

# A Generalization of the Concavity of Rényi Entropy Power

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## Abstract

Recently, Savaré-Toscani proved that the Rényi entropy power of general probability densities solving the  $p$ -nonlinear heat equation in  $\mathbb{R}^n$  is always a concave function of time, which extends Costa's concavity inequality for Shannon's entropy power to Rényi entropies. In this paper, we give a generalization of Savaré-Toscani's result by giving a class of sufficient conditions of the parameters under which the concavity of the Rényi entropy power is still valid. These conditions are quite general and include the parameter range given by Savaré-Toscani as special cases. Also, the conditions are obtained with a systematical approach.

**Keywords.** Rényi entropy, entropy power inequality, nonlinear heat equation.

## 1 Introduction

The  $p$ -th Rényi entropy [1, 2] of a probability density function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined as

$$H_p(f) := \frac{1}{1-p} \log \int_{\mathbb{R}^n} f^p(x) dx, \quad (1)$$

for  $0 < p < +\infty$ ,  $p \neq 1$ . The  $p$ -th Rényi entropy power is given by

$$N_p(f) := \exp\left(\frac{\mu}{n} H_p(f)\right), \quad (2)$$

where  $\mu$  is a real valued parameter. The Rényi entropy for  $p = 1$  is defined as the limit of  $H_p(f)$  as  $p \rightarrow 1$ . It follows from definition (1) that

$$H_1(f) = \lim_{p \rightarrow 1} H_p(f) = - \int_{\mathbb{R}^n} f(x) \log f(x) dx,$$

which is Shannon's entropy. In this case, the proposed Rényi entropy power of index  $p = 1$ ,  $\mu = 2$ , given by (2), coincides with Shannon's entropy power

$$N_1(f) := \exp\left(\frac{2}{n} H_1(f)\right). \quad (3)$$

Shannon's *entropy power inequality (EPI)* is one of the most important information inequalities [3], which has many proofs, generalizations, and applications [4, 5, 6, 7, 8, 9, 10, 11, 12]. In particular, Costa presented a stronger version of the EPI [13].

Let  $X_t \triangleq X + N_n(0, tI)$  be the  $n$ -dimensional random vector introduced by Costa [13, 19, 20] and  $u(x_t)$  the *probability density* of  $X_t$ , which solves the heat equation in the whole space  $\mathbb{R}^n$ ,

$$\frac{\partial}{\partial t}u(x_t) = \Delta u(x_t). \quad (4)$$

Costa's *differential entropy* is defined to be

$$H(u(x_t)) = - \int_{\mathbb{R}^n} u(x_t) \log u(x_t) dx_t. \quad (5)$$

Costa [13] proved that the *Shannon entropy power*  $N(u) = \frac{1}{2\pi e} e^{(2/n)H(u)}$  is a concave function in  $t$ , that is  $(d/dt)N(u) \geq 0$  and  $(d^2/d^2t)N(u) \leq 0$ . Several new proofs and generalizations for Costa's EPI were given [14, 15, 16].

Savaré-Toscani [21] proved that the *concavity of entropy power* is a property which is not restricted to the Shannon entropy power (3) in connection with the heat equation (4), but it holds for the  $p$ -th Rényi entropy power (2). They put it in connection with the solution to the *nonlinear heat equation*

$$\frac{\partial}{\partial t}u(x_t) = \Delta u(x_t)^p \quad (6)$$

posed in the whole space  $\mathbb{R}^n$  and  $p \in \mathbb{R}_{>0}$ .

In this paper, we give a generalization for *concavity of  $p$ -th Rényi entropy power (CREP)*. Precisely, we give a propositional logic formula  $\Phi(n, p, \mu)$  such that if  $n \in \mathbb{N}, p, \mu \in \mathbb{R}$  satisfy this formula, then the CREP holds. The condition  $\Phi(n, p, \mu)$  extends the parameter's range of the CREP given by Savaré-Toscani [21] and contains much more cases.

The formula  $\Phi$  is obtained using a systematic procedure which can be considered as a parametric version of that given in [19, 20, 17], where parameters  $n, p, \mu$  exist in the formulas. The procedure reduces the proof of the CREP to check the semi-positiveness of a quadratic form whose coefficients are polynomials in the parameters  $n, p, \mu$ . In principle, a necessary and sufficient condition for the parameters to satisfy this property can be computed with the quantifier elimination [22]. In this paper, the problem is special and an explicit proof is given.

The rest of this paper is organized as follows. In Section 2, we give the proof procedure and to prove concavity of entropy powers in the parametric case. In Section 3, we present the generalized version of CREP using the proof procedure. In Section 4, conclusions are presented.

## 2 Proof Procedure

In this section, we give a procedure to prove the CREP. To make the paper concise, we only give those steps that are needed in this paper.

### 2.1 Notations

Let  $x_t = [x_{1,t}, x_{2,t}, \dots, x_{n,t}]$  be a set of variables depending on  $t$  and

$$d^{(i)}x_t = dx_{1,t}dx_{2,t} \dots dx_{i-1,t}dx_{i+1,t} \dots dx_{n,t}, i = 1, 2, \dots, n.$$

Let  $[n]_0 = \{0, 1, \dots, n\}$  and  $[n] = \{1, \dots, n\}$ . To simplify the notations, we use  $u$  to denote  $u(x_t)$  in the rest of the paper. Denote

$$\mathcal{P}_n = \left\{ \frac{\partial^h u}{\partial^{h_1} x_{1,t} \cdots \partial^{h_n} x_{n,t}} : h = \sum_{i=1}^n h_i, h_i \in \mathbb{N} \right\}$$

to be the set of all derivatives of  $u$  with respect to the differential operators  $\frac{\partial}{\partial x_{i,t}}, i = 1, \dots, n$  and

$$\mathcal{R} = \mathbb{R}[n, p, \mu][\mathcal{P}_n]$$

to be the set of polynomials in  $\mathcal{P}_n$  with coefficients in  $\mathbb{R}[n, p, \mu]$ , where  $n, p, \mu$  are parameters. For  $v \in \mathcal{P}_{h,n}$ , let  $\text{ord}(v)$  be the order of  $v$ . For a monomial  $\prod_{i=1}^r v_i^{d_i}$  with  $v_i \in \mathcal{P}_n$ , its *degree*, *order*, and *total order* are defined to be  $\sum_{i=1}^r d_i$ ,  $\max_{i=1}^r \text{ord}(v_i)$ , and  $\sum_{i=1}^r d_i \cdot \text{ord}(v_i)$ , respectively.

A polynomial in  $\mathcal{R}$  is called a *k*th-order *differentially homogenous polynomial* or simply a *k*th-order *differential form*, if all its monomials have degree  $k$  and total order  $k$ . Let  $\mathcal{M}_{k,n}$  be the set of all monomials which have degree  $k$  and total order  $k$ . Then the set of *k*th-order differential forms is an  $\mathbb{R}$ -linear vector space generated by  $\mathcal{M}_{k,n}$ , which is denoted as  $\text{Span}_{\mathbb{R}}(\mathcal{M}_{k,n})$ . We will use Gaussian elimination in  $\text{Span}_{\mathbb{R}}(\mathcal{M}_{k,n})$  by treating the monomials as variables. We always use the *lexicographic order for the monomials* defined in [19, 20].

## 2.2 The proof procedure

In this section, we give the procedure to prove the *CREP*. The property  $\frac{d}{dt} N_p(u) \geq 0$  can be easily proved [21]. We focus on proving  $\frac{d^2}{dt^2} N_p(u) \leq 0$ . The procedure consists of four steps.

In step 1, we reduce the proof of *CREP* into the proof of an integral inequality, as shown by the following lemma whose proof is given in section 2.4.

**Lemma 2.1.** *Proof of  $\frac{d^2}{dt^2} N_p(u) \leq 0$  can be reduced to show*

$$\int_{\mathbb{R}^n} u^{3p-6} E_{2,n} dx_t \geq 0 \tag{7}$$

*under the condition  $p \geq 1 - \frac{\mu}{n}$ , where  $E_{2,n} = \sum_{a=1}^n \sum_{b=1}^n E_{2,n,a,b}$  is a 4th-order differential form in  $\mathbb{R}[n, p, \mu][\mathcal{P}_{2,n}]$  and*

$$\mathcal{P}_{2,n} = \left\{ \frac{\partial^h u}{\partial^{h_1} x_{a,t} \partial^{h_2} x_{b,t}} : h \in [3]_0; a, b \in [n] \right\}. \tag{8}$$

Then the problem  $\frac{d^2}{dt^2} N_p(u) \leq 0$  can be transformed to  $\int u^{3p-6} E_{2,n} dx_t \geq 0$ . Thus, Lemma 2.1 is proved.

In step 2, we compute the constraints which are relations satisfied by the probability density  $u$  of  $X_t$ . Since  $E_{2,n}$  in (7) is a 4th-order differential form, we need only the constraints which are 4th-order differential forms. A 4th-order differential form  $R$  is called an *equational or inequality constraint* if

$$\int_{\mathbb{R}^n} u^{3p-6} R dx_t = 0 \text{ or } \int_{\mathbb{R}^n} u^{3p-6} R dx_t \geq 0. \tag{9}$$

The method to compute the constraints is given in section 2.3. Suppose that the equational and inequality constraints are respectively

$$\mathcal{C}_E = \{R_i, | i = 1, \dots, N_1\} \text{ and } \mathcal{C}_I = \{I_i, | i = 1, \dots, N_2\}. \tag{10}$$

In step 3, we find a propositional formula  $\Phi(n, p, \mu)$  such that when  $n, p, \mu \in \mathbb{R}$  satisfy  $\Phi$ ,

$$\exists c_j, e_i \in \mathbb{R}, s.t. E_{2,n} - \sum_{i=1}^{N_1} e_i R_i - \sum_{j=1}^{N_2} c_j I_j = S \geq 0 \text{ and } c_j \geq 0, j = 1, \dots, n_2. \quad (11)$$

Details of this step and the formula  $\Phi(n, p, \mu)$  are given in section 3.

To summarize the proof procedure, we have

**Theorem 2.2.** *The CREP is true if  $\Phi(n, p, \mu)$  is valid.*

*Proof.* By Lemma 2.1, we have the following proof:

$$\begin{aligned} & \int_{\mathbb{R}} u^{3p-6} E_{2,n} dx_t \\ \stackrel{(11)}{=} & \int_{\mathbb{R}} u^{3p-6} \left( \sum_{i=1}^{N_1} e_i R_i + \sum_{j=1}^{N_2} c_j I_j + S \right) dx_t \\ \stackrel{S1}{=} & \int_{\mathbb{R}} u^{3p-6} \left( \sum_{j=1}^{N_2} c_j I_j + S \right) dx_t \\ \stackrel{S2}{\geq} & \int_{\mathbb{R}} u^{3p-6} S dx_t \stackrel{S3}{\geq} 0. \end{aligned} \quad (12)$$

Equality S1 is true, because  $R_i$  is a equational constraint. Inequality S2 is true, because  $I_j$  is an inequality constraint. Inequality S3 is true, because  $S \geq 0$  under the condition  $\Phi(n, p, \mu)$ .  $\square$

### 2.3 The equational constraints

In this section, we show how to find the second order equational constraints. A *second order equational constraint* is a 4th-order differential form in  $\mathbb{R}[n, p, \mu][\mathcal{P}_{2,n}]$  such that  $\int_{\mathbb{R}^n} u^{3p-6} R dx_t = 0$ .

We introduce the following notations

$$\mathcal{V}_{a,b} = \left\{ \frac{\partial^h u}{\partial^{h_1} x_{a,t} \partial^{h_2} x_{b,t}} : h = h_1 + h_2 \in [3]_0 \right\}, \quad (13)$$

where  $a, b$  are variables taking values in  $[n]$ . Then  $\mathcal{P}_{2,n} = \cup_{a=1}^n \cup_{b=1}^n \mathcal{V}_{a,b}$ .

We need the following property.

**Property 2.3.** *Let  $a, r, m_i, k_i \in \mathbb{N}_{>0}$  and  $u^{(m_i)}$  an  $m_i$ th-order derivative of  $u$ . If  $u(x_t)$  is a smooth, strictly positive and rapidly decaying probability density, then*

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} u^{3p-2} \left[ \prod_{i=1}^r \frac{[u^{(m_i)}]^{k_i}}{u^{k_i}} \right] \Big|_{x_{a,t}=-\infty}^{\infty} d^{(a)} x_t = 0, \quad (14)$$

with  $\sum_{i=1}^r k_i m_i = 4$ ,  $\sum_{i=1}^r k_i = 4$ .

When  $p \geq 2$ , Property 2.3 follows from [23]. While  $0 < p < 2, p \neq 1$ , we make the assumption that  $u(x_t)$  also satisfies Property 2.3.

Using Property 2.3, we can compute the second order equational constraints using the method given in [19, 20]:

$$\mathcal{C}_{2,n} = \{R_{i,a,b} : i = 1, \dots, 28\} \subset \mathbb{R}[n, p, \mu][\mathcal{V}_{a,b}], \quad (15)$$

where  $R_{i,a,b}$  can be found in the Appendix. Note that  $a, b$  are variables taking values in  $[n]$ .

We use an example to show how to obtain these constraints. Starting from a monomial  $u \frac{\partial^2 u}{\partial^2 x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2$  with degree 4 and total order 4, by integral by parts, we have

$$\begin{aligned} & \int u^{3p-6} u \frac{\partial^2 u}{\partial^2 x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2 dx_t \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [u^{3p-5} \frac{\partial u}{\partial x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2] \Big|_{x_{a,t}=-\infty}^{\infty} d^{(a)} x_t \\ & - \int \frac{\partial u}{\partial x_{a,t}} [\frac{\partial}{\partial x_{a,t}} (u^{3p-5} (\frac{\partial u}{\partial x_{a,t}})^2)] dx_t \\ & \stackrel{(14)}{=} - \int \frac{\partial u}{\partial x_{a,t}} [\frac{\partial}{\partial x_{a,t}} (u^{3p-5} (\frac{\partial u}{\partial x_{a,t}})^2)] dx_t. \end{aligned} \tag{16}$$

Then,

$$\begin{aligned} & \int u^{3p-6} u \frac{\partial^2 u}{\partial^2 x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2 + \frac{\partial u}{\partial x_{a,t}} [\frac{\partial}{\partial x_{a,t}} (u^{3p-5} (\frac{\partial u}{\partial x_{a,t}})^2)] dx_t \\ &= \int u^{3p-6} [3p (\frac{\partial u}{\partial x_{a,t}})^4 + 3u \frac{\partial^2 u}{\partial^2 x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2 - 5 (\frac{\partial u}{\partial x_{a,t}})^4] dx_t = 0. \end{aligned} \tag{17}$$

We obtain a 2th-order constraint:  $R_{1,a,b} = 3p (\frac{\partial u}{\partial x_{a,t}})^4 + 3u \frac{\partial^2 u}{\partial^2 x_{a,t}} (\frac{\partial u}{\partial x_{a,t}})^2 - 5 (\frac{\partial u}{\partial x_{a,t}})^4$ . The other 27 constraints in  $\mathcal{C}_{2,n}$  are obtained in the same way.

## 2.4 Proof of Lemma 2.1

We first prove several lemmas.

**Lemma 2.4.**

$$\frac{dH_p(u)}{dt} = \frac{p}{1-p} \frac{\int u^{p-1} \Delta u^p dx_t}{\int u^p dx_t}, \tag{18}$$

$$\frac{d^2 H_p(u)}{d^2 t} = \frac{p}{1-p} \frac{\frac{\partial}{\partial t} (\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t - p (\int u^{p-1} \frac{\partial u}{\partial t} dx_t)^2}{(\int u^p dx_t)^2}. \tag{19}$$

*Proof.* By the definition of  $p$ -Rényi entropy (1), we have

$$\begin{aligned} \frac{dH_p(u)}{dt} &= \frac{p}{1-p} \frac{\int u^{p-1} \frac{\partial u}{\partial t} dx_t}{\int u^p dx_t} = \frac{p}{1-p} \frac{\int u^{p-1} \Delta u^p dx_t}{\int u^p dx_t}, \\ \frac{d^2 H_p(u)}{d^2 t} &= \frac{p}{1-p} \frac{\frac{\partial}{\partial t} (\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t - \int u^{p-1} \frac{\partial u}{\partial t} dx_t \frac{\partial}{\partial t} (\int u^p dx_t)}{(\int u^p dx_t)^2} \\ &= \frac{p}{1-p} \frac{\frac{\partial}{\partial t} (\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t - \int u^{p-1} \frac{\partial u}{\partial t} dx_t \int p u^{p-1} \frac{\partial u}{\partial t} dx_t}{(\int u^p dx_t)^2} \\ &= \frac{p}{1-p} \frac{\frac{\partial}{\partial t} (\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t - p (\int u^{p-1} \frac{\partial u}{\partial t} dx_t)^2}{(\int u^p dx_t)^2}. \end{aligned}$$

□

**Lemma 2.5.** *We have*

$$\int u^{p-1} \Delta u^p dx_t = \int \Delta u^{p-1} u^p dx_t. \tag{20}$$

*Proof.* Integrating by parts[21], we have

$$\int u^{p-1} \Delta u^p dx_t = - \int \nabla u^{p-1} \nabla u^p dx_t = \int \Delta u^{p-1} u^p dx_t.$$

□

**Lemma 2.6.** *By Cauchy-Schwarz inequality we have*

$$\left(\int \Delta u^{p-1} u^p dx_t\right)^2 \leq \int u^p dx_t \int (\Delta u^{p-1})^2 u^p dx_t. \quad (21)$$

Then, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} N_p(u) &= \frac{\mu}{n} \frac{d^2 H_p(u)}{dt^2} e^{\frac{\mu}{n} H_p(u)} + \left(\frac{\mu}{n} \frac{dH_p(u)}{dt}\right)^2 e^{\frac{\mu}{n} H_p(u)} \\ &= \frac{\mu}{n} e^{\frac{\mu}{n} H_p(u)} I_{2,n}, \end{aligned} \quad (22)$$

where  $I_{2,n} = \frac{d^2 H_p(u)}{dt^2} + \frac{\mu}{n} \left(\frac{dH_p(u)}{dt}\right)^2$ . So, by (18),(19),we have

$$\begin{aligned} I_{2,n} &= \frac{p}{1-p} \frac{\frac{\partial}{\partial t}(\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t - p(\int u^{p-1} \frac{\partial u}{\partial t} dx_t)^2}{(\int u^p dx_t)^2} \\ &\quad + \frac{\mu}{n} \left(\frac{p}{1-p} \frac{\int u^{p-1} \Delta u^p dx_t}{\int u^p dx_t}\right)^2 \\ &= \frac{\mu p^2}{n(1-p)^2} \frac{(\int u^{p-1} \Delta u^p dx_t)^2}{(\int u^p dx_t)^2} + \frac{p}{1-p} \frac{\frac{\partial}{\partial t}(\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t}{(\int u^p dx_t)^2} \\ &\quad - \frac{p^2}{1-p} \frac{(\int u^{p-1} \Delta u^p dx_t)^2}{(\int u^p dx_t)^2} \\ &= \left(\frac{\mu p^2}{n(1-p)^2} - \frac{p^2}{1-p}\right) \frac{(\int u^{p-1} \Delta u^p dx_t)^2}{(\int u^p dx_t)^2} \\ &\quad + \frac{p}{1-p} \frac{\frac{\partial}{\partial t}(\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t}{(\int u^p dx_t)^2} \\ &\stackrel{(20)}{=} \frac{(\mu - n(1-p))p^2}{n(1-p)^2} \frac{(\int \Delta u^{p-1} u^p dx_t)^2}{(\int u^p dx_t)^2} + \frac{p}{1-p} \frac{\frac{\partial}{\partial t}(\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t}{(\int u^p dx_t)^2} \\ &\stackrel{(i)}{\leq} \frac{(\mu - n(1-p))p^2}{n(1-p)^2} \frac{\int u^p dx_t \int (\Delta u^{p-1})^2 u^p dx_t}{(\int u^p dx_t)^2} \\ &\quad + \frac{p}{1-p} \frac{\frac{\partial}{\partial t}(\int u^{p-1} \frac{\partial u}{\partial t} dx_t) \int u^p dx_t}{(\int u^p dx_t)^2} \\ &= \frac{1}{\int u^p dx_t} \left( \frac{(\mu - n(1-p))p^2}{n(1-p)^2} \int (\Delta u^{p-1})^2 u^p dx_t + \frac{p}{1-p} \frac{\partial}{\partial t} \left( \int u^{p-1} \frac{\partial u}{\partial t} dx_t \right) \right) \\ &= \frac{1}{\int u^p dx_t} \int \left( \frac{(\mu - n(1-p))p^2}{n(1-p)^2} (\Delta u^{p-1})^2 u^p + \frac{p}{1-p} \frac{\partial}{\partial t} \left( u^{p-1} \frac{\partial u}{\partial t} \right) \right) dx_t \\ &= \frac{1}{\int u^p dx_t} \int F_{2,n} dx_t, \end{aligned} \quad (23)$$

where  $F_{2,n} = \frac{(\mu - n(1-p))p^2}{n(1-p)^2} (\Delta u^{p-1})^2 u^p + \frac{p}{1-p} \frac{\partial}{\partial t} \left( u^{p-1} \frac{\partial u}{\partial t} \right)$ .

**Remark:** In (23), the step (i) is according to (21), and  $\frac{(\mu - n(1-p))p^2}{n(1-p)^2} \geq 0$  should be satisfied, which is true under condition  $p \geq 1 - \frac{\mu}{n}$ . When  $\mu := 2 + n(p-1)$ ,  $\frac{(\mu - n(1-p))p^2}{n(1-p)^2} \geq 0$  yields  $p \geq 1 - \frac{1}{n}$ . Savaré-Toscani [21] also used the inequality (21), but ignore the nonnegativity of the coefficient  $\frac{(\mu - n(1-p))p^2}{n(1-p)^2}$ , thus the parameter's range  $p > 1 - \frac{2}{n}$  in [21] should be corrected to  $p \geq 1 - \frac{1}{n}$ .

Further, we can get

$$\begin{aligned}
F_{2,n} &= \frac{(\mu - n(1-p))p^2}{n(1-p)^2} u^p \sum_{a=1}^n \sum_{b=1}^n \left[ \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^{p-1} \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^{p-1} \right) \right] \\
&+ \frac{p}{1-p} \frac{\partial}{\partial t} \left[ u^{p-1} \sum_{a=1}^n \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^p \right) \right] \\
&= \frac{(\mu - n(1-p))p^2}{n(1-p)^2} u^p \sum_{a=1}^n \sum_{b=1}^n \left[ \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^{p-1} \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^{p-1} \right) \right] \\
&+ \frac{p}{1-p} \sum_{a=1}^n \left[ (p-1) u^{p-2} \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^p \right) \frac{\partial u}{\partial t} + p u^{p-1} \frac{\partial^2}{\partial^2 x_{a,t}} \left( u^{p-1} \frac{\partial u}{\partial t} \right) \right] \\
&= \frac{(\mu - n(1-p))p^2}{n(1-p)^2} u^p \sum_{a=1}^n \sum_{b=1}^n \left[ \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^{p-1} \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^{p-1} \right) \right] \\
&+ \frac{p}{1-p} \sum_{a=1}^n \left[ (p-1) u^{p-2} \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^p \right) \sum_{b=1}^n \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right. \\
&\quad \left. + p u^{p-1} \frac{\partial^2}{\partial^2 x_{a,t}} \left( u^{p-1} \sum_{b=1}^n \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right) \right] \\
&= \frac{(\mu - n(1-p))p^2}{n(1-p)^2} u^p \sum_{a=1}^n \sum_{b=1}^n \left[ \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^{p-1} \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^{p-1} \right) \right] \\
&+ \frac{p}{1-p} \sum_{a=1}^n \sum_{b=1}^n \left[ (p-1) u^{p-2} \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^p \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right. \\
&\quad \left. + p u^{p-1} \frac{\partial^2}{\partial^2 x_{a,t}} \left( u^{p-1} \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right) \right] \\
&= \sum_{a=1}^n \sum_{b=1}^n \mathcal{T}_{a,b},
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\mathcal{T}_{a,b} &= \frac{(\mu - n(1-p))p^2}{n(1-p)^2} u^p \left[ \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^{p-1} \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^{p-1} \right) \right] \\
&+ \frac{p}{1-p} \left[ (p-1) u^{p-2} \left( \frac{\partial^2}{\partial^2 x_{a,t}} u^p \right) \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right. \\
&\quad \left. + p u^{p-1} \frac{\partial^2}{\partial^2 x_{a,t}} \left( u^{p-1} \left( \frac{\partial^2}{\partial^2 x_{b,t}} u^p \right) \right) \right].
\end{aligned} \tag{25}$$

For convenience, introduce the notation  $u_{i,j} := \frac{\partial^{i+j} u}{\partial^i x_{a,t} \partial^j x_{b,t}}$ . By simple computation, we have

$\mathcal{T}_{a,b} = -\frac{u^{3p-6}p^2}{(p-1)^n}T_{a,b}$ , and

$$\begin{aligned}
T_{a,b} = & 4np^4u_{0,1}^2u_{1,0}^2 + 2np^3uu_{0,1}^2u_{2,0} + 8np^3uu_{0,1}u_{1,0}u_{1,1} \\
& + 4np^3uu_{0,2}u_{1,0}^2 - 15np^3u_{0,1}^2u_{1,0}^2 - \mu p^3u_{0,1}^2u_{1,0}^2 + 2np^2u^2u_{0,1}u_{2,1} \\
& + 2np^2u^2u_{0,2}u_{2,0} + 4np^2u^2u_{1,0}u_{1,2} + 2np^2u^2u_{1,1}^2 - 3np^2uu_{0,1}^2u_{2,0} \\
& - 20np^2uu_{0,1}u_{1,0}u_{1,1} - 8np^2uu_{0,2}u_{1,0}^2 - \mu p^2uu_{0,1}^2u_{2,0} - \mu p^2uu_{0,2}u_{1,0}^2 \\
& + 16np^2u_{0,1}^2u_{1,0}^2 + 5\mu p^2u_{0,1}^2u_{1,0}^2 + npu^3u_{2,2} - 2npu^2u_{0,1}u_{2,1} \\
& - npu^2u_{0,2}u_{2,0} - 4npu^2u_{1,0}u_{1,2} - 2npu^2u_{1,1}^2 - \mu pu^2u_{0,2}u_{2,0} \\
& - npuu_{0,1}^2u_{2,0} + 12npuu_{0,1}u_{1,0}u_{1,1} + 2npuu_{0,2}u_{1,0}^2 + 3\mu puu_{0,1}^2u_{2,0} \\
& + 3\mu puu_{0,2}u_{1,0}^2 - npu_{0,1}^2u_{1,0}^2 - 8\mu pu_{0,1}^2u_{1,0}^2 - nu^2u_{0,2}u_{2,0} \\
& + \mu u^2u_{0,2}u_{2,0} + 2nuu_{0,1}^2u_{2,0} + 2nuu_{0,2}u_{1,0}^2 - 2\mu uu_{0,1}^2u_{2,0} \\
& - 2\mu uu_{0,2}u_{1,0}^2 - 4nu_{0,1}^2u_{1,0}^2 + 4\mu u_{0,1}^2u_{1,0}^2,
\end{aligned} \tag{26}$$

which is a 4th-order differential form.

From (22), (23), (24), and (25), we have

$$\frac{d^2}{dt^2}N_p(u) \leq -\frac{p^2\mu}{n^2}e^{\frac{\mu}{n}H_p(u)}\frac{1}{\int u^p dx_t} \int u^{3p-6}E_{2,n}dx_t \tag{27}$$

where  $E_{2,n} = \sum_{a=1}^n \sum_{b=1}^n \frac{T_{a,b}}{p-1}$  and  $T_{a,b}$  is defined in (26). Then the problem  $\frac{d^2}{dt^2}N_p(u) \leq 0$  can be transformed to  $\int u^{3p-6}E_{2,n}dx_t \geq 0$ . Thus, Lemma 2.1 is proved.

### 3 A generalized version of CREP

In this section, we prove a generalized CREP using the procedure given in section 2.

**Theorem 3.1.** *Let  $u(x_t)$  be a probability density in  $\mathbb{R}^n$  solving (6) and satisfying (14). Then we give a formula  $\Phi(n, p, \mu)$  such that the  $p$ -th Rényi entropy power defined in (2) satisfies*

$$\frac{d^2}{dt^2}N_p(x_t) \leq 0, \tag{28}$$

under the condition  $\Phi(n, p, \mu)$ , that is  $N_p(x_t)$  is concave under  $\Phi(n, p, \mu)$ .

The proof of the above theorem consists three steps which will be given in the following subsections.

#### 3.1 Reduce to a finite problem

We first give an inequality constraint. Denote that  $|\nabla^2 f|^2 = \sum_{i,j} (\frac{\partial^2 f}{\partial x_i \partial x_j})^2$ . Then, based on the trace inequality  $|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2$ , we give a nonnegative constraint:

$$I_1 = \frac{u^p}{u^{3p-6}} \left[ |\nabla^2 u^{p-1}|^2 - \frac{1}{n}(\Delta u^{p-1})^2 \right] = \sum_{a=1}^n \sum_{b=1}^n I_{1,a,b} \geq 0 \tag{29}$$



where  $I_{1,a,b} = u^{6-2p} \left[ \left( \frac{\partial^2 u^{p-1}}{\partial x_{a,t} \partial x_{b,t}} \right)^2 - \frac{1}{n} \frac{\partial^2 u^{p-1}}{\partial^2 x_{a,t}} \frac{\partial^2 u^{p-1}}{\partial^2 x_{b,t}} \right]$ .

From (27) and (29), in order for (28) to be true, it suffices to solve

**Problem I.** Find a formula  $\Phi(n, p, \mu)$  such that

$$E_{2,n} \geq \tilde{E}_{2,n} = E_{2,n} + c_1 I_1 = \sum_{a=1}^n \sum_{b=1}^n \left( \frac{1}{p-1} T_{a,b} + c_1 I_{1,a,b} \right) \geq 0$$

under the conditions  $c_1 \leq 0$ ,  $p \geq 1 - \frac{\mu}{n}$ ,  $R_{i,a,b} = 0$ ,  $i = 1, \dots, 28$  given in (15).

Since  $\sum_{a=1}^n \sum_{b=1}^n T_{a,b} = \sum_{a=1}^n \sum_{b=1}^n T_{b,a}$  and  $I_{1,a,b} = R_{1,b,a}^{(I)}$ , we have

$$\tilde{E}_{2,n} = \frac{1}{2} \sum_{a=1}^n \sum_{b=1}^n \left[ \frac{1}{p-1} (T_{a,b} + T_{b,a}) + c_1 (I_{1,a,b} + I_{1,b,a}) \right] = \frac{1}{2} \sum_{a=1}^n \sum_{b=1}^n L_{a,b}, \quad (30)$$

where  $L_{a,b} = \frac{1}{p-1} (T_{a,b} + T_{b,a}) + c_1 (I_{1,a,b} + I_{1,b,a})$ .

### 3.2 Simplify the problem with the constraints

From (30), to solve **Problem I**, it suffices to solve

**Problem II.** Find a formula  $\Phi(n, p, \mu)$  such that  $L_{a,b} \geq 0$  under the conditions  $c_1 \leq 0$ ,  $p \geq 1 - \frac{\mu}{n}$ , and  $R_{i,a,b} = 0$ ,  $i = 1, \dots, 28$ .

In this section, we simplify  $L_{a,b}$  in **Problem II** with the constraints. Note that the subscripts  $a$  and  $b$  are fixed and will be treated as symbols.

Our goal is to reduce  $L_{a,b}$  into a quadratic form in certain new variables. The new variables are all the monomials in  $\mathbb{R}[\mathcal{V}_{a,b}]$  with degree 2 and total order 2, where  $\mathcal{V}_{a,b}$  is defined in (13).

$$\begin{aligned} m_1 &= uu_{0,2}, \quad m_2 = uu_{1,1}, \quad m_3 = uu_{2,0} \\ m_4 &= u_{0,1}^2, \quad m_5 = u_{1,0}u_{0,1}, \quad m_6 = u_{1,0}^2. \end{aligned}$$

We will simplify the constraints in (15) as follows. A quadratic monomial in  $m_i$  is called a *quadratic monomial*. Write monomials in  $\mathcal{C}_{2,n} = \{R_i, i = 1, \dots, N_1\}$  as quadratic monomials if possible. Doing Gaussian elimination to  $\mathcal{C}_{2,n}$  by treating the monomials as variables and according to a monomial order such that a quadratic monomial is less than a non-quadratic monomial, we obtain

$$\tilde{\mathcal{C}}_{2,n} = \mathcal{C}_{2,n,1} \cup \mathcal{C}_{2,n,2},$$

where  $\mathcal{C}_{2,n,1}$  is the set of quadratic forms in  $m_i$ ,  $\mathcal{C}_{2,n,2}$  is the set of non-quadratic forms, and  $\text{Span}_{\mathbb{R}}(\mathcal{C}_{2,n}) = \text{Span}_{\mathbb{R}}(\tilde{\mathcal{C}}_{2,n})$ . We obtain  $\mathcal{C}_{2,n,1} = \{\hat{R}_i, i = 1, \dots, 9\}$  and  $\mathcal{C}_{2,n,2} = \{\tilde{R}_i, i = 1, \dots, 13\}$ , where

$$\begin{aligned}
\widehat{R}_1 &= 2m_1m_5 + \frac{2(3p-5)}{3}m_4m_5, \widehat{R}_2 = m_2m_6 + \frac{3p-5}{3}m_5m_6, \\
\widehat{R}_3 &= -6m_3m_5 + 2(5-3p)m_5m_6, \widehat{R}_4 = (3p-5)m_4^2 + 3m_1m_4, \\
\widehat{R}_5 &= (3p-5)m_6^2 + 3m_3m_6, \widehat{R}_6 = (3p-5)m_4m_5 + 3m_2m_4, \\
\widehat{R}_7 &= (3p-5)m_5^2 + 2m_2m_5 + m_3m_4, \\
\widehat{R}_8 &= m_1m_3 - m_2^2 + \frac{9p-12}{2}m_3m_4 + \frac{9p^2-27p+20}{2}m_5^2, \\
\widehat{R}_9 &= m_1m_6 - m_3m_4. \\
\widetilde{R}_1 &= u^3u_{0,4} + (3-3p)m_1^2 + (9p^3-36p^2+47p-20)m_4^2, \\
\widetilde{R}_2 &= u^3u_{1,3} + (3-3p)m_1m_2 + (9p^3-36p^2+47p-20)m_4m_5, \\
\widetilde{R}_3 &= u^3u_{3,1} + (3-3p)m_2m_3 + (-9p^2+21p-12)m_3m_5, \\
\widetilde{R}_4 &= u^3u_{4,0} + (3-3p)m_3^2 + (9p^3-36p^2+47p-20)m_6^2, \\
\widetilde{R}_5 &= u^2u_{0,1}u_{0,3} + m_1^2 + \frac{-9p^2+27p-20}{3}m_4^2, \\
\widetilde{R}_6 &= u^2u_{0,1}u_{1,2} + m_1m_2 + \frac{-9p^2+27p-20}{3}m_4m_5, \\
\widetilde{R}_7 &= u^2u_{0,1}u_{3,0} + m_2m_3 + \frac{-9p^2+27p-20}{3}m_5m_6, \\
\widetilde{R}_8 &= u^2u_{1,0}u_{0,3} + m_1m_2 + \frac{-9p^2+27p-20}{3}m_4m_5, \\
\widetilde{R}_9 &= u^2u_{1,0}u_{2,1} + m_2m_3 + \frac{-9p^2+27p-20}{3}m_5m_6, \\
\widetilde{R}_{10} &= u^2u_{1,0}u_{3,0} + m_3^2 + \frac{-9p^2+27p-20}{3}m_6^2, \\
\widetilde{R}_{11} &= u^3u_{2,2} + (3-3p)m_2^2 + \frac{9p^2-21p+12}{2}m_3m_4 + \frac{27p^3-108p^2+141p-60}{2}m_5^2, \\
\widetilde{R}_{12} &= u^2u_{0,1}u_{2,1} + m_2^2 + \frac{4-3p}{2}m_3m_4 + \frac{-9p^2+27p-20}{2}m_5^2, \\
\widetilde{R}_{13} &= u^2u_{1,0}u_{1,2} + m_2^2 + \frac{4-3p}{2}m_3m_4 + \frac{-9p^2+27p-20}{2}m_5^2.
\end{aligned}$$

We now simplify  $L_{a,b}$  using  $\mathcal{C}_{2,n,1}$  and  $\mathcal{C}_{2,n,2}$ . Eliminating the non-quadratic monomials in  $L_{a,b}$  using  $\mathcal{C}_{2,n,2}$ , and doing further reduction by  $\mathcal{C}_{2,n,1}$ , we have

$$\begin{aligned}
\widehat{L}_{a,b} &= L_{a,b} - 2(p^3c_1 + 4np^2 - 4p^2c_1 - 6np + 5pc_1 - 2c_1)\widehat{R}_7 \\
&\quad - \frac{2}{n}(2n^2p - p^2c_1 + n^2 - n\mu + 2pc_1 - c_1)\widehat{R}_8 \\
&\quad - \frac{1}{n}(6n^2p^2 - 2p^3c_1 - 5n^2p - 2np\mu + 8p^2c_1 - 4n^2 + 4n\mu - 10pc_1 + 4c_1)\widehat{R}_9 \\
&\quad - \frac{2np}{p-1}\widetilde{R}_{11} - 6np\widetilde{R}_{12} - 6np\widetilde{R}_{13} \\
&= (2np + 2n - 2\mu)m_2^2 + (5np - 5np^2 + 5p\mu + 4n - 4\mu)m_3m_4 \\
&\quad + (18np^2 - 7np^3 + 7p^2\mu - 3np - 19p\mu - 12n + 12\mu)m_5^2 \\
&\quad + \frac{c_1}{n}[(2n-2)(p^2-2p+1)m_2^2 + (4n-2np+5p-4)(p^2-2p+1)m_3m_4 \\
&\quad + (14np-4np^2+7p^2-12n-19p+12)(p^2-2p+1)m_5^2]
\end{aligned} \tag{31}$$

In order for  $\widehat{L}_{a,b} \geq 0$  to be true, we need to eliminate the monomial  $m_3m_4$  from  $\widehat{L}_{a,b}$ , which can be done with  $\widehat{R}_7$  as follows.

$$\widehat{L}_{a,b} + p_7\widehat{R}_7 = A_1m_2^2 + A_2m_2m_5 + A_3m_5^2 \tag{32}$$

where

$$\begin{aligned}
p_7 &= (2np^3c_1 + 5n^2p^2 - 8np^2c_1 - 5p^3c_1 - 5n^2p - 5np\mu + 10npc_1 \\
&\quad + 14p^2c_1 - 4n^2 + 4n\mu - 4nc_1 - 13pc_1 + 4c_1)/n. \\
A_1 &= -2c_1p^2/n + 4c_1p/n + 2np + 2c_1 + 2c_1p^2 - 4c_1p - 2c_1/n - 2\mu + 2n, \\
A_2 &= 4c_1p^3 - 16c_1p^2 - 10c_1p^3/n - 10p\mu + 20c_1p + 28c_1p^2/n - 26c_1p/n \\
&\quad + 10np^2 - 10np + 8\mu - 8c_1 + 8c_1/n - 8n, \\
A_3 &= -8\mu + 8n + 26c_1p^2 - 24c_1p - 8c_1/n - 12c_1p^3 + 2c_1p^4 + 8c_1 - 52c_1p^2/n \\
&\quad + 34c_1p/n + 34c_1p^3/n - 8c_1p^4/n + 18p\mu + 8np^3 - 22np^2 - 8p^2\mu + 10np.
\end{aligned}$$

### 3.3 Compute $\Phi(n, p, \mu)$

From (32), in order to solve **Problem II**, it suffices to solve

**Problem III:** Find a formula  $\Phi(n, p, \mu)$  such that

$$\Phi(n, p, \mu) = \exists c_1 \forall m_1, m_2, m_3 (c_1 \leq 0 \wedge p \geq 1 - \frac{\mu}{n} \wedge A_1m_2^2 + A_2m_2m_5 + A_3m_5^2 \geq 0). \quad (33)$$

In principle, **Problem III** can be solved with the quantifier elimination [22]. In this paper, the problem is special and an explicit proof is given.

By the knowledge of linear algebra, we have  $A_1m_2^2 + A_2m_2m_5 + A_3m_5^2 \geq 0$  is equivalent to  $\Delta_1 = A_1 = \frac{2}{n}s_1 \geq 0$ ,  $\Delta_2 = A_3 = \frac{2}{n}s_2 \geq 0$ ,  $\Delta_3 = A_1A_3 - \frac{1}{4}A_2^2 = \frac{p}{n^2}s_3 \geq 0$ , where

$$\begin{aligned}
s_1 &= (p-1)^2(n-1)c_1 + n^2(p+1) - n\mu, \\
s_2 &= (p-1)^2(n(p-2)^2 - 4p^2 + 9p - 4)c_1 \\
&\quad + n^2(4p^3 - 11p^2 + 5p + 4) - (4p^2 - 9p + 4)n\mu, \\
s_3 &= (4 - 9p)n^2(\mu - \mu_3)(\mu - \mu_4),
\end{aligned}$$

where  $\mu_3$  and  $\mu_4$  are defined in (37) and  $p \neq \frac{4}{9}$  is assumed in  $\mu_4$ . We thus proved

**Lemma 3.2.** *We can eliminate  $m_1, m_2, m_3$  in (33):*

$$\Phi(n, p, \mu) = \exists c_1 (s_1 \geq 0 \wedge s_2 \geq 0 \wedge s_3 \geq 0 \wedge c_1 \leq 0 \wedge p \geq 1 - \mu/n). \quad (34)$$

We will give an explicit formula for  $\Phi$  in (34). First, introduce the following parameters.

$$\begin{aligned}
n_1 &= \frac{9-\sqrt{17}}{8}, & n_2 &= \frac{9+\sqrt{17}}{8}, & n_3 &= (\sqrt{17}+1)/2, \\
\theta_1 &= -\frac{2n}{(p-1)^2}, & \theta_2 &= \frac{2n^2p(9p-13)}{(p-1)^2(4n+9p-4)}, & \theta_3 &= (\sqrt{17}-9)n, \\
\theta_4 &= \frac{n^2p-n^2-n\mu}{(p-1)^2}, & \theta_5 &= \frac{n^2(9p^2-13p-4)-n(9p-4)\mu}{(p-1)^2(4n+9p-4)}, & \theta_6 &= \frac{-4n(\sqrt{17}-1)\mu+8n^2}{\sqrt{17}+1}, \\
\theta_7 &= n(1-p), & \theta_8 &= 5n/9, & \theta_9 &= -\frac{162n}{25}, \\
\theta_{10} &= \frac{64n}{\sqrt{17}-9}, & \theta_{11} &= \frac{8(9\sqrt{17}+23)n^2}{-32n-49+9\sqrt{17}}, & \theta_{12} &= -\frac{16(11\sqrt{17}+47)n\mu+152n^2}{26\sqrt{17}n+73\sqrt{17}+118n+305}, \\
\theta_{13} &= -\frac{4n(\mu\sqrt{17}+2n+\mu)}{\sqrt{17}-1}, & \theta_{14} &= -\frac{8n(22\mu\sqrt{17}-19n-94\mu)}{26\sqrt{17}n+73\sqrt{17}-118n-305}, & \theta_{15} &= -\frac{9}{5}n^2 - \frac{81}{25}\mu n, \\
\phi_1 &\triangleq p \geq 1 - \frac{1}{n}, & \phi_2 &\triangleq \mu = 2 + n(p-1), & \phi_3 &\triangleq p > 1 - \frac{1}{n}.
\end{aligned} \quad (35)$$

With these notations, we introduce the conditions for defining  $\Phi$  in Table 1, where \* means  $\emptyset$ . Define  $T(i, j)$  to be the formula in the  $i$ -th row and the  $j$ -th column in Table 1. Then we denote

$$\mathbb{T}(i, j) \triangleq T(i, 1) \wedge T(i, j) \text{ for } i = 1, \dots, 8, j = 2, \dots, 4. \quad (36)$$

$p > n_2$	$\theta_4 > \theta_1 \wedge \theta_5 \leq 0$	*	$\phi_1 \wedge \phi_2$
$p = n_2$	$\theta_6 > \theta_3 \wedge \theta_{12} \leq 0$	*	$\phi_1 \wedge \phi_2$
$\frac{13}{9} < p < n_2$	$\theta_4 > \theta_1 \wedge \theta_5 \leq 0$	*	$\phi_1 \wedge \phi_2$
$n_1 < p \leq \frac{13}{9}, p \neq 1$	$\phi_3 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq \theta_2$	$\phi_3 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_2$	$\phi_1 \wedge \phi_2$
$p = n_1$	$n < n_3 \wedge \theta_{13} > \theta_{10} \wedge \theta_{14} \leq \theta_{11}$	$n < n_3 \wedge \theta_{13} > \theta_{11} \wedge \mu \geq \theta_{16}$	$\phi_1 \wedge \phi_2$
$\frac{4}{9} < p < n_1$	$\phi_3 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq \theta_2$	$\phi_3 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_2$	$\phi_1 \wedge \phi_2$
$p = \frac{4}{9}$	$n = 1 \wedge \theta_{15} > \theta_9 \wedge \mu \geq \theta_8$	*	$n = 1 \wedge -p \leq \mu - 1 \leq p$
$0 < p < \frac{4}{9}$	$n = 1 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_1$	$n = 1 \wedge \theta_5 < \theta_1 \wedge \mu \geq \theta_7$	$n = 1 \wedge -p \leq \mu - 1 \leq p$

Table 1: The description for  $\Phi$  in (34)

For examples,  $\mathbb{T}(1, 2) = (p > n_2 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq 0)$ , which means that if  $p, n, \mu$  satisfy  $\mathbb{T}(1, 2)$  then there exists a  $c_1 \leq 0$  such that (33) is true and the CREP is valid.  $\mathbb{T}(1, 3) = \emptyset$ , which means that there exist no values for  $p, n, \mu$  such that (33) and the CREP are true.

We now give the main result of the paper, which implies Theorem 3.1. The proof for the theorem can be found in section 3.5. Also, from the proof we can see that for the cases  $\mathbb{T}(i, 4), i = 1, \dots, 8$ , the corresponding value for  $c_1$  is  $c_1 = \theta_1$ .

**Theorem 3.3.** *The sufficient and necessary condition for **Problem III**, that is, (33) to be true, is*

$$\Phi(n, p, \mu) = \bigvee_{i=1}^8 \bigvee_{j=2}^4 \mathbb{T}(i, j),$$

where  $\mathbb{T}(i, j)$  is defined in (36) and  $\vee$  means disjunction.

### 3.4 Compare with existing results

We will show that our result includes the result proved in [21], and essentially more results.

In [21], CREP was proved under the conditions  $\mu = 2 + n(p - 1)$  and  $p \geq 1 - \frac{1}{n}$ . Obviously, the result proved in [21] corresponds to  $\mathbb{T}(i, 4), i = 1, \dots, 8$  in Table 1.

We can also prove the result in [21] directly as follows. Set  $\mu = 2 + n(p - 1)$  and  $c_1 = \theta_1 \leq 0$  in (31), we obtain  $\widehat{L}_{a,b} = 0$ . Also, the condition  $p \geq 1 - \frac{1}{n}$  implies  $p \geq 1 - \frac{1}{n}$ . So, when  $\mu = 2 + n(p - 1)$  and  $p \geq 1 - \frac{1}{n}$ , the CREP is proved based on our proof procedure.

We can use the SDP code in [19][APPENDIX B] to verify the result in Table 1. For instance, for  $\mu = 2, p = \frac{11}{5}, n = 2$ , the condition  $p \geq 1 - \frac{1}{n}$  is satisfied naturally. With the SDP code in [19], we obtain  $\widehat{L}_{a,b} + \frac{172}{25}\widehat{R}_7 = (2\sqrt{2}m_2 + \frac{344}{100\sqrt{2}}m_5)^2 + \frac{22}{625}m_5^2 \geq 0$  with  $c_1 = -\frac{5}{9}$ . Thus, the CREP is proved when  $\mu = 2, p = \frac{11}{5}, n = 2$ . This case  $[\mu = 2, p = \frac{11}{5}, n = 2, c_1 = -\frac{5}{9}]$  is included in  $\mathbb{T}(1, 1)$  in Table 1. Note that  $\mu = 2 + n(p - 1)$  is not satisfied for these parameters.

### 3.5 Proof of Theorem 3.3

Introduce more parameters.

$$\begin{aligned}
\mu_1 &= ((p-1)^2(n-1)c_1 + n^2(p+1))/n, \\
\mu_2 &= ((p-1)^2(n(p-2)^2 - 4p^2 + 9p - 4)c_1 + n^2(4p^3 - 11p^2 + 5p + 4))/(n(4p^2 - 9p + 4)) \\
\mu_3 &= (n^2p - p^2c_1 - n^2 + 2pc_1 - c_1)/n, \\
\mu_4 &= (9n^2p^2 - 4np^2c_1 - 9p^3c_1 - 13n^2p + 8npc_1 + 22p^2c_1 - 4n^2 - 4nc_1 - 17pc_1 + 4c_1)/(n(9p - 4)), \\
\mu_5 &= -(nc_1\sqrt{17} - c_1\sqrt{17} + 136n^2 + 17nc_1 - 17c_1)/(4n(\sqrt{17} - 17)), \\
\mu_6 &= -(c_1\sqrt{17} - 8n^2 + c_1)/(4n(\sqrt{17} - 1)), \\
\mu_7 &= -(26nc_1\sqrt{17} + 73c_1\sqrt{17} + 152n^2 + 118nc_1 + 305c_1)/(16n(11\sqrt{17} + 47)), \\
\mu_8 &= -(nc_1\sqrt{17} - c_1\sqrt{17} - 136n^2 - 17nc_1 + 17c_1)/(4n(\sqrt{17} + 17)), \\
\mu_9 &= -(c_1\sqrt{17} + 8n^2 - c_1)/(4n(\sqrt{17} + 1)), \\
\mu_{10} &= -(26nc_1\sqrt{17} + 73c_1\sqrt{17} - 152n^2 - 118nc_1 - 305c_1)/(16n(11\sqrt{17} - 47)), \\
\mu_{11} &= (117n^2 + 25nc_1 - 25c_1)/(81n), \\
\mu_{12} &= (7218n^2 + 1225nc_1 - 400c_1)/(1296n), \\
\mu_{13} &= -(5(9n^2 + 5c_1))/(81n), \\
\eta_1 &= \frac{2n^2}{p-1}, \eta_2 = -\frac{16n^2}{\sqrt{17}-1}, \eta_3 = -\frac{18}{5}n^2.
\end{aligned} \tag{37}$$

We first treat the three inequalities  $s_1 \geq 0$ ,  $s_2 \geq 0$ ,  $s_3 \geq 0$ . Firstly,  $s_1 \geq 0$  is equivalent to  $\mu \leq \mu_1$ . Secondly, since the roots of  $4p^2 - 9p + 4 = 0$  are  $n_1$  and  $n_2$ , we have  $s_2 \geq 0 \Leftrightarrow \mu \leq \mu_2$  if  $p < n_1$  or  $p > n_2$ ; and  $s_2 \geq 0 \Leftrightarrow \mu \geq \mu_2$  if  $n_1 < p < n_2$ . In order to analyse  $s_3 \geq 0$ , we first compute

$$\mu_3 - \mu_4 = \frac{4((p-1)^2c_1 + 2n)}{9p-4}. \tag{38}$$

Therefore,  $s_3 \geq 0$  can be divided into four cases:  $s_3 \geq 0 \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$  if  $p > \frac{4}{9}$  and  $\theta_1 < c_1$ ;  $s_3 \geq 0 \Leftrightarrow \mu_3 \leq \mu \leq \mu_4$  if  $p > \frac{4}{9}$  and  $c_1 < \theta_1$ ;  $s_3 \geq 0 \Leftrightarrow \mu \geq \mu_3$  or  $\mu \leq \mu_4$  if  $p < \frac{4}{9}$  and  $c_1 < \theta_1$ ;  $s_3 \geq 0 \Leftrightarrow \mu \geq \mu_4$  or  $\mu \leq \mu_3$  if  $p < \frac{4}{9}$  and  $\theta_1 < c_1$ . Finally,  $p \geq 1 - \frac{\mu}{n}$  is equivalent to  $\mu \geq \theta_7$ .

Based on the above analysis and (34),  $\Phi(n, p, \mu)$  can be divided into six cases:

$$\begin{aligned}
\Phi(n, p, \mu) &\Leftrightarrow \max(\mu_4, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2, \mu_3), \text{ if } (p \in (\frac{4}{9}, n_1) \text{ or } p > n_2) \text{ and } \theta_1 < c_1 \leq 0; \\
\Phi(n, p, \mu) &\Leftrightarrow \max(\mu_2, \mu_4, \theta_7) \leq \mu \leq \min(\mu_1, \mu_3), \text{ if } p \in (n_1, n_2) \text{ or } \theta_1 < c_1 \leq 0; \\
\Phi(n, p, \mu) &\Leftrightarrow \max(\mu_3, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2, \mu_4), \text{ if } (p \in (\frac{4}{9}, n_1) \text{ or } p > n_2) \text{ or } c_1 < \theta_1; \\
\Phi(n, p, \mu) &\Leftrightarrow \max(\mu_2, \mu_3, \theta_7) \leq \mu \leq \min(\mu_1, \mu_4), \text{ if } p \in (n_1, n_2) \text{ or } c_1 < \theta_1; \\
\Phi(n, p, \mu) &\Leftrightarrow \theta_7 \leq \mu \leq \min(\mu_1, \mu_2, \mu_4) \text{ or } \max(\mu_3, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2), \text{ if } p < \frac{4}{9} \text{ or } c_1 < \theta_1; \\
\Phi(n, p, \mu) &\Leftrightarrow \theta_7 \leq \mu \leq \min(\mu_1, \mu_2, \mu_3) \text{ or } \max(\mu_4, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2), \text{ if } p < \frac{4}{9} \text{ or } \theta_1 < c_1 \leq 0.
\end{aligned} \tag{39}$$

The special cases  $p = \frac{4}{9}, n_1, n_2$  and  $c = \theta_1$  need to be considered separately. Also, we omit  $\exists c_1$  in the above formulations.

In what below, we will give detailed analysis of the above six cases which leads to the results in Table 1. We first have the formulas:

$$\mu_1 - \mu_3 = (p-1)^2c_1 + 2n, \tag{40}$$

$$\mu_1 - \mu_4 = \frac{9p((p-1)^2c_1 + 2n)}{9p-4}, \tag{41}$$

$$\mu_2 - \mu_3 = \frac{2(p-2)^2((1/2)(p-1)^2c_1 + n)}{(4p^2 - 9p + 4)}, \tag{42}$$

$$\mu_2 - \mu_4 = \frac{p(3p-4)^2((p-1)^2c_1 + 2n)}{(4p^2 - 9p + 4)(9p-4)}, \tag{43}$$

$$\mu_4 - \theta_7 = \frac{-(p-1)^2(4n+9p-4)c_1 + 2n^2p(9p-13)}{n(9p-4)}, \quad (44)$$

$$\mu_3 - \theta_7 = \frac{(p-1)(2n^2 - pc_1 + c_1)}{n}, \quad (45)$$

$$\theta_2 - \theta_1 = \frac{18n(np-n+1)(p-\frac{4}{9})}{(p-1)^2(4n+9p-4)}, \quad (46)$$

$$\theta_1 - \eta_1 = -\frac{2n(np-n+1)}{(p-1)^2}, \quad (47)$$

$$\eta_1 - \theta_2 = \frac{8n^2(np-n+1)}{(p-1)^2(4n+9p-4)}. \quad (48)$$

We also have the following formulas which will be used to eliminate  $c_1$  in the proof.

$$\begin{aligned} \mu \leq \mu_3 &\Leftrightarrow c_1 \leq \theta_4, & \mu \geq \mu_4 &\Leftrightarrow c_1 \geq \theta_5, \text{ if } p > \frac{4}{9}, \\ \mu \geq \mu_4 &\Leftrightarrow c_1 \leq \theta_5, \text{ if } p < \frac{4}{9}, & \mu \leq \mu_4 &\Leftrightarrow c_1 \leq \theta_5, \text{ if } p > \frac{4}{9}, \\ \mu \leq \mu_4 &\Leftrightarrow c_1 \geq \theta_5, \text{ if } p < \frac{4}{9}, & \mu \leq \mu_6 &\Leftrightarrow c_1 \leq \theta_6, \\ \mu \geq \mu_7 &\Leftrightarrow c_1 \geq \theta_{12}, & \mu \leq \mu_9 &\Leftrightarrow c_1 \leq \theta_{13}, \\ \mu \geq \mu_{10} &\Leftrightarrow c_1 \geq \theta_{14}, & \mu \leq \mu_{13} &\Leftrightarrow c_1 \leq \theta_{15}. \end{aligned} \quad (49)$$

We divide the proof into several cases, first according to the values of  $c_1$  and then according to the values of  $n$ .

**Case 1:**  $\theta_1 < c_1 \leq 0$ . From (40), we have  $\mu_1 > \mu_3$  in this case and from (39),  $\Phi(n, p, \mu)$  simplifies to three cases:

$$\begin{aligned} \Phi(n, p, \mu) &\Leftrightarrow \max(\mu_4, \theta_7) \leq \mu \leq \min(\mu_2, \mu_3), \text{ if } p \in (\frac{4}{9}, n_1) \text{ or } p > n_2; \\ \Phi(n, p, \mu) &\Leftrightarrow \max(\mu_2, \mu_4, \theta_7) \leq \mu \leq \mu_3, \text{ if } p \in (n_1, n_2); \\ \Phi(n, p, \mu) &\Leftrightarrow \theta_7 \leq \mu \leq \min(\mu_2, \mu_3) \text{ or } \max(\mu_4, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2), \text{ if } p < \frac{4}{9}. \end{aligned} \quad (50)$$

According to the vales of  $p$ , we consider seven cases below.

**Case 1.1:**  $\theta_1 < c_1 \leq 0$  and  $p > n_2$ . In this case, from (42) and (44), we have  $\mu_2 \geq \mu_3$  and  $\mu_4 > \theta_7$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p > n_2 \wedge \theta_1 < c_1 \leq 0 \wedge \mu_4 \leq \mu \leq \mu_3)$ . By (49),  $\mu_4 \leq \mu \leq \mu_3$  is equivalent to  $\theta_5 \leq c_1 \leq \theta_4$ .  $\exists c_1(\theta_5 \leq c_1 \leq \theta_4 \wedge \theta_1 < c_1 \leq 0)$  is equivalent to  $(\theta_4 > \theta_1 \wedge \theta_5 \leq 0)$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p > n_2 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq 0)$ , and  $\mathbb{T}(1, 2)$  is proved.

**Case 1.2:**  $\theta_1 < c_1 \leq 0$  and  $p = n_2$ . When  $p = n_2$ , we have  $\theta_1 = \theta_3$ ,  $s_2 = -\frac{1}{1024}(7\sqrt{17} - 33)(c_1\sqrt{17} + 64n + 9c_1)n$ . Then  $s_2 \geq 0 \Leftrightarrow c_1 \geq \theta_3$ . Because  $\theta_1 < c_1 \leq 0$  and  $p = n_2 > \frac{4}{9}$ , we have  $s_3 \geq 0 \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$ . By (44), we have  $\mu_4 > \theta_7$ . When  $p = n_2$ , we have  $\mu_3 = \mu_6$  and  $\mu_4 = \mu_7$ . Thus  $\Phi(n, p, \mu) \Leftrightarrow (\theta_3 < c_1 \leq 0, \mu_7 \leq \mu \leq \mu_6)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p = n_2 \wedge \theta_3 < c_1 \leq 0 \wedge \mu_7 \leq \mu \leq \mu_6)$ . By (49),  $\mu_7 \leq \mu \leq \mu_6$  is equivalent to  $\theta_{12} \leq c_1 \leq \theta_6$ .  $\exists c_1(\theta_{12} \leq c_1 \leq \theta_6 \wedge \theta_3 < c_1 \leq 0)$  is equivalent to  $\theta_6 > \theta_3$  and  $\theta_{12} \leq 0$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p = n_2 \wedge \theta_6 > \theta_3 \wedge \theta_{12} \leq 0)$ , and  $\mathbb{T}(2, 2)$  is proved.

**Case 1.3:**  $\theta_1 < c_1 \leq 0$  and  $p \in (n_1, n_2)$ ,  $p \neq 1$ . Due to (44), this case is divided into two sub-cases.

**Case 1.3.1:**  $\theta_1 < c_1 \leq 0$  and  $p \in (\frac{13}{9}, n_2)$ . By (43) and (44), we have  $\mu_4 > \mu_2$  and  $\mu_4 > \theta_7$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (\frac{13}{9}, n_2) \wedge \theta_1 < c_1 \leq 0 \wedge \mu_4 \leq \mu \leq \mu_3)$ . Similar to Case 1.1, we have  $\Phi(n, p, \mu) \Leftrightarrow (p \in (\frac{13}{9}, n_2) \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq 0)$ ,  $\mathbb{T}(3, 2)$  is proved.

**Case 1.3.2:**  $\theta_1 < c_1 \leq 0$  and  $p \in (n_1, \frac{13}{9}]$ ,  $p \neq 1$ . By (43), (45), (44) and (46), we have  $\mu_4 \geq \mu_2$ , ( $\mu_3 \geq \theta_7 \Leftrightarrow c_1 \leq \eta_1$ ), ( $\mu_4 \geq \theta_7 \Leftrightarrow c_1 \leq \theta_2$ ) and ( $\theta_2 > \theta_1 \Leftrightarrow \phi_3$ ). Hence  $\Phi(n, p, \mu) \Leftrightarrow \max(\mu_4, \theta_7) \leq \mu \leq \mu_3$ . This case is further divided into two sub-cases.

**Case 1.3.2.1:** If  $c_1 \leq \theta_2$ , then  $\mu_4 \geq \mu_5$ , and  $\Phi(n, p, \mu) \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$ . So we need  $\theta_1 < \theta_2$ , which yields  $\phi_3$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_1 < c_1 \leq \theta_2, \phi_3, \mu_4 \leq \mu \leq \mu_3)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (n_1, \frac{13}{9}) \wedge p \neq 1 \wedge \phi_3 \wedge \theta_1 < c_1 \leq \theta_2 \wedge \mu_4 \leq \mu \leq \mu_3)$ . Like Case 1.1, we have  $\Phi(n, p, \mu) \Leftrightarrow (p \in (n_1, \frac{13}{9}) \wedge p \neq 1 \wedge \phi_3 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq \theta_2)$ , and  $\mathbb{T}(4, 2)$  is proved

**Case 1.3.2.2:** If  $c_1 \geq \theta_2$ , then  $\mu_4 \leq \theta_7$ , and  $\Phi(n, p, \mu) \Leftrightarrow \theta_7 \leq \mu \leq \mu_3$ . So we need  $\theta_7 \leq \mu_3$ , which yields  $c_1 \leq \eta_1$ . By (47), we know  $\eta_1 > \theta_1$  results  $\phi_3$ , which yields  $\theta_1 < \theta_2 < \eta_1$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_2 < c_1 \leq \min(0, \eta_1), \phi_3, \theta_7 \leq \mu \leq \mu_3)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (n_1, \frac{13}{9}) \wedge p \neq 1 \wedge \phi_3 \wedge \theta_2 < c_1 \leq \min(0, \eta_1) \wedge \theta_7 \leq \mu \leq \mu_3)$ .  $\theta_7 \leq \mu \leq \mu_3$  is equivalent to  $\theta_7 \leq \mu$  and  $c_1 \leq \theta_4$ .  $\exists c_1(c_1 \leq \theta_4 \wedge \theta_2 < c_1 \leq \min(0, \eta_1))$  is equivalent to  $\theta_4 > \theta_2$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p \in (n_1, \frac{13}{9}) \wedge p \neq 1 \wedge \phi_3 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_2)$ , and  $\mathbb{T}(4, 3)$  is proved.

**Case 1.4:**  $\theta_1 < c_1 \leq 0$  and  $p = n_1$ . When  $p = n_1$ , we have  $\theta_1 = \theta_{10}$ ,  $\theta_2 = \theta_{11}$ ,  $\eta_1 = \eta_2$ ,  $s_2 = -\frac{1}{1024}(33 + 7\sqrt{17})(c_1\sqrt{17} - 64n - 9c_1)n$ . Then  $s_2 \geq 0 \Leftrightarrow c_1 \geq \theta_{10}$ . Because  $\theta_1 < c_1 \leq 0$  and  $p = n_1 > \frac{4}{9}$ , we have  $s_3 \geq 0 \Leftrightarrow \mu_4 \leq \mu \leq \mu_3$ . By (44), we have  $\mu_4 \geq \theta_7 \Leftrightarrow c_1 \leq \theta_2$ . This case is divided into two sub-cases.

**Case 1.4.1:** Similar to Case 1.3.2.1,  $\Phi(n, p, \mu) \Leftrightarrow (\theta_1 < c_1 \leq \theta_2 \wedge \phi_3 \wedge \mu_4 \leq \mu \leq \mu_3)$ . When  $p = n_1$ , we have  $\mu_3 = \mu_9$ ,  $\mu_4 = \mu_{10}$ ,  $\phi_3 \Leftrightarrow n < n_3$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_{10} < c_1 \leq \theta_{11}, n < n_3, \mu_{10} \leq \mu \leq \mu_9)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p = n_1 \wedge \theta_{10} < c_1 \leq \theta_{11} \wedge n < n_3 \wedge \mu_{10} \leq \mu \leq \mu_9)$ .  $\mu_{10} \leq \mu \leq \mu_9$  is equivalent to  $\theta_{14} \leq c_1 \leq \theta_{13}$ .  $\exists c_1(\theta_{14} \leq c_1 \leq \theta_{13} \wedge \theta_{10} < c_1 \leq \theta_{11})$  is equivalent to  $\theta_{13} > \theta_{10}$  and  $\theta_{14} \leq \theta_{11}$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p = n_1 \wedge n < n_3 \wedge \theta_{13} > \theta_{10} \wedge \theta_{14} \leq \theta_{11})$ , and  $\mathbb{T}(5, 2)$  is proved.

**Case 1.4.2:** Similar to Case 1.3.2.2,  $\Phi(n, p, \mu) \Leftrightarrow (\theta_2 < c_1 \leq \min(0, \eta_1) \wedge \phi_3 \wedge \mu_5 \leq \mu \leq \mu_3)$ . When  $p = n_1$ , we have  $\theta_7 = \theta_{16}$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_{11} < c_1 \leq \eta_2, n < n_3, \theta_{16} \leq \mu \leq \mu_9)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p = n_1 \wedge \theta_{11} < c_1 \leq \eta_2 \wedge n < n_3 \wedge \theta_{16} \leq \mu \leq \mu_9)$ .  $\theta_{16} \leq \mu \leq \mu_9$  is equivalent to  $c_1 \leq \theta_{13}$  and  $\mu \geq \theta_{16}$ .  $\exists c_1(c_1 \leq \theta_{13} \wedge \theta_{11} < c_1 \leq \eta_2)$  is equivalent to  $\theta_{13} > \theta_{11}$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p = n_1 \wedge n < n_3 \wedge \theta_{13} > \theta_{11} \wedge \mu \geq \theta_{16})$ , and  $\mathbb{T}(5, 3)$  is proved.

**Case 1.5:**  $\theta_1 < c_1 \leq 0$  and  $p \in (\frac{4}{9}, n_1)$ . By (42) and (44), we have  $\mu_2 > \mu_3$  and ( $\mu_4 \geq \theta_7 \Leftrightarrow c_1 \leq \theta_2$ ). Hence  $\Phi(n, p, \mu) \Leftrightarrow \max(\mu_4, \theta_7) \leq \mu \leq \mu_3$ . This case is divided into two sub-cases.

**Case 1.5.1:** Similar to Case 1.3.2.1, we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_1 < c_1 \leq \theta_2 \wedge \phi_3 \wedge \mu_4 \leq \mu \leq \mu_3)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (\frac{4}{9}, n_1) \wedge \phi_3 \wedge \theta_1 < c_1 \leq \theta_2 \wedge \mu_4 \leq \mu \leq \mu_3)$ . Like Case 1.3.2.1, we have  $\Phi(n, p, \mu) \Leftrightarrow (p \in (\frac{4}{9}, n_1) \wedge \phi_3 \wedge \theta_4 > \theta_1 \wedge \theta_5 \leq \theta_2)$ , and  $\mathbb{T}(6, 2)$  is proved

**Case 1.5.2:** Similar to Case 1.3.2.2, we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_2 < c_1 \leq \eta_1 \wedge \phi_3 \wedge \mu_5 \leq \mu \leq \mu_3)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (\frac{4}{9}, n_1) \wedge \phi_3 \wedge \theta_2 < c_1 \leq \eta_1 \wedge \theta_7 \leq \mu \leq \mu_3)$ . Similar to Case 1.3.2.2, we have  $\Phi(n, p, \mu) \Leftrightarrow (p \in (\frac{4}{9}, n_1) \wedge \phi_3 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_2)$ , and  $\mathbb{T}(6, 3)$  is proved.

**Case 1.6:**  $\theta_1 < c_1 \leq 0$  and  $p = \frac{4}{9}$ . When  $p = \frac{4}{9}$ , we have  $\theta_1 = \theta_9$ ,  $\eta_1 = \eta_3$ ,  $\theta_7 = \theta_8$ ,  $s_3 =$

$-\frac{4n}{6561}(162n + 25c_1)(45n^2 + 81nu + 25c_1)$ . Then  $s_3 \geq 0 \Leftrightarrow \mu \leq \mu_{13}$  if  $c_1 \geq \theta_9$ . By (45), we know  $\mu_3 \geq \theta_7 \Leftrightarrow c_1 \leq \eta_1$ . And  $\theta_9 < \eta_3 \Leftrightarrow n = 1$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_9 < c_1 \leq \eta_3, n = 1, \theta_8 \leq \mu \leq \mu_{13})$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p = \frac{4}{9} \wedge \theta_9 < c_1 \leq \eta_3 \wedge n = 1 \wedge \theta_8 \leq \mu \leq \mu_{13})$ .  $\theta_8 \leq \mu \leq \mu_{13}$  is equivalent to  $c_1 \leq \theta_{15}$  and  $\mu \geq \theta_8$ .  $\exists c_1(c_1 \leq \theta_{15} \wedge \theta_9 < c_1 \leq \eta_3)$  is equivalent to  $\theta_{15} > \theta_9$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p = \frac{4}{9} \wedge n = 1 \wedge \theta_{15} > \theta_9 \wedge \mu \geq \theta_8)$ , and  $\mathbb{T}(7, 2)$  is proved

**Case 1.7:**  $\theta_1 < c_1 \leq 0$  and  $0 < p < \frac{4}{9}$ . This case is divided into two sub-cases.

**Case 1.7.1:** If we select  $\theta_7 \leq \mu \leq \min(\mu_2, \mu_3)$ , by (42), we have  $\mu_2 > \mu_3$ . Thus  $\Phi(n, p, \mu) \Leftrightarrow \theta_7 \leq \mu \leq \mu_3$ . So we need  $\theta_7 \leq \mu_3$ , which yields  $c_1 \leq \eta_1$ . By (47), we know  $\eta_1 > \theta_1$  results  $\phi_3$ , which yields  $n = 1$  with  $0 < p < \frac{4}{9}$ . From (50), we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_1 < c_1 \leq \eta_1, n = 1, \theta_7 \leq \mu \leq \mu_3)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (0, \frac{4}{9}) \wedge n = 1 \wedge \theta_1 < c_1 \leq \eta_1 \wedge \theta_7 \leq \mu \leq \mu_3)$ . Similar to Case 1.3.2.2, we have  $\Phi(n, p, \mu) \Leftrightarrow (p \in (0, \frac{4}{9}) \wedge n = 1 \wedge \theta_7 \leq \mu \wedge \theta_4 > \theta_1)$ , and  $\mathbb{T}(8, 2)$  is proved.

**Case 1.7.2:** If we select  $\max(\mu_4, \theta_7) \leq \mu \leq \min(\mu_1, \mu_2)$ , by (41), we have  $\mu_1 < \mu_4$ , which yields contradiction.

**Case 2:**  $c_1 < \theta_1$ . From (40), we have  $\mu_1 < \mu_3$  in this case and from (35),  $\Phi(n, p, \mu)$  simplifies to one case:

$$\Phi(n, p, \mu) \Leftrightarrow \exists c_1(\theta_7 \leq \mu \leq \min(\mu_1, \mu_2, \mu_4), \text{ if } 0 < p < \frac{4}{9} \text{ and } c_1 < \theta_1).$$

Since  $p$  satisfies  $0 < p < \frac{4}{9}$ , we need only consider the following cases.

**Case 2.1:**  $c_1 < \theta_1$  and  $0 < p < \frac{4}{9}$ . By (41), (43) and (44), we have  $\mu_1 > \mu_4$ ,  $\mu_2 > \mu_4$  and  $(\mu_4 \geq \theta_7 \Leftrightarrow c_1 \geq \theta_2)$ . Then we need  $\theta_2 < \theta_1$ , which yields  $\phi_3$  by (46). Because  $\phi_3$  means  $n = 1$  with  $0 < p < \frac{4}{9}$ , we have  $\Phi(n, p, \mu) \Leftrightarrow (\theta_2 \leq c_1 < \theta_1, n = 1, \theta_7 \leq \mu \leq \mu_4)$ .

We now eliminate  $c_1$  from  $\Phi(n, p, \mu) \Leftrightarrow \exists c_1(p \in (0, \frac{4}{9}) \wedge n = 1 \wedge \theta_2 \leq c_1 < \theta_1 \wedge \theta_7 \leq \mu \leq \mu_4)$ .  $\theta_7 \leq \mu \leq \mu_4$  is equivalent to  $c_1 \geq \theta_5$  and  $\mu \geq \theta_7$ .  $\exists c_1(c_1 \geq \theta_5 \wedge \theta_2 \leq c_1 < \theta_1)$  is equivalent to  $\theta_5 < \theta_1$ . Therefore, in this case  $\Phi(n, p, \mu) \Leftrightarrow (p \in (0, \frac{4}{9}) \wedge n = 1 \wedge \theta_5 < \theta_1 \wedge \mu \geq \theta_7)$ , and  $\mathbb{T}(8, 3)$  is proved

**Case 2.2:**  $c_1 < \theta_1$  and  $p = n_2$ . In Case 1.2, we know  $\theta_1 = \theta_3$  with  $p = n_2$ , and  $s_2 \geq 0 \Leftrightarrow c_1 \geq \theta_3$ , which yields contradiction.

**Case 2.3:**  $c_1 < \theta_1$  and  $p = n_1$ . In Case 1.4, we know  $\theta_1 = \theta_{10}$  with  $p = n_1$ , and  $s_2 \geq 0 \Leftrightarrow c_1 \geq \theta_{10}$ , which yields contradiction.

**Case 2.4:**  $c_1 < \theta_1$  and  $p = \frac{4}{9}$ . We have  $\theta_1 = \theta_9$ ,  $\mu_2 = \mu_{12}$ ,  $\mu_3 = \mu_{13}$  based on  $p = \frac{4}{9}$ . Then we have  $(s_2 \geq 0 \Leftrightarrow \mu \leq \mu_{12})$  and  $(s_3 \geq 0 \Leftrightarrow \mu \geq \mu_{13} \text{ if } c_1 \leq \theta_9)$ . So we need  $\mu_{12} \geq \mu_{13}$ . By (42), we have  $\mu_{12} < \mu_{13}$ , which yields contradiction.

**Case 3:**  $c_1 = \theta_1$ . When  $c_1 = \theta_1$ , we have  $s_1 = n(np - n - \mu + 2)$ ,  $s_2 = n(4p^2 - 9p + 4)(np - n - \mu + 2)$  and  $s_3 = -n^2(9p - 4)(np - n - \mu + 2)^2$ . Thus,  $s_1 \geq 0 \Leftrightarrow \mu \leq 2 + n(p - 1)$  and  $s_3 \geq 0 \Leftrightarrow (p \leq \frac{4}{9} \text{ or } \mu = 2 + n(p - 1))$ . This case is divided into two sub-cases.

**Case 3.1:** If  $\mu = 2 + n(p - 1)$ , then  $s_1 = s_2 = s_3 = 0$ . And  $p \geq 1 - \frac{\mu}{n} \Leftrightarrow \phi_1$ . Thus  $\Phi(n, p, \mu) \Leftrightarrow (c_1 = \theta_1 \wedge \phi_1 \wedge \phi_2)$ , and  $\mathbb{T}(i, 4)$ ,  $i = 1, \dots, 6$  are proved.

**Case 3.2:** If  $p \leq \frac{4}{9}$ , then  $s_2 \geq 0 \Leftrightarrow \mu \leq 2 + n(p - 1)$ . Then we need  $2 + n(p - 1) \geq \mu_5$ , which yields  $\phi_1$ . And  $\phi_1$  implies  $n = 1$  with  $p \leq \frac{4}{9}$ . Thus  $\Phi(n, p, \mu) \Leftrightarrow (c_1 = \theta_1 \wedge n = 1 \wedge -p \leq \mu - 1 \leq p)$ , and  $\mathbb{T}(7, 4)$ ,  $\mathbb{T}(8, 4)$  are proved.



## 4 Conclusion

This paper is an extension of the work [19, 20] to the case where the entropy power involves parameters. The basic idea is to prove entropy power inequalities in a systematical way. Precisely, the concavity of Rényi entropy power is considered, where the probability density  $u_t$  solve the nonlinear heat equation with two parameters  $p$  and  $\mu$ . Our procedure reduces the proof of the CREP to check the semi-positiveness of a quadratic form (33) whose coefficients are polynomials in the parameters  $n, p, \mu$ . In principle, a necessary and sufficient condition on parameters  $n, p, \mu$  for this can be computed with the quantifier elimination [22].

Based on the above method, we give a sufficient condition  $\Phi(n, p, \mu)$  for CREP, which extends the parameter's range of the CREP given by Savaré-Toscani [21]. In fact, our results give the necessary and sufficient condition for CREP under certain conditions. But in the general case, Theorem 2.2 only gives a sufficient condition for the following reasons: 1. we use inequalities (21) in step (23); 2. there might exist more constraints; 3. **Problem II** may not equivalent to **Problem III**.

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## Appendix. Constraints in (15)

In this appendix, we give the constraints in (15), where  $u_{h_1, h_2} = \frac{\partial^{h_1+h_2} u}{\partial^{h_1} x_{a,t} \partial^{h_2} x_{b,t}}$ .

$$\begin{aligned}
R_{1,a,b} &= 3pu_{1,0}^4 + 3uu_{2,0}u_{1,0}^2 - 5u_{1,0}^4, \\
R_{2,a,b} &= 3pu_{0,1}^4 + 3uu_{0,2}u_{0,1}^2 - 5u_{0,1}^4, \\
R_{3,a,b} &= 3pu^2u_{1,0}u_{3,0} + u^3u_{4,0} - 3u^2u_{1,0}u_{3,0}, \\
R_{4,a,b} &= 3pu^2u_{0,1}u_{3,0} + u^3u_{3,1} - 3u^2u_{0,1}u_{3,0}, \\
R_{5,a,b} &= 3pu^2u_{1,0}u_{2,1} + u^3u_{3,1} - 3u^2u_{1,0}u_{2,1}, \\
R_{6,a,b} &= 3pu^2u_{0,1}u_{2,1} + u^3u_{2,2} - 3u^2u_{0,1}u_{2,1}, \\
R_{7,a,b} &= 3pu^2u_{1,0}u_{1,2} + u^3u_{2,2} - 3u^2u_{1,0}u_{1,2}, \\
R_{8,a,b} &= 3pu^2u_{0,1}u_{1,2} + u^3u_{1,3} - 3u^2u_{0,1}u_{1,2}, \\
R_{9,a,b} &= 3pu^2u_{1,0}u_{0,3} + u^3u_{1,3} - 3u^2u_{1,0}u_{0,3}, \\
R_{10,a,b} &= 3pu^2u_{0,1}u_{0,3} + u^3u_{0,4} - 3u^2u_{0,1}u_{0,3}, \\
R_{11,a,b} &= 3pu_{0,1}u_{1,0}^3 + 3uu_{1,1}u_{1,0}^2 - 5u_{1,0}^3u_{0,1}, \\
R_{12,a,b} &= 3pu_{0,1}^3u_{1,0} + 3uu_{1,1}u_{0,1}^2 - 5u_{0,1}^3u_{1,0}, \\
R_{13,a,b} &= 3pu_{0,1}^2u_{1,0}^2 + 2uu_{0,1}u_{1,0}u_{1,1} + uu_{1,0}^2u_{0,2} - 5u_{1,0}^2u_{0,1}^2, \\
R_{14,a,b} &= 3puu_{1,0}^2u_{2,0} + u^2u_{1,0}u_{3,0} + u^2u_{2,0}^2 - 4uu_{2,0}u_{1,0}^2, \\
R_{15,a,b} &= 3puu_{0,1}u_{1,0}u_{2,0} + u^2u_{0,1}u_{3,0} + u^2u_{1,1}u_{2,0} - 4uu_{2,0}u_{1,0}u_{0,1}, \\
R_{16,a,b} &= 3puu_{1,0}^2u_{1,1} + u^2u_{1,0}u_{2,1} + u^2u_{1,1}u_{2,0} - 4uu_{1,1}u_{1,0}^2, \\
R_{17,a,b} &= 3puu_{0,1}u_{1,0}u_{2,0} + u^2u_{1,0}u_{2,1} + u^2u_{1,1}u_{2,0} - 4uu_{2,0}u_{1,0}u_{0,1},
\end{aligned}$$

$$\begin{aligned}
R_{18,a,b} &= 3puu_{0,1}^2u_{2,0} + u^2u_{0,1}u_{2,1} + u^2u_{0,2}u_{2,0} - 4uu_{0,1}^2u_{2,0}, \\
R_{19,a,b} &= 3puu_{0,1}u_{1,0}u_{1,1} + u^2u_{0,1}u_{2,1} + u^2u_{1,1}^2 - 4uu_{0,1}u_{1,0}u_{1,1}, \\
R_{20,a,b} &= 3puu_{1,0}^2u_{0,2} + u^2u_{1,0}u_{1,2} + u^2u_{0,2}u_{2,0} - 4uu_{1,0}^2u_{0,2}, \\
R_{21,a,b} &= 3puu_{0,1}u_{1,0}u_{1,1} + u^2u_{1,0}u_{1,2} + u^2u_{1,1}^2 - 4uu_{0,1}u_{1,0}u_{1,1}, \\
R_{22,a,b} &= 3puu_{0,1}^2u_{1,1} + u^2u_{0,1}u_{1,2} + u^2u_{0,2}u_{1,1} - 4uu_{1,1}u_{0,1}^2, \\
R_{23,a,b} &= 3puu_{0,1}u_{1,0}u_{0,2} + u^2u_{0,1}u_{1,2} + u^2u_{0,2}u_{1,1} - 4uu_{0,2}u_{1,0}u_{0,1}, \\
R_{24,a,b} &= 3puu_{0,1}u_{1,0}u_{0,2} + u^2u_{1,0}u_{0,3} + u^2u_{0,2}u_{1,1} - 4uu_{0,2}u_{1,0}u_{0,1}, \\
R_{25,a,b} &= 3puu_{0,1}^2u_{0,2} + u^2u_{0,1}u_{0,3} + u^2u_{0,2}^2 - 4uu_{0,2}u_{0,1}^2, \\
R_{26,a,b} &= 3pu_{0,1}u_{1,0}^3 + 2uu_{2,0}u_{1,0}u_{0,1} + uu_{1,1}u_{1,0}^2 - 5u_{1,0}^3u_{0,1}, \\
R_{27,a,b} &= 3pu_{0,1}^2u_{1,0}^2 + uu_{0,1}^2u_{2,0} + 2uu_{0,1}u_{1,0}u_{1,1} - 5u_{1,0}^2u_{0,1}^2, \\
R_{28,a,b} &= 3pu_{0,1}^3u_{1,0} + uu_{1,1}u_{0,1}^2 + 2uu_{0,2}u_{1,0}u_{0,1} - 5u_{0,1}^3u_{1,0}.
\end{aligned}$$

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