccc}
2a_0 & a_1 & \dots & 0 & \dots & 0 & 0\\
 & & & \dots\dots & & & \\
0 & 0 & \dots & 0 & \dots & \delta a_4 & \delta a_5\\
0 & 0 & \dots & a_0 & \dots & 0 & 0\\
 & & & \dots\dots & & & \\
0 & 0 & \dots & 0 & \dots & a_4 & a_5\\
2b_0 & b_1 & \dots & 0 & \dots & 0 & 0\\
 & & & \dots\dots & & & \\
0 & 0 & \dots & 0 & \dots & \delta b_4 & \delta b_5\\
0 & 0 & \dots & b_0 & \dots & 0 & 0\\
 & & & \dots\dots & & & \\
0 & 0 & \dots & 0 & \dots & b_4 & b_5
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} B_3^3 \delta g_1\\
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} (B_2^3 \cup y_1 B_2^2 \cup y_2 B_2^2) g_1\\
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} B_3^3 \delta g_2\\
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} (B_2^1 \cup y_1 B_2^1 \cup y_2 B_2^1) g_2
\end{array}
\end{array}
$$

As shown in the proof of Lemma 2, $(a_0)^{10}(a_2)^{10}(b_5)^6(\delta b_0)^{10}$ is a unique monomial in the expansion of the determinant of $D_{2,2}$. Hence, the differential resultant of g_1 and g_2 is a factor of $\det(D_{2,2})$. Note that in Carrà Ferro's construction for g_1, g_2, $M(\delta, 1, 1)$ is an 80×56 matrix, which is larger than $D_{2,2}$.

In particular, suppose $a_0 = b_0 = 1$ and a_i, b_i are differential constants, i.e., $\delta a_i = \delta b_i = 0$, $i = 1, \dots, 5$. Then $D_{2,2}$ can be expanded as a polynomial and the differential resultant of g_1 and g_2 can be found with Corollary 6, which is a polynomial of degree 12 and contains 3,210 terms. This is the same as the result obtained in [22].

4 Differential Resultant as the Algebraic Sparse Resultant

In this section, we show that differential resultant of f_1 and f_2 is a factor of the algebraic sparse resultant of the system $\{f_1, f_2, \delta f_1$ and $\delta f_2\}$.

4.1 Results About Algebraic Sparse Resultant

In this subsection, notions of algebraic sparse resultants are introduced. Details can be found in [4, 7, 11, 20].

A set S in $\mathbb{R}^n$ is said to be convex if it contains the line segment connecting any two points in S. If a set is not itself convex, its convex hull is the smallest convex set containing it and denoted by Conv(S). A set $V = \{a_1, \ldots, a_m\}$ is called a vertex set of a convex set Q if each point $q \in Q$ can be expressed as:

$$q = \sum_{j=1}^{m} \lambda_j a_j, \quad \text{with} \sum_{j=1}^{m} \lambda_j = 1 \text{ and } \lambda_j \geq 0,$$

and each a_j is called a *vertex* of Q.

Consider $n + 1$ generic sparse polynomials in the algebraic indeterminates $x_1, \ldots, x_n$:

$$p_i = u_{i0} + u_{i1}M_{i1} + \cdots + u_{il_i}M_{il_i}, \quad i = 1, \ldots, n+1,$$

where u_{ij} are indeterminates and $M_{ik} = \prod_{s=1}^{n} x_s^{e^{iks}}$ are monomials in $\mathbb{Q}[x_1, \ldots, x_n]$ with exponent vectors $a_{ik} = (e^{ik_1}, \ldots, e^{ik_n}) \in \mathbb{Z}^n$. Note that we assume each p_i contains a constant term u_{i0}. For $a = (e^1, \ldots, e^n) \in \mathbb{Z}^n$, the corresponding monomial is denoted as $M(a) = \prod_{s=1}^{n} x_s^{e^s}$.

The finite set $\mathscr{A}_i \subset \mathbb{Z}^n$ of all monomial exponents appearing in p_i is called the *support* of p_i, denoted by supp(p_i). Its cardinality is $l_i = |\mathscr{A}_i|$. The *Newton polytope* $Q_i \subset \mathbb{R}^n$ of p_i is the *convex hull* of $\mathscr{A}_i$, denoted by $Q_i = \text{Conv}(\mathscr{A}_i)$. Since Q_i is the convex hull for a finite set of points, it must have a vertex set. For simplicity, we assume that each $\mathscr{A}_i$ is of dimension n as did in [11, p. 252]. Let $\mathbf{u}$ be the set of coefficients of $p_i, i = 0, \ldots, n$. Then, the ideal

$$(p_1, p_2, \ldots, p_{n+1}) \cap \mathbb{Q}[\mathbf{u}] = (\mathscr{R}(\mathbf{u})) \tag{15}$$

is principal and the generator $\mathscr{R}$ is defined to be the sparse resultant of $p_1, \ldots, p_{n+1}$ [11, p. 252]. When the coefficients $\mathbf{u}$ of p_i are specialized to certain values $\mathbf{v}$, the sparse resultant for the specialized polynomials is defined to be $\mathscr{R}(\mathbf{v})$. The matrix representation of $\mathscr{R}$ is associated with the decomposition of the Minkowski sum of the Newton polytopes Q_i.

The *Minkowski sum* of the convex polytopes Q_i

$$Q = Q_1 + \cdots + Q_{n+1} = \{q_1 + \cdots + q_{n+1} | q_i \in Q_i\}.$$

is still convex and of dimension n.

Choose sufficiently small numbers $\delta_i > 0$ and let $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ be a perturbed vector. Then the points which lie in the interior of the perturbed district $\mathscr{E} = \mathbb{Z}^n \cap (Q + \delta)$ are chosen as the *column monomial set* [4] to construct the matrix for the sparse resultant.

Choose $n + 1$ sufficiently generic linear lifting functions $l_1, \ldots, l_{n+1} \in \mathbb{Z}[x_1, \ldots, x_n]$ and define the lifted Newton polytopes $\widehat{Q}_i = \{\widehat{q}_i = (q_i, l_i(q_i)) : q_i \in Q_i\} \subset \mathbb{R}^{n+1}$. Let

$$\widehat{Q} = \sum_{i=1}^{n+1} \widehat{Q}_i \subset \mathbb{R}^{n+1}$$

which is an $(n + 1)$-dimensional convex polytope. The *lower envelope* of $\widehat{Q}$ with respect to vector $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is the union of all the n-dimensional faces of $\widehat{Q}$, whose inner normal vector has positive last component.

Let $\pi : (q_1, \ldots, q_{n+1}) \mapsto (q_1, \ldots, q_n)$ be the projection to the first n coordinates from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Then π is a one to one map between the lower envelope of $\widehat{Q}$ and Q [4]. The genericity requirements on l_i assure that every point $\widehat{q}$ on the lower envelope can be uniquely expressed as $\widehat{q} = \widehat{q}_1 + \cdots + \widehat{q}_{n+1}$ with $\widehat{q}_i \in \widehat{Q}_i$, such that the sum of the projections under π of these points leads to a unique sum of $q = q_1 + \cdots + q_{n+1} \in Q \subset \mathbb{R}^n$ with $q_i \in Q_i$, which is called the *optimal (Minkowski) sum* of q. For $\mathscr{F}_i \subset Q_i$, $R = \sum_{i=1}^{n+1} \mathscr{F}_i$ is called an optimal sum, if each element of R can be written as a unique optimal sum $\sum_{i=0}^{n} p_i$ for $p_i \in \mathscr{F}_i$.

A polyhedral *subdivision* of an n-dimensional polytope Q consists of finitely many n-dimensional polytopes $R_1, \ldots, R_s$, called the cells of the subdivision, such that $Q = R_1 \cup \cdots \cup R_s$ and for $i \neq j$ and $R_i \cap R_j$ is either empty or a face of both R_i and R_j. A polyhedral subdivision is called a *mixed subdivision* if each cell R_l can be written as an optimal sum $R_l = \sum_{i=1}^{n+1} \mathscr{F}_i$, where each $\mathscr{F}_i$ is a face of Q_i and $n = \sum_{i=1}^{n+1} \dim(\mathscr{F}_i)$. Furthermore, if $R_j = \sum_{i=1}^{n+1} \mathscr{F}'_i$ is another cell in the subdivision, then $R_l \cap R_j = \sum_{i=1}^{n+1} (\mathscr{F}_i \cap \mathscr{F}'_i)$. A cell $R_l = \sum_{i=1}^{n+1} \mathscr{F}_i$ is called *mixed* if $\dim(\mathscr{F}_i) \leq 1$ for all i; otherwise, it is a *nonmixed cell*. As a result of $n = \sum_{i=1}^{n+1} \dim(\mathscr{F}_i)$, a mixed cell has one unique vertex, which satisfies $\dim(\mathscr{F}_{i_0}) = 0$, while a nonmixed cell has at least two vertices.

Recall $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, where $0 < \delta_i < 1$. If $Q = R_1 \cup \cdots \cup R_s$ is a subdivision of Q, then $\delta + Q = (\delta + R_1) \cup \cdots \cup (\delta + R_s)$ is a subdivision of $\delta + Q$.

Let $q \in \mathbb{Z}^n \cap (Q + \delta)$ lie in the interior of a cell $\delta + \mathscr{F}_1 + \cdots + \mathscr{F}_{n+1}$ of a mixed subdivision for $Q + \delta$, where $\mathscr{F}_i$ is a face of Q_i. The *row content function* of q is defined as the largest integer such that $\mathscr{F}_i$ is a vertex. In fact, all the vertices in the optimal sum of p can be selected as the row content functions. Hence, we define *generalized row content functions* (GRC for brief) as one of the integers, not necessary largest, such that $\mathscr{F}_i$ is a vertex.

Suppose that we have a mixed subdivision of Q. With a fixed GRC, a sparse resultant matrix can be constructed as follows. For each $i = 1, \ldots, n+1$, define the subset S_i of $\mathscr{E}$ as follows:

$$S_i = \{q \in \mathscr{E} \mid \mathrm{GRC}(q) = (i, j_0)\}, \tag{16}$$

where $j_0 \in \{1, \ldots, m_i\}$, m_i is the number of vertexes of Q_i, and we obtain a disjoint union for $\mathscr{E}$:

$$\mathscr{E} = S_1 \cup \cdots \cup S_{n+1}. \tag{17}$$

For $q \in S_i$, let $q = q_1 + \cdots + q_{n+1} \in Q$ be an optimal sum of q. Then, q_i is a vertex of Q_i and the corresponding monomial $M(q_i)$ is called the *main monomial* of p_i and denoted by $\mathrm{mm}(p_i)$, similar to what we did in Sect. 3. Main monomials have the following important property [7, p. 350].

Lemma 3 *If $q \in S_i$, then the monomials in $(M(q)/\mathrm{mm}(p_i))p_i$ are contained in $\mathscr{E}$.*

Now consider the following equation systems

$$(M(q)/\mathrm{mm}(p_i))p_i, \quad q \in S_i, i = 1, \ldots, n+1. \tag{18}$$

Treating the monomials in $\mathscr{E}$ as variables, by Lemma 3, the coefficient matrix for the equations in (18) is an $|\mathscr{E}| \times |\mathscr{E}|$ square matrix, called the *sparse resultant matrix*. The sparse resultant of $p_i, i = 1, \ldots, n+1$ is a factor of the determinant of this matrix.

In [3, 4], Canny and Emiris used linear programming algorithms to find the row content functions and to construct S_i. We briefly describe this procedure below.

Now assume Q_i has the vertex set $V_i = \{a_{i1}, \ldots, a_{i\,m_i}\}$. A point $q \in \mathbb{Z}^n \cap (Q+\delta)$ implies that $q \in \sigma + \delta$ with a cell $\sigma \in Q$. In order to obtain the generalized row content functions of q, we wish to find the optimal sum of $q - \delta$ in terms of the vertexes in V_i. Introducing variables $\lambda_{ij}, i = 1, \ldots, n+1, j = 1, \ldots, m_i$, one has

$$q - \delta = \sum_{i=1}^{n+1} q_i = \sum_{i=1}^{n+1}\sum_{j=1}^{m_i} \lambda_{ij}\, a_{ij}, \quad \text{with } \sum_{j=1}^{m_i} \lambda_{ij} = 1 \text{ and } \lambda_{ij} \ge 0. \tag{19}$$

On the other hand, in order to make the lifted points lie on the lower envelope of $\widehat{Q}$, one must force the "height" of the listed points minimal, thus requiring to find λ_{ij} such that

$$\sum_{i=1}^{n+1}\sum_{j=1}^{m_i} \lambda_{ij}\, l_i(a_{ij}) \text{ to be minimized} \tag{20}$$

under the linear constraint conditions (19), where $l_i(a_{ij})$ is a random linear function in a_{ij}.

For $q \in \mathscr{E}$, let λ^*_{ij} be an optimal solution for the linear programming problem (20). Then $q - \delta = \sum_{i=1}^{n+1} q_i^*$ where $q_i^* = \sum_{j=1}^{m_i} \lambda^*_{ij} l_i(a_{ij})$. $a^*_{ij_0}$ is a vertex of Q_i if and only if there exists a j_0 such that $\lambda^*_{ij_0} = 1$ and $\lambda^*_{ij} = 0$ for $j \neq j_0$. In this case, the generalized row content function of q is (i, j_0) and $\mathrm{mm}(p_i)$ is $M(a^*_{ij_0})$. It is shown that when the lift functions l_i are general enough, all S_i can be computed in the above way [4].

In order to study the linear programming problem (20), we need to recall a lemma about the optimality criterion for the general linear programming problem

$$\begin{aligned} &\min_x \qquad z = c^T x \\ &\text{subject to} \quad Ax = b, \quad \text{with } \ l \le x \le u, \end{aligned} \tag{21}$$

where A is an $m \times n$ rectangular matrix, b is a column vector of dimension m, c and x are column vectors of dimension n, and the superscript T stands for transpose. In order for the linear programming problem to be meaningful, the row rank of A must be less than the column rank of A. We thus can assume A to be row full rank. Let $n_1, \ldots, n_m$ be linear independent columns of A. Then the corresponding $x_{n_1}, \ldots, x_{n_m}$ are called *basic variables of* x. Let B be the matrix consisting of the $n_1, \ldots, n_m$ columns of A. Then B is an $m \times m$ invertible matrix. Lemma 4 below gives an optimality criterion for the linear programming problem (21).

Lemma 4 [10] *Let x_B be a basic variables set of x, where B is the corresponding coefficient matrix of x_B. If the corresponding basic feasible solution $x_B = B^{-1}b \ge 0$ and the conditions $c_B B^{-1} A - c \le 0$ hold, where c_B is the row vector obtained by listing the coefficients of x_B in the object function, then an optimal solution for the linear programming problem* (21) *can be given as $x_B = B^{-1}b$ and all other x_i equals zero, which is called the optimal solution determined by the basic variables x_B.*

4.2 Algebraic Sparse Resultant Matrix

In this subsection, we show that the sparse resultant for f_1, f_2, δf_1 and δf_2 is nonzero and contains the differential resultant of f_1 and f_2 as a factor.

For the differential polynomials f_1 and f_2 given in (3), consider the $p_1 = \delta f_1$, $p_2 = \delta f_2$, $p_3 = f_1$, $p_4 = f_2$ as algebraic polynomials in y, y_1, y_2. The monomial sets of $\delta f_1, \delta f_2, f_1$ and f_2 are $B_3^{d_1} \cup y_2 * B_3^{d_1-1}$, $B_3^{d_2} \cup y_2 * B_3^{d_2-1}$, $B_3^{d_1}$, and $B_3^{d_2}$ respectively. For convenience, we will not distinguish a monomial M and its exponential vector when there exists no confusion. Then the Newton polytopes for $\delta f_1, \delta f_2, f_1$ and f_2 are respectively,

$$Q_1 = \mathrm{Conv}(\sup(\delta f_1)) = \mathrm{Conv}(B_3^{d_1} \cup y_2 * B_3^{d_1-1}) \subset \mathbb{R}^3,$$
$$Q_2 = \mathrm{Conv}(\sup(\delta f_2)) = \mathrm{Conv}(B_3^{d_2} \cup y_2 * B_3^{d_2-1}) \subset \mathbb{R}^3,$$
$$Q_3 = \mathrm{Conv}(\sup(f_1)) = \mathrm{Conv}(B_3^{d_1}) \subset \mathbb{R}^3, \tag{22}$$
$$Q_4 = \mathrm{Conv}(\sup(f_2)) = \mathrm{Conv}(B_3^{d_2}) \subset \mathbb{R}^3,$$

The Newton polytopes Q_1 and Q_3 are shown in Fig. 1 (for $d_1 = 5$) while Q_2 and Q_4 have similar polytopes as Q_1 and Q_3 but with different sizes respectively.

Let the Minkowski sum $Q = Q_1 + Q_2 + Q_3 + Q_4$. In order to compute the column monomial set, we choose a perturbed vector $\delta = (\delta_1, \delta_2, \delta_3)$ with $0 < \delta_i < 1$ with $i = 1, 2, 3$. Then the points in $\mathbb{Z}^3 \cap (Q + \delta)$ is easily shown to be $yy_1y_2\mathscr{E}$ where $\mathscr{E}$ is given in (4). Note that using $\mathscr{E}$ or $yy_1y_2\mathscr{E}$ as the column monomial set will lead to the same matrix.

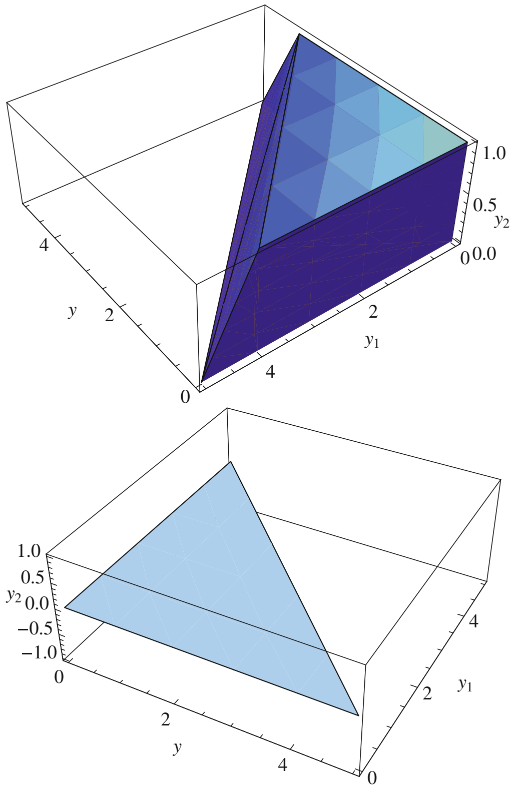

Fig. 1 The Newton polytopes Q_1 and Q_3

The vertex sets of Q_i, denoted by V_i, are respectively

$$\begin{aligned}
\mathrm{V}_1 &:= \{(0,0,0),(0,0,1),(0,d_1-1,1),(0,d_1,0),(d_1-1,0,1),(d_1,0,0)\},\\
\mathrm{V}_2 &:= \{(0,0,0),(0,0,1),(0,d_2-1,1),(0,d_2,0),(d_2-1,0,1),(d_2,0,0)\},\\
\mathrm{V}_3 &:= \{(0,0,0),(0,d_1,0),(d_1,0,0)\},\\
\mathrm{V}_4 &:= \{(0,0,0),(0,d_2,0),(d_2,0,0)\}.
\end{aligned}$$

Let the lifting functions be $l_i = (L_{i1}, L_{i2}, L_{i3})$, $i = 1, \ldots, 4$, where L_{ij} are parameters to be determined later. From (20), the object function of the linear programming problem to be solved is

$$\begin{aligned}
\min_{\lambda_{ij}}(&\lambda_{12}L_{13} + \lambda_{13}[L_{12}(d_1-1) + L_{13}] + \lambda_{14}L_{12}d_1 + \lambda_{15}[L_{11}(d_1-1) + L_{13}]\\
&+\lambda_{16}L_{11}d_1 + \lambda_{22}L_{23} + \lambda_{23}[L_{22}(d_2-1) + L_{23}] + \lambda_{24}L_{22}d_2\\
&+\lambda_{25}[L_{21}(d_2-1) + L_{23}] + \lambda_{26}L_{21}d_2 + \lambda_{32}L_{32}d_1 + \lambda_{33}L_{31}d_1\\
&+\lambda_{42}L_{42}d_2 + \lambda_{43}L_{41}d_2)
\end{aligned} \tag{23}$$

under the constraints

$$\begin{aligned}
&A_1 = \lambda_{15}(d_1-1) + \lambda_{16}d_1 + \lambda_{25}(d_2-1) + \lambda_{26}d_2 + \lambda_{33}d_1 + \lambda_{43}d_2,\\
&A_2 = \lambda_{13}(d_1-1) + \lambda_{14}d_1 + \lambda_{23}(d_2-1) + \lambda_{24}d_2 + \lambda_{32}d_1 + \lambda_{42}d_2,\\
&A_3 = \lambda_{12} + \lambda_{13} + \lambda_{15} + \lambda_{22} + \lambda_{23} + \lambda_{25},\\
&\sum_{j=1}^{m_i}\lambda_{ij} = 1,\ i = 1, \ldots, 4,\\
&\lambda_{ij} \geq 0, i = 1, \ldots, 4,\ j = 1, \ldots, m_i \ \text{ with } m_1 = m_2 = 6, m_3 = m_4 = 3,
\end{aligned} \tag{24}$$

where $A_1 = \varepsilon_1 - \delta_1, A_2 = \varepsilon_2 - \delta_2, A_3 = \varepsilon_3 - \delta_3$ with $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{Z}^3 \cap (Q + \delta)$. According to the procedure given in Sect. 4.1, we need to solve the linear programming problem (23) for each $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{Z}^n \cap (Q + \delta)$. Note that L_{ij} are parameters. What we need to do is to show that there exist L_{ij} such that the solutions of (23) make the corresponding main monomials to be the ones selected by us in (5). More precisely, we need to determine L_{ij} such that for each $q \in \mathbb{Z}^n \cap (Q + \delta)$, the optimal solution for the linear programming problem (23) consists one of the following cases:

$$\begin{aligned}
&\lambda_{13} = 1 \text{ implies } \mathrm{GRC}(q) = (1,3),\ \text{the vertex is } (0, d_1-1, 1),\ \text{and } \mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1},\\
&\lambda_{24} = 1 \text{ implies } \mathrm{GRC}(q) = (2,4),\ \text{the vertex is } (0, d_2, 0),\ \text{and } \mathrm{mm}(\delta f_2) = y_1^{d_2},\\
&\lambda_{33} = 1 \text{ implies } \mathrm{GRC}(q) = (3,3),\ \text{the vertex is } (d_1, 0, 0),\ \text{and } \mathrm{mm}(f_1) = y^{d_1},\\
&\lambda_{41} = 1 \text{ implies } \mathrm{GRC}(q) = (4,1),\ \text{the vertex is } (0,0,0),\ \text{and } \mathrm{mm}(f_2) = 1.
\end{aligned}$$

The following lemma proves that the above statement is valid.

Lemma 5 *There exist L_{ij} such that the optimal solution of the corresponding linear programming problem (23) can be chosen such that the corresponding main monomials for $f_1, f_2, \delta f_1$ and δf_2 are $\mathrm{mm}(f_1) = y^{d_1}$, $\mathrm{mm}(f_2) = 1$, $\mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}$, $\mathrm{mm}(\delta f_2) = y_1^{d_2}$ respectively and $\mathscr{E}$ can be written as a disjoint union $\mathscr{E} = S_1 \cup S_2 \cup S_3 \cup S_4$, where S_i is defined in (16).*

Proof We write the linear programming problem as the standard form (21). It is easy to see

$$\begin{aligned} c = \; & (0, L_{13}, (d_1-1)L_{12} + L_{13}, d_1 L_{12}, (d_1-1)L_{11} + L_{13}, d_1 L_{11}, \\ & 0, L_{23}, (d_2-1)L_{22} + L_{23}, d_2 L_{22}, (d_2-1)L_{21} + L_{23}, d_2 L_{21}, \\ & 0, d_1 L_{32}, d_1 L_{31}, 0, d_2 L_{42}, d_2 L_{41}). \end{aligned}$$

Let $\delta = (\delta_1, \delta_2, \delta_3)$ be a sufficiently small vector in sufficiently generic position. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \widetilde{d_1} & d_1 & 0 & 0 & 0 & 0 & \widetilde{d_2} & d_2 & 0 & 0 & d_1 & 0 & 0 & d_2 \\ 0 & 0 & \widetilde{d_1} & d_1 & 0 & 0 & 0 & 0 & \widetilde{d_2} & d_2 & 0 & 0 & 0 & d_1 & 0 & 0 & d_2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

where $\widetilde{d_1} = d_1 - 1$, $\widetilde{d_2} = d_2 - 1$, which is a 7×18 matrix and $b = (A_1, A_2, A_3, 1, 1, 1, 1)$. It is easy to see that the rank of A is 7, since $d_1 \geq 1$.

From (4), we have $\mathscr{E} = \mathbb{Z}^3 \cap (Q + \delta) = y y_1 y_2 (B_3^D \cup y_2 B_3^{D-1})$, where $D = 2d_1 + 2d_2 - 3$. We will construct a disjoint union $\mathscr{E} = S_1 \cup S_2 \cup S_3 \cup S_4$ like (17) such that the corresponding main monomials are respectively $\mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}$, $\mathrm{mm}(\delta f_2) = y_1^{d_1}$, $\mathrm{mm}(f_1) = y^{d_1}$, $\mathrm{mm}(f_2) = 1$.

Four cases will be considered.

Case 1. We will give the conditions about L_{ij} under which $\mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}$, or equivalently, the linear programming problem (23) has an optimal solution where $\lambda_{13} = 1$. As a consequence, S_1 will also be constructed.

As shown by Lemma 4, an optimal solution for a linear programming problem can be uniquely determined by a set of basic variables. We will construct the required optimal solutions by choosing different sets of basic variables. Four subcases are considered.

1.1. Selecting basic variables as $\mathrm{vet}_{11} = \{\lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{32}, \lambda_{33}, \lambda_{41}, \lambda_{43}\}$ while other variables are nonbasic variables and equal to zero. Due the constraint $\lambda_{11} + \lambda_{12} + \cdots + \lambda_{16} = 1$, we have $\lambda_{13} = 1$. Then for any such an optimal solution of the linear programming problem (23), in the optimal sum of any element

$q = q_1 + q_2 + q_3 + q_4, q_1 = (0, d_1 - 1, 1)$ is a vertex of Q_1, $\mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}$, and the corresponding q belongs to S_1 as defined in (16).

We claim that the basic feasible solutions in vet_{11} must be nondegenerate meaning that all basic variables are positive, that is, $x_B = B^{-1}b > 0$. A mixed cell $R = \sum_{i=1}^{4} \mathscr{F}_i$, where $\mathscr{F}_i$ is a face of Q_i, must satisfy the dimension constraint $\sum_{i=1}^{4} \dim(\mathscr{F}_i) = 3$. In the cell corresponding to the basic variables vet_{11}, $\mathscr{F}_1 = (0, d_1 - 1, 1)$ is a vertex of Q_1, $\mathscr{F}_4$ is a one dimensional face of Q_4 of the form $\lambda_{41} V_{41} + \lambda_{43} V_{43}$, where $V_{41} = (0, 0, 0)$, $V_{43} = (d_2, 0, 0)$, and $\lambda_{41} + \lambda_{43} = 1$. $\mathscr{F}_2$ and $\mathscr{F}_3$ are one dimensional faces of Q_2 and Q_3 respectively. In order for the dimension constraint $\sum_{i=1}^{4} \dim(\mathscr{F}_i) = 3$ to be valid, the claim must be true. For otherwise, one of the variables in vet_{11} must be zero, say $\lambda_{41} = 0$. Then $\lambda_{43} = 1$ and $\mathscr{F}_4$ becomes a vertex, which implies $\sum_{i=1}^{4} \dim(\mathscr{F}_i) < 3$, a contradiction.

From Lemma 4, the coefficient matrix of basic variables in (24) is

$$B_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & d_1 & 0 & d_2 \\ d_1 - 1 & d_2 - 1 & d_2 & d_1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

with $\mathrm{rank}(B_{11}) = 7$. For all (A_1, A_2, A_3) and $b = (A_1, A_2, A_3, 1, 1, 1, 1)$, the requirement $B_{11}^{-1} b > 0$ in Lemma 4 gives

$$\begin{aligned} &1 < A_3 < 2, d_1 + d_2 < A_2 + A_3 < 2d_1 + d_2, \\ &2d_1 + d_2 < A_1 + A_2 + A_3 < 2d_1 + 2d_2. \end{aligned}$$

Substituting $A_1 = \varepsilon_1 - \delta_1, A_2 = \varepsilon_2 - \delta_2, A_3 = \varepsilon_3 - \delta_3$ into the above inequalities and considering that $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are integer points, we have

$$\begin{aligned} &\varepsilon_3 = 2, \quad \varepsilon_2 = d_1 + d_2 - 1, \ldots, 2d_1 + d_2 - 2, \\ &\varepsilon_1 + \varepsilon_2 = 2d_1 + d_2 - 1, \ldots, 2d_1 + 2d_2 - 2. \end{aligned} \tag{25}$$

On the other hand, $c_{B_{11}} = ((d_1-1)L_{12}+L_{13}, (d_2-1)L_{22}+L_{23}, d_2 L_{22}, d_1 L_{32}, d_1 L_{31}, 0, d_2 L_{41})$. After simplification and rearrangement, the condition $c_{B_{11}} B_{11}^{-1} A - c \leq 0$ in Lemma 4 becomes

$$\begin{aligned} &\{L_{12} - L_{11} + L_{31} - L_{32}, L_{12} + L_{31} - L_{32} - L_{41}, \\ &L_{13} - L_{12} + L_{22} - L_{23}, L_{22} - L_{21} + L_{31} - L_{32}, \\ &L_{22} + L_{31} - L_{32} - L_{41}, L_{31} - L_{41}, L_{32} - L_{31} + L_{41} - L_{42}\} \leq 0 \end{aligned} \tag{26}$$

where, hereinafter, $\{w_1, \ldots, w_s\} \leq 0$ means $w_i \leq 0$ for $i = 1, \ldots, s$.

By Lemma 4, if (25) and (26) are valid, we obtain an optimal solution of the linear programming problem (23) which is determined by the basic variables vet_{11}. Hence, if (26) is valid, the corresponding $q = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in and (25) are in S_1, since in the optimal decomposition of $q = q_1 + q_2 + q_3 + q_4$, $q_1 = (0, d_1 - 1, 1)$ is a vertex.

1.2. Similarly, choosing the basic variables as $\mathrm{vet}_{12} = \{\lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{41}\}$, which generates a new basic matrix B_{12}, and from $B_{12}^{-1} b > 0$, we obtain

$$0 < A_1, d_1 + d_2 < A_2 + A_3, 1 < A_3 < 2, A_1 + A_2 + A_3 < 2d_1 + d_2,$$

which in turn lead to the following values for $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\begin{aligned} &\varepsilon_3 = 2, \quad \varepsilon_1 = 1, \ldots, \quad \varepsilon_2 = d_1 + d_2 - 1, \ldots \\ &\varepsilon_1 + \varepsilon_2 = d_1 + d_2, \ldots, 2d_1 + d_2 - 2. \end{aligned} \tag{27}$$

The condition $c_{B_{12}} B_{12}^{-1} A - c \leq 0$ leads to the following constraints on L_{ij}, $i = 1, \ldots, 4$, $j = 1, 2, 3$,

$$\begin{aligned} &\{L_{12} - L_{11} + L_{31} - L_{32}, L_{12} - L_{32}, \\ &L_{13} - L_{12} + L_{22} - L_{23}, L_{22} - L_{32}, \\ &L_{22} - L_{21} + L_{31} - L_{32}, L_{31} - L_{41}, L_{32} - L_{42}\} \leq 0. \end{aligned} \tag{28}$$

1.3. Similarly, the basic variables $\mathrm{vet}_{13} = \{\lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{32}, \lambda_{33}, \lambda_{41}, \lambda_{42}\}$ lead to

$$\begin{aligned} &\varepsilon_3 = 2, \quad \varepsilon_1 = 1, \ldots, d_1, \\ &\varepsilon_1 + \varepsilon_2 = 2d_1 + d_2 - 1, \ldots, 2d_1 + 2d_2 - 2, \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\{L_{12} - L_{42}, L_{12} - L_{11} + L_{31} - L_{32}, \\ &L_{13} - L_{12} + L_{22} - L_{23}, L_{22} - L_{21} + L_{31} - L_{32}, \\ &L_{22} - L_{42}, L_{31} - L_{32} - L_{41} + L_{42}, L_{32} - L_{42}\} \leq 0. \end{aligned} \tag{30}$$

1.4. Similarly, the basic variables $\mathrm{vet}_{14} = \{\lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{33}, \lambda_{41}, \lambda_{42}, \lambda_{43}\}$ lead to

$$\begin{aligned} &\varepsilon_3 = 2, \quad \varepsilon_1 = d_1 + 1, \ldots, \quad \varepsilon_2 = d_1 + d_2 - 1, \ldots \\ &\varepsilon_1 + \varepsilon_2 = 2d_1 + d_2, \ldots, 2d_1 + 2d_2 - 2. \end{aligned} \tag{31}$$

and

$$\begin{aligned}&\{L_{12}-L_{11}+L_{41}-L_{42}, L_{12}-L_{42},\\&L_{13}-L_{12}+L_{22}-L_{23}, L_{22}-L_{21}+L_{41}-L_{42},\\&L_{22}-L_{42}, L_{31}-L_{41}, L_{31}-L_{32}-L_{41}+L_{42}\}\le 0.\end{aligned} \tag{32}$$

Let S_1 be the set $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ defined in (25), (27), (29), and (31). Then for $\eta \in S_1$ and an optimal sum of $\eta = q_1 + q_2 + q_3 + q_4$, since $\lambda_{13} = 1$, q_1 must be the vertex $(0, d_1 - 1, 1)$ of Q_1. Therefore, $\mathrm{mm}(\delta f_1) = y_2 y_1^{d_1-1}$. Of course, in order for this statement to be valid, L_{ij} must satisfy constraints (26), (28), (30), and (32). We will show later that these constraints indeed have common solutions.

The following three cases can be treated similarly, and we only list the conditions for $\varepsilon_1, \varepsilon_2, \varepsilon_3$ while the concrete requirements for L_{ij} are listed at the end of the proof.

Case 2. In order for $\mathrm{mm}(\delta f_2) = y_1^{d_1}$, we choose the basic variables

$$\begin{aligned}\mathrm{vet}_{21} &= \{\lambda_{13}, \lambda_{14}, \lambda_{24}, \lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{41}\},\\ \mathrm{vet}_{22} &= \{\lambda_{13}, \lambda_{14}, \lambda_{24}, \lambda_{33}, \lambda_{41}, \lambda_{42}, \lambda_{43}\},\\ \mathrm{vet}_{23} &= \{\lambda_{13}, \lambda_{14}, \lambda_{24}, \lambda_{32}, \lambda_{33}, \lambda_{41}, \lambda_{43}\},\\ \mathrm{vet}_{24} &= \{\lambda_{13}, \lambda_{14}, \lambda_{24}, \lambda_{32}, \lambda_{33}, \lambda_{41}, \lambda_{42}\},\\ \mathrm{vet}_{25} &= \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{24}, \lambda_{31}, \lambda_{33}, \lambda_{41}\},\\ \mathrm{vet}_{26} &= \{\lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{24}, \lambda_{33}, \lambda_{41}, \lambda_{43}\},\\ \mathrm{vet}_{27} &= \{\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{24}, \lambda_{33}, \lambda_{41}\},\end{aligned}$$

which lead to the following elements of S_2

$$\begin{aligned}&\varepsilon_3 = 1,\ \ 0 < \varepsilon_1,\ \ d_1+d_2-1 < \varepsilon_2, \varepsilon_1+\varepsilon_2 = d_1+d_2+1, \ldots, 2d_1+d_2-1;\\ &\varepsilon_3 = 1,\ \ d_1 < \varepsilon_1,\ \ d_1+d_2-1 < \varepsilon_2, \varepsilon_1+\varepsilon_2 = 2d_1+d_2+1, \ldots, 2d_1+2d_2-1;\\ &\varepsilon_3 = 1,\ \ \varepsilon_2 = d_1+d_2, \ldots, 2d_1+d_2-1,\ \ \varepsilon_1+\varepsilon_2 = 2d_1+d_2, \ldots, 2d_1+2d_2-1;\\ &\varepsilon_3 = 1,\ \ \varepsilon_1 = 1, \ldots, d_1,\ \ \varepsilon_1+\varepsilon_2 = 2d_1+d_2, \ldots, 2d_1+2d_2-1;\\ &\varepsilon_3 = 1,\ \ \varepsilon_1 = 1, \ldots, d_1,\ \ \varepsilon_2 = d_2+1, \ldots, d_1+d_2-1;\\ &\varepsilon_3 = 1,\ \ \varepsilon_2 = d_2+1, \ldots, d_1+d_2-1,\ \ \varepsilon_2 = 2d_1+d_2, \ldots, 2d_1+2d_2-1;\\ &\varepsilon_3 = 1,\ \ \varepsilon_1 = d_1+1, \ldots,\ \ \varepsilon_2 = d_2+1, \ldots, \varepsilon_1+\varepsilon_2 = d_1+d_2+2, \ldots, 2d_1+d_2-1.\end{aligned}$$

Case 3. In order for $\mathrm{mm}(f_1) = y_1^{d_1}$, we choose the basic variables

$$\begin{aligned}\mathrm{vet}_{31} &= \{\lambda_{15}, \lambda_{16}, \lambda_{24}, \lambda_{26}, \lambda_{33}, \lambda_{41}, \lambda_{43}\},\\ \mathrm{vet}_{32} &= \{\lambda_{13}, \lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{33}, \lambda_{41}, \lambda_{43}\},\\ \mathrm{vet}_{33} &= \{\lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{33}, \lambda_{41}, \lambda_{43}\},\\ \mathrm{vet}_{34} &= \{\lambda_{12}, \lambda_{13}, \lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{33}, \lambda_{41}\},\end{aligned}$$

which lead to the following results about $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in S_3

$$\begin{aligned}
&\varepsilon_3 = 1,\ \ \varepsilon_2 = 1, \ldots, d_2,\ \ \varepsilon_1 + \varepsilon_2 = 2d_1 + d_2, \ldots, 2d_1 + 2d_2 - 1;\\
&\varepsilon_3 = 2,\ \ \varepsilon_2 = d_2, \ldots, d_1 + d_2 - 2, \varepsilon_1 + \varepsilon_2 = 2d_1 + d_2 - 1, \ldots, 2d_1 + 2d_2 - 2;\\
&\varepsilon_3 = 2,\ \ \varepsilon_2 = 1, \ldots, d_2 - 1,\ \ \varepsilon_1 + \varepsilon_2 = 2d_1 + d_2 - 1, \ldots, 2d_1 + 2d_2 - 2;\\
&\varepsilon_3 = 2,\ \ \varepsilon_1 = d_1 + 1, \ldots,\ \ \varepsilon_2 = d_2, \ldots, \varepsilon_1 + \varepsilon_2 = d_1 + d_2 + 1, \ldots, 2d_1 + d_2 - 2.
\end{aligned} \tag{33}$$

Case 4. In order for $\mathrm{mm}(f_2) = 1$, we choose the following basic variables

$$\begin{aligned}
\mathrm{vet}_{41} &= \{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{24}, \lambda_{26}, \lambda_{31}, \lambda_{41}\},\\
\mathrm{vet}_{42} &= \{\lambda_{11}, \lambda_{12}, \lambda_{24}, \lambda_{26}, \lambda_{31}, \lambda_{33}, \lambda_{41}\},\\
\mathrm{vet}_{43} &= \{\lambda_{11}, \lambda_{12}, \lambda_{15}, \lambda_{24}, \lambda_{26}, \lambda_{33}, \lambda_{41}\},\\
\mathrm{vet}_{44} &= \{\lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{31}, \lambda_{33}, \lambda_{41}\},\\
\mathrm{vet}_{45} &= \{\lambda_{12}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{31}, \lambda_{33}, \lambda_{41}\},\\
\mathrm{vet}_{46} &= \{\lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{25}, \lambda_{31}, \lambda_{41}\},\\
\mathrm{vet}_{47} &= \{\lambda_{12}, \lambda_{15}, \lambda_{23}, \lambda_{25}, \lambda_{26}, \lambda_{33}, \lambda_{41}\},
\end{aligned}$$

which correspond to the elements in S_4

$$\begin{aligned}
&\varepsilon_3 = 1,\ \ 0 < \varepsilon_1,\ \ 0 < \varepsilon_2,\ \ \varepsilon_1 + \varepsilon_2 = 2, \ldots, d_2;\\
&\varepsilon_3 = 1,\ \ \varepsilon_2 = 1, \ldots, d_2,\ \ \varepsilon_1 + \varepsilon_2 = d_2 + 1, \ldots, d_1 + d_2;\\
&\varepsilon_3 = 1,\ \ \varepsilon_2 = 1, \ldots, d_2,\ \ \varepsilon_1 + \varepsilon_2 = d_1 + d_2 + 1, \ldots, 2d_1 + d_2 - 1;\\
&\varepsilon_3 = 2,\ \ \varepsilon_1 = 1, \ldots, d_1,\ \ \varepsilon_2 = d_2, \ldots, d_1 + d_2 - 2;\\
&\varepsilon_3 = 2,\ \ \varepsilon_2 = 1, \ldots, d_2 - 1,\ \ \varepsilon_1 + \varepsilon_2 = d_2, \ldots, d_1 + d_2 - 1;\\
&\varepsilon_3 = 2,\ \ \varepsilon_1 = 1, \ldots,\ \ \varepsilon_2 = 1, \ldots, d_2 - 2,\ \ \varepsilon_1 + \varepsilon_2 = 2, \ldots, d_2 - 1;\\
&\varepsilon_3 = 1,\ \ \varepsilon_2 = 1, \ldots, d_2 - 1, \varepsilon_1 + \varepsilon_2 = d_1 + d_2 - 1, \ldots, 2d_1 + d_2 - 2.
\end{aligned}$$

Merge all the constraints for L_{ij}, we obtain

$$\begin{aligned}
&L_{11} - L_{12} - L_{21} + L_{22} \le 0,\\
&L_{13} \le L_{23}, L_{21} \le L_{31} \le L_{11} \le L_{41},\\
&L_{22} \le L_{12} \le L_{32} \le L_{42}, L_{31} = L_{32} + L_{41} - L_{42}.
\end{aligned} \tag{34}$$

The solution set for system (34) is nonempty. For example, $l_1 = (7, -4, -5)$, $l_2 = (5, -9, 5), l_3 = (6, 2, 1), l_4 = (8, 4, 7)$, which will be used for example (1), satisfy the conditions in (34).

We can also check that $\mathscr{E} = S_1 \cup S_2 \cup S_3 \cup S_4$ is a disjoint union for $\mathscr{E}$. The lemma is proved. □

We now have the main result of this section.

Theorem 7 *The sparse resultant of $f_1, f_2, \delta f_1, \delta f_2$ as polynomials in y, y_1, y_2 is not identically zero and contains the differential resultant of f_1 and f_2 as a factor.*

Proof Note that $a_0, b_0, \delta a_0, \delta b_0$, which are the zero degree terms of $f_1, f_2, \delta f_1, \delta f_2$ respectively, are algebraic indeterminates. As a consequence,

$$J_1 = (f_1, f_2, \delta f_1, \delta f_2)$$

is a prime ideal in $\mathbb{Q}[\mathbf{u}, y, y_1, y_2]$, where $\mathbf{u}$ is the set of the coefficients of f_1, f_2 are their first order derivatives. Let

$$J_2 = J_1 \cap \mathbb{Q}[\mathbf{u}].$$

Then J_2 is also a prime ideal. We claim that

$$J_2 = (\mathbf{R}) \tag{35}$$

where $\mathbf{R}$ is the differential resultant of f_1 and f_2. From (c) of Theorem 3, $\mathbf{R} \in J_2$. Let $T \in J_2$. Then $T \in J_1 \subset [f_1, f_2]$. From (2), the pseudo remainder of T with respect to $\mathbf{R}$ is zero. Also note that the order of T in a_i, b_i is less than or equal to 1. From (a) and (b) of Theorem 3, $\mathbf{R}$ must be a factor of T, which proves (35).

From Lemma 5, the main monomials for $f_1, f_2, \delta f_1, \delta f_2$ are the same as those used to construct $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \mathscr{S}_4$ in (5). As a consequence, we have $S_1 \subset \mathscr{S}_1$. For $q \in \mathscr{S}_1 \setminus S_1$, q must be in some S_i, say $q \in S_2$. Then from Lemma 5, the monomials in $(M(q)/\mathrm{mm}(\delta f_2))\delta f_2$ are contained in $\mathscr{E}$. By Corollary 4, the sparse resultant matrix of $f_1, f_2, \delta f_1, \delta f_2$ obtained after move q from S_2 to S_1 is still nonsingular. Doing such movements repeatedly will lead to $\mathscr{S}_1 = S_1, \mathscr{S}_2 = S_2, \mathscr{S}_3 = S_3, \mathscr{S}_4 = S_4$. As a consequence, the sparse resultant is not identically zero.

From (15), we have $\mathscr{R} \in J_1$ which implies $\mathscr{R} \in J_2$. Since $\mathbf{R}$ is irreducible, $\mathbf{R}$ must be a factor of $\mathscr{R}$. □

4.3 Example (1) Revisited

We show how to construct a nonsingular algebraic sparse resultant matrix of the system $\{g_1, g_2, \delta g_1, \delta g_2\}$, where g_1, g_2 are from (1).

Using the algorithm for sparse resultant in [3, 4], we choose perturbed vector $\delta = (0.01, 0.01, 0.01)$ and the lifting functions $l_1 = (7, -4, -5), l_2 = (5, -9, 5)$, $l_3 = (6, 2, 1), l_4 = (8, 4, 7)$, where l_i corresponds to Q_i defined in (22) with $d_1 = d_2 = 2$. These lift functions satisfy the conditions (34).

By Lemma 5, the main monomials for $g_1, g_2, \delta g_1, \delta g_2$ are identical with those given in Sect. 3.3. Let S_1, S_2, S_3, S_4 be those constructed as in the proof of Lemma 5. After the following changes

move $\{y_2 y_1 y^3, y_2 y_1 y^2\}$ in S_3 to S_1,
move $\{y_2 y^2, y_1 y^2, y^3, y^2\}$ in S_4 to S_3,
move $\{y_2 y_1 y, y_2 y_1\}$ in S_4 to S_1,

we have $\mathscr{S}_i = S_i, i = 1, \ldots, 4$. Then by Corollary 4, the sparse resultant matrix constructed with the original S_1, S_2, S_3, S_4 is nonsingular and contains the differential resultant as a factor.

5 Conclusion and Discussion

In this paper, a matrix representation for two first order nonlinear generic ordinary differential polynomials f_1, f_2 is given. That is, a nonsingular matrix is constructed such that its determinant contains the differential resultant as a factor. The constructed matrix is further shown to be an algebraic sparse matrix of $f_1, f_2, \delta f_1$ and δf_2 when certain special lift functions are used. Combining the two results, we show that the sparse resultant of $f_1, f_2, \delta f_1$ and δf_2 is not zero and conatins the differential resultant of f_1 and f_2 as a factor.

It can be seen that to give a matrix representation for $n + 1$ generic polynomials in n variables is far from solved, even in the case of $n = 1$. Based on what is proved in this paper, we propose the following conjecture.

Conjecture Let $\mathscr{P} = \{f_1, f_2, \ldots, f_{n+1}\}$ be $n+1$ generic differential polynomials in n indeterminates, $\mathrm{ord}(f_i) = s_i$, and $s = \sum_{i=0}^{n} s_i$.

Then the sparse resultant of the algebraic polynomial system

$$f_1, \delta f_1, \ldots \delta^{s-s_0} f_1, \ldots, f_{n+1}, \delta f_{n+1}, \ldots \delta^{s-s_n} f_{n+1} \tag{36}$$

is not zero and contains the differential resultant of $\mathscr{P}$ as a factor.

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