Robust and Information-theoretically Safe Bias Classifier against Adversarial Attacks^{*}

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Abstract

In this paper, the bias classifier is introduced, that is, the bias part of a DNN with Relu as the activation function is used as a classifier. The work is motivated by the fact that the bias part is a piecewise constant function with zero gradient and hence cannot be directly attacked by gradient-based methods to generate adversaries, such as FGSM. The existence of the bias classifier is proved and an effective training method for the bias classifier is given. It is proved that by adding a proper random first-degree part to the bias classifier, an information-theoretically safe classifier against the original-model gradient attack is obtained in the sense that the attack will generate a totally random attacking direction. This seems to be the first time that the concept of information-theoretically safe classifier is proposed. Several attack methods for the bias classifier are proposed and numerical experiments are used to show that the bias classifier is more robust than DNNs with similar size against these attacks in most cases.

Keywords. Robust DNN, adversarial samples, bias classifier, information-theoretically safe, gradient-based attack.

1 Introduction

The deep neural network (DNN) [19] has become the most powerful machine learning method, which has been successfully applied in computer vision, natural language processing, game playing, protein structure prediction, and many other fields.

A major weakness of DNNs is the existence of adversarial samples [28], that is, it is possible to intentionally make small modifications to an input such that human can still recognize the input clearly, but the DNN outputs a wrong label or even any label given by the adversary. Existence of adversary samples makes the DNN vulnerable in safety-critical applications. Although many effective methods to defend adversaries were proposed [21, 1, 4, 38], it was shown that adversaries are still inevitable for current DNNs [3, 24]. In [5], it was proved that adversarial attacks always exist for any successful DNNs under certain conditions. In this paper, we present a new approach by using the bias part of the DNN as the classifier and show that the bias classifier is safe against gradient-based attacks.

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1.1 Contributions

Let $\mathbb{I} = [0,1] \subset \mathbb{R}$ and $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ a classification DNN for *m* objects, using Relu as the activation function. For any $x \in \mathbb{I}^n$, there exist $W_x \in \mathbb{R}^{m \times n}$ and $B_x \in \mathbb{R}^m$ such that

$$\mathcal{F}(x) = W_x x + B_x$$

where $W_x x$ is called the *first-degree part* and B_x the *bias part* of \mathcal{F} . From the definition of Relu, the bias part

$$B_{\mathcal{F}}:\mathbb{I}^n\to\mathbb{R}^m$$

defined as $B_{\mathcal{F}}(x) = B_x$ is a piecewise constant function with a finite number of values.

The most popular and effective methods to generate adversaries, such as FGSM [12] or PGD [21], use $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ to make the loss function bigger. An attack on DNNs only using the values of $\mathcal{F}(x)$ and $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ is called a *gradient-based attack*. Since *n* is generally quite large, using $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ to find adversaries in the high-dimensional space \mathbb{R}^n seems inevitable.

Motivated by the above observation, the *bias classifier* is introduced in this paper, that is, the bias part $B_{\mathcal{F}} : \mathbb{I}^n \to \mathbb{R}^m$ of \mathcal{F} is used to classify the *m* objects. Since $B_{\mathcal{F}}$ is a piecewise constant function, it has zero gradients and is safe against direct gradient-based attacks. The contributions of this paper are summarized below.

First, the existence of the bias classifier is proved. Precisely, it is shown that for any classification problem, there exists a DNN \mathcal{F} such that its bias part $B_{\mathcal{F}}$ gives the correct label with arbitrarily high probability.

Second, an effective training method for the bias classifier is proposed. It is observed that the adversarial training method introduced in [21] significantly increases the classification power of the bias part. Furthermore, using the adversarial training to the loss function $L_{\text{CE}}(B_{\mathcal{F}}(x), y) + \gamma L_{\text{CE}}(\mathcal{F}(x), y)$ increases the classification power of the bias part and decreases the classification power of first-degree part of \mathcal{F} , and hence is used to train the bias classifier.

Third, an information-theoretically safe bias classifier against gradient-based attacks is given. A network \mathcal{F} is called information-theoretically safe against an attack \mathcal{A} , if when generate an adversary for a sample x with \mathcal{A} , a random attack direction is given. In other words, the rate to generate adversaries with \mathcal{A} equals the rate of random samples to be adversaries. Let $W \in \mathbb{R}^{m \times n}$ be a matrix with certain random entries, $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ a trained bias classifier, and $\widetilde{\mathcal{F}}(x) = \mathcal{F}(x) + Wx$. Then, it is shown that $B_{\widetilde{\mathcal{F}}}$ is information-theoretically safe against the gradient-based attack of $\widetilde{\mathcal{F}}$, if the structure and parameters of \mathcal{F} are kept secret. The notion of information-theoretically safe is borrowed from cryptography [10], which means that the ciphertext yields no information regarding the plaintext for cyphers which are perfectly random.

Fourth, several methods to attack the bias classifier are proposed. Experiments with MNIST and CIFAR-10 show that the bias classifier has comparable accuracies with DNNs on the test sets and is more robust than DNNs of similar sizes against these adversarial attacks in most cases.

1.2 Related work

There exist two main approaches to obtain more robust DNNs: using a better training method or a better structured DNN. Of course, the two approaches can be combined.

Many effective methods were proposed to train more robust DNNs to defend adversaries [1, 4, 38, 34]. The adversarial training method proposed by Madry et al [21] can reduce adversaries

significantly, where the value of the loss function of the worst adversary in a small neighborhood of the training sample is minimized. A similar approach is to generate adversaries and add them to the training set [12]. A fast adversarial training algorithm was proposed, which improves the training efficiency by reusing the backward pass calculations [25]. A less direct approach to resist adversaries is to make the DNN more stable by introducing the Lipschitz constant or $L_{p,\infty}$ regulations of each layer [6, 28, 36, 35]. Adding noises to the training data is an effective way to increase the robustness [11]. Knowledge distilling is also used to enhance robustness and defend adversaries [16].

In this paper, the adversarial training [21] is used to a new loss function to train the bias classifier.

Many effective new structures for DNNs were proposed to defend adversaries. The ensembler adversarial training [29] was introduced for CNN models, which can apply to large datasets such as ImageNet. In [33], a denoising layer is added to each hidden layer to defend adversarial attack. In [17], difference-privacy noise layers are added to defend adversaries. In [23], a low-rank DNN is shown to be more robust. In [37], a classification-autoencoder was proposed, which is robust against outliers and adversaries. In [7], it was observed that by taking average values of points in a small neighbourhood of an input can give a larger robust region for the input. In [13, 32], strategies to defend adversarial attacks by modifying the input were given.

In this paper, a new idea to obtain robust DNNs is given, that is, the bias part is used as the classifier to avoid gradient-based attacks. Another advantage of using the bias part as the classifier is that, an information-theoretically safe classifier can be constructed. Our network does not deliberately hide the gradient like the method in [2]. Our network does not have gradient, so the white box attack method for the gradient hiding method in [2] does not work for our model.

The rest of this paper is organized as follows. In section 2, the existence of the bias classifier is proved and the training method is given. In section 3, several attack methods for the bias classifier are given. In section 4, the bias classifier is shown to be information-theoretically safe against the original-model gradient-based attack. In section 5, numerical experimental results are given to show that the bias classifier indeed improves robustness to resist adversaries. In section 6, conclusions are given.

2 Bias Classifier

In this section, we prove the existence of a DNN \mathcal{F} such that the bias part of \mathcal{F} can be used as a classifier. We also give a training algorithm for the new classifier.

2.1 The standard DNN

Let $\mathbb{I} = [0,1] \subset \mathbb{R}$ and $[n] = \{1,\ldots,n\}$ for $n \in \mathbb{N}_{>0}$. Let $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ be a classification DNN with L hidden layers and the label set $\mathbb{L} = [m]$. Each hidden-layer of \mathcal{F} uses Relu as activity functions and the output layer does not have activity functions. We write $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ as

$$\begin{aligned}
x_0 \in \mathbb{I}^n, & n_0 = n, n_{L+1} = m, \\
x_l = \text{Relu}(W_l x_{l-1} + b_l) \in \mathbb{R}^{n_l}, l \in [L], \\
\mathcal{F}(x_0) = x_{L+1} = W_{L+1} x_L + b_{L+1},
\end{aligned} \tag{1}$$

where $W_l \in \mathbb{R}^{n_l \times n_{l-1}}, b_l \in \mathbb{R}^{n_l}$. Denote $\Theta_{\mathcal{F}} = \{W_l, b_l\}_{l=1}^{L+1}$ to be the parameter set of \mathcal{F} . Given a training set \mathcal{S} , the network \mathcal{F} can be trained by solving the following optimization problem with

BP

$$\min_{\Theta} \sum_{(x,y)\in\mathcal{S}} L_{\rm CE}(\mathcal{F}(x), y).$$
(2)

For any $x \in \mathbb{I}^n$, there exist $W_x \in \mathbb{R}^{m \times n}$ and $B_x \in \mathbb{R}^m$, such that $\mathcal{F}(x) = W_x x + B_x$. We define the first-degree part of \mathcal{F} to be $W_{\mathcal{F}} : \mathbb{I}^n \to \mathbb{R}^m$, that is $W_{\mathcal{F}}(x) = W_x x$; and the bias part of \mathcal{F} to be $B_{\mathcal{F}} : \mathbb{I}^n \to \mathbb{R}^m$, that is $B_{\mathcal{F}}(x) = B_x$. It is easy to see that

$$\mathcal{F}(x) = W_{\mathcal{F}}(x) + B_{\mathcal{F}}(x) = W_x x + B_x. \tag{3}$$

For a label $i \in [m]$ and $x \in \mathbb{I}^n$, denote $\mathcal{F}_i(x)$ to be the *i*-th coordinate of $\mathcal{F}(x)$.

A linear region of \mathcal{F} is a maximal connected open subset of the input space \mathbb{I}^n , on which \mathcal{F} is linear [12]. On each linear region A of \mathcal{F} , there exist $W_A \in \mathbb{R}^{m \times n}$ and $B_A \in \mathbb{R}^m$, such that $\mathcal{F}(x) = W_A x + B_A$ for $x \in A$. Due to the property of Relu function, it is clear that \mathcal{F} has a finite number of disjoint linear regions and \mathbb{I}^n is the union of the closures of these linear regions.

2.2 Existence of bias classifier

In this section, we will prove the existence of the bias classifier. Let $\mathbb{O} \subset \mathbb{I}^n$ be the objects to be classified. For $x \in \mathbb{O}$ and $r \in \mathbb{R}_{>0}$, when r is small enough, all images in

$$\mathbb{B}(x,r) = \{x + \eta \mid \eta \in \mathbb{R}^n, ||\eta|| < r\}$$

can be considered to have the same label with x. Therefore, the object \mathbb{O} to be classified may be considered as bounded open sets in \mathbb{I}^n . This observation motivates the following existence theorem, whose proof is given in Appendix A.

Theorem 2.1. Let $\mathbb{O} = \bigcup_{i=1}^{m} O_i \subset \mathbb{I}^n$ be the elements to be classified and $\mathbb{L} = \{l\}_{l=1}^{m}$ the label set, where $O_i \subset \mathbb{I}^n$ is an open set, $O_i \bigcap O_j = \phi$ if $i \neq j$, and x has label l for $x \in O_l$. Then for any $\epsilon > 0$, there exist a DNN \mathcal{F} and an open set $D \subset \mathbb{I}^n$ with volume $V(D) < \epsilon$, such that $B_{\mathcal{F}}(x)$ gives the correct label for $x \in \mathbb{O} \setminus D$, that is, the l-th coordinate of $\mathcal{F}(x)$ has the biggest value for $x \in O_l$.

A network \mathcal{F} satisfying the conditions of Theorem 2.1 gives a *bias classifier* $B_{\mathcal{F}}$, which can be computed from \mathcal{F} as follows:

$$B_{\mathcal{F}}(x) = \mathcal{F}(x) - W_{\mathcal{F}}(x) = \mathcal{F}(x) - \frac{\nabla \mathcal{F}(x)}{\nabla x} \cdot x.$$
(4)

2.3 Training the bias classifier

In order to increase the robustness of the network, we will use the adversarial training introduced in [21], which is one of the best practical training method to defend adversaries. Let (x, y) be a data in the training set S. Then the adversarial training is to solve

$$\min_{\Theta} \max_{||\zeta|| < \varepsilon} \sum_{(x,y) \in S} L_{CE}(\mathcal{F}(x+\zeta), y)$$
(5)

where $\varepsilon \in \mathbb{R}_{>0}$ is a given small real number. In order to increase the power of the bias part $B_{\mathcal{F}}$, we use the following training method

$$\min_{\Theta} \max_{||\zeta|| < \varepsilon} \sum_{(x,y) \in \mathcal{S}} [L_{CE}(B_{\mathcal{F}}(x+\zeta), y) + \gamma L_{CE}(\mathcal{F}(x+\zeta), y)]$$
(6)

where γ is a super parameter. The training procedure is given in Algorithm 1.

We first use a simple example to show that the adversarial training can increase the classification power of $B_{\mathcal{F}}$. The accuracies of \mathcal{F} , $W_{\mathcal{F}}$, $B_{\mathcal{F}}$ on the test set for three kinds of training methods are given in Table 1, respectively. More comprehensive numerical experiments are given in section 5.

	$W_{\mathcal{F}}$	$B_{\mathcal{F}}$	\mathcal{F}
Normal training (2)	98.80%	15.62%	99.09%
Adversarial training (5)	90.61%	98.77%	99.19%
Adversarial training (6)	0.28%	99.09~%	99.43%

Table 1: Accuracies of network Lenet-5 for MNIST

Algorithm 1 BCTrain

Require:

The set of training data: $S = \{(x_i, y_i)\}$; The initial value of the parameter set Θ : Θ_0 ; The super parameter: M_s, M_b, M_n . **Ensure:** The trained parameters $\widetilde{\Theta}$. In each iteration: Input Θ_k Let $L(x, y, \Theta) = L_{CE}(B_{\mathcal{F}_{\Theta}}(x), y)$. Let $L_1(x, y, \Theta) = L_{CE}(\mathcal{F}_{\Theta}(x), y)$. For $(x, y) \in S$, do $i=0, x_0 = x$ While $i < M_s$: $x_{i+1} = x_i - M_b \frac{\partial L(x_i, y, \Theta_k)}{\partial x_i}$ i = i + 1 $x = x_{i+1}$ Let $L(\Theta_k) = \frac{1}{|S|} \sum_{(x,y) \in S} L(x, y, \Theta_k)$. Let $L_1(\Theta_k) = \frac{1}{|S|} \sum_{(x,y) \in S} L_1(x, y, \Theta_k)$. Let $\nabla L = \frac{\partial (L(\Theta_k) + M_n L_1(\Theta_k))}{\partial \Theta_k}$. Output $\Theta_{k+1} = \Theta_k + \gamma_k \nabla L$; γ_k is the stepsize at iteration k.

3 Attack methods for the bias classifier

In this section, several possible methods to attack the bias classifier are given.

3.1 Safety against gradient-based attack

The most popular methods to generate adversaries, such as FGSM [12] or PGD [21], use $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ to make the loss function bigger. More precisely, adversaries are generated as follows

$$x \to x + \varepsilon \operatorname{sign}(\frac{\nabla L_{CE}(\mathcal{F}(x), y)}{\nabla x})$$
 (7)

for a small parameter $\varepsilon \in \mathbb{R}_{>0}$. It is easy to see that, $\frac{\nabla L_{CE}(\mathcal{F}(x),y)}{\nabla x}$ can be obtained from $\frac{\nabla \mathcal{F}(x)}{\nabla x}$. So, in the above attack, only the values of $\mathcal{F}(x)$ and $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ are needed and the detailed structure of \mathcal{F}

is not needed. Motivated by this fact, we introduce the concept of gradient-based attack. A DNN model is called a *gradient-based model*, if for $x \in \mathbb{I}^n$, the values of $\mathcal{F}(x)$ and $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ are known, but the detailed structure of \mathcal{F} is not known. Correspondingly, an attack only uses the values of $\mathcal{F}(x)$ and $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ is called *gradient-based attack*.

Since the derivative of $B_{\mathcal{F}}$ is always zero, a gradient-based attack against $B_{\mathcal{F}}$ becomes a blackbox attack, and in this sense we say that the bias classifier is safe against the gradient-based attack.

In the gradient-based model, we do not know the structure of \mathcal{F} , but we can calculate $B_{\mathcal{F}}(x)$ from $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ using (4), and the bias classifier still works.

3.2 Original-model attack

An obvious attack for the bias classifier is to create adversaries of $B_{\mathcal{F}}$ using the gradients of \mathcal{F} , which is called *original-model attack*. The attack is given in Algorithm 2, where $\widehat{B}_{\mathcal{F}}(x)$ is the label of $B_{\mathcal{F}}(x)$.

Algorithm 2 OAttack

Require: The value of the parameter set Θ of \mathcal{F} ; The super parameters: $\epsilon \in \mathbb{R}, N \in \mathbb{N}$; A sample x_0 and its label y_0 . **Ensure:** An adversarial sample x_a . $x = x_0$ For $i = 1, \dots, N$: If $\hat{B}_{\mathcal{F}}(x) \neq y_0$: Break. $x = x + \epsilon \operatorname{sign}(\frac{\nabla L_{CE}(\mathcal{F}(x), y)}{\nabla x})$ If $\hat{B}_{\mathcal{F}}(x) \neq y_0, x_a = x$ output: x_a Output: No adversary for x_0

3.3 Correlation attack on the bias classifier

From numerical experiments, we have the following observations. For a network $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ trained with (6) and a small vector $\epsilon \in \mathbb{R}^n$, the following fact happens with high probability: $W_{\mathcal{F}}(x)[l] \ge W_{\mathcal{F}}(x')[l]$ is valid if and only if $B_{\mathcal{F}}(x)[l] \le B_{\mathcal{F}}(x')[l]$ is valid, where $l \in \mathbb{L}$ and $x' = x + \epsilon$. In other words, $W_{\mathcal{F}}$ and $B_{\mathcal{F}}$ are co-related and we thus can decrease $B_{\mathcal{F}}[l]$ by increasing $W_{\mathcal{F}}[l]$, which is called the *correlation attack*.

In the correlation attack, we create adversaries by making $W_{\mathcal{F}}(x)[y] - W_{\mathcal{F}}(x)[i]$ bigger, where y is the label of $x, i \in [m]$ and $i \neq y$. The attack is given in Algorithm 3.

Algorithm 3 CAttack

Require: The value of the parameter set Θ of \mathcal{F} ; The super parameters: $\epsilon \in \mathbb{R}$, $N \in \mathbb{N}$; A samples x_0 and its label y. **Ensure:** An adversarial sample x_a . For $i \in \mathbb{L}$ and $i \neq y$: $x = x_0$, j=0While j < N: $U_a = \frac{\nabla \mathcal{F}_y(x)}{\nabla x} - \frac{\nabla \mathcal{F}_i(x)}{\nabla x}$, $x = x + \epsilon U_a$ If $\hat{B}_{\mathcal{F}}(x) \neq y$, break; else: j=j+1If $\hat{B}_{\mathcal{F}}(x) \neq y$, $x_a = x$ output: x_a Output: No adversary for x_0

4 Information-theoretically safety against original-model gradientbased attack

By the original-model gradient-based attack, we mean using the gradient of \mathcal{F} to generate adversaries for $B_{\mathcal{F}}$. In this section, we show that it is possible to make the bias classifier safe against this kind of attack. The idea is to make $\frac{\nabla \mathcal{F}(x)}{\nabla x}$ random and $B_{\mathcal{F}}$ still gives the correct classification.

4.1 Information-theoretically safety

In this section, we will define the concept of information-theoretically safety of a DNN against an attack.

Let \mathcal{F} be a DNN defined in (1). Motivated by the FGSM attack (7), we assume that the attack $\mathcal{A}(x, \mathcal{F}, \rho) : \mathbb{R}^n \to \mathbb{R}^n$ generates an adversary of x as below:

$$\mathcal{A}(x, \mathcal{F}, \rho) = x + \rho V \tag{8}$$

where $\rho \in \mathbb{R}_{>0}$ and $V \in \{-1, 1\}^n$ is the sign vector of certain quantity related with the gradient of $\mathcal{F}(x)$.

The attack $\mathcal{A}(x, \mathcal{F}, \rho)$ is called *information-theoretically safe*, if $V = (\mathcal{A}(x, \mathcal{F}, \rho) - x)/\rho$ is a random vector in $\{-1, 1\}^n$ for any input x.

We now show how to build an information-theoretically safe bias classifier. First train a DNN $\mathcal{F} : \mathbb{I}^n \to \mathbb{R}^m$ with the method in Section 2.3. Let $W_R \in \mathbb{R}^{m \times n}$ satisfy a given distribution \mathcal{M} of random matrices in $\mathbb{R}^{m,n}$ and

$$\widetilde{\mathcal{F}}(x) = \mathcal{F}(x) + W_R x = (W_x + W_R)x + B_x$$

$$B_{\widetilde{\mathcal{F}}}(x) = \widetilde{\mathcal{F}}(x) - \frac{\nabla \widetilde{\mathcal{F}}(x)}{\nabla x} \cdot x.$$
(9)

It is easy to see that $B_{\widetilde{\mathcal{F}}} = B_{\mathcal{F}}$, that is, the bias classifiers for \mathcal{F} and $\widetilde{\mathcal{F}}$ are the same. On the other hand, $\frac{\nabla \widetilde{\mathcal{F}}(x)}{\nabla x} = \frac{\nabla \mathcal{F}(x)}{\nabla x} + W_R$ is random in certain sense.

The safety of $B_{\widetilde{\mathcal{F}}}$ against the attack $\mathcal{A}(x, \widetilde{\mathcal{F}}, \rho)$ can be measured by the following adversary creation rate

$$\mathcal{C}(B_{\widetilde{\mathcal{F}}},\mathcal{A},\mathcal{M}) = \mathbb{E}_{W_R \sim \mathcal{M}}[\mathbb{E}_{x \sim \mathcal{D}_0}[\mathbf{I}(\widehat{B}_{\mathcal{F}}(\mathcal{A}(x,\widetilde{\mathcal{F}},\rho)) \neq \widehat{B}_{\mathcal{F}}(x))]]$$
(10)

where $\widehat{B}_{\mathcal{F}}$ is the label of the classification and $\mathcal{D}_{\mathbb{O}}$ is the distribution of the objects to be classified.

If $B_{\widetilde{\mathcal{F}}}$ is information-theoretically safe against the attack $\mathcal{A}(x, \widetilde{\mathcal{F}}, \rho)$, then it is easy to show that $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}, \mathcal{M})$ equals

$$\mathcal{C}(\mathcal{F},\rho) = \frac{1}{2^n} \mathbb{E}_{x \sim \mathcal{D}_{\mathbb{O}}} \sum_{V \in \{-1,1\}^n} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x+\rho V) \neq \widehat{B}_{\mathcal{F}}(x))]$$
(11)

which depends only on \mathcal{F} and ρ and will be used as a measure of the robustness of the bias classifier.

Note that $\mathcal{C}(\mathcal{F}, \rho)$ is the rate of adversaries in certain random samples. In other words, if $B_{\tilde{\mathcal{F}}}$ is information-theoretically safe under attack \mathcal{A} , then the adversary creation rate of $B_{\tilde{\mathcal{F}}}$ under attack \mathcal{A} is equal to the rate of random samples to be adversaries, which is very small as shown in section 5.3.1.

If $B_{\widetilde{\mathcal{F}}}$ is not information-theoretically safe, we can use the value $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}, \mathcal{M})/\mathcal{C}(\mathcal{F}, \rho)$ to measure the safety of $B_{\widetilde{\mathcal{F}}}$ relative to the information-theoretically safety.

4.2 Safety against direct attack

In this section, we show that $B_{\widetilde{\mathcal{F}}}$ defined in (9) is safe against the direct attack [40] of $\widetilde{\mathcal{F}}$.

Let $\mathcal{U}(a,b)$ be the uniform distribution in $[a,b] \subset \mathbb{R}$. For $\lambda \in \mathbb{R}_{>0}$, denote $\mathcal{M}_{m,n}(\lambda)$ to be the random matrices such that the elements of their *i*-row are in $(\mathcal{U}(-2i\lambda, -(2i-1)\lambda) \cup \mathcal{U}((2i-1)\lambda, 2i\lambda))^{m \times n}$.

Let $||x||_{-\infty} = \min_{i \in [n]} \{|x_i|\}$ for $x \in \mathbb{R}^n$. It is easy to see that for $W_R \sim \mathcal{M}_{m,n}(\lambda)$, we have $||W_{R,i} - W_{R,j}||_{-\infty} > \lambda$ for $i \neq j$, where $W_{R,i}$ is the *i*-th row of W_R .

For $\rho \in \mathbb{R}_{>0}$, consider the following gradient-based *direct attack* [40] for the network \mathcal{F} :

$$\mathcal{A}_1(x, \mathcal{F}, \rho) = x + \rho \operatorname{sign}\left(\frac{\nabla \mathcal{F}_{n_x}(x)}{\nabla x} - \frac{\nabla \mathcal{F}_y(x)}{\nabla x}\right)$$
(12)

where y is the label of x and $n_x = \arg \max_{i \neq y} \{ \mathcal{F}_i(x) \}.$

Theorem 4.1. Let $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \lambda/2$ and $W_R \in \mathcal{M}_{m,n}(\lambda)$. If the structure and parameters of \mathcal{F} are kept secret, then $B_{\widetilde{\mathcal{F}}}$ is information-theoretically safe against the attack $\mathcal{A}_1(x, \widetilde{\mathcal{F}}, \rho)$.

Proof. From (3) and (9), $\frac{\nabla \tilde{\mathcal{F}}(x)}{\nabla x} = W_x + W_R$. Let $W_{R,i}$ and $W_{x,i}$ be the *i*-rows of W_R and W_x , respectively. If $W_R \sim \mathcal{M}_{m,n}(\lambda)$, then $||W_{R,i} - W_{R,j}||_{-\infty} > \lambda$ for $i \neq j$. Since $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} = |W_x|_{\infty} < \lambda/2$, we have $||W_{x,i} - W_{x,j}||_{\infty} < \lambda$ for $i \neq j$. Then,

$$\begin{aligned}
\mathcal{A}_{1}(x, \widetilde{\mathcal{F}}, \rho) \\
&= x + \rho \operatorname{sign}\left(\frac{\nabla \widetilde{\mathcal{F}}_{n_{x}}(x)}{\nabla x} - \frac{\nabla \widetilde{\mathcal{F}}_{y}(x)}{\nabla x}\right) \\
&= x + \rho \operatorname{sign}(W_{x,n_{x}} - W_{x,y} + W_{R,n_{x}} - W_{R,y}) \\
&= x + \rho \operatorname{sign}(W_{R,n_{x}} - W_{R,y}).
\end{aligned} \tag{13}$$

Since $W_R \in \mathcal{M}_{m,n}(\lambda)$, $\widehat{W} = W_{R,n_x} - W_{R,y}$ is a random vector whose entries having values in two intervals of the form $[-b_2, -b_1] \cup [b_1, b_2]$, $\operatorname{sign}(\widehat{W})$ is a random vector in $\{-1, 1\}^n$ and the theorem is proved.

4.3 Safety against FGSM attack

In this section, we show that the result in section 4.2 holds for the FGSM attack if m = 2. Here is the FGSM attack:

$$\mathcal{A}_2(x, \mathcal{F}, \rho) = x + \rho \operatorname{sign}(\frac{\nabla L(\mathcal{F}(x), y)}{\nabla x}).$$
(14)

Theorem 4.2. If $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \lambda/2$, $W_R \sim \mathcal{M}_{m,n}(\lambda)$, and m = 2, then $B_{\widetilde{\mathcal{F}}}$ is informationtheoretically safe against the attack $\mathcal{A}_2(x, \widetilde{\mathcal{F}}, \rho)$.

Proof. Let $y \in \{0, 1\}$ be the label of x. Use the notations introduced in the proof of Theorem 4.1. Since the loss function is L_{CE} and m = 2, we have

$$=\frac{\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x}}{\sum_{i=1}^{m} e^{\tilde{\mathcal{F}}_{i}}(W_{x,i}-W_{x,y}+W_{R,i}-W_{R,y})}}{\sum_{i=1}^{m} e^{\tilde{\mathcal{F}}_{i}(x)}}$$

$$=\frac{e^{\tilde{\mathcal{F}}_{1-y}(x)}}{\sum_{i=1}^{m} e^{\tilde{\mathcal{F}}_{i}(x)}}(W_{x,1-y}-W_{x,y}+W_{R,1-y}-W_{R,y}).$$
(15)

The last equality comes from m = 2. Since $||W_{x,i} - W_{x,j}||_{\infty} < \lambda$ and $||W_{R,i} - W_{R,j}||_{-\infty} > \lambda$ for $i \neq j$, we have $\operatorname{sign}(\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x}) = \operatorname{sign}(W_{R,1-y} - W_{R,y})$ which is a random vector in $\{-1,1\}^n$, similar to the proof of Theorem 4.1. The theorem is proved.

When m > 2, we have the following result, whose proof is given in Appendix B.

Theorem 4.3. Assume $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$, $|B_{\mathcal{F}}(x)|_{\infty} < \beta$, and $\lambda \in \mathbb{R}_{>0}$ satisfying $(\lambda - \mu)e^{-2\beta - n\mu + \sqrt{\lambda}} > (2m\lambda + \mu)m$. Furthermore, assume the samples are normalized, that is, $|x|_{\infty} = 1$. If $W_R \sim \mathcal{M}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{M}_{m,n}(\lambda)) \leq (m-1)\mathcal{C}(\mathcal{F}, \rho) + \frac{(m-2)^2}{\sqrt{\lambda}}$.

We can choose a large λ to make the term $\frac{(m-2)^2}{2\sqrt{\lambda}\eta}$ small. So from Theorem 4.3, $B_{\tilde{\mathcal{F}}}$ is approximately safe if m is small.

4.4 Safety against direct attack under simpler distribution

Let $\mathcal{U}_{m,n}(\lambda)$ be the random matrices whose entries are in $\mathcal{U}(-\lambda,\lambda)$. In this section, we show that the result in section 4.2 is approximately valid for the simpler distribution $\mathcal{U}_{m,n}(\lambda)$. We consider the k-step direct attack:

$$\begin{aligned}
x^{(0)} &= x \\
x^{(i)} &= x^{(i-1)} + \frac{\rho}{k} \operatorname{sign}(\frac{\nabla \mathcal{F}_{n_x}(x^{(i-1)})}{\nabla x^{(i-1)}} - \frac{\nabla \mathcal{F}_y(x^{(i-1)})}{\nabla x^{(i-1)}}), i \in [k] \\
\mathcal{A}_3(x, \mathcal{F}, \rho) &= x^{(k)}
\end{aligned} \tag{16}$$

where y is the label of x and $n_x = \arg \max_{i \neq y} \{ \mathcal{F}_i(x) \}.$

Theorem 4.4. If $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$ and $W_R \sim \mathcal{U}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq \mathcal{C}(\mathcal{F}, \rho) + \mu n/\lambda$. Furthermore, if $\lambda > n\mu/\epsilon$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq \mathcal{C}(\mathcal{F}, \rho) + \epsilon$ for any $\epsilon \in \mathbb{R}_{>0}$, and in particular, if $\lambda > na/(\epsilon \mathcal{C}(\mathcal{F}, \rho))$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq (1 + \epsilon)\mathcal{C}(\mathcal{F}, \rho)$.

Proof of Theorem 4.4 is given in Appendix C. Theorem 4.4 implies that $B_{\widetilde{\mathcal{F}}}$ can be made close to information-theoretically safe under attack $\mathcal{A}_3(x, \widetilde{\mathcal{F}}, \rho)$.

4.5 Safety against FGSM under simpler distribution

In this section, we show that the result in section 4.3 is approximately valid for the simpler distribution $\mathcal{U}_{m,n}(\lambda)$. Let \mathcal{A}_2 be the attack in (14). Then we have

Theorem 4.5. If $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$, $W_R \sim \mathcal{U}_{m,n}(\lambda)$, and m = 2, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq e^{n\mu/\lambda}\mathcal{C}(\mathcal{F}, \rho)$. Furthermore, if $\lambda > n\mu/\ln(1+\epsilon)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq (1+\epsilon)\mathcal{C}(\mathcal{F}, \rho)$.

For the general m, we have

Theorem 4.6. Assume $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/4$, $|B_{\mathcal{F}}(x)|_{\infty} < \beta$, and $\lambda \in \mathbb{R}_{>0}$ satisfying $\mu e^{-2\beta - n\mu/2 + \sqrt{\lambda}} > 2(2\lambda + \mu)m$. Furthermore, assume the samples are normalized, that is, $|x|_{\infty} = 1$. If $W_R \sim \mathcal{U}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq (m-1)\mathcal{C}(\mathcal{F}, \rho) + \frac{(m-1)n\mu}{\lambda} + \frac{(m-2)^2}{\sqrt{\lambda}}$.

Proofs of Theorems 4.5 and 4.6 are given in Appendixes D and E, respectively. Theorem 4.5 shows that, for binary classifications, $B_{\tilde{\mathcal{F}}}$ is close to information-theoretically safe against FGSM under distribution $\mathcal{U}_{m,n}(\lambda)$. Theorem 4.6 shows that the result is approximately valid in the general case under certain conditions.

5 Experiments

5.1 Accuracy of the bias classifier

In this section, we give the accuracy of the bias classifier using the MNIST and CIFAR-10 data sets. We compare two DNN models:

$$\mathcal{F}^{(1)}: \text{ trained with adversarial trianing (5)}$$

$$\mathcal{F}^{(2)}: \text{ trained with Algorithm 1}$$
(17)

whose detailed structure can be found in Appendix F.

We give the accuracy on the test set (TS) and the strong adversaries (SA, see [37]) and the results are given in Table 2.

From the table, we can see that the bias classifier has comparable accuracies with $\mathcal{F}^{(1)}$ on the test set, but achieves significant higher accuracies than $\mathcal{F}^{(1)}$ for the strong adversaries, which implies that the bias classifier is more robust against adversaries than DNNs of similar size and trained with adversarial training.

DNN	TS/MNIST	SA/MNIST	TS/CIFAT-10	SA/CIFAT-10
$\mathcal{F}^{(1)}$	99.19%	51.5%	81.23%	19%
$B_{\mathcal{F}^{(2)}}$	99.12%	87.5%	82.84%	42%

Table 2: Accuracies for MNIST and CI	IFAR-10
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Moreover, for CIFAR-10, we compare the accuracy of our network and two other networks ResNet18 and VGG19, all using adversarial training. From the results in Table 3, our network $\mathcal{F}^{(1)}$ performs better than ResNet18 and VGG 19.

DNN	Test Set	Strong Adversaries
$\mathcal{F}^{(1)}$	81.23%	19%
ResNet18	80.64%	9%
VGG19	78.92%	12%

Table 3: Accuracies for three networks on CIFAR-10.

As pointed out in [39], networks trained with adversarial training usually have lower accuracies, and the accuracies given in Tables 2 and 3 are about the best ones for DNNs of similar sizes, which implies that the models $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are appropriate for MNIST and CIFAR-10.

5.2 Robustness of the bias classifier against original-model attack

In this section, we check the robustness of the bias classifier against the original-model attack given in Algorithm 2.

5.2.1 Experimental results

We use two more networks: $\mathcal{F}^{(3)}$ has the same structure with $\mathcal{F}^{(1)}$ given in (17), but trained with the first-order regulation method [26], and $\mathcal{F}^{(4)}$ has the same structure with $\mathcal{F}^{(1)}$, but trained with TRADES [39]. Six kinds of adversaries are used:

 l_{∞} adversaries: 1-*i* (*i* = 1, 2, 3). Each pixel of the sample changes at most 0.*i*. PGD [21] is used to attack: each step changes 0.01 and moves 10*i* steps.

 l_0 adversaries: 2-*i* (*i* = 40, 60, 80). Change at most *i* pixels of the sample. JSMA [22] is used to attack: change *i* pixels and each pixel can change up to 1.

The adversary creation rates are given in Tables 4 and 5. The results in the last two rows are obtained with the original-model attack.

DNN	1-1	1-2	1-3	2-40	2-60	2-80
$\mathcal{F}^{(1)}$	3%	17%	55%	55%	79%	87%
$\mathcal{F}^{(2)}$	4%	22%	77%	62%	82%	90%
$\mathcal{F}^{(3)}$	22%	78%	99%	75%	98%	99%
$\mathcal{F}^{(4)}$	4%	15%	53%	62%	77%	88%
$B_{\mathcal{F}^{(1)}}$	3%	14%	49%	48%	67%	92%
$B_{\mathcal{F}^{(2)}}$	2%	6%	22%	41%	56%	79%

Table 4: Creation rates of adversaries for MNIST

DNN	1-1	1-2	1-3	2-40	2-60	2-80
$\mathcal{F}^{(1)}$	54%	77%	90%	72%	85%	96%
$\mathcal{F}^{(2)}$	54%	72%	85%	69%	88%	97%
$\mathcal{F}^{(3)}$	88%	92%	99%	89%	99%	99%
$\mathcal{F}^{(4)}$	49%	73%	85%	70%	89%	97%
$B_{\mathcal{F}^{(1)}}$	67%	70%	86%	70%	84%	91%
$B_{\mathcal{F}^{(2)}}$	41%	58%	77%	49%	73%	84%

Table 5: Creation rates of adversaries for CIFAR-10

From the tables, we can see that the bias classifiers $B_{\mathcal{F}^{(2)}}$ has significant lower adversary creation rates than all other networks. For l_{∞} adversaries of MNIST, $B_{\mathcal{F}^{(2)}}$ achieves near optimal results and the adversaries almost disappear. For CIFAR-10, the adversary creation rates are still quite high comparing to that of MNIST. We will explain the reason in Section 5.2.2.

Also, $B_{\mathcal{F}^{(2)}}$ achieves much better results than $B_{\mathcal{F}^{(1)}}$, which implies that our training method (6) is better than the usual adversarial training (5).

5.2.2 Influence of adversarial training on the bias classifier

In this section, we give an intuitive explanation for the results in Tables 4 and 5. Let x be a sample and y its label. We use PGD [21] to create adversaries and show how $B_{\mathcal{F}_y}(x)$ and $\mathcal{F}_y(x)$ change along with the steps of the adversarial training to explain the results in Tables 4 and 5.

In Figure 1, we give the data of using the network Lenet-5 [18] for a sample x with label y in MNIST. The blue, orange, green lines in the first picture are Softmax $\mathcal{F}_y(x)$, Softmax $\mathcal{B}_{\mathcal{F}_y}(x)$, Softmax $\mathcal{W}_{\mathcal{F}_y}(x)$, respectively. The blue, orange, green lines in the second picture are $\mathcal{F}_y(x)$, $\mathcal{B}_{\mathcal{F}_y}(x)$, $\mathcal{W}_{\mathcal{F}_y}(x)$, respectively.

When the blue line decreases, we obtain an adversary for \mathcal{F} , which is not an adversary of $B_{\mathcal{F}}$, because the orange line does not reduce significantly. For most samples from MNIST, the pictures are almost like this one, and this explains why the values in lines 5-6 of Table 4 are low.



Figure 1: Values of $\mathcal{F}_y(x)$, $B_{\mathcal{F}_y}(x)$ and $W_{\mathcal{F}_y}(x)$ along with the adversarial training steps

Similar results are given in Figures 2 for CIFAR-10 and network VGG-19 [27]. In this case, the orange and the blue lines both decrease, and the adversary of \mathcal{F} is also an adversary of $B_{\mathcal{F}}$. This explains why the values in lines 5-6 of Table 5 are higher than that of Table 4.



Figure 2: Values of $\mathcal{F}_y(x)$, $B_{\mathcal{F}_y}(x)$ and $W_{\mathcal{F}_y}(x)$ along with the adversarial training steps

5.3 Safety against original-model gradient-based attack

In this section, we use experimental results to validate the results in Section 4.

5.3.1 Rates of adversaries in random samples

In this section, we give the rates of a random point near a sample to be an adversary. Two ways to select random points near a sample x are used:

R₁: Randomly change 60 pixels of x from b to 1 - b.

 R_2 : Add a random number in [-0.2, 0.2] to each pixel of x.

Two networks are used:

 \mathcal{N}_1 : Lenet-5 for MNIST and VGG-19 for CIFAR-10, with normal training (2).

 \mathcal{N}_2 : Lenet-5 for MNIST and VGG-19 for CIFAR-10, with adversarial training (5).

In Table 6, we give the average rates of adversaries. From the table, we can see that the rates for random samples to be adversaries are quite low for networks \mathcal{N}_2 and $\mathcal{B}_{\mathcal{F}^{(2)}}$ trained with the adversarial training.

DNN	$R_1/MNIST$	$R_2/MNIST$	$R_1/CIFAR-10$	$R_2/CIFAR-10$
\mathcal{N}_1	0.77%	1.47%	13.31%	11.81%
\mathcal{N}_2	1.00%	1.02%	4.69%	2.49%
$B_{\mathcal{F}^{(2)}}$	0.85%	1.64%	4.28%	1.67%

Table 6: Rates for random samples to be adversaries

In Table 7, we give the values of $\mathcal{C}(\mathcal{F}, \rho)$ defined in (11) for two values of ρ . We can see that $\mathcal{C}(\mathcal{F}, \rho)$ is a little bit smaller than the values in Table 6, as expected.

DNN	$\rho = 0.1/M$	$\rho = 0.2/M$	$\rho = 0.1/C$	$\rho = 0.2/C$
\mathcal{N}_1	1.00%	1.77%	5.26%	9.92%
\mathcal{N}_2	0.88%	1.01%	1.84%	2.04%
$B_{\mathcal{F}^{(2)}}$	0.72%	0.97%	1.59%	1.71%

Table 7: $\mathcal{C}(\mathcal{F}, \rho)$. M means MNIST, C means CIFAR-10.

5.3.2 Safety of the bias classifier

For MNIST, let $\mathcal{F}^{(5)} = \mathcal{F}^{(2)} + W_5 x$, where $\mathcal{F}^{(2)}$ is given in (17) and $W_5 \in \mathbb{R}^{10 \times 784}$ is from $\mathcal{U}_{10,784}(\lambda)$ for $\lambda = 100$.

For CIFAR-10, let $\mathcal{F}^{(6)} = \mathcal{F}^{(2)} + W_6 x$, where $\mathcal{F}^{(2)}$ is in (17) and $W_5 \in \mathbb{R}^{10 \times 3072}$ is from $\mathcal{U}_{10,3072}(\lambda)$ for $\lambda = 100$.

The adversary creation rates are given in Table 8, where the adversaries are introduced in Section 5.2.

DNN	1-1	1-2	1-3	2-40	2-60	2-80
$B_{\mathcal{F}^{(5)}}$ for MNIST	1%	2%	2%	2%	3%	4%
$B_{\mathcal{F}^{(6)}}$ for CIFAR-10	19%	20%	22%	21%	22%	24%

Table 8: Original-model gradient-based attack

From Table 8, the bias classifier is safe against the original-model gradient-based attack for MNIST, and the adversarial creation rates in Table 8 are close to those in Table 6.

From Table 8, the results are also near optimal for CIFAR-10. First, comparing to the results in Table 5, the adversary creation rates are decreased by half and are about 20%. Second, from Table 3, the accuracy of the bias classifier is about 82%. Comparing these data, the real adversary creation rates are about 1% - 6% which are just above the rates of random samples to be adversaries in Table 6.

5.4 Black-box attack on the bias classifier

In this section, we use the transfer-based black-box attack [30] to compare four networks: $\mathcal{F}^{(1)}$, $\mathcal{F}^{(3)}$, $\mathcal{F}^{(4)}$, $B_{\mathcal{F}^{(2)}}$ defined in Sections 5.1 and 5.2.

The black-box attack for \mathcal{F} works as follows. A new network $\overline{\mathcal{F}}$ is trained with the training set $\{(x, \mathcal{F}(x))\}$ for certain samples x. Then, we use PGD and JSMA to create adversaries for $\overline{\mathcal{F}}$ and check wether they are adversaries of \mathcal{F} . The adversary creation rates are given in Table 9. We can see that, the bias classifier performs better for most adversaries and in particular for l_{∞} adversaries. Also, the adversary creation rates are about half of that of the original-model attack in Tables 4 and 5. So the bias classifier has better robustness for the black-box attack in most cases.

DNN	1-1	1-2	1-3	2-40	2-60	2-80
$\mathcal{F}^{(1)}$	1%	2%	18%	28%	35%	40%
$\mathcal{F}^{(3)}$	6%	12%	28%	38%	45%	50%
$\mathcal{F}^{(4)}$	1%	3%	21%	24%	39%	46%
$B_{\mathcal{F}^{(2)}}$	3%	5%	13%	24%	30%	37%

Table 9: Black-box attack of MNIST

DNN	1-0.1	1-0.2	1-0.3	2-40	2-60	2-80
$\mathcal{F}^{(1)}$	22%	23%	28%	35%	36%	41%
$\mathcal{F}^{(3)}$	27%	29%	36%	40%	43%	50%
$\mathcal{F}^{(4)}$	21%	24%	28%	33%	39%	44%
$B_{\mathcal{F}^{(2)}}$	21%	23%	24%	33%	36%	41%

Table 10: Black-box attack of CIFAR-10

5.5 Correlation attack

In this section, it is shown that the bias classifier is safe against the correlation attack proposed in Section 3.3. The network used here is $\mathcal{F}^{(2)}$ given in (17) and the data set is CIFAR-10. In Table 11, we give the adversary creation rates for samples which are given the correct label by $B_{\mathcal{F}^{(2)}}$. Comparing to results in Tables 5 and 10, we can see that the bias classifier is quite safe against the correlation arrack.

Network	1-0.1	1-0.2	1-0.3	2-40	2-60	2-80
$B_{\mathcal{F}^{(2)}}$	4%	11%	12%	8%	17%	21%

Table 11: Adversary creation rates for the correlation attack

In Figure 3, we give the attack procedure. It can be seen that when $(W_{x,y} - W_{x,i})x$ increases $B_{x,y} - B_{x,i}$ indeed decreases, but $B_{x,y} - B_{x,i}$ does not decrease enough to change the label, where y is the label of x and $i \neq y$.



Figure 3: The input is an image for 9 from MNIST. The x-axis is the number of steps of the attack. The blue line is $(W_{x,9} - W_{x,0})x$ and the orange line $B_{x,9} - B_{x,0}$.

5.6 Comparison with other methods

In this section, we compare our model with several existing models to defend adversaries. We use PGD-20 with l_{∞} bound $\epsilon = 8/255$ to create adversaries on the test set of CIFAR-10.

In Table 12, we give the adversary creation rates for various attacks. Our models are $\mathcal{F}^{(2)}$ in (17) and $\mathcal{F}^{(6)}$ in Section 5.3.2. ResNet-10 [14] is used in other cases. The results for other networks are from the cited papers. Gradient-based attacks cannot be used for the bias classifier, so we use the original-model attack given in Section 3.2.

Attack Method	Adv. creation rates
ADV [21]	57.1%
TRADE $[39]$	54.7%
MMA [31]	62.7%
FOAR [26]	67.7%
SOAR [20]	44.0%
$B_{\mathcal{F}^{(2)}}$ in Sec. 5.1	41.1%
$B_{\mathcal{F}^{(6)}}$ in Sec. 5.3.2	20%

Table 12: Adversary creation rates for CIFAR-10

Although the DNN models and the attacks are not the same, this comparison gives a rough idea of the performance that can be achieved for various methods of defending adversaries.

From Table 12, we see that the attack method SOAR [20] and the bias classifier $B_{\mathcal{F}_B^{(2)}}$ achieve the best results for creating lower rates of adversaries, besides $B_{\mathcal{F}_B^{(6)}}$. As explained in Section 5.3.2, the optimal adversary creation rate is about 20% and is achieved by $B_{\mathcal{F}_B^{(6)}}$.

5.7 Summary of the experiments

We give a summary of the experiments in this section.

From Tables 2 and 3, we can see that the bias classifier achieves comparable accuracies with DNNs of similar sizes.

From Table 6, we can see that the bias classifier with a random first-degree part is safe against gradient-based attacks, as proved in Section 4.

From Tables 2, 4, 5, 9, 10, and 12, we can see that the bias classifier is more robust than DNNs with similar sizes against adversarial attacks .

From Tables 5, 10, 11, the original-model attack, the black-box attack, and the correlation attack become weaker for creating adversaries, and the original model attack is the best available attack for the bias classifier.

6 Concluding remarks

In this paper, we show that the bias part of a DNN can be effectively trained as a classifier. The motivation to use the bias part as the classifier is that gradients of the DNN seems to be inevitable to generate adversaries efficiently and the bias part of a DNN with Relu as activation functions is a piecewise constant function with zero gradient and is safe against direct gradient-based attacks such as FGSM.

The bias classifier can be effectively trained with the adversarial training method [21], which increases the classification power of the bias part and decreases the classification power of firstdegree part. Experimental results are used to show the robustness of the bias classifier over the standard DNNs

Further, by adding a random first-degree part to the bias classifier, an information-theoretically safe classifier against gradient-based attacks is obtained, that is, the adversary creation rate is almost the same as the rate of certain random samples to be adversaries.

For further research, the estimations in Theorems 4.3, 4.5, 4.6 are not optimal, and better estimations are desirable.

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Appendix

Appendix A. Proof of Theorem 2.1

Theorem 2.1. Let $\mathbb{O} = \bigcup_{i=1}^{m} O_i \subset \mathbb{I}^n$ be the elements to be classified and $\mathbb{L} = \{l\}_{l=1}^{m}$ the label set, where $O_i \subset \mathbb{I}^n$ is an open set, $O_i \cap O_j = \phi$ if $i \neq j$, and x has label l for $x \in O_l$. Then for any $\epsilon > 0$, there exist a DNN \mathcal{F} and an open set $D \subset \mathbb{I}^n$ with volume $V(D) < \epsilon$, such that $B_{\mathcal{F}}(x)$ gives the correct label for $x \in \mathbb{O} \setminus D$, that is, the l-th coordinate of $\mathcal{F}(x)$ has the biggest value for $x \in O_l$.

We first prove several lemmas. In this section, the notations $\mathbb{O}, O_l, \mathbb{L}$ introduced in Theorem 2.1 will be used.

Let $\Gamma : \mathbb{R} \to \mathbb{R}$ be another activation function:

$$\Gamma(x) = \begin{cases} 0 & \text{if } x \le 0\\ 1 & \text{if } x > 0 \end{cases}$$

Lemma 2.1 (Theorem 5 in [8]). Let $l \in \mathbb{L}$ and $F_l : \mathbb{R} \to \mathbb{R}$ be a function such that $F_l(x) = 1$ if $x \in O_l$ and $F_l(x) = -1$ otherwise. Then for any $\epsilon > 0$, there exist $N \in \mathbb{N}_{>0}$, $W \in \mathbb{R}^{N \times n}$, $b \in \mathbb{R}^N$, $U \in \mathbb{R}^{1 \times N}$, and an open set $D \subset \mathbb{I}^n$ with $V(D) < \epsilon$, such that

$$G(x) = U \cdot \Gamma(Wx + b) : \mathbb{I}^n \to \mathbb{R}$$

and $|G(x) - F_l(x)| < \epsilon$ for $x \in \mathbb{O} \setminus D$.

The following lemma shows that there exists a DNN with one hidden layer and using Γ as the activation function, which can be used as a classifier for \mathbb{O} .

Lemma 2.2. For any $\epsilon > 0$, there exist $N \in \mathbb{N}_{>0}$, $W \in \mathbb{R}^{N \times n}$, $b \in \mathbb{R}^N$, $U \in \mathbb{R}^{m \times N}$, and an open set $D \subset \mathbb{I}^n$ with $V(D) < \epsilon$, such that

$$\mathcal{G}(x) = U \cdot \Gamma(Wx + b) : \mathbb{I}^n \to \mathbb{R}^m$$

gives the correct label for $x \in \mathbb{O} \setminus D$.

Proof. By Lemma 2.1, for $l \in \mathbb{L} = [m]$, there exist $N_a \in \mathbb{N}_{>0}$, $W_l \in \mathbb{R}^{N_a \times n}$, $b_l \in \mathbb{R}^{N_a}$, $U_l \in \mathbb{R}^{1 \times N_a}$, and $D_l \subset \mathbb{I}^n$ with $V(D_l) < \epsilon/m$ such that

$$G_l(x) = U_l \cdot \Gamma(W_l x + b_l)$$
 and $|G_l(x) - F_l(x)| < \epsilon$

for $x \in \mathbb{O} \setminus D_l$, where F_l is defined in Lemma 2.1.

Let $N = N_a m$, $W \in \mathbb{R}^{N \times n}$, $b \in \mathbb{R}^N$, where the *l*-th row of W is the l_2 -th row of W_{l_1} and the *l*-th row of *b* is the l_2 -th row of b_{l_1} , where $l = l_1 N_a + l_2$, $0 \le l_2 < N_a$, and $0 \le l_1 < m$.

Let $U \in \mathbb{R}^{m \times N}$ be formed as follows: for $j \in [m]$, the *j*-th row of U are zeros except the $(j-1)N_a$ -th to the $((j-1)N_a + N_a - 1)$ -th rows, and the values of the $((j-1)N_a + k)$ -th place of the *j*-th row of U equal to the values of the *k*-th place of U_l , where $k = 0, 1, \ldots, N_a - 1$.

Now we have $N = N_a m \in \mathbb{N}_{>0}$, $W \in \mathbb{R}^{N \times n}$, $b \in \mathbb{R}^N$, $U \in \mathbb{R}^{m \times N}$, and

$$\mathcal{G}(x) = U \cdot \Gamma(Wx + b)$$

satisfies $\mathcal{G}(x)_l = G_l(x)$, where $\mathcal{G}(x)_l$ is the *l*-th coordinate of $\mathcal{G}(x)$. Let $D = \bigcup_{i=1}^m D_l \subset \mathbb{I}^n$ with $V(D) < \epsilon$. Then $\mathcal{G}(x)_y > 1 - \epsilon$ and $\mathcal{G}(x)_l < -1 + \epsilon$ for $x \in O_y$ and $l \neq y$. Since ϵ can be as small as possible, we have $\mathcal{G}(x)_y > \mathcal{G}(x)_l$ for $l \neq y$, and $\mathcal{G}(x)$ give label y for $x \in O_y \setminus D$. Hence, $\mathcal{G}(x)$ gives the correct label for $x \in \mathbb{O} \setminus D$.

Lemma 2.3. Let $W \in \mathbb{R}^{1 \times n}$ have nonzero entries and $b \in \mathbb{R}$. For any a > 0, let $Z_a = \{x \in \mathbb{I}^n \mid |Wx+b| < a\}$. Then $V(Z_a) \leq 2a\sqrt{n^{n-1}}/||W||_2$.

Proof. Let $U = \{U_1, \ldots, U_n\}$ be a unit orthogonal basis of \mathbb{R}^n and $U_1 = \frac{W}{||W||_2}$. If $T_i = \max_{x,y \in Z_a} \{\langle x - y, U_i \rangle\}$, then we have $V(Z_a) \leq \prod_{i=1}^n T_i$.

For i > 1, we have $T_i \leq \max_{x,y \in Z_a} ||x - y||_2 \leq \sqrt{n}$. Moreover, for any $x, y \in Z_a$, we have

$$\begin{array}{l} \langle x - y, U_1 \rangle \\ = \langle x - y, W \rangle / ||W||_2 \\ = (Wx + b - Wy - b) / ||W||_2 \\ \leq 2a / ||W||_2 \end{array}$$

which means $T_1 \leq 2a/||W||_2$. Then we have

$$V(Z_a) \le \prod_{i=1}^n T_i \le 2a\sqrt{n}^{n-1}/||W||_2.$$

The lemma is proved.

Lemma 2.4. The bias vector b in Lemma 2.2 can be chosen to consist of nonzero values.

Proof. By Lemma 2.2, there exist $N \in \mathbb{N}_{>0}$, $W \in \mathbb{R}^{N \times n}$, $b \in \mathbb{R}^N$, $U \in \mathbb{R}^{m \times N}$, and $D_1 \subset \mathbb{I}^n$ with $V(D_1) < \epsilon/2$, such that

$$\mathcal{G}(x) = U \cdot \Gamma(Wx + b)$$

gives the correct label for $x \in \mathbb{O} \setminus D_1$.

Let $\gamma = \frac{\epsilon W_m}{4N\sqrt{n^{n-1}}}$, where $W_m = ||W||_{2,\infty}$. Assume $\tilde{b} = b - I_0(|b|)\gamma$, where $I_0(x) = 1 - \operatorname{sign}(x)$ and when I_0 is treated as a map of a vector, it acts on each entry of the vector, respectively. From the construction, \tilde{b} does not have zero entries, because $\tilde{b}_i = b_i$ if $b_i \neq 0$, and $\tilde{b}_i = \gamma$ if $b_i = 0$, where b_i and \tilde{b}_i are respectively the *i*-th rows of *b* and \tilde{b} .

Let W_i be the *i*-th row of W and $Z_i = \{z \in \mathbb{R}^n | |W_i z + b_i| < \gamma\}$. By Lemma 2.3, we have $V(Z_i \cap \mathbb{I}^n) < 2\gamma \sqrt{n^{n-1}}/W_m$. We write $C_n = \sqrt{n^{n-1}}/W_m$.

Let $Z = \{x \in \mathbb{R}^n | \Gamma(Wx+b) \neq \Gamma(Wx+\widetilde{b})\}$. We will show that $Z = \bigcup_{i=1}^N Z_i$. If $\Gamma(Wx+b) \neq \Gamma(Wx+\widetilde{b})$, then there exists an $i \in [N]$ such that $W_ix + b_i > 0$ and $W_ix + \widetilde{b}_i < 0$, or $W_ix + b_i < 0$ and $W_ix + \widetilde{b}_i > 0$. If $W_ix + b_i > 0$ and $W_ix + \widetilde{b}_i < 0$, then $W_ix + \widetilde{b}_i = W_ix + b_i - I_0(|b|)\gamma < 0$ and hence $|W_ix + b_i| \leq \gamma$. Similarly, if $W_ix + b_i < 0$ and $W_ix + \widetilde{b}_i > 0$, we also have $|W_ix + b_i| \leq \gamma$, which implies $x \in Z_i$. As a consequence $Z = \bigcup_{i=1}^N Z_i$.

From $Z = \bigcup_{i=1}^{N} Z_i$, we have $V(Z \cap \mathbb{I}^n) < 2\gamma NC_n < \epsilon/2$, since $\gamma = \frac{\epsilon}{4NC_n}$. Let $D = D_1 \bigcup (Z \cap \mathbb{I}^n) \subset \mathbb{I}^n$. Then $V(D) < V(D_1) + V(Z \cap \mathbb{I}^n) < \epsilon$.

Finally, let

$$\widetilde{\mathcal{G}}(x) = U \cdot \Gamma(Wx + \widetilde{b}).$$

Then, for $x \in \mathbb{O} \setminus D$, we have $\Gamma(Wx + b) = \Gamma(Wx + \tilde{b})$ and hence $\widetilde{\mathcal{G}}(x) = \mathcal{G}(x)$. That is, $\widetilde{\mathcal{G}}$ satisfies the conditions of the lemma.

Lemma 2.5. Let $\mathcal{G} : \mathbb{I}^n \to \mathbb{R}^m$ be a one-hidden-layer DNN with activation function $\Gamma(x)$, and any coordinate of its bias vector is nonzero. Then there exists a DNN \mathcal{F} , which has the same structure as \mathcal{G} , except that the activation function of \mathcal{F} is Relu, such that $B_{\mathcal{F}}(x) = \mathcal{G}(x)$ for all $x \in \mathbb{I}^n$.

Proof. Assume $\mathcal{G}(x) = U \cdot \Gamma(Wx + b) + c$. Let $\mathcal{F}(x) = U^{\mathcal{F}} \operatorname{Relu}(Wx + b) + c$, where $U^{\mathcal{F}} = U \operatorname{diag}(\frac{1}{b_i})$ and b_i is the *i*-th entry of *b*. We will show that \mathcal{F} satisfies the condition of the lemma. By the definition of Γ , the constant part of $\operatorname{Relu}(Wx + b)$ is $b \circ \Gamma(Wx + b)$, where \circ is the point-wise product. So, $B_{\mathcal{F}}(x) = U^{\mathcal{F}}(b \circ \Gamma(Wx + b)) + c = U \operatorname{diag}(\frac{1}{b_i})(b \circ \Gamma(Wx + b)) + c = U \Gamma(WX + b) + c$. $B_{\mathcal{F}}(x) = \mathcal{G}(x)$ and the lemma is proved.

Proof of Theorem 2.1. By Lemma 2.2, there exist a $D \subset \mathbb{I}^n$ with $V(D) < \epsilon$ and a network \mathcal{G} with one-hidden-layer and with activation function $\Gamma(x)$, such that $\mathcal{G}(x)$ gives the correct label for $x \in \mathbb{O} \setminus D$. By Lemma 2.4, all the parameters of \mathcal{G} are nonzero. Then by Lemma 2.5, we can obtain a network \mathcal{F} with Relu as the activation function such that $B_{\mathcal{F}} = \mathcal{G}(x)$, and the theorem is proved.

Appendix B. Proof of Theorem 4.3

We first prove two lemmas.

Lemma 4.1. Let $\{u_i\}_{i=1}^n$ be a set of iid random variables with values in $[-\lambda, \lambda]$ and $u = \sum_{i=1}^n x_i u_i$, where $x_i \in \mathbb{R}$ such that $|x_i| > a > 0$ for some *i*. Let the density function of *u* be f(x). Then $f(x) < \frac{1}{2\lambda a}$ for all *x*.

Proof. Assume $|x_n| > a$ and $f_n(x)$ is the distribution function of $x_n u_n$. We have

$$P(u < m) = \int_{\{-\lambda |x_i|\}_{i=1}^{n-1}}^{\{\lambda |x_i|\}_{i=1}^{n-1}} (\prod_{i=1}^{n-1} \frac{1}{2\lambda |x_i|}) f_n(m - \sum_{i=1}^{n-1} t_i) dt_1 t_2 \dots t_{n-1}.$$

Since $0 < f'_n(x) \le \frac{1}{2\lambda |x_n|}$ and $f(x) = \frac{\nabla P(u < x)}{\nabla x}$, we have

$$\begin{aligned} &f(x) \\ &= \frac{\nabla P(u < x)}{\nabla x} \\ &= \frac{\nabla \int_{\{-\lambda | x_i |\}_{i=1}^{n-1}}^{\{\lambda | x_i |\}_{i=1}^{n-1}} (\Pi_{i=1}^{n-1} \frac{1}{2\lambda | x_i |}) f_n(x - \sum_{i=1}^{n-1} t_i) dt_1 t_2 \dots t_{n-1} \\ &= \int_{\{-\lambda | x_i |\}_{i=1}^{n-1}}^{\{\lambda | x_i |\}_{i=1}^{n-1}} (\Pi_{i=1}^{n-1} \frac{1}{2\lambda | x_i |}) \frac{\nabla f_n(x - \sum_{i=1}^{n-1} t_i)}{\nabla x} dt_1 t_2 \dots t_{n-1} \\ &\leq \int_{\{-\lambda | x_i |\}_{i=1}^{n-1}}^{\{\lambda | x_i |\}_{i=1}^{n-1}} (\Pi_{i=1}^{n-1} \frac{1}{2\lambda | x_i |}) \frac{1}{2\lambda | x_n |} dt_1 t_2 \dots t_{n-1} \\ &\leq \frac{1}{2\lambda | x_n |} \\ &\leq \frac{1}{2\lambda a}. \end{aligned}$$

The lemma is proved.

Lemma 4.2. Let $\{u_i\}_{i=1}^n$ be a set of iid variables, f_i the density function of u_i , and $f_i(x) < a$ for all $x \in \mathbb{R}$. Then we have

$$P(|u_i - u_j| > \psi \text{ for } \forall i \neq j) > \prod_{i=0}^{n-1} (1 - 2i\psi a).$$

Proof. Let D_k be the event $|u_i - u_j| > \psi$ for $\forall i, j \leq k$, and $F_k : \mathbb{R}^k \to \mathbb{R}$ the joint probability density function of $\{u_i\}_{i=1}^k$ under condition D_k . Then we have

$$\begin{split} &P(D_k) \\ = & P(D_k, D_{k-1}) \\ = & P(D_k \| D_{k-1}) P(D_{k-1}) \\ = & P(|u_k - u_i| > \psi \ for \ \forall i < k \| D_{k-1}) P(D_{k-1}) \\ = & P(D_{k-1}) \int_{-\infty^{k-1}}^{\infty^{k-1}} \int_{-\infty}^{\infty} F_{k-1}(t_1, \dots, t_{k-1}) \\ & f_k(t_k) I(|t_k - t_i| > \psi \ \forall i < k) dt_k dt_1 \dots t_{k-1} \\ > & P(D_{k-1}) \int_{-\infty^{k-1}}^{\infty^{k-1}} \int_{-\infty}^{\infty} F_{k-1}(t_1, \dots, t_{k-1}) \\ & (f_k(t_k) - aI(|t_k - t_i| < \psi \ \exists i < k)) dt_k dt_1 \dots t_{k-1}) \\ = & P(D_{k-1})(1 - \int_{-\infty^{k-1}}^{\infty^{k-1}} \int_{-\infty}^{\infty} aF_{k-1}(t_1, \dots, t_{k-1}) \\ & I(|t_k - t_i| < \psi \ \exists i < k) dt_k dt_1 \dots t_{k-1}) \\ > & P(D_{k-1})(1 - \\ & \int_{-\infty^{k-1}}^{\infty^{k-1}} 2a(k-1) \psi F_{k-1}(t_1, \dots, t_{k-1}) dt_1 \dots t_{k-1}) \\ = & P(D_{k-1})(1 - 2a(k-1)\psi). \end{split}$$

Since $P(D_0) = 1$, we have

$$P(|u_{i} - u_{j}| > \psi \text{ for } \forall i \neq j)$$

$$= P(D_{n})$$

$$> P(D_{n-1})(1 - 2(n-1)\psi a)$$

$$> P(D_{n-2})(1 - 2(n-1)\psi a)(1 - 2(n-2)\psi a)$$

$$> \dots$$

$$> \prod_{i=0}^{n-1}(1 - 2i\psi a).$$

The lemma is proved.

Theorem 4.3. Assume $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$, $|B_{\mathcal{F}}(x)|_{\infty} < \beta$, and $\lambda \in \mathbb{R}_{>0}$ satisfying $(\lambda - \mu)e^{-2\beta - n\mu + \sqrt{\lambda}} > (2m\lambda + \mu)m$. Furthermore, assume the samples are normalized, that is, $|x|_{\infty} = 1$. If $W_R \sim \mathcal{M}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{M}_{m,n}(\lambda)) \leq (m-1)\mathcal{C}(\mathcal{F}, \rho) + \frac{(m-2)^2}{\sqrt{\lambda}}$.

Proof. From (3) and (9), we have $\mathcal{F}(x) = W_x x + B_x$ and $\widetilde{\mathcal{F}}(x) = (W_x + W_R)x + B_x$. Let x be a sample with label y. From equation (15), we have

$$\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x} = \frac{\sum_{i=1}^{m} (W_{R,i} - W_{R,y} + W_{x,i} - W_{x,y}) e^{\tilde{\mathcal{F}}_i(x)}}{\sum_{i=1}^{m} e^{\tilde{\mathcal{F}}_i(x)}}.$$

Let $m_x = \arg \max_{i \neq y} \{ \langle W_{R,i}, x \rangle \}$ and consider the condition:

Condition C_1 : $\langle W_{R,m_x}, x \rangle > \langle W_{R,j}, x \rangle + \sqrt{\lambda}$ for all $j \in [m] \setminus \{y, m_x\}$.

We first give the probability for condition C_1 to be valid. By Lemmas 4.1 and 4.2 and due to $|x|_{\infty} = 1$, we have

$$P_{W_{R}\sim\mathcal{M}_{m,n}(\lambda)}(C_{1})$$

$$\geq P_{W_{R}\sim\mathcal{M}_{m,n}(\lambda)}(|\langle W_{R,j},x\rangle - \langle W_{R,i},x\rangle| > \sqrt{\lambda},$$

$$\forall i, j \in [m]/\{y\}, \ i \neq j)$$

$$\geq \Pi_{i=1}^{m-2}(1 - \frac{2i\sqrt{\lambda}}{2\lambda|x|_{\infty}})$$

$$\geq (1 - \frac{m-2}{\sqrt{\lambda}})^{m-2}$$

$$\geq 1 - \frac{(m-2)^{2}}{\sqrt{\lambda}}.$$
(18)

Let $||x||_{-\infty} = \min_{i \in [n]} \{|x|_i\}$ for $x \in \mathbb{R}^n$. Since $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$ and $W_R \sim \mathcal{M}_{m,n}(\lambda)$, we have $||W_{R,i} + W_{R,j}||_{-\infty} > \lambda$, $||W_{R,i} + W_{R,j}||_{\infty} < 2m\lambda$ and $||W_{x,i} + W_{x,j}||_{\infty} < \mu$ for any $i \neq j$. If condition C_1 is satisfied, then for any $j \in [m] \setminus \{y, m_x\}$, we have

$$\begin{aligned} \widetilde{\mathcal{F}}_{m_x}(x) &- \widetilde{\mathcal{F}}_j(x) \\ &= (W_{R,m_x} + W_{x,m_x} - W_{R,j} - W_{x,j})x + B_{x,m_x} - B_{x,j} \\ &= (W_{R,m_x} - W_{R,j})x + (W_{x,m_x} - W_{x,j})x + B_{x,m_x} - B_{x,j} \\ &> \sqrt{\lambda} - n\mu - 2\beta. \end{aligned}$$

Further considering the hypothesis $(\lambda - \mu)e^{-2\beta - n\mu + \sqrt{\lambda}} > (2m\lambda + \mu)m$, we have

$$||W_{R,m_x} - W_{R,y} + W_{x,m_x} - W_{x,y}||_{-\infty} e^{\tilde{\mathcal{F}}_{m_x}(x)}$$

$$> (\lambda - \mu) e^{\tilde{\mathcal{F}}_{m_x}(x)}$$

$$> (\lambda - \mu) e^{\tilde{\mathcal{F}}_j(x) + \sqrt{\lambda} - 2\beta - n\mu}$$

$$= (\lambda - \mu) e^{-2\beta - n\mu} e^{\sqrt{\lambda}} e^{\tilde{\mathcal{F}}_j(x)}$$

$$> (2m\lambda + \mu) m e^{\tilde{\mathcal{F}}_j(x)}$$

$$> m||(W_{R,j} - W_{R,y} + W_{x,j} - W_{x,y})||_{\infty} e^{\tilde{\mathcal{F}}_j(x)}$$

which means

$$sign(\sum_{i=1}^{m} (W_{R,i} - W_{R,y} + W_{x,i} - W_{x,y})e^{\widetilde{\mathcal{F}}_{i}(x)}) = sign((W_{R,m_{x}} - W_{R,y} + W_{x,m_{x}} - W_{x,y})e^{\widetilde{\mathcal{F}}_{m_{x}}(x)}).$$

Because of this, we have:

$$sign(\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x}))$$

$$= sign(\frac{\sum_{i=1}^{m}(W_{R,i}-W_{R,y}+W_{x,i}-W_{x,y})e^{\tilde{\mathcal{F}}_{i}(x)}}{\sum_{i=1}^{m}e^{\tilde{\mathcal{F}}_{i}(x)}}))$$

$$= sign(\sum_{i=1}^{m}(W_{R,i}-W_{R,y}+W_{x,i}-W_{x,y})e^{\tilde{\mathcal{F}}_{i}(x)}))$$

$$= sign((W_{R,m_{x}}-W_{R,y}+W_{x,m_{x}}-W_{x,y})e^{\tilde{\mathcal{F}}_{m_{x}}(x)})$$

$$= sign((W_{R,m_{x}}-W_{R,y}+W_{x,m_{x}}-W_{x,y}))$$

$$= sign(W_{R,m_{x}}-W_{R,y}).$$

Let V be a random vector in $\{0,1\}^n$. Then the probability for the sign of $W_{R,m_x} - W_{R,y}$ to be V is

$$P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V, C_1)$$

$$\leq P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V)$$

$$= \sum_{i < y} P(m_x = i, \operatorname{sign}(W_{R,y}) = V) + \sum_{i > y} P(m_x = i, \operatorname{sign}(W_{R,i}) = V)$$

$$\leq \sum_{i < y} P(\operatorname{sign}(W_{R,y}) = V)$$

$$\sum_{i > y} P(\operatorname{sign}(W_{R,i}) = V)$$

$$= \frac{m-1}{2^n}.$$

So we have

$$\mathbb{E}_{W_R \sim \mathcal{M}_{m,n}(\lambda)} \\ [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x})) \neq \widehat{B}_{\mathcal{F}}(x))\mathbf{I}(C_1)] \\ = \mathbb{E}_{W_R \sim \mathcal{M}_{m,n}(\lambda)} \\ [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(W_{R,m_x} - W_{R,y})) \neq \widehat{B}_{\mathcal{F}}(x))\mathbf{I}(C_1)] \\ = \sum_{V \in \{-1,1\}^n} P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V, C_1) \\ \mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho V) \neq \widehat{B}_{\mathcal{F}}(x)) \\ \leq \sum_{V \in \{-1,1\}^n} (m - 1)/(2^n)\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho V) \neq \widehat{B}_{\mathcal{F}}(x)) \\ = (m - 1)\mathcal{C}(\mathcal{F}, \rho). \end{aligned}$$

Finally, from (18) we have

$$C(B_{\widetilde{\mathcal{F}}}\mathcal{A}_{2}, \mathcal{M}_{m,n}(\lambda))$$

$$= \mathbb{E}_{x \sim D_{0}} \mathbb{E}_{W_{R} \sim \mathcal{M}_{m,n}(\lambda)} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x})) \neq \widehat{B}_{\mathcal{F}}(x))]$$

$$\leq \mathbb{E}_{x \sim D_{0}} \mathbb{E}_{W_{R} \sim \mathcal{M}_{m,n}(\lambda)} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x})) \neq \widehat{B}_{\mathcal{F}}(x))\mathbf{I}(C_{1}) + (1 - \mathbf{I}(C_{1}))]$$

$$\leq (m - 1)\mathcal{C}(\mathcal{F}, \rho) + \mathbb{E}_{x \sim D_{0}} \mathbb{E}_{W_{R} \sim \mathcal{M}_{m,n}(\lambda)}[(1 - \mathbf{I}(C_{1}))]$$

$$\leq (m - 1)\mathcal{C}(\mathcal{F}, \rho) + \mathbb{E}_{x \sim D_{0}}[1 - P_{W_{R} \sim \mathcal{M}_{m,n}(\lambda)}(C_{1})]$$

$$\leq (m - 1)\mathcal{C}(\mathcal{F}, \rho) + \frac{(m - 2)^{2}}{\sqrt{\lambda}}.$$

The theorem is proved.

Appendix C. Proof of Theorem 4.4

We first prove a lemma.

Lemma 4.3. Let $x_1, x_2 \sim \mathcal{U}(-\lambda, \lambda)$ and $z = x_1 - x_2$. Then for $a \in [0, 2\lambda]$, we have $P(z < a) = P(z > -a) = 1 - \frac{(2\lambda - a)^2}{8\lambda^2}$, which is denoted as $T(\lambda, a) = 1 - \frac{(2\lambda - a)^2}{8\lambda^2}$.

Proof. Let f(z) be the density function of z. Then f(z) = 0, if $z \ge 2\lambda$ or $z \le -2\lambda$; $f(z) = \frac{2\lambda+z}{4\lambda^2}$, if $0 \ge z \ge -2\lambda$; $f(z) = \frac{2\lambda-z}{4\lambda^2}$, if $0 \le z \le 2\lambda$. Hence, $P(z < a) = P(z > -a) = 1 - \frac{(2\lambda-a)^2}{8\lambda^2}$.

Note that $T(\lambda, a)$ increases with a and $T(\lambda, a) \in [0.5, 1]$.

Theorem 4.4. If $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$ and $W_R \sim \mathcal{U}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq \mathcal{C}(\mathcal{F}, \rho) + \mu n/\lambda$. Furthermore, if $\lambda > n\mu/\epsilon$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq \mathcal{C}(\mathcal{F}, \rho) + \epsilon$ for any $\epsilon \in \mathbb{R}_{>0}$, and in particular, if $\lambda > na/(\epsilon \mathcal{C}(\mathcal{F}, \rho))$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_3, \mathcal{U}_{m,n}(\lambda)) \leq (1 + \epsilon)\mathcal{C}(\mathcal{F}, \rho)$.

Proof. Similar to (13), if $||W_{R,n_x} - W_{R,y}||_{-\infty} > \mu$, then we have

$$\mathcal{A}_3(x, \widetilde{\mathcal{F}}, \rho)$$

$$= x + \frac{\rho}{k} \sum_{i=1}^k \operatorname{sign}(W_{x^{i-1}, n_x} - W_{x^{i-1}, y} + W_{R, n_x} - W_{R, y})$$

$$= x + \rho \operatorname{sign}(W_{R, n_x} - W_{R, y}).$$

Since $V = W_{R,n_x} - W_{R,y}$ is a random variable in $[-2\lambda, 2\lambda]$, sign(V) is a random variable in $\{-1, 1\}^n$. By Lemma 4.3,

$$C(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_{3}, \mathcal{U}_{m,n}(\lambda))$$

$$= \mathbb{E}_{x \sim \mathcal{D}_{0}} \mathbb{E}_{W_{R} \sim \mathcal{U}_{m,n}(\lambda)} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(\mathcal{A}_{3}(x, \widetilde{\mathcal{F}})) \neq \widehat{B}_{\mathcal{F}}(x))]$$

$$\leq \mathbb{E}_{x \sim \mathcal{D}_{0}} \mathbb{E}_{W_{R} \sim \mathcal{U}_{m,n}(\lambda)} [\mathbf{I}(||W_{R,n_{x}} - W_{R,y}||_{-\infty} \leq \mu) + \mathbf{I}(||W_{R,n_{x}} - W_{R,y}||_{-\infty} > \mu) [\mathbf{I}(\widehat{B}_{\mathcal{F}}(\mathcal{A}_{3}(x, \widetilde{\mathcal{F}})) \neq \widehat{B}_{\mathcal{F}}(x))]$$

$$\leq (1 - 2^{n}(1 - T(\lambda, \mu))^{n}) + \mathbb{E}_{x \sim \mathcal{D}_{0}} \sum_{V \in \{-1,1\}^{n}} \mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho V) \neq \widehat{B}_{\mathcal{F}}(x))]$$

$$\leq (1 - 2^{n}(1 - T(\lambda, \mu))^{n}) + \mathcal{C}(\mathcal{F}, \rho)$$

where $T(\lambda, \mu) = 1 - \frac{(2\lambda - \mu)^2}{8\lambda^2}$. We have $2^n (1 - T(\lambda, \mu))^n = (2 - 2 + \frac{4\lambda^2 + \mu^2 - 4\lambda\mu}{4\lambda^2})^n = (1 - \frac{4\lambda\mu - \mu^2}{4\lambda^2})^n \ge 1 - n\frac{4\lambda\mu - \mu^2}{4\lambda^2} \ge 1 - n\mu/\lambda$. So,

$$\begin{array}{rcl} \mathcal{C}(B_{\widetilde{\mathcal{F}}},\mathcal{A}_{3},\mathcal{U}_{m,n}(\lambda)) \\ \leq & 1-2^{n}(1-T(\lambda,\mu))^{n}+\mathcal{C}(\mathcal{F},\rho) \\ \leq & \mathcal{C}(\mathcal{F},\rho)+n\mu/\lambda. \end{array}$$

The theorem is proved.

Appendix D. Proof of Theorem 4.5

Theorem 4.5. If $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/2$, $W_R \sim \mathcal{U}_{m,n}(\lambda)$, and m = 2, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq e^{n\mu/\lambda} \mathcal{C}(\mathcal{F}, \rho)$. Furthermore, if $\lambda > n\mu/\ln(1+\epsilon)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq (1+\epsilon)\mathcal{C}(\mathcal{F}, \rho)$.

Proof. Let $y \in \{0, 1\}$ be the label of x. Denote $U = W_{x,1-y} - W_{x,y} \in \mathbb{R}^{1 \times n}$ and $Z = W_{R,1-y} - W_{R,y} \in \mathbb{R}^{1 \times n}$. We have

$$sign(\frac{\nabla L(\mathcal{F}(x),y)}{\nabla x}))$$

$$= sign(\frac{e^{\mathcal{F}_{1-y}(x)}(\frac{\nabla(\mathcal{F}_{1-y}(x))}{\nabla x} - \frac{\nabla(\mathcal{F}_{y}(x))}{\nabla x})}{e^{\mathcal{F}_{y}(x)} + e^{\mathcal{F}_{1-y}(x)}})$$

$$= sign(\frac{\nabla(\mathcal{F}_{1-y}(x))}{\nabla x} - \frac{\nabla(\mathcal{F}_{y}(x))}{\nabla x})$$

$$= sign(W_{x,1-y} - W_{x,y})$$

$$= sign(U).$$

From equation (15), we have

$$\operatorname{sign}\left(\frac{\nabla L(\mathcal{F}(x),y)}{\nabla x}\right) \\ = \operatorname{sign}\left(\frac{e^{\tilde{\mathcal{F}}_{1-y}(x)}}{\sum_{i=1}^{m} e^{\tilde{\mathcal{F}}_{i}(x)}} (W_{x,1-y} - W_{x,y} + W_{R,1-y} - W_{R,y})\right). \\ = \operatorname{sign}(U+Z).$$

For $i \in [n]$, $\operatorname{sign}(U_i) = \operatorname{sign}(U_i + Z_i)$ if and only if $(Z_i \leq -U_i \text{ when } U_i \leq 0)$ or $(Z_i \geq -U_i \text{ when } U_i \geq 0)$, where Z_i, U_i are respectively the *i*-th coordinates of Z, U. Since $W_R \sim \mathcal{U}_{m,n}(\lambda), Z = W_{R,1-y} - W_{R,y}$ is the difference of two uniform distributions in $[-\lambda, \lambda]$. By Lemma 4.3, $U_i > 0$ implies $P(Z_i \geq -U_i) = T(\lambda, |U_i|) < T(\lambda, \mu)$, and $U_i < 0$ implies $P(Z_i \leq -U_i) = T(\lambda, |U_i|) < T(\lambda, \mu)$. Hence, no matter what is the value of U, we always have $P(\operatorname{sign}(U) = \operatorname{sign}(U + Z)) < T(\lambda, \mu)^n$, where $T(\lambda, \mu) = 1 - \frac{(2\lambda - \mu)^2}{8\lambda^2}$.

Moreover, for $i \in [n]$, if $\operatorname{sign}(U_i) \neq \operatorname{sign}(U_i + Z_i)$, we have $(Z_i > 0$ when $U_i < 0)$ or $(Z_i < 0$ when $U_i > 0$). So, $P(\operatorname{sign}(U_i) \neq \operatorname{sign}(U_i + Z_i)) < 1/2 < T(\lambda, \mu)$, since $T(\lambda, \mu)$ is always $\geq 1/2$.

Since $\{Z_i\}_{i \in [n]}$ is iid, by Lemma 4.3, for a random vector $V \in \{-1, 1\}^n$ we have

$$\begin{split} P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(\operatorname{sign}(\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x}) &= V) \\ &= P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(\operatorname{sign}(U+Z) = V) \\ &= \prod_{i=1}^n P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(\operatorname{sign}(U_i + Z_i) = V_i) \\ &= \prod_{i=1}^n (\mathbf{I}(\operatorname{sign}(U_i) = V_i) P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(\operatorname{sign}(U_i) = \operatorname{sign}(U_i + Z_i))) \\ &\quad + \mathbf{I}(\operatorname{sign}(U_i) \neq V_i) P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(\operatorname{sign}(U_i) \neq \operatorname{sign}(U_i + Z_i))) \\ &\leq \prod_{i=1}^n (\mathbf{I}(\operatorname{sign}(U_i) = V_i) T(\lambda, \mu) + \mathbf{I}(\operatorname{sign}(U_i) \neq V_i) T(\lambda, \mu)) \\ &= T(\lambda, \mu)^n. \end{split}$$

For $V \in \{-1,1\}^n$, denote $Q(x,V,\rho) = \mathbf{I}(\widehat{B}_{\mathcal{F}}(x+\rho V) \neq \widehat{B}_{\mathcal{F}}(x))$. We have

$$\begin{aligned} \mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_{2}, \mathcal{U}_{m,n}(\lambda)) \\ &= \mathbb{E}_{x \sim D_{\mathbb{O}}} \mathbb{E}_{W_{R} \sim \mathcal{U}_{m,n}(\lambda)} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x})) \neq \widehat{B}_{\mathcal{F}}(x))]] \\ &= \mathbb{E}_{x \sim D_{\mathbb{O}}} [\sum_{V \in \{-1,1\}^{n}} P_{W_{R} \sim \mathcal{U}_{m,n}(\lambda)} (\operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x}) = V) Q(x, V, \rho)] \\ &\leq (T(\lambda, \mu))^{n} \mathbb{E}_{x \sim D_{\mathbb{O}}} [(\sum_{V \in \{-1,1\}^{n}} Q(x, V, \rho))] \\ &\leq (2T(\lambda, \mu))^{n} \mathcal{C}(\mathcal{F}, \rho) \end{aligned}$$

where $T(\lambda,\mu) = 1 - \frac{(2\lambda-\mu)^2}{8\lambda^2}$. We have $(2T(\lambda,\mu))^n = (2 - \frac{4\lambda^2 + \mu^2 - 4\lambda\mu}{4\lambda^2})^n = (1 + \frac{4\lambda\mu-\mu^2}{4\lambda^2})^n \le (1 + \frac{\mu}{\lambda})^n \le e^{n\mu/\lambda}$. Hence, $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) < e^{n\mu/\lambda}\mathcal{C}(\mathcal{F}, \rho)$. The theorem is proved.

Appendix E. Proof of Theorem 4.6

Theorem 4.6. Assume $|\frac{\nabla \mathcal{F}(x)}{\nabla x}|_{\infty} < \mu/4$, $|B_{\mathcal{F}}(x)|_{\infty} < \beta$, and $\lambda \in \mathbb{R}_{>0}$ satisfying $\mu e^{-2\beta - n\mu/2 + \sqrt{\lambda}} > 2(2\lambda + \mu)m$. Furthermore, assume the samples are normalized, that is, $|x|_{\infty} = 1$. If $W_R \sim \mathcal{U}_{m,n}(\lambda)$, then $\mathcal{C}(B_{\widetilde{\mathcal{F}}}, \mathcal{A}_2, \mathcal{U}_{m,n}(\lambda)) \leq (m-1)\mathcal{C}(\mathcal{F}, \rho) + \frac{(m-1)n\mu}{\lambda} + \frac{(m-2)^2}{\sqrt{\lambda}}$.

Proof. The proof is similar to that of Theorem 4.3. So certain details of the proof are omitted. From equation (15), we have

$$\frac{\nabla L(\widetilde{\mathcal{F}}(x),y)}{\nabla x} = \frac{\sum_{i=1}^{m} (W_{R,i} + W_{x,i} - W_{R,y} - W_{x,y}) e^{\widetilde{\mathcal{F}}_i(x)}}{\sum_{i=1}^{m} e^{\widetilde{\mathcal{F}}_i(x)}}.$$

Let $m_x = \arg \max_{i \neq y} \{ \langle W_{R,i}, x \rangle \}$ and consider two conditions C_1 and C_2 : **Conditions** C_1 : $\langle W_{R,m_x}, x \rangle > \langle W_{R,j}, x \rangle + \sqrt{\lambda}$ for all $j \in [m] \setminus \{y, m_x\}$. **Conditions** C_2 : $||W_{R,m_x} - W_{R,y}||_{-\infty} > \mu$. Note that condition C_2 implies $\operatorname{sign}((W_{R,i} - W_{R,y} + W_{x,i} - W_{x,y})) = \operatorname{sign}(W_{R,i} - W_{R,y})$.

We give the probabilities for conditions C_1 and C_2 to be valid. From the proof of Theorem 4.3,

$$(m - 2)^2$$

$$P_{W_R \sim \mathcal{M}_{m,n}(\lambda)}(C_1) \ge 1 - \frac{(m-2)^2}{\sqrt{\lambda}}$$

Let f(x) be the density function of W_{R,m_x} . Then

$$P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(C_2) \geq P_{W_R \sim \mathcal{U}_{m,n}(\lambda)}(||W_{R,i} - W_{R,y}||_{-\infty} > \mu, \forall i \neq y) \geq (1 - \frac{(m-1)\mu}{\lambda})^n > 1 - \frac{(m-1)n\mu}{\lambda}.$$

For $V \in \{-1, 1\}^n$, it is also easy to see

$$P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V, C_1, C_2)$$

$$\leq P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V)$$

$$= \sum_{i < y} P(m_x = i, \operatorname{sign}(W_{R,y}) = V) + \sum_{i > y} P(m_x = i, \operatorname{sign}(W_{R,i}) = V)$$

$$\leq \sum_{i < y} P(\operatorname{sign}(W_{R,y}) = V) + \sum_{i > y} P(\operatorname{sign}(W_{R,i}) = V)$$

$$= \frac{m-1}{2^n}.$$

If conditions C_1 and C_2 are satisfied, then for any $y \in [m] \setminus \{y, m_x\}$, we have

$$\begin{split} ||W_{R,m_{x}} + W_{x,m_{x}} - W_{R,y} - W_{x,y}||_{-\infty} e^{\tilde{\mathcal{F}}_{m_{x}}(x)} \\ > & \mu/2e^{\tilde{\mathcal{F}}_{m_{x}}(x)} \\ > & \mu/2e^{\tilde{\mathcal{F}}_{j}(x) + \sqrt{\lambda} - 2\beta - n\mu/2} \\ = & \mu/2e^{-2b - n\mu/2}e^{\sqrt{\lambda}}e^{\tilde{\mathcal{F}}_{j}(x)} \\ > & (2\lambda + \mu)me^{\tilde{\mathcal{F}}_{j}(x)} \\ > & m||W_{R,j} + W_{x,j} - W_{R,y} - W_{x,y}||_{\infty}e^{\tilde{\mathcal{F}}_{j}(x)} \end{split}$$

which means

$$sign(\sum_{i=1}^{m} (W_{R,i} + W_{x,i} - W_{R,y} - W_{x,y})e^{\widetilde{\mathcal{F}}_{i}(x)}) \\ = sign((W_{R,m_{x}} + W_{x,m_{x}} - W_{R,y} - W_{x,y})e^{\widetilde{\mathcal{F}}_{m_{x}}(x)}),$$

and hence

$$sign(\frac{\nabla L(\tilde{\mathcal{F}}(x),y)}{\nabla x})) = sign(\frac{\sum_{i=1}^{m}(W_{R,i}+W_{x,i}-W_{R,y}-W_{x,y})e^{\tilde{\mathcal{F}}_{i}(x)}}{\sum_{i=1}^{m}e^{\tilde{\mathcal{F}}_{i}(x)}}) = sign(\sum_{i=1}^{m}(W_{R,i}+W_{x,i}-W_{R,y}-W_{x,y})e^{\tilde{\mathcal{F}}_{i}(x)}) = sign((W_{R,m_{x}}+W_{x,m_{x}}-W_{R,y}-W_{x,y})e^{\tilde{\mathcal{F}}_{m_{x}}(x)}) = sign(W_{R,m_{x}}+W_{x,m_{x}}-W_{R,y}-W_{x,y}) = sign(W_{R,m_{x}}-W_{R,y}).$$

Hence

$$\mathbb{E}_{W_R \sim \mathcal{U}_{m,n}(\lambda)} [\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x), y)}{\nabla x})) \neq \widehat{B}_{\mathcal{F}}(x))\mathbf{I}(C_1, C_2)]$$

$$= \mathbb{E}_{W_R \sim \mathcal{U}_{m,n}(\lambda)}[\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho \operatorname{sign}(W_{R,m_x} - W_{R,y})) \neq \widehat{B}_{\mathcal{F}}(x))$$

$$\mathbf{I}(C_1, C_2)]$$

$$= \sum_{V \in \{-1,1\}^n} P(\operatorname{sign}(W_{R,m_x} - W_{R,y}) = V, C_1, C_2)$$

$$\mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho V) \neq \widehat{B}_{\mathcal{F}}(x))$$

$$\leq \frac{m-1}{2^n} \sum_{V \in \{-1,1\}^n} \mathbf{I}(\widehat{B}_{\mathcal{F}}(x + \rho V) \neq \widehat{B}_{\mathcal{F}}(x))$$

$$= (m-1)\mathcal{C}(\mathcal{F}, \rho).$$

Finally, we have

$$\begin{split} \mathcal{C}(B_{\widetilde{\mathcal{F}}}\mathcal{A}_{2},\mathcal{U}_{m,n}(\lambda)) \\ &= \mathbb{E}_{x\sim D_{0}}\mathbb{E}_{W_{R}\sim\mathcal{U}_{m,n}(\lambda)}[\mathbf{I}(\widehat{B}_{\mathcal{F}}(x+\rho\mathrm{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x),y)}{\nabla x}))\neq\widehat{B}_{\mathcal{F}}(x))] \\ &\leq \mathbb{E}_{x\sim D_{0}}\mathbb{E}_{W_{R}\sim\mathcal{U}_{m,n}(\lambda)}[\mathbf{I}(\widehat{B}_{\mathcal{F}}(x+\rho\mathrm{sign}(\frac{\nabla L(\widetilde{\mathcal{F}}(x),y)}{\nabla x}))\neq\widehat{B}_{\mathcal{F}}(x)) \\ &\mathbf{I}(C_{1}, C_{2}) + (1-\mathbf{I}(C_{1})) + (1-\mathbf{I}(C_{2}))] \\ &\leq (m-1)\mathcal{C}(\mathcal{F},\rho) + \mathbb{E}_{x\sim D_{0}}\mathbb{E}_{W_{R}\sim\mathcal{U}_{m,n}(\lambda)}[(1-\mathbf{I}(C_{1})) + (1-\mathbf{I}(C_{2}))] \\ &\leq (m-1)\mathcal{C}(\mathcal{F},\rho) + \mathbb{E}_{x\sim D_{0}}[1-P_{W_{R}\sim\mathcal{U}_{m,n}(\lambda)}(C_{1})] + \\ &\mathbb{E}_{x\sim D_{0}}[1-P_{W_{R}\sim\mathcal{U}_{m,n}(\lambda)}(C_{2})] \\ &\leq (m-1)\mathcal{C}(\mathcal{F},\rho) + \frac{(m-1)n\mu}{\lambda} + \frac{(m-2)^{2}}{\sqrt{\lambda}}. \end{split}$$

The theorem is proved.

Appendix F. Structures of DNN models used in the experiments

The networks in section 5.1:

Networks $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ for MNIST have the same structure:

Input layer: $N \times 1 \times 28 \times 28$, where N is steps of training.

Hidden layer 1: a convolution layer with kernel $1 \times 32 \times 3 \times 3$ with padding= $1 \rightarrow$ do a batch normalization \rightarrow do Relu \rightarrow use max pooling with step=2.

Hidden layer 2: a convolution layer with kernel $32 \times 64 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu \rightarrow use max pooling with step=2.

Hidden layer 3: a convolution layer with kernel $64 \times 128 \times 3 \times 3$ with padding= $1 \rightarrow do$ a batch normalization $\rightarrow do$ Relu \rightarrow use max pooling with step=2.

Hidden layer 4: draw the output as $N \times 128 \times 3 \times 3 \rightarrow$ use a full connection with output size $N \times 128 \times 2 \rightarrow$ do Relu.

Hidden layer 4: use a full connection with output size $N \times 100 \rightarrow \text{do Relu}$.

Output layer: a full connection layer with output size $N \times 10$.

Networks $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ for CIFAR-10 have the same structure:

Input layer: $N \times 3 \times 32 \times 32$, where N is steps of training.

Hidden layer 1: a convolution layer with kernel $3 \times 64 \times 3 \times 3$ with padding= $1 \rightarrow$ do a batch normalization \rightarrow do Relu.

Hidden layer 2: a convolution layer with kernel $64 \times 64 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 3: a convolution layer with kernel $64 \times 128 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 4: a convolution layer with kernel $128 \times 128 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu \rightarrow use max pooling with step=2.

Hidden layer 5: a convolution layer with kernel $128 \times 256 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 6: a convolution layer with kernel $256 \times 256 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 7: a convolution layer with kernel $256 \times 256 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu \rightarrow use max pooling with step=2.

Hidden layer 8: a convolution layer with kernel $256 \times 512 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 9: a convolution layer with kernel $512 \times 512 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 10: a convolution layer with kernel $512 \times 512 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu \rightarrow use max pooling with step=2.

Hidden layer 11: a convolution layer with kernel $512 \times 512 \times 3 \times 3$ with padding= $1 \rightarrow do$ a batch normalization $\rightarrow do$ Relu.

Hidden layer 12: a convolution layer with kernel $512 \times 512 \times 3 \times 3$ with padding= 1 \rightarrow do a batch normalization \rightarrow do Relu.

Hidden layer 13: a convolution layer with kernel $512 \times 512 \times 3 \times 3$ with padding= $1 \rightarrow do$ a batch normalization $\rightarrow do$ Relu \rightarrow use max pooling with step=2.

Hidden layer 14: draw the output as $N \times 2048 \rightarrow$ a full connection layer with output size $N \times 1024 \rightarrow$ do Relu.

Hidden layer 15: a full connection layer with output size $N \times 512 \rightarrow \text{do Relu}$.

Hidden layer 16: a full connection layer with output size $N \times 128 \rightarrow$ do Relu.

Output layer: a full connection layer with output size $N \times 10$.