

BIT COMPLEXITY OF POLYNOMIAL GCD ON SPARSE REPRESENTATION

QIAO-LONG HUANG AND XIAO-SHAN GAO

ABSTRACT. An input- and output-sensitive GCD algorithm for multi-variate polynomials over finite fields is proposed by combining the modular method with the Ben-Or/Tiwari sparse interpolation. The bit complexity of the algorithm is given and is sensitive to the sparse representation, while for previous sparse GCD algorithms, the complexities were given only in some special cases. It is shown that the new algorithm is superior both in theory and in practice comparing with existing GCD algorithms: the complexity in the degree is decreased from quadratic to linear and the running times are decreased by 1-3 orders of magnitude in various benchmarks.

1. INTRODUCTION

Multivariate polynomial GCD computation is one of the central problems in algebraic and symbolic computation. In 1967, Collins [4] gave the first major advance by proposing a refined Euclidean style algorithm. Such direct computational algorithms lack scalability due to the so-called intermediate expression swell phenomenon. The effective idea to solve the intermediate expression swell problem is modular algorithms, that is, by substituting some of the variables by certain integers, multivariate polynomial GCD computation becomes univariate GCD computation over finite fields, and the true GCD will be recovered from these univariate GCDs either by interpolations or by the Chinese Remainder Theorem. In 1971, Brown [3] gave the first modular algorithm based on interpolation for dense polynomials.

The above algorithms are for dense polynomials. In 1973, Moses and Yun proposed the EZ-GCD algorithm [14], where Hensel lifting instead of interpolation was used to recover the GCD. In 1980, Wang [17] proposed an enhanced EZ-GCD algorithm, called EEZ-GCD algorithm, which improved the EZ-GCD algorithm by solving the leading coefficient problem, bad-zero problem, unlucky evaluation problem, and the common divisor problem.

In 1979, Zippel [18] developed the first modular sparse GCD algorithm based on sparse polynomial interpolations, which interpolates the GCD one variable at a time. Zippel's algorithm is probabilistic and its correctness relies on the Schwartz-Zippel lemma. In 1988, Kaltofen [10] gave a GCD algorithm for polynomials given by straight-line programs. In 1990, Kaltofen and Trager [12] gave a GCD algorithm for polynomials given by black boxes. In 2008, Cuyt and Lee [5] proposed another improved technique. For one evaluation, their algorithm reduces a multivariate

1991 *Mathematics Subject Classification.* Primary 68W30; Secondary 68Q25, 68Q20 .

Key words and phrases. Multivariate polynomial GCD, bit complexity, sparse interpolation, finite field.

The first author was supported in part by NSFC Grant No.12001321.

The second author was supported in part by NSFC Grant No. 12288201.

polynomial into a univariate polynomial. In 2016, Hu and Monagan [7] presented a parallel GCD algorithm for sparse GCD computation, which combined a Kronecker substitution with a Ben-Or/Tiwari sparse interpolation [2] modulo a smooth prime to determine the support of the GCD. In 2018, Tang, Li, and Zeng [16] proposed two methods based on variations of Zippel’s method and Ben-Or/Tiwari’s interpolation algorithm [2], respectively.

Despite of the vast literatures, explicit bit complexities for sparse multivariate polynomial GCD algorithms seems not given. In previous work, the bit complexities were either mentioned to be polynomial in the number of variables, degrees, and the number of terms of the input polynomials or given under certain conditions. In this paper, we will give a new GCD algorithm and its bit complexity, which is sensitive for the sparse representation. The given algorithm is shown to have better complexities and much better practical performance than existing algorithms.

1.1. Main results. Let \mathbb{F}_q be a finite field with q elements, where q is a prime or prime power. In this paper, we focus on GCD computation over finite fields. Let A and B be two polynomials in $\mathbb{F}_q[x_1, \dots, x_n]$ and $G = \gcd(A, B)$. In the following, T_A, T_B , and T_G are respectively the numbers of terms in A, B and G . D (and d) is the degree (and partial degree) bound of A and B . The algorithm is randomized, so we assume that we can obtain a random bit with bit-cost $O(1)$. The main result of the paper is given below.

Theorem 1.1. *Let A, B be in $\mathbb{F}_q[x_1, \dots, x_n]$, and suppose that a primitive root ω of \mathbb{F}_q is given. For any $\varepsilon \in (0, 1)$, there exists an algorithm that takes as inputs A, B and returns $G = \gcd(A, B)$ with probability at least $1 - \varepsilon$ using $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations.*

Our algorithm may fail to find the correct number T_G , which leads to an endless run. At this point, we force quit when the wrong T_G reaches $(d + 1)^n$. But luckily, this case only happens with probability $\leq \varepsilon$. So if we choose ε small enough, for example $\varepsilon = \frac{1}{nD(d+1)^n(T_A+T_B)\log^2 q}$, then the expected complexity is $O^\sim(n^3DT_G(T_A + T_B) \log^2 q)$ bit operations.

The algorithm is implemented in Maple and extensive numerical experiments show that the new algorithm outperforms the default GCD in Maple by 1-3 orders of magnitudes as shown by Table 1. Details of the experiments can be found in section 4.

Experiment Settings	Maple GCD	our GCD
$n = 6, D = 30, t$ varies, $t_l \leq 60$ s	$t_m \approx 18$	$t_m \approx 150$
$t = 30, D = 100, n$ varies, $t_l \leq 60$ s	$n_m \approx 3$	$n_m \approx 200$
$n = 6, t = 30, D$ varies, $t_l \leq 100$ s	$D_m \approx 23$	$D_m \approx 29525$

TABLE 1. Experimental results for computing $G = \text{GCD}(A, B)$, where $D = \deg A = \deg B = \deg G$, $t = \#A = \#B = \#G$, t_l is the running time threshold in seconds. We use t_m, n_m, D_m to denote the maximum terms, numbers of variables, degrees that can be computed within the given time threshold t_l .

At top level, the algorithm is a combination of the modular method with the Ben-Or/Tiwari sparse interpolation [2]. Main ingredients of the algorithm include: a new variable substitution is introduced to isolate the leading coefficient of the

GCD, that is, the leading coefficient of substituted GCD is a monomial; the concept of diverse polynomials introduced by Giesbrecht and Roche [6] is modified to give a Ben-Or/Tiwari sparse interpolation algorithm over finite fields; the early termination introduced by Kaltofen and Lee [11] is used to estimate the terms bound for the coefficients of the substituted GCD; a new type of good points is introduced to recover the GCD from its modular images by using only primes with small sizes. Combination of these ingredients leads to the lower binary complexity and the practical efficiency of the algorithm.

1.2. Related work and comparison. The EZ-GCD [14] appears to have a computing bound which in most cases is a polynomial function of T and n . But in some cases, the complexity is increased, for example, when the number of terms in the expanded series form of $B(x_1, x_2 - b_2, \dots, x_n - b_n)$ has larger order than that in $B(x_1, x_2, \dots, x_n)$ for some (b_2, \dots, b_n) or when the input polynomials are not monic with respect to any variable. In Zippel’s algorithm [18], $O(ndT)$ images of the GCDs are needed, while our algorithm only need $O(nT)$ images and has a better complexity. In Table 2, we list the complexities for the GCD algorithms compared with Zippel algorithm. Here we assume the probability of failure ε is fixed. The complexity is analysed by the authors of this paper.

Zippel’s algorithm was originally designed for GCDs which are monic w.r.t. the main variable. The complexity of Zippel’s algorithm is also sensitive to the sparse representation. The main advantage of our algorithm is that its complexity is linear in D , while the complexity of Zippel’s algorithm is quadratic in d .

Algorithms	Bit complexity	Condition
Zippel [18]	$nd^2T \log q + ndT(T_A + T_B) \log d \log q$	Monic GCD
This paper	$nDT(T_A + T_B) \log^2 q$	All cases

TABLE 2. A “soft-Oh” complexity comparison for GCD algorithms over $\mathbb{F}_q[x_1, \dots, x_n]$. T is the number of terms of the GCD. Condition means under what condition the result is valid.

In the EEZ-GCD algorithm [17], a factorization of the GCD of the leading coefficient is computed. However, this step may lead to high complexity, because the number of terms of the factors may be very large. Our algorithm predetermines the leading coefficient of the GCD by isolating the maximum term instead of factorization, which was similar to the method introduced by Cuyt and Lee [5] and had controllable complexity. Furthermore, inspired by the work of Klivans and Spielman [13], we introduce a new variable substitution such that the leading coefficient of the substituted GCD is a monomial, which greatly decreases the computation cost. Compared to the algorithm in [7], our algorithm also uses Ben-Or/Tiwari algorithm to interpolate the coefficients of GCD, but our algorithm was based on a new diversification method. Also, we do not use the Kronecker substitution and use only primes with small size. These techniques allow us to give an exact bit complexity, while in their algorithm the smooth prime has size $O(D^n)$ in theory. Compared to the algorithm in [16], our algorithm uses the method of isolating maximum term instead of the shifted homogenization introduced in [5] and our algorithm works for any finite field even if the degree is large.

2. BASIC CONCEPTS AND PRELIMINARY RESULTS

2.1. Notations. Let $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$ and

$$A(\mathbb{X}) = c_1M_1 + c_2M_2 + \dots + c_tM_t \in \mathcal{F}[\mathbb{X}],$$

where \mathcal{F} is any field, $c_i \neq 0$ and M_i 's are monomials. Let the exponent vector of M_i be $\mathbf{e}_i = (e_{i,1}, \dots, e_{i,n})$ and the monomials M_i 's are arranged in lexicographically increasing order of \mathbf{e}_i 's. Then c_tM_t is called the *leading term* and c_t is called the *leading coefficient*, denoted as $\text{LC}(A)$.

We first introduce the concept of monomial content.

Definition 2.1. Let $f \in \mathcal{F}[\mathbb{X}]$, where \mathcal{F} is any field. Assume $f = \sum_{i=1}^t c_iM_i$, $c_i \neq 0$ and M_i are distinct monomials. Then $\text{gcd}(M_1, \dots, M_t)$ is called the *monomial content* of f , denoted by $\text{MoCont}(f)$. We call $f/\text{MoCont}(f)$ the *monomial primitive part* of f and denote it by $\text{MoPrim}(f)$.

Since $\text{MoCont}(f)$ is the greatest common factor of M_1, \dots, M_t , $\text{MoCont}(f)$ is a monomial and $\text{MoPrim}(f) = f/\text{MoCont}(f)$ is coprime to any monomial. In particular, $\text{MoPrim}(f)$ is relatively prime to $\text{MoCont}(f)$.

Let A, B be nonzero elements of $\mathcal{F}[\mathbb{X}]$. Then $G = \text{gcd}(A, B)$ is uniquely determined by assuming $\text{LC}(G) = 1$. We say that A is similar to B , denoted as $A \approx B$, if there exists an $a \in \mathcal{F}^*$ such that $aA = B$.

Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}_+^n$ be an integer vector. Define

$$(2.1) \quad A_{(\mathbf{s}, y)} := \frac{A(x_1y^{s_1}, \dots, x_ny^{s_n})}{y^k}$$

where k is the smallest exponent of y in $A(x_1y^{s_1}, \dots, x_ny^{s_n})$. $A_{(\mathbf{s}, y)}$ separates the terms of A by degrees of y .

Let $A = A_1 + \dots + A_\ell$, where A_i is a part of A such that $\deg_y A_i(x_1y^{s_1}, \dots, x_ny^{s_n}) = d_i$. Assume $d_1 < d_2 < \dots < d_\ell$. Then $A_{(\mathbf{s}, y)} = A_1 + A_2y^{d_2-d_1} + \dots + A_\ell y^{d_\ell-d_1}$. Here A_ℓ is the leading coefficient of $A_{(\mathbf{s}, y)}$ w.r.t y if we regard $\mathcal{F}[\mathbb{X}]$ as the coefficients domain.

For any $A \in \mathbb{F}_q[\mathbb{X}]$ and $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}_q^n$, denote $A(\vec{\alpha}) = A(\alpha_1, \dots, \alpha_n)$. For any $i \in \mathbb{N}$, denote $\vec{\alpha}^i = (\alpha_1^i, \dots, \alpha_n^i)$. The main idea of the modular GCD algorithm is to interpolate $G = \text{gcd}(A, B)$ from a sequence of evaluations. Pick a sequence of evaluation points $\vec{\alpha}_1, \vec{\alpha}_2, \dots$ from \mathbb{F}_q^n , compute the images of G , then interpolate each part $G_i(\mathbb{X})$ of G from the scaled images.

\mathbb{F}_q may not have enough elements, and in this case we work in a suitable extension $\mathbb{F}_q \subset \mathbb{F}_{q^m}$, where the latter one is represented as $\mathbb{F}_q[z]/\langle \Phi(z) \rangle$, for a degree- m irreducible polynomial Φ over \mathbb{F}_q . With this representation, arithmetic operations in \mathbb{F}_{q^m} can be done in $O^\sim(m)$ arithmetic operations in \mathbb{F}_q , and thus in $O^\sim(m \log q)$ bit operations.

The cost of sparse polynomial interpolations is determined mainly by the number of points $\vec{\alpha}_1, \vec{\alpha}_2, \dots$, needed and the size of the prime power q needed.

2.2. Preliminary results. We show that the monomial content and the monomial primitive part of the GCD can be computed separately.

Lemma 2.2. *Let $A, B, G \in \mathbb{F}_q[\mathbb{X}]$ and assume $G = \text{gcd}(A, B)$. Then*

- (i) $\text{MoCont}(G) = \text{gcd}(\text{MoCont}(A), \text{MoCont}(B))$, and
- (ii) $\text{MoPrim}(G) = \text{gcd}(\text{MoPrim}(A), \text{MoPrim}(B))$.

$\deg A, \deg b_j \leq \deg B$, the degree of R is no more than $\ell \cdot \deg A + d \cdot \deg B$, which is $\leq 2\|\mathbf{s}\|_\infty \deg A \deg B$. \square

2.3. Isolating the leading coefficient. In this section, we will show how to find an \mathbf{s} such that the leading coefficient of $\gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ in y is a monomial.

2.3.1. Generalized homogenization technique. Let $A, B \in \mathbb{F}_q[\mathbb{X}]$. Instead of directly computing the GCD of A and B , we compute the GCD of the *generalized homogenizing polynomials* $A_{(\mathbf{s},y)}$ and $B_{(\mathbf{s},y)}$ (see (2.1)) by introducing a new variable y . Then $A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)} \in \mathbb{F}_q[\mathbb{X}, y]$. Denote $C = \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ and $G = \gcd(A, B)$. The following lemma shows that $C \approx G_{(\mathbf{s},y)}$, which means C and $G_{(\mathbf{s},y)}$ are the same up to a non-zero constant.

Denote $y^{\mathbf{s}}\mathbb{X} = (x_1 y^{s_1}, \dots, x_n y^{s_n})$ and $y^{-\mathbf{s}}\mathbb{X} = (x_1/y^{s_1}, \dots, x_n/y^{s_n})$.

Lemma 2.7. *Let $A, B \in \mathbb{F}_q[\mathbb{X}]$, $G = \gcd(A, B)$, and $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$. Then $\gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)}) \approx G_{(\mathbf{s},y)}$.*

Proof. First we claim that $\gcd(A(y^{\mathbf{s}}\mathbb{X}), B(y^{\mathbf{s}}\mathbb{X})) \approx y^m G(y^{\mathbf{s}}\mathbb{X})$ for some integer $m \geq 0$. Proof of the claim: Assume $P = \gcd(A(y^{\mathbf{s}}\mathbb{X}), B(y^{\mathbf{s}}\mathbb{X}))$. $G|A$ and $G|B$ imply that $G(y^{\mathbf{s}}\mathbb{X})|A(y^{\mathbf{s}}\mathbb{X})$ and $G(y^{\mathbf{s}}\mathbb{X})|B(y^{\mathbf{s}}\mathbb{X})$, and then we have $G(y^{\mathbf{s}}\mathbb{X})|P$.

We prove the reverse direction. Since $P|A(y^{\mathbf{s}}\mathbb{X})$, there exists a $Q \in \mathbb{F}_q[y, \mathbb{X}]$ such that $A(y^{\mathbf{s}}\mathbb{X}) = P(y, \mathbb{X})Q(y, \mathbb{X})$. Replacing $x_i y^{s_i}$ by x_i , we have $A(\mathbb{X}) = P(y, y^{-\mathbf{s}}\mathbb{X})Q(y, y^{-\mathbf{s}}\mathbb{X})$. Then, there exists an integer k such that $Q(y, y^{-\mathbf{s}}\mathbb{X}) = y^k(Q_\ell y^{d_\ell} + \dots + Q_1 y^{d_1} + Q_0)$, where $Q_i \in \mathbb{F}_q[\mathbb{X}]$ and $d_i > 0$. So $y^k P(y, y^{-\mathbf{s}}\mathbb{X})$ is a polynomial in $\mathbb{F}_q[y, \mathbb{X}]$ and $y^k P(y, y^{-\mathbf{s}}\mathbb{X})|A(\mathbb{X})$. If $k > 0$, then $P(y, y^{-\mathbf{s}}\mathbb{X})|A(\mathbb{X})$. So we can always assume $k \leq 0$. For the same reason, there exists an integer $u \leq 0$ such that $y^u P(y, y^{-\mathbf{s}}\mathbb{X})|B(\mathbb{X})$. Now let $m' = \min(-k, -u)$. Without loss of generality, assume $m' = -k$. Then $y^{-m'} P(y, y^{-\mathbf{s}}\mathbb{X})|A(\mathbb{X})$ and $y^{-m'} P(y, y^{-\mathbf{s}}\mathbb{X})|y^{k-u} B(\mathbb{X})$. So $y^{-m'} P(y, y^{-\mathbf{s}}\mathbb{X})|\gcd(A(\mathbb{X}), y^{k-u} B(\mathbb{X})) = \gcd(A, B)$, which implies $P(y, y^{-\mathbf{s}}\mathbb{X})|y^{m'} G$. Replace x_i/y^{s_i} by x_i , we have $P(y, \mathbb{X})|y^{m'} G(y^{\mathbf{s}}\mathbb{X})$. So there exists an integer $m \geq 0$ such that $P \approx y^m G(y^{\mathbf{s}}\mathbb{X})$. The claim is proved.

Since $C = \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ and $A_{(\mathbf{s},y)} = \frac{A(y^{\mathbf{s}}\mathbb{X})}{y^{d_A}}$, $C y^{d_A} | A(y^{\mathbf{s}}\mathbb{X})$. Here d_A is the integer k in (2.1). For the same reason, $C y^{d_B} | B(y^{\mathbf{s}}\mathbb{X})$. So

$$C y^{\min\{d_A, d_B\}} | \gcd(A(y^{\mathbf{s}}\mathbb{X}), B(y^{\mathbf{s}}\mathbb{X})).$$

By the claim, $\gcd(A(y^{\mathbf{s}}\mathbb{X}), B(y^{\mathbf{s}}\mathbb{X})) \approx y^m G(y^{\mathbf{s}}\mathbb{X})$ for some integer $m \geq 0$, so

$$C y^{\min\{d_A, d_B\}} | y^m G(y^{\mathbf{s}}\mathbb{X}).$$

Then $C | \frac{y^m G(y^{\mathbf{s}}\mathbb{X})}{y^{\min\{d_A, d_B\}}}$. Clearly, $d_C \leq d_A$ and $d_C \leq d_B$. So $d_C \leq \min\{d_A, d_B\}$ and $C | \frac{y^m G(y^{\mathbf{s}}\mathbb{X})}{y^{d_C}} = y^m G_{(\mathbf{s},y)}$. For the reverse direction, since $G|A$ and $G|B$, we have $G(y^{\mathbf{s}}\mathbb{X})|A(y^{\mathbf{s}}\mathbb{X})$ and $G(y^{\mathbf{s}}\mathbb{X})|B(y^{\mathbf{s}}\mathbb{X})$. Since $G(y^{\mathbf{s}}\mathbb{X}) = G_{(\mathbf{s},y)} \cdot y^{d_G}$ and $A(y^{\mathbf{s}}\mathbb{X}) = A_{(\mathbf{s},y)} \cdot y^{d_A}$, $G_{(\mathbf{s},y)} | A_{(\mathbf{s},y)}$. For the same reason, we have $G_{(\mathbf{s},y)} | B_{(\mathbf{s},y)}$. So we have

$$G_{(\mathbf{s},y)} | \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)}) = C.$$

So there exists an integer m' such that $C \approx y^{m'} G_{(\mathbf{s},y)}$. Regard C and $G_{(\mathbf{s},y)}$ as polynomials in y with coefficients in $\mathbb{F}_q[\mathbb{X}]$, we know both C and $G_{(\mathbf{s},y)}$ have non-zero constants, so $m' = 0$. The lemma is proved. \square

Once $C = \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ is computed, the polynomial $C(1, \mathbb{X})$ is similar to $\gcd(A, B)$.

2.3.2. *Isolating the leading coefficient.* In previous work on GCD computation, $A_{(\mathbf{1},y)}$ instead of $A_{(\mathbf{s},y)}$ is used, where $\mathbf{1}$ is the vector all of whose entries are 1. Suppose $G = 5x_1^3x_2 + 7x_1^5x_2^8 + 4x_1^9x_2^4$ is the GCD to be computed. Then $G_{(\mathbf{1},y)} = 5x_1^3x_2 + (7x_1^5x_2^8 + 4x_1^9x_2^4)y^9$. Regarding y as the main variable, $G_{(\mathbf{1},y)}$ is not monic. In this case, the sparse modular GCD algorithm of Zippel cannot be applied directly as the leading coefficient in the univariate images of G in y cannot be known in advance. In the computing of GCD, how to find such a leading coefficient of $A_{(\mathbf{1},y)}$ is a key and bottleneck step. On the other hand, let $\mathbf{s} = (1, 2)$. Then the leading coefficient of $G_{(\mathbf{s},y)} = 5x_1^3x_2 + 4x_1^9x_2^4y^{12} + 7x_1^5x_2^8y^{16}$ in y is a monomial, which will be used to greatly simplify the GCD computation.

In this section, we will introduce a new method to solve this leading coefficient problem. We know that $\text{LC}_y(G)$ divides $\text{gcd}(\text{LC}_y(A), \text{LC}_y(B))$. If a new variable y is constructed so that $\text{gcd}(\text{LC}_y(A), \text{LC}_y(B))$ is only a monomial, then $\text{LC}_y(G)$ must also be a monomial. In order to make $\text{gcd}(\text{LC}_y(A), \text{LC}_y(B))$ a monomial, the simplest case is that $\text{LC}_y(A)$ or $\text{LC}_y(B)$ is a monomial.

Before our description, we define the concept of the maximum isolated term.

Definition 2.8. Let $F = f_\ell y^{e_\ell} + f_{\ell-1} y^{e_{\ell-1}} + \dots + f_1 y^{e_1} \in \mathbb{F}_q[\mathbb{X}, y]$, where $f_i \in \mathbb{F}_q[\mathbb{X}]$, $f_i \neq 0$ and $e_\ell > \dots > e_1 \geq 0$. If f_ℓ is a single term in $\mathbb{F}_q[\mathbb{X}]$, then we say F has a maximum isolated term w.r.t y .

The following lemma says that if a polynomial has a maximum isolated term w.r.t y , then so do its factors.

Lemma 2.9. *If $F \in \mathbb{F}_q[\mathbb{X}, y]$ has a maximum isolated term w.r.t y , then its factor polynomials also have maximum isolated terms w.r.t y .*

Proof. Assume $F = G \cdot H$ and $G = g_\ell y^{d_\ell} + \dots + g_1 y^{d_1}$ and $H = h_t y^{e_t} + \dots + h_1 y^{e_1}$. Then the leading coefficient of F is $g_\ell \cdot h_t$. If the number of terms of g_ℓ or h_t exceeds one, so does $g_\ell \cdot h_t$, which contradicts to assumption of F . \square

The following theorem gives a probabilistic method to construct \mathbf{s} , so that the new polynomial has a maximum isolated term.

Theorem 2.10. *Let $A(\mathbb{X}) \in \mathbb{F}_q[\mathbb{X}]$, $T \geq \#A$, $N = 2(T - 1)$. If we choose a vector $\mathbf{s} = (s_1, \dots, s_n) \in [1, N]^n$ uniformly at random, then $A_{(\mathbf{s},y)}$ has a maximum isolated term w.r.t y with probability $\geq \frac{1}{2}$.*

Proof. Assume $A = \sum_{i=1}^t a_i x_1^{e_{i,1}} \dots x_n^{e_{i,n}}$. The degrees of y of terms in $A(y^{\mathbf{s}}\mathbb{X})$ for $\mathbf{s} = (s_1, \dots, s_n)$ are $d_{\mathbf{s},i} = e_{i,1}s_1 + \dots + e_{i,n}s_n$, $i = 1, \dots, t$. Let $d_{\mathbf{s},\max} = \max_{i=1}^t d_{\mathbf{s},i}$ and call $(s_1, \dots, s_n, d_{\mathbf{s},\max})$ the maximum point of \mathbf{s} .

Considering s_1, \dots, s_n, z as variables, we have t hyperplanes $P_i : z = e_{i,1}s_1 + \dots + e_{i,n}s_n$, $i = 1, \dots, t$ in \mathbb{R}^{n+1} . Let $\bar{S} = \{\mathbf{s} \in \mathbb{R}^n : s_i > 0, i = 1, \dots, n\}$ be the open first octant. Define $\mathcal{C} \subset \mathbb{R}_+^{t+1}$ as follows.

$$\mathcal{C} = \{(\mathbf{s}, d_{\mathbf{s},i_m}) : \mathbf{s} \in \bar{S} \text{ and } i_m \in \text{argmax}_{i=1}^t d_{\mathbf{s},i}\}$$

that is, \mathcal{C} consists of maximum points over \bar{S} .

We claim that $\mathcal{C} = \cup_{i=1}^\ell Q_i$ is an open n -dimensional polyhedral cone, where $\ell \leq t$, $Q_i \subset P_{\mu_i}$ is a convex polyhedral cone, $P_{\mu_i} \neq P_{\mu_j}$ for $i \neq j$, $Q_i \cap Q_{i+1} \subset \mathcal{C}$ for $i = 1, \dots, \ell - 1$. Furthermore, the map $\mathcal{D}(\mathbf{s}) = d_{\mathbf{s},i_m} : \bar{S} \rightarrow \mathbb{R}$ for $(\mathbf{s}, d_{\mathbf{s},i_m}) \in \mathcal{C}$ is a concave function.

We prove the claim by induction. The claim is easily seen to be true for $t = 1$. For $t = 2$, let the projection of the intersection of P_1 and P_2 to the \mathbf{s} -coordinate space be $R_1 = \{\mathbf{s} : P_1(\mathbf{s}) = P_2(\mathbf{s})\}$ which is a linear subspace of the \mathbf{s} -space \mathbb{R}^n . If R_1 is outside \bar{S} , then we have either $P_1(\mathbf{s}) > P_2(\mathbf{s})$ for all $\mathbf{s} \in \bar{S}$ or $P_1(\mathbf{s}) < P_2(\mathbf{s})$ for all $\mathbf{s} \in \bar{S}$. We can set $\mathcal{C} = \{(\mathbf{s}, P_1(\mathbf{s})) : \mathbf{s} \in \bar{S}\}$ in the first case and $\mathcal{C} = \{(\mathbf{s}, P_2(\mathbf{s})) : \mathbf{s} \in \bar{S}\}$ in the second case, and the claim is proved. If R_1 is inside \bar{S} , then \bar{S} is divided into two convex polyhedral cones: $C_1 = \{\mathbf{s} : d_{\mathbf{s},1} \geq d_{\mathbf{s},2}\}$ and $C_2 = \{\mathbf{s} : d_{\mathbf{s},2} \geq d_{\mathbf{s},1}\}$ by R_1 . It is clear that $C_1 \cap C_2 = R_1$. Let $Q_i = \{(\mathbf{s}, d_{\mathbf{s},i}) : \mathbf{s} \in C_i\}$, which are clearly convex polyhedral cones. Then it is easy to see that $\mathcal{C} = Q_1 \cup Q_2$. Since all coordinates of \mathbf{s} are positive, $\mathcal{D}(\mathbf{s})$ is clearly concave. Also note that $A_{(\mathbf{s},y)}$ has a maximum isolated term for $\mathbf{s} \in \bar{S} \setminus R_1$.

Suppose the claim is valid for t and A has $t + 1$ monomials. Then for the first t monomials of A , $\mathcal{C}_t = \cup_{i=1}^t Q_i$ with $\ell \leq t$. Let i_1 be the smallest index such that P_{t+1} intersects Q_{i_1} and $i_2 \geq i_1 + 1$ be the next smallest index such that P_{t+1} intersects Q_{i_2} . Here, we consider the generic case, that is $Q_{i_1} \cap Q_{i_1+1} \subset P_{t+1}$ and $Q_{i_2} \cap Q_{i_2+1} \subset P_{t+1}$ are not valid. If one of them is valid, the claim can be proved similarly. Also, P_{t+1} may intersect only one Q_i , and this case can also be proved similarly.

Since $\mathcal{D}_t(\mathbf{s})$ is concave, P_{t+1} intersects no Q_i for $i \geq i_2 + 1$, that is P_{t+1} intersects essentially at most two Q_i s. Let $E_i (i = 1, \dots, t)$, R_1 , and R_2 be the projections of Q_i , $P_{t+1} \cap Q_{i_1}$ and $P_{t+1} \cap Q_{i_2}$ to the \mathbf{s} -coordinate space. Further let

$$\begin{aligned} C_1 &= \{\mathbf{s} \in E_{i_1} : d_{\mathbf{s},\mu_{i_1}} \geq d_{\mathbf{s},t+1}\} & \tilde{Q}_{i_1} &= \{(\mathbf{s}, d_{\mathbf{s},\mu_{i_1}}) : \mathbf{s} \in C_1\} \\ C_2 &= \{\mathbf{s} \in E_{i_2} : d_{\mathbf{s},\mu_{i_2}} \geq d_{\mathbf{s},t+1}\} & \tilde{Q}_{i_2} &= \{(\mathbf{s}, d_{\mathbf{s},\mu_{i_2}}) : \mathbf{s} \in C_2\} \\ C_3 &= \{\mathbf{s} \in \cup_{i=i_1}^{i_2} E_i : d_{\mathbf{s},t+1} \geq d_{\mathbf{s},\mu_i}\} & \tilde{Q}_{i_3} &= \{(\mathbf{s}, d_{\mathbf{s},t+1}) : \mathbf{s} \in C_3\}. \end{aligned}$$

Since $\mathcal{D}_t(\mathbf{s})$ is concave, we have $d_{\mathbf{s},t+1} \geq d_{\mathbf{s},\mu_i}$ for $i = i_1 + 1, \dots, i_2 - 1$. Then, it can be shown that the following decomposition satisfies the properties in the claim

$$\mathcal{C}_{t+1} = Q_1 \cup \dots \cup Q_{i_1-1} \cup \tilde{Q}_{i_1} \cup \tilde{Q}_{i_3} \cup \tilde{Q}_{i_2} \cup Q_{i_2+1} \cup \dots \cup Q_\ell.$$

Let $F_i (i = 1, \dots, \ell - 1)$ be the projections of $Q_i \cap Q_{i+1}$ to the \mathbf{s} -coordinate space. Then, for $\mathbf{s} \in \bar{S} \setminus \cup_{i=1}^{\ell-1} F_i$, $A_{(\mathbf{s},y)}$ has a maximum isolated term. Define the polynomial $B(\mathbf{s}) = \prod_{i=1}^{\ell-1} (d_{\mathbf{s},\mu_i} - d_{\mathbf{s},\mu_{i+1}})$. Then $\deg B(\mathbf{s}) \leq T - 1$. By Lemma 2.4, if randomly choose $\mathbf{s} \in [1, N]^n$, with the probability $\geq 1 - \frac{\ell}{N} \geq 1 - \frac{T-1}{2(T-1)} = \frac{1}{2}$, \mathbf{s} is not a zero of the $B(\mathbf{s})$, and in this case, $A_{(\mathbf{s},y)}$ has a maximum isolated term. \square

Example 2.11. Let $A = 2x_1^7x_2^3 + 3x_1^5x_2^8 + 5x_1x_2^9 \in \mathbb{F}_{11}[x_1, x_2]$. For $\mathbf{s} \in \mathbb{N}^2$, we have the degrees of y in $A(x_1y^{s_1}, x_2y^{s_2})$ are $d_1 = 7s_1 + 3s_2, d_2 = 5s_1 + 8s_2, d_3 = s_1 + 9s_2$. Regarding s_1, s_2, z as variables, we obtain three hyperplanes:

$$\begin{cases} P_1 : z = 7s_1 + 3s_2 \\ P_2 : z = 5s_1 + 8s_2 \\ P_3 : z = s_1 + 9s_2 \end{cases}$$

As shown Figure 1, P_1, P_2, P_3 form an open polyhedral cone \mathcal{C} , which is concave as a function of (s_1, s_2) . The projection of the edges of \mathcal{C} to the s_1s_2 -coordinate plane are two lines $7s_1 + 3s_2 = 5s_1 + 8s_2$ and $5s_1 + 8s_2 = s_1 + 9s_2$, shown in Figure 2. Over these two lines, two of P_1, P_2, P_3 achieve the same maximum value for a given \mathbf{s} . Thus $A_{(\mathbf{s},y)}$ has a maximum isolated term if and only if $7s_1 + 3s_2 \neq 5s_1 + 8s_2$ and $5s_1 + 8s_2 \neq s_1 + 9s_2$.

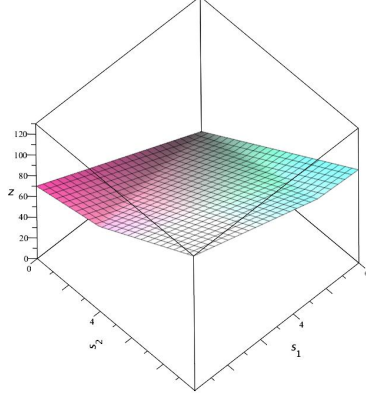


FIGURE 1. The open polyhedral cone \mathcal{C} formed by P_1, P_2, P_3 .

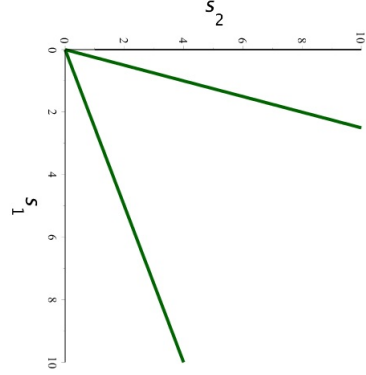


FIGURE 2. Projection of edges of the open polyhedral cone

For polynomials A and B , we can always choose a vector \mathbf{s} such that $A_{(\mathbf{s},y)}$ or $B_{(\mathbf{s},y)}$ has a maximum isolated term. As $N = 2(T_A - 1)$ or $N = 2(T_B - 1)$, the degrees of $A_{(\mathbf{s},y)}$ or $B_{(\mathbf{s},y)}$ in y are $O(T_A D)$ or $O(T_B D)$. Once one of A and B has a maximum isolated term, so does their GCD.

2.4. Diverse polynomial and sparse interpolation over finite fields. In this section, we give the Ben-Or/Tiwari sparse interpolation over finite fields.

2.4.1. Diverse polynomials. We use the following concept of diverse polynomials, introduced by Giesbrecht and Roche [6].

Definition 2.12. Let \mathcal{R} be any ring. If a polynomial $f \in \mathcal{R}[\mathbb{X}]$ has all coefficients distinct; that is, $f = \sum_{i=1}^t c_i M_i$ and $c_i = c_j \Rightarrow i = j$, then we say f is *diverse*.

We define the following more loosely concept: diverse w.r.t. y .

Definition 2.13. Let $F \in \mathcal{R}[y, \mathbb{X}]$. Assume $F = \sum_{i=0}^d a_i y^i$. F is called *diverse* w.r.t. y if each $a_i \in \mathcal{R}[\mathbb{X}]$ is diverse.

The following is an illustrative example for this concept.

Example 2.14. Let $F = (4x_1x_2^2 + 6x_1^2x_2^5)y^3 + (3x_1^2x_2 + 2x_1^2x_2^2)y \in \mathbb{F}_7[y, x_1, x_2]$. Regard y as the main variable in F . The coefficients of y^3 and y are $4x_1x_2^2 + 6x_1^2x_2^5$ and $3x_1^2x_2 + 2x_1^2x_2^2$. Both of them have the pair-wise different coefficients, so F is *diverse* w.r.t. y . As a counter-example, if $F = (6x_1x_2^2 + 6x_1^2x_2^5)y^3 + (3x_1^2x_2 + 2x_1^2x_2^2)y \in \mathbb{F}_7[y, x_1, x_2]$, $6x_1x_2^2 + 6x_1^2x_2^5$, the coefficient of y^3 , has the same coefficient 6. So F is not diverse w.r.t. y .

Giesbrecht and Roche [6] introduced a method of *diversification*, which converted a non-diverse polynomial into a diverse polynomial with high probability. If $\zeta_1, \dots, \zeta_n \in \mathcal{R}^*$, the polynomials $f(\zeta_1x_1, \dots, \zeta_nx_n) \leftrightarrow f(\mathbb{X})$ are one-to-one corresponding. We can interpolate $f(\zeta_1x_1, \dots, \zeta_nx_n)$ instead of $f(\mathbb{X})$. If $f(\mathbb{X}) = \sum_{i=1}^t c_i x_1^{e_{i,1}} \cdots x_n^{e_{i,n}}$, then

$$f(\zeta_1x_1, \dots, \zeta_nx_n) = \sum_{i=1}^t c_i \zeta_1^{e_{i,1}} \cdots \zeta_n^{e_{i,n}} x_1^{e_{i,1}} \cdots x_n^{e_{i,n}} = \sum_{i=1}^t \tilde{c}_i x_1^{e_{i,1}} \cdots x_n^{e_{i,n}},$$

where $\tilde{c}_i = c_i \zeta_1^{e_{i,1}} \cdots \zeta_n^{e_{i,n}}$. Now the coefficients of $f(\zeta_1 x_1, \dots, \zeta_n x_n)$ are \tilde{c}_i 's. Giesbrecht and Roche [6] proved that if \mathcal{R} has enough many elements and $(\zeta_1, \dots, \zeta_n)$ are randomly chosen from \mathcal{R}^{*n} , then $f(\zeta_1 x_1, \dots, \zeta_n x_n)$ is diverse with high probability. Once $g = f(\zeta_1 x_1, \dots, \zeta_n x_n)$ is known, so $f = g(\zeta_1^{-1} x_1, \dots, \zeta_n^{-1} x_n)$.

The following theorem states that diversification of F_1 and F_2 leads to diversification of their GCD. Denote $\vec{\zeta} \mathbb{X} = (\zeta_1 x_1, \dots, \zeta_n x_n)$ and $\vec{\zeta}^{-1} = (\zeta_1^{-1}, \dots, \zeta_n^{-1})$.

Lemma 2.15. *Let $F_1, F_2 \in \mathbb{F}_q[y, \mathbb{X}]$, $C = \gcd(F_1, F_2)$, and $\vec{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathcal{K}^{*n}$, where \mathcal{K} is an extension field of \mathbb{F}_q . Then $C(y, \vec{\zeta} \mathbb{X}) \approx \gcd(F_1(y, \vec{\zeta} \mathbb{X}), F_2(y, \vec{\zeta} \mathbb{X}))$.*

Proof. Assume $P = \gcd(F_1(y, \vec{\zeta} \mathbb{X}), F_2(y, \vec{\zeta} \mathbb{X}))$. Since $C = \gcd(F_1, F_2)$, we have $C(y, \vec{\zeta} \mathbb{X}) | F_1(y, \vec{\zeta} \mathbb{X})$ and $C(y, \vec{\zeta} \mathbb{X}) | F_2(y, \vec{\zeta} \mathbb{X})$. So we have $C(y, \vec{\zeta} \mathbb{X}) | P$. We prove the reverse direction. From $P | F_1(y, \vec{\zeta} \mathbb{X})$, we have $P(y, \vec{\zeta}^{-1} \mathbb{X}) | F_1$. For the similar reason, $P(y, \vec{\zeta}^{-1} \mathbb{X}) | F_2$, which implies $P(y, \vec{\zeta}^{-1} \mathbb{X}) | \gcd(F_1, F_2)$. So $P(y, \vec{\zeta}^{-1} \mathbb{X}) | C$ and then $P | C(y, \vec{\zeta} \mathbb{X})$. The lemma is proved. \square

2.4.2. Sparse interpolation over finite fields. We generalize the Ben-Or and Tiwari algorithm to polynomials over finite fields. Compared with the original Ben-Or and Tiwari algorithm over fields with characteristic 0, the following assumption need to be satisfied.

Assumption 2.16. Let $f = \sum_i c_i M_i$, $M_i = x_1^{e_{i,1}} \cdots x_n^{e_{i,n}}$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}_{q^m}^n$, and ω a primitive root of \mathbb{F}_q .

- (1) f is a diverse polynomial.
- (2) The polynomial $\prod_{1 \leq i < j \leq t} (M_i - M_j)$ is not zero at point $\vec{\alpha}$.
- (3) The polynomial $\prod_{k=1}^n \prod_{1 \leq i < j \leq t} (\omega^{e_{i,k}} M_i - \omega^{e_{j,k}} M_j)$ is not zero at point $\vec{\alpha}$.

The algorithm is listed below for ease of the call of other algorithms. For details, see [9].

Algorithm 2.17. Interpolation

Input:

- $2T$ evaluations $f(\vec{\alpha}^i) = f(\alpha_1^i, \dots, \alpha_n^i)$, $i = 1, 2, \dots, 2T$, where Assumptions 2.16 are satisfied.
- A primitive root ω of \mathbb{F}_q , where $q > \max_{i=1}^n \deg_{x_i} f$.
- $2nT$ evaluations $f(\vec{\alpha}_k^i) = f(\alpha_1^i, \dots, \alpha_{k-1}^i, (\alpha_k \omega)^i, \alpha_{k+1}^i, \dots, \alpha_n^i)$, $i = 1, 2, \dots, 2T$, $k = 1, 2, \dots, n$.

Output: The polynomial $f = \sum_{i=1}^t c_i M_i$.

We choose the points in the extension field $\mathbb{F}_{q^m} = \mathbb{F}_q[z]/(\Phi)$ with irreducible polynomial $\Phi(z)$ of degree m .

Theorem 2.18. [9] *Algorithm 2.17 needs $O \sim (nm^2 T \log^2 q + nT \sqrt{d} \log q)$ bit operations.*

Remark 2.19. The complexity $nT \sqrt{d} \log q$ comes from the computing of discrete logarithms.

2.5. Early termination for the terms bound. We show how to estimate a tight terms bound for a polynomial. Kalfoten and Lee [11] proposed the technique *early termination*, which can be used to detect the number of terms of f with high

probability. Based on this idea, a method that to test whether f is t -sparse is given. Hu and Monagan [8] also applied this method to determine the terms bound of the GCD. Let $v_i = f(x_1^i, \dots, x_n^i)$ be the symbolic evaluations of f at powers, and define the Hankel matrices of polynomials

$$\text{HK}_s = \begin{pmatrix} v_1 & v_2 & \cdots & v_s \\ v_2 & v_3 & \cdots & v_{s+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_s & v_{s+1} & \cdots & v_{2s-1} \end{pmatrix}$$

Kaltofen and Lee [11] proved that if $s > t$, then HK_s is singular; if $s \leq t$, HK_s has full rank. For any $\vec{\alpha}$ with components taken from the algebraic completion of \mathbb{F}_q , we have

$$\det \text{HK}_s(\vec{\alpha}) \begin{cases} = 0, & \text{if } s > t, \\ \neq 0 \text{ with high probability,} & \text{if } s \leq t. \end{cases}$$

The degree of $\det \text{HK}_s$ is bounded by $s^2 \deg f$ [11, Theorem 5]. If $s \leq t$, for $\vec{\alpha}$ chosen uniformly at random from $\mathbb{F}_{q^m}^n$, $\det \text{HK}_s(\vec{\alpha})$ is nonzero with probability at least $1 - s^2 \deg f / q^m$ by Lemma 2.4. Choose a random point $\vec{\alpha} \in \mathbb{F}_{q^m}^n$, if we test whether $\det \text{HK}_s(\vec{\alpha}) = 0$ for $s = 1, 2, 3, \dots, t+1$, the probability that $\det \text{HK}_{t+1}(\vec{\alpha})$ is not the first singular Hankel matrix is at most $\frac{\deg f}{q^m} \sum_{s=1}^t s^2 = \frac{t(t+1)(2t+1) \deg f}{6q^m}$.

In this paper, an estimation for t that is tight up to a constant factor is enough. As shown by Arnold [1], in this case one can employ a technique called repeated doubling. We make an initial guess $s = 1$, and test $\det \text{HK}_s(\vec{\alpha}) = 0$ for $s = 1, 2, 2^2, \dots$, until $\det \text{HK}_s(\vec{\alpha}) = 0$. In this case, the probability that the first instance of $\det \text{HK}_s(\vec{\alpha}) = 0$ is for $t < s \leq 2t$ is $\frac{\deg f}{q^m} \sum_{i=0}^{\lfloor \log_2 t \rfloor} (2^i)^2 < \frac{4t^2 \deg f}{3q^m}$.

2.6. Good point. The main idea of our algorithm is mapping the entire problem to a simpler domain via homomorphisms. Assume

$$\Phi_{\vec{\alpha}} : \mathbb{F}_q[\mathbb{X}, y] \rightarrow \mathbb{F}_q[y]$$

is a homomorphism of rings by evaluating $x_i = \alpha_i, i = 1, \dots, n$, where $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$. Let $F_1, F_2 \in \mathbb{F}_q[\mathbb{X}, y]$ and $C = \gcd(F_1, F_2)$. Then $\Phi_{\vec{\alpha}}(F_1) = F_1(y, \vec{\alpha})$ and $\Phi_{\vec{\alpha}}(F_2) = F_2(y, \vec{\alpha})$. Compute the GCD of univariate polynomials $\Phi_{\vec{\alpha}}(F_1)$ and $\Phi_{\vec{\alpha}}(F_2)$, and it is easy to see that

$$\Phi_{\vec{\alpha}}(C) \mid \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2)).$$

If $\Phi_{\vec{\alpha}}(C) \approx \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$, $\gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$ retains parts of the information of C to solve the problem in the original domain. The leading coefficient $\text{LC}_y(C)(\vec{\alpha})$ of the GCD is not zero if $\Phi_{\vec{\alpha}}(F_1)$ and $\Phi_{\vec{\alpha}}(F_2)$ do not decrease in degree, which leads to the following definition.

Definition 2.20. Let $F_1, F_2 \in \mathbb{F}_q[\mathbb{X}, y]$ and $C = \gcd(F_1, F_2)$. Let $\vec{\alpha} \in \mathbb{F}_{q^m}^n$. We say $\vec{\alpha}$ is a *good point* for F_1, F_2 if $\text{LC}_y(F_1)(\vec{\alpha}) \neq 0$, $\text{LC}_y(F_2)(\vec{\alpha}) \neq 0$ and $\deg \Phi_{\vec{\alpha}}(C) = \deg \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$.

Since $\Phi_{\vec{\alpha}}(C) \mid \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$, if $\vec{\alpha}$ is a good point for F_1, F_2 , we always have $\Phi_{\vec{\alpha}}(C) \approx \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$.

Lemma 2.21. *Let $F_1, F_2 \in \mathbb{F}_q[\mathbb{X}, y]$ and $\vec{\alpha} \in \mathbb{F}_{q^m}^n$. Assume*

$$R = \text{res}_y(F_1/\gcd(F_1, F_2), F_2/\gcd(F_1, F_2)).$$

Let $L = R \cdot \text{LC}_y(F_1) \cdot \text{LC}_y(F_2)$. Then $\vec{\alpha}$ is a good point for F_1, F_2 if and only if $L(\vec{\alpha}) \neq 0$.

Proof. First, we have $R = \text{res}_y(F_1/\gcd(F_1, F_2), F_2/\gcd(F_1, F_2)) \in \mathbb{F}_q[\mathbb{X}]$. Since $\gcd(F_1/\gcd(F_1, F_2), F_2/\gcd(F_1, F_2)) = 1$, $R \neq 0$.

Assume $L(\vec{\alpha}) \neq 0$. Then $\text{LC}_y(F_1)(\vec{\alpha}) \neq 0, \text{LC}_y(F_2)(\vec{\alpha}) \neq 0$ and $R(\vec{\alpha}) \neq 0$. Assume $C = \gcd(F_1, F_2)$. As the leading coefficients of F_1, F_2 are not zero at point $\vec{\alpha}$, by the definition of resultant, $R(\alpha) = \text{res}_y(F_1(y, \vec{\alpha})/C(y, \vec{\alpha}), F_2(y, \vec{\alpha})/C(y, \vec{\alpha}))$. Since $R(\alpha) \neq 0$, $\gcd(F_1(y, \vec{\alpha})/C(y, \vec{\alpha}), F_2(y, \vec{\alpha})/C(y, \vec{\alpha})) = 1$. So $C(y, \vec{\alpha}) \approx \gcd(F_1(y, \vec{\alpha}), F_2(y, \vec{\alpha}))$, which implies $\deg \Phi_{\vec{\alpha}}(C) = \deg \gcd(\Phi_{\vec{\alpha}}(F_1), \Phi_{\vec{\alpha}}(F_2))$. So $\vec{\alpha}$ is a good point for F_1, F_2 . For the other direction, assume $\vec{\alpha}$ is a good point for F_1, F_2 , then $\text{LC}(F_1)(\vec{\alpha}) \neq 0, \text{LC}(F_2)(\vec{\alpha}) \neq 0$. The remaining proof can be traced back directly. \square

Of course, a single $\Phi_{\vec{\alpha}}$ does not usually retain all the information necessary to solve the problem in the original domain. For a fixed primitive root ω of \mathbb{F}_q , we have the following definition.

Definition 2.22. Let $\vec{\alpha} \in \mathbb{F}_{q^m}^n$. If the points $\vec{\alpha}^i, i = 1, 2, \dots, 2T$ and $\vec{\alpha}_k^i, i = 1, 2, \dots, 2T, k = 1, 2, \dots, n$ (the element at the k -th row and i -th column of Table 3) are all *good points* for A and B , then $\vec{\alpha}$ is called a $2T$ - ω good point for A and B .

Base	$(\alpha_1, \alpha_2, \dots, \alpha_n)$	$(\alpha_1^2, \alpha_2^2, \dots, \alpha_n^2)$...	$(\alpha_1^{2T}, \alpha_2^{2T}, \dots, \alpha_n^{2T})$
1	$(\alpha_1\omega, \alpha_2, \dots, \alpha_n)$	$((\alpha_1\omega)^2, \alpha_2^2, \dots, \alpha_n^2)$...	$((\alpha_1\omega)^{2T}, \alpha_2^{2T}, \dots, \alpha_n^{2T})$
2	$(\alpha_1, \alpha_2\omega, \dots, \alpha_n)$	$(\alpha_1^2, (\alpha_2\omega)^2, \dots, \alpha_n^2)$...	$(\alpha_1^{2T}, (\alpha_2\omega)^{2T}, \dots, \alpha_n^{2T})$
\vdots	\vdots	\vdots	\vdots	\vdots
n	$(\alpha_1, \alpha_2, \dots, \alpha_n\omega)$	$(\alpha_1^2, \alpha_2^2, \dots, (\alpha_n\omega)^2)$...	$(\alpha_1^{2T}, \alpha_2^{2T}, \dots, (\alpha_n\omega)^{2T})$

TABLE 3. The evaluation points for the $2T$ - ω good point

Our GCD algorithm cannot reconstruct $\gcd(F_1, F_2)$ using the images if $\vec{\alpha}$ is not a $2T$ - ω good point for F_1 and F_2 .

3. A GCD ALGORITHM OVER FINITE FIELD

In this section, we present a GCD algorithm for polynomials over finite fields. Lemma 2.2 shows that

$$(3.1) \quad G = \gcd(A, B) = \gcd(\text{MoCont}(A), \text{MoCont}(B)) \cdot \gcd(\text{MoPrim}(A), \text{MoPrim}(B)),$$

where $\gcd(\text{MoCont}(A), \text{MoCont}(B))$ and $\gcd(\text{MoPrim}(A), \text{MoPrim}(B))$ are the monomial content of G and the monomial primitive part of G , respectively. Based on Equ. (3.1), to compute G , our algorithm is mainly divided into three parts.

Part 1: compute the monomial content of G by computing

$$\text{MoCont}(G) = \gcd(\text{MoCont}(A), \text{MoCont}(B)).$$

This part is trivial as all polynomials appearing in the computing are monomials.

Part 2: compute the monomial primitive part of G by computing

$$\text{MoPrim}(G) = \gcd(\text{MoPrim}(A), \text{MoPrim}(B)).$$

We explain the framework of this part more details in Section 3.1 and summarize it as a subroutine algorithm (given in Section 3.2).

Part 3: multiply the two parts to get the final result $G = \text{MoCont}(G) \cdot \text{MoPrim}(G)$. This part can be converted to the additions of exponents, as $\text{MoCont}(G)$ is a monomial.

3.1. Framework of our GCD algorithm for monomial primitive polynomials. Assume $A, B \in \mathbb{F}_q[\mathbb{X}]$ are monomial primitive. To compute $G = \gcd(A, B)$, we first compute $C = \gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ for some suitable vector $\mathbf{s} \in \mathbb{N}^n$, and then let $y = 1$ to obtain G from C . The algorithm is mainly divided into following parts:

Part a: find a vector $\mathbf{s} \in \mathbb{N}^n$ such that $G_{(\mathbf{s}, y)}$ has a maximum isolated term w.r.t y using Theorem 2.10.

Part b: compute the polynomial $H = \Delta \cdot \gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ by evaluation-interpolation scheme. Here H is the GCD of $A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)}$ up to a monomial Δ , and the details will be given below.

Part c: delete Δ from H to obtain $\gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ and then $\gcd(A, B)$.

We will explain Part **b** in more details. For matching different evaluations, we need to know the leading coefficient (or some other coefficients in a fixed degree). But it is hard to know in advance before we know the exact form of G . Luckily, in Part **a**, we have found a vector \mathbf{s} such that $G_{(\mathbf{s}, y)}$ has a maximum isolated term, so the leading coefficient of G w.r.t y is a factor of $(x_1 x_2 \cdots x_n)^d$, where d is the partial degree bound of A, B . So we regard the leading coefficient of $\gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ as $(x_1 x_2 \cdots x_n)^d$. The result is only different from $\gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ by a monomial factor which can be removed easily.

Denote $F_1 = A_{(\mathbf{s}, y)}$ and $F_2 = B_{(\mathbf{s}, y)}$. Assume $\vec{\alpha}$ is a good point for $F_1, F_2 \in \mathbb{F}_q[\mathbb{X}, y]$. Regard $C = \gcd(F_1, F_2)$ as the polynomial in y with coefficients in $\mathbb{F}_q[\mathbb{X}]$ and assume

$$C = C_\ell y^{e_\ell} + \cdots + C_1 y^{e_1},$$

where $C_i \in \mathbb{F}_q[\mathbb{X}]$ and $e_\ell > e_{\ell-1} > \cdots > e_1$. As C has a maximum isolated term w.r.t y , C_ℓ is a monomial. The image has the form

$$\gcd(F_1(y, \vec{\alpha}), F_2(y, \vec{\alpha})) = y^{e_\ell} + \frac{C_{\ell-1}(\vec{\alpha})}{C_\ell(\vec{\alpha})} y^{e_{\ell-1}} + \cdots + \frac{C_1(\vec{\alpha})}{C_\ell(\vec{\alpha})} y^{e_1}.$$

Let

$$(3.2) \quad H = \Delta \cdot C = H_\ell y^{e_\ell} + H_{\ell-1} y^{e_{\ell-1}} + \cdots + H_1 y^{e_1},$$

where $\Delta = \frac{(x_1 \cdots x_n)^d}{C_\ell}$, $H_i = \frac{(x_1 \cdots x_n)^d}{C_\ell} C_i$, and in particular $H_\ell = (x_1 \cdots x_n)^d$. As d is a partial degree bound of A, B , C_ℓ divides $(x_1 \cdots x_n)^d$ and Δ is a monomial. So we have

$$(3.3) \quad H(y, \vec{\alpha}) = (\alpha_1 \cdots \alpha_n)^d \cdot \gcd(F_1(y, \vec{\alpha}), F_2(y, \vec{\alpha})).$$

We can evaluate $H_i \in \mathbb{F}_q[\mathbb{X}], i = 1, 2, \dots, \ell - 1$ at the point $\vec{\alpha}$. Varying $\vec{\alpha}$, we can recover all H_i 's from the evaluations by interpolation.

Part **b** can be divided mainly into four steps.

- compute all term bounds T_i 's of all coefficients of $\gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ w.r.t. y using the technique of early termination in section 2.5;

- diversify all the coefficients of $\gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ w.r.t. y using the method given in section 2.4.1 and find a good point $\vec{\alpha}$ using the method given in section 2.6;
- evaluate all the coefficients of $H(y, \vec{\alpha}^i), H(y, \vec{\alpha}_k^i)$ in (3.3), which is possible because $\gcd(F_1(y, \vec{\alpha}^i), F_2(y, \vec{\alpha}^i)), \gcd(F_1(y, \vec{\alpha}_k^i), F_2(y, \vec{\alpha}_k^i))$ are monic;
- interpolate all the coefficients of $H = \Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ in (3.2) from the evaluations by sparse polynomial interpolation.

3.2. Primitive GCD algorithm. We give a GCD algorithm in $\mathbb{F}_q[\mathbb{X}]$ for primitive polynomials.

Algorithm 3.1. Primitive GCD over finite fields

Input:

- Two monomial primitive polynomials $A, B \in \mathbb{F}_q[\mathbb{X}]$.
- A primitive element ω of \mathbb{F}_q .
- A tolerance ϵ .

Output: $G = \gcd(A, B)$ with probability $\geq 1 - \epsilon$; or “Failure.”

Initial

Step 0: Let $d = \max\{\deg_{x_i} A, \deg_{x_i} B, i = 1, 2, \dots, n\}$ and $D = \max\{\deg A, \deg B\}$.

Step 1: If $q < D$, then find an irreducible polynomial $\Upsilon(z)$ over $\mathbb{F}_q[z]$ of degree $k \geq \frac{\log D}{\log q}$. Construct finite field \mathbb{F}_{q^k} as $\mathbb{F}_q[z]/(\Upsilon)$. For the convenience of description, in the following, we still denote \mathbb{F}_{q^k} as \mathbb{F}_q . Find a primitive root of \mathbb{F}_{q^k} and still denote it ω .

Stage I: Find a vector \mathbf{s} such that at least one of $A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)}$ has a maximum isolated term.

Step 2: Let $N = 2 \min\{T_A - 1, T_B - 1\}$. Randomly choose $\mathbf{s} \in [1, N]^n$. If both of $A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)}$ do not have a maximum isolated term, then repeat Step 2.

Step 3: Set $F_1 := A_{(\mathbf{s},y)}, F_2 := B_{(\mathbf{s},y)}$.

Stage II: Find terms bound for all coefficients of $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$.

Step 4: Find an irreducible polynomial $\Phi(z)$ over $\mathbb{F}_q[z]$ of degree

$$r = \left\lceil \frac{\log \frac{1}{\epsilon} + \log 86 + 2n \log(d+1) + 2 \log(nd) + \log \|\mathbf{s}\|_\infty}{\log q} \right\rceil.$$

Construct finite field \mathbb{F}_{q^r} as $\mathbb{F}_q[z]/(\Phi)$.

Step 5: Set $T := 1$. Randomly choose $\vec{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{F}_{q^r}^{*n}$.

Loop:

Step 6: For $i = T, T+1, \dots, 2T-1$

- a: If one of $\text{LC}_y(F_1)(\vec{\sigma}^i), \text{LC}_y(F_2)(\vec{\sigma}^i)$ is zero, then return “Failure.”
- b: Compute the monic GCD of $F_1(y, \vec{\sigma}^i)$ and $F_2(y, \vec{\sigma}^i)$. Let

$$\eta'_i := \gcd(F_1(y, \vec{\sigma}^i), F_2(y, \vec{\sigma}^i)).$$

If one of η'_i has different degree with others, then return “Failure.”

Step 7: Multiply η'_i by the leading coefficient $(\sigma_1 \cdots \sigma_n)^{i \cdot d}$.

For $i = T, T+1, \dots, 2T-1$ do

$$\eta_i := \eta'_i \cdot (\sigma_1 \cdots \sigma_n)^{i \cdot d}.$$

Assume

$$\eta_i := c_{i,1}y^{e_1} + \cdots + c_{i,\ell}y^{e_\ell}, i = 1, \dots, 2T - 1.$$

- Step 8:** Construct Hankel matrices $\mathbf{H}_k := (c_{i+j-1,k})_{i,j=1,\dots,T}, k = 1, 2, \dots, \ell - 1$.
 1. If one of $\det(\mathbf{H}_j)$ is not zero for $j = 1, 2, \dots, \ell - 1$, then $T := 2T$, and goto **Loop**. Write down T_i for each y^{e_i} , which is the first T for $\det(\mathbf{H}_i) = 0$.
Step 9: Let $T := T - 1$; and $T_i = T_i - 1, i = 1, 2, \dots, \ell - 1$.

Stage III: Choose good evaluation points and diversify the GCD.

Step 10: Find an irreducible polynomial $\Phi'(z)$ over $\mathbb{F}_q[z]$ of degree

$$m \geq \left\lceil \frac{\log \frac{1}{\varepsilon} + \log 42 + \log(n+1) + 2 \log(ndT)}{\log q} \right\rceil.$$

Construct finite field \mathbb{F}_{q^m} as $\mathbb{F}_q[z]/(\Phi')$.

Step 11: Randomly choose $\vec{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathbb{F}_{q^m}^n$ and $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}_{q^m}^n$. /* It will be proved that H is diverse w.r.t. y by $\vec{\zeta}$ and $\vec{\alpha}$ is a good point. */

Step 12: Compute $\tilde{A} = A(\zeta_1 x_1, \dots, \zeta_n x_n)$, $\tilde{B} = B(\zeta_1 x_1, \dots, \zeta_n x_n)$. Compute $\tilde{F}_1 = (\tilde{A})_{(s,y)}$, $\tilde{F}_2 = (\tilde{B})_{(s,y)}$.

Stage IV: Evaluate the GCD.

Step 13: For $i = 1, 2, \dots, 2T$

- a: If one of $\text{LC}_y(\tilde{F}_1)(\vec{\alpha}^i)$, $\text{LC}_y(\tilde{F}_2)(\vec{\alpha}^i)$, $\text{LC}_y(\tilde{F}_1)(\vec{\alpha}_k^i)$, and $\text{LC}_y(\tilde{F}_2)(\vec{\alpha}_k^i), k = 1, 2, \dots, n$ is zero, then return “Failure.”
 b: Compute the monic univariate GCD of $\tilde{F}_1(y, \vec{\alpha}^i)$ and $\tilde{F}_2(y, \vec{\alpha}^i)$.

$$f'_i := \gcd(\tilde{F}_1(y, \vec{\alpha}^i), \tilde{F}_2(y, \vec{\alpha}^i)).$$

- c: Compute the monic univariate GCD of $\tilde{F}_1(y, \vec{\alpha}_k^i)$ and $\tilde{F}_2(y, \vec{\alpha}_k^i)$ for $k = 1, 2, \dots, n$

$$g'_{i,k} := \gcd(\tilde{F}_1(y, \vec{\alpha}_k^i), \tilde{F}_2(y, \vec{\alpha}_k^i)).$$

Step 14: Multiply $f'_i, g'_{i,k}$ by the leading coefficients $(\alpha_1 \cdots \alpha_n)^{i \cdot d}$ and $(\alpha_1 \cdots \alpha_n \cdot \omega)^{i \cdot d}$.

For $i = 1, 2, \dots, 2T$ do

$$f_i := f'_i \cdot (\alpha_1 \cdots \alpha_n)^{i \cdot d} = c_{i,1}y^{e_1} + \cdots + c_{i,\ell}y^{e_\ell}$$

$$g_{i,k} := g'_{i,k} \cdot (\alpha_1 \cdots \alpha_n \cdot \omega)^{i \cdot d} = r_{i,k,1}y^{e_1} + \cdots + r_{i,k,\ell}y^{e_\ell}$$

for all $k = 1, 2, \dots, n$.

Stage V: Compute GCD by interpolation.

Step 15: For $j = 1, 2, \dots, \ell - 1$, compute the polynomials by Algorithm 2.17:

$$H'_j := \text{Interpolation}(\omega, c_{i,j}, r_{i,k,j}, i = 1, \dots, 2T_i, k = 1, 2, \dots, n).$$

$$\text{Set } H_j := H'_j(\zeta_1^{-1}x_1, \dots, \zeta_n^{-1}x_n) \text{ and } H_\ell := (x_1 \cdots x_n)^d.$$

Stage VI: Compute the monomial primitive part.

Step 16: For $i = 1, \dots, n$, let $k_i = \min\{\deg(H_j, x_i), j = 1, \dots, \ell\}$

/* $x_1^{k_1} \cdots x_n^{k_n}$ is the monomial content Δ of $\sum_{i=1}^\ell H_i$. */

Step 17: Return the primitive part $(\sum_{i=1}^\ell H_i) / (x_1^{k_1} \cdots x_n^{k_n})$.

Theorem 3.2. *Let $A, B \in \mathbb{F}_q[\mathbb{X}]$ be monomial primitive polynomials and $\omega \in \mathbb{F}_q$ a fixed primitive root. Then Algorithm 3.1 is correct.*

- (1) With probability $\geq 1 - \varepsilon$, it returns the correct GCD and the complexity is $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations.
- (2) The expected complexity is $O^\sim(n^3DT_G(T_A + T_B) \log^2 q)$ bit operations.

Proof. The proof is given in Section 5. \square

In Stage I, we find a suitable vector \mathbf{s} for A, B such that at least one of $A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)}$ has a maximum isolated term. In Stage II, we compute the terms bound T_i of coefficients of $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ in y by using the technique of early termination. In Stage III, we diversify all the coefficients of $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ w.r.t. y by using $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ and choose the good evaluation point $\vec{\alpha}$. In Stage IV, we evaluate all the coefficients of $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ at points $\vec{\alpha}^i, i = 1, 2, \dots, 2T$ and $\vec{\alpha}_k^i, i = 1, \dots, 2T, k = 1, \dots, n$. In Stage V, we interpolate all the coefficients of $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ at points $\vec{\alpha}^i, i = 1, 2, \dots, 2T$ and $\vec{\alpha}_k^i, i = 1, \dots, 2T, k = 1, \dots, n$ by sparse polynomial interpolation. In Stage VI, we remove the factor Δ from $\Delta \cdot \gcd(A_{(\mathbf{s},y)}, B_{(\mathbf{s},y)})$ by computing the monomial content and return the GCD of A, B .

3.3. GCD algorithm for polynomials over finite fields. Based on Algorithm 3.1, we give the complete GCD algorithm polynomials over finite fields.

Algorithm 3.3. GCD over finite fields

Input:

- $A, B \in \mathbb{F}_q[\mathbb{X}]$.
- A primitive element ω of \mathbb{F}_q .
- A tolerance ε .

Output: $G = \gcd(A, B)$ with probability $\geq 1 - \varepsilon$; or “Failure.”

Step 1: Compute the monomial contents and the primitive parts of A and B , and denote them by $C_A := \text{MoCont}(A), C_B := \text{MoCont}(B), P_A := \text{MoPrim}(A), P_B := \text{MoPrim}(B)$.

Step 2: Compute the GCD $G' = \gcd(P_A, P_B)$ by Algorithm 3.1 with tolerance ε .

Step 3: Compute the GCD $C' = \gcd(C_A, C_B)$.

Step 4: Return $G' \cdot C'$.

Theorem 3.4. Let $A, B \in \mathbb{F}_q[\mathbb{X}]$ and $\omega \in \mathbb{F}_q$ a fixed primitive root. Then Algorithm 3.3 is correct.

- (1) With probability $\geq 1 - \varepsilon$, it returns the correct GCD $G = \gcd(A, B)$ and the complexity is $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations.
- (2) The expected complexity is $O^\sim(n^3DT_G(T_A + T_B) \log^2 q)$ bit operations.

Proof. The correctness comes from the Equ. (3.1). Once G' is computed correctly in Step 2, Algorithm 3.3 returns the correct polynomial. According to Theorem 3.2, we compute the correct $\gcd(P_A, P_B)$ with probability $\geq 1 - \varepsilon$. So the correctness is proved.

Now we analyse the complexity. In Step 1, since $\text{MoCont}(A)$ is a monomial, to compute $\text{MoCont}(A)$, it suffices to find each exponent of x_i , which is equivalent to finding the minimum degrees of x_i 's in A . So the cost is $O(nT_A \log D)$ bit operations. As $\text{MoPrim}(A) = A/\text{MoCont}(A)$, to compute $\text{MoPrim}(A)$, just subtract the exponents of $\text{MoCont}(A)$ from the exponents of each term of A , which costs

$O(nT_A \log D)$ bit operations. Similarly, the cost of computing $\text{MoCont}(B)$ and $\text{MoPrim}(B)$ is $O(nT_B \log D)$. So the total cost is $O(n(T_A + T_B) \log D)$ bit operations. In Step 2, by Theorem 3.2, the complexity is $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations. In Step 3, computing the GCD of two monomials is equivalent to comparing the exponent of each x_i of C_A and C_B , and the smaller one is the exponent of the GCD about x_i . So the cost is $O(n \log D)$ bit operations. In Step 4, to compute the product, we can directly add the exponents of C' to the exponents of all terms of G' , which requires $O(nT_G \log D)$ bit operations. \square

4. EXPERIMENTAL RESULTS

In this section, the practical performance of our GCD algorithm for polynomials over finite fields are given. We compare with the default implementation of the GCD algorithm in Maple 2018. The data are collected on a desktop with Windows system, 2.50GHz Core i5 processor and 8GB RAM memory. The codes can be found in <https://github.com/huangqiaolong/Maple-Codes-GCD>.

To test the average running times of the algorithm, we use the Maple command *randpoly* to construct five pairs of random co-prime polynomials $A, B \in \mathbb{F}_p[\mathbb{X}]$ and a polynomial G within the given terms bound and degree bound, then expand $A \cdot G$ and $B \cdot G$ and compute $G = \text{gcd}(A \cdot G, B \cdot G)$ with our algorithm and the Maple command $\text{Gcd}(A, B) \bmod p$. The average times are collected. In our testing, we fix $p = 10000019$ and use the primitive element $\omega = 6$. In our code, we do not use the expansion of finite fields. A simple analysis shows that the success rate is $\geq 1 - \frac{86n^2t^2d^2 \min\{t_A, t_B\} + 168(n+1)t^2n^2d^2 \min\{t_A, t_B\}}{p}$, where $t_A := \#(A \cdot G)$, $t_B = \#(B \cdot G)$, $t = \#G$ and d is the partial degree bound of $A \cdot G$ and $B \cdot G$.

Three benchmarks are used for the experiments and the results are given in Figures 3-6, where the red lines are the timings of our algorithm and the black lines are the timings of the Maple code.

Benchmark 1. For the first benchmark, we fix the degrees of A, B, G for $n = 6$, $\deg A = \deg B = \deg G = 30$, and change $\#A = \#B = \#G = T$ from 2 to 152. The computing times are shown in Figure 3, where we take 60 seconds as the threshold: once exceeding 60 seconds, we terminate the computing. From this figure, we can see that the Maple code can compute GCDs with terms up to 18 and our code can compute GCDs with terms up to 150.

Benchmark 2. For the second benchmark, we fix the terms of A, B, G for $\#A = \#B = \#G = 30$, and the degrees of A, B, G for $\deg A = \deg B = \deg G = 100$, and change n from 1 to 200. The computing times are shown in Figure 4, where the threshold is set to be 60 seconds. From this figure, we can see that the Maple code can compute GCDs with numbers of variables up to 3 and our code can compute GCDs with numbers of variables up to 200, so the new algorithm has about 2-orders of magnitude improvement. The computing time of the Maple code increases rapidly when $n > 3$ and is $\geq 600s$ for $n \geq 10$.

Benchmark 3. For the third benchmark, we fix the terms of A, B, G for $n = 6$, $\#A = \#B = \#G = 30$, and change $D = \deg(A) = \deg(B) = \deg(G)$ from 5 to 29525 in increments of 500. The computing times are shown in Figure 5, where the threshold is set to be 100 seconds. From this figure, we can see that the Maple code can compute GCDs with degrees up to 23 and our code

can compute GCDs with degrees up to 29525, so the new algorithm has about 3-orders of magnitude improvement. In Figure 6, we give more details by changing $D = \deg(A) = \deg(B) = \deg(G)$ from 5 to 14. Figure 6 shows that $D = 8$ is the intersection point. When $D < 8$, the Maple code is better than ours, but when $D \geq 8$, our algorithm costs less and increase slowly until $d = 29525$. The timings for the Maple code increase drastically after $D \geq 23$. These experimental results also validate the complexities in Table 2.

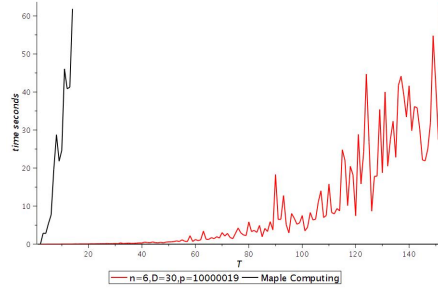


FIGURE 3. Average running time with varying terms

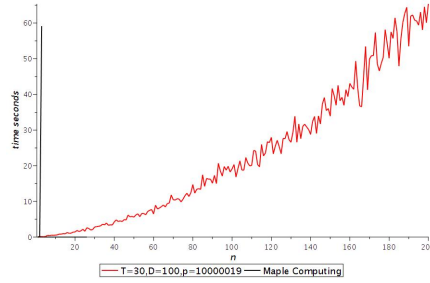


FIGURE 4. Average running time with varying number of variables

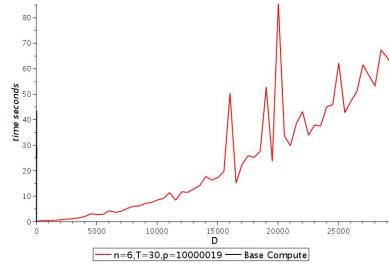


FIGURE 5. Average running time with varying degree: the global picture for all degrees

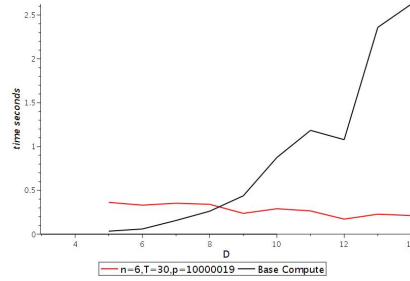


FIGURE 6. Average running time with varying degree: the picture with degrees ≤ 14

Remark 4.1. In our Maple code, in Step 2 of Algorithm 3.1, to isolate the maximum term of A or B , we actually let $N = 1, 2, 2^2, 2^3, \dots$ because the success rate of isolation is very high in practice even if N is small. In Step 8, we actually let $T = 1, 2, 3, \dots$ and increase T by only 1 each time, as the cost of computing the GCD of two univariate polynomials is more expensive than testing the Hankel matrices.

5. PROOF OF THEOREM 3.2

We first prove some lemmas.

Lemma 5.1. *Let $A \in \mathbb{F}_q[\mathbb{X}]$, $D = \deg A$, and $t = \#A$. If A contains all variables x_i 's, then $Dt \geq n$.*

Proof. Assume $A = c_1M_1 + \dots + c_tM_t$ and each M_i contains k_i different variables. Then $Dt \geq k_1 + k_2 + \dots + k_t$. As A contains all variables x_i 's, $k_1 + k_2 + \dots + k_t \geq n$. So we have $Dt \geq n$. \square

Let $H = \Delta \cdot \gcd(A_{(\mathbf{s}, y)}, B_{(\mathbf{s}, y)})$ defined in (3.2) have the form $H = H_\ell y^{e_\ell} + H_{\ell-1} y^{e_{\ell-1}} + \dots + H_1 y^{e_1}$, where $H_i \in \mathbb{F}_q[\mathbb{X}]$, $e_1 < e_2 < \dots < e_\ell$ and $\deg H_i \leq D + nd$.

Lemma 5.2. *In Stage II, Algorithm 3.1 returns correct bounds T_i 's, which satisfy $\#H_i \leq T_i < 2\#H_i$, with probability $\geq 1 - \frac{\varepsilon}{2}$.*

Proof. We use $\vec{\sigma}$ to test the term bounds of all the coefficients of H w.r.t y . Assume $t_i = \#H_i$ and $t = \max\{t_i, i = 1, \dots, \ell\}$. Without loss of generality, consider H_1 . We test if $\lfloor \log t_1 \rfloor + 2$ determinants are zero, whose orders are $2^0, 2^1, \dots, 2^{\lfloor \log t_1 \rfloor + 1}$. Assume the corresponding determinants are $\mathbf{DetH}_i, i = 0, 1, \dots, \lfloor \log t_1 \rfloor + 1$, by [11], $\deg \mathbf{DetH}_i \leq (2^i)^2(D + nd)$. The success of early termination is decided by the selection of $\vec{\sigma}$ such that all $\mathbf{DetH}_i(\vec{\sigma}) \neq 0, i = 0, 1, \dots, \lfloor \log t_1 \rfloor$. Multiply all of them into one polynomial $\Gamma_1 := \prod_{i=0}^{\lfloor \log t_1 \rfloor} \mathbf{DetH}_i$, with degree $\leq \sum_{i=0}^{\lfloor \log t_1 \rfloor} 2^{2i}(D + nd) \leq \frac{4t_1^2}{3}(D + nd)$. For the same reason, for each H_i , there exists a non-zero condition polynomial Γ_i , such that if $\Gamma_i(\vec{\sigma}) \neq 0$. Then Step 9 returns a correct term bound T_i for H_i . The degree of $\Gamma_i \leq \frac{4t_i^2}{3}(D + nd)$.

As the evaluations of H_i come from the images of GCD of F_1 and F_2 , $\vec{\sigma}^i, i = 1, 2, \dots, 4t$ should all be good points for F_1 and F_2 .

Let $R = \text{res}_y(F_1/\gcd(F_1, F_2), F_2/\gcd(F_1, F_2))$. Set $L = \text{LC}_y(F_1) \cdot \text{LC}_y(F_2) \cdot R \in \mathbb{F}_q[\mathbb{X}]$. As L is a non-zero polynomial, $L(x_1^i, \dots, x_n^i)$ is also a non-zero polynomial. So $(\sigma_1^i, \dots, \sigma_n^i)$ is a good point if $(\sigma_1, \dots, \sigma_n)$ is not a zero of $L(x_1^i, \dots, x_n^i)$. Define a non-zero polynomial

$$L_{\text{term}} = \prod_{i=1}^{4t} L(x_1^i, \dots, x_n^i) \prod_{i=1}^{\ell-1} \Gamma_i.$$

So Step 9 returns term bounds T_i satisfying $\#H_i \leq T_i < 2\#H_i$ if $\vec{\sigma}$ satisfies $L_{\text{term}}(\vec{\sigma}) \neq 0$. By Lemma 2.6, $\deg R \leq 2\|\mathbf{s}\|_\infty \deg A \deg B \leq 2\|\mathbf{s}\|_\infty D^2$, so we have $\deg L \leq \deg(\text{LC}_y(A)) + \deg(\text{LC}_y(B)) + \deg R \leq 2D + 2\|\mathbf{s}\|_\infty D^2$. So $\deg L(x_1^i, \dots, x_n^i) \leq i(2D + 2\|\mathbf{s}\|_\infty D^2)$, which implies $\deg \prod_{i=1}^{4t} L(x_1^i, \dots, x_n^i) \leq \sum_{i=1}^{4t} i(2D + 2\|\mathbf{s}\|_\infty D^2) = 2t(4t+1)(2D + 2\|\mathbf{s}\|_\infty D^2)$. As $D \leq nd$ and $\ell \leq \|\mathbf{s}\|_\infty D + 1$, we have $\deg L_{\text{term}} \leq 2t(4t+1)(2D + 2\|\mathbf{s}\|_\infty D^2) + \frac{4t^2}{3}(D + nd)(\ell-1) \leq 2t(4t+1)(2D + 2\|\mathbf{s}\|_\infty D^2) + \frac{4t^2}{3}(D + nd)\|\mathbf{s}\|_\infty D < 43n^2t^2d^2\|\mathbf{s}\|_\infty$. Since $t \leq (d+1)^n$, $\deg L_{\text{term}} < 43(d+1)^{2n}n^2d^2\|\mathbf{s}\|_\infty$. Since $r \geq \frac{\log \frac{1}{\varepsilon} + \log 86 + 2n \log(d+1) + 2 \log(nd) + \log \|\mathbf{s}\|_\infty}{\log q}$, $q^r \geq \frac{86}{\varepsilon}(d+1)^{2n}n^2d^2\|\mathbf{s}\|_\infty$. So by Lemma 2.4, Stage II returns correct term bounds with probability

$$\geq 1 - \frac{\deg L_{\text{term}}}{q^r - 1} \geq 1 - \frac{43(d+1)^{2n}n^2d^2\|\mathbf{s}\|_\infty - 1}{\frac{86}{\varepsilon}(d+1)^{2n}n^2d^2\|\mathbf{s}\|_\infty - 1} \geq 1 - \frac{\varepsilon}{2}.$$

\square

Lemma 5.3. *In Stages III, IV, V and VI, if $T_i \geq \#G_i$, Algorithm 3.1 returns the correct $G = \gcd(A, B)$, with probability $\geq 1 - \frac{\varepsilon}{2}$.*

Proof. Once $H = \sum_{i=1}^{\ell} H_i y^{e_i}$ is computed correctly in Stage V, we can obtain the correct $G = \gcd(A, B)$. So we analyse the probability of obtaining correct H_j 's in Step 15. We will prove that H is diverse w.r.t. y by $\vec{\zeta}$ and $\vec{\alpha}$ is a good point.

In our algorithm, H should be diverse w.r.t. y . So a point $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ should be chosen, such that $H_1(\vec{\zeta} \mathbb{X}), \dots, H_{\ell-1}(\vec{\zeta} \mathbb{X})$ are all diverse. Considering H_1 and assume that $H_1 = \Delta \cdot (c_1 M_1 + \dots + c_u M_u)$, $\vec{\zeta}$ diversifying H_1 means $\prod_{i \neq j} (c_i M_i(\vec{\zeta}) - c_j M_j(\vec{\zeta})) \neq 0$. Set $U_1 = \prod_{i \neq j} (c_i M_i - c_j M_j)$. Then $\vec{\zeta}$ diversifies H_1 if it is not a zero of U_1 . For the same reason, we can set polynomials $U_2, \dots, U_{\ell-1}$ for $H_2, \dots, H_{\ell-1}$ and set $U = U_1 \cdots U_{\ell-1}$. If $U(\vec{\zeta}) \neq 0$, then the point $\vec{\zeta}$ satisfies our diversification condition. Estimate the degree bound of U ,

$$\deg U = \deg U_1 + \dots + \deg U_{\ell-1} \leq \frac{T(T-1)}{2} D(\ell-1) \leq \frac{T(T-1)}{2} D^2 \|\mathbf{s}\|_{\infty}.$$

Now we evaluate the GCD in Step 13. Consider point $(\alpha_1, \dots, \alpha_n)$. As F_1 and F_2 are diversified, consider the polynomial $R_{\vec{\zeta}} = \text{res}_y(\tilde{F}_1 / \gcd(\tilde{F}_1, \tilde{F}_2), \tilde{F}_2 / \gcd(\tilde{F}_1, \tilde{F}_2))$. Set $Q = \text{LC}_y(\tilde{F}_1) \cdot \text{LC}_y(\tilde{F}_2) \cdot R_{\vec{\zeta}} \in \mathbb{F}_{q^m}[\mathbb{X}]$. As Q is a non-zero polynomial, $Q(x_1^i, \dots, x_n^i)$ is also a non-zero polynomial. So $(\alpha_1^i, \dots, \alpha_n^i)$ is a good point if $(\alpha_1, \dots, \alpha_n)$ is not a zero of $Q(x_1^i, \dots, x_n^i)$. Clearly, $Q(x_1^i, \dots, \omega^i x_k^i, \dots, x_n^i)$ is a non-zero polynomial. For the same reason, $(\alpha_1^i, \dots, (\alpha_k \omega)^i, \dots, \alpha_n^i)$ is a good point if $(\alpha_1, \dots, \alpha_n)$ is not a zero of $Q(x_1^i, \dots, \omega^i x_k^i, \dots, x_n^i)$. Define a non-zero polynomial

$$Q_{\text{good}} = \prod_{i=1}^{2T} Q(x_1^i, \dots, x_n^i) \prod_{k=1}^n \prod_{j=1}^{2T} Q(x_1^j, \dots, x_k^j \omega^j, \dots, x_n^j).$$

So $(\alpha_1, \dots, \alpha_n)$ is a $2T$ - ω good point for \tilde{F}_1, \tilde{F}_2 if and only if

$$Q_{\text{good}}(\vec{\alpha}, \vec{\zeta}) \neq 0.$$

If $\vec{\zeta}$ are constants, then $\deg R_{\vec{\zeta}} \leq 2\|\mathbf{s}\|_{\infty} \deg A \deg B \leq 2\|\mathbf{s}\|_{\infty} D^2$. Now regard $\vec{\zeta}$ as variables. So $\deg_{\mathbb{X}} R_{\vec{\zeta}} \leq 2\|\mathbf{s}\|_{\infty} \deg_{\mathbb{X}} A \deg_{\mathbb{X}} B \leq 2\|\mathbf{s}\|_{\infty} D^2$ and $\deg_{\vec{\zeta}} R_{\vec{\zeta}} \leq 2\|\mathbf{s}\|_{\infty} \deg_{\vec{\zeta}} A \deg_{\vec{\zeta}} B \leq 2\|\mathbf{s}\|_{\infty} D^2$. We thus have $\deg_{\mathbb{X}} Q \leq \deg_{\mathbb{X}}(\text{LC}_y(\tilde{F}_1)) + \deg_{\mathbb{X}} R + \deg_{\mathbb{X}}(\text{LC}_y(\tilde{F}_2)) \leq 2D + 2\|\mathbf{s}\|_{\infty} D^2$ and $\deg_{\vec{\zeta}} Q \leq \deg_{\vec{\zeta}}(\text{LC}_y(\tilde{F}_1)) + \deg_{\vec{\zeta}}(\text{LC}_y(\tilde{F}_2)) + \deg_{\vec{\zeta}} R_{\vec{\zeta}} \leq 2D + 2\|\mathbf{s}\|_{\infty} D^2$. Then $\deg Q(x_1^i, \dots, x_n^i) \leq (2D + 2\|\mathbf{s}\|_{\infty} D^2) + i \cdot (2D + 2\|\mathbf{s}\|_{\infty} D^2) = (2D + 2\|\mathbf{s}\|_{\infty} D^2)(i+1)$, which implies $\deg \prod_{i=1}^{2T} Q(x_1^i, \dots, x_n^i) \leq \sum_{i=1}^{2T} (i+1)(2D + 2\|\mathbf{s}\|_{\infty} D^2) = T(2T+3)(2D + 2\|\mathbf{s}\|_{\infty} D^2)$. So $\deg Q_{\text{good}} \leq (n+1)T(2T+3)(2D + 2\|\mathbf{s}\|_{\infty} D^2)$.

Now we turn to Stage V. The correctness of H_i comes from the correctness of the interpolation. According to Assumption 2.16, as $\deg H_i \leq D + nd$, $\vec{\alpha}$ should not vanish a polynomial with degree $\leq (n+1) \frac{T(T-1)}{2} (D+nd)(\ell-1) \leq (n+1) \frac{T(T-1)}{2} (D+nd)(\|\mathbf{s}\|_{\infty} D)$. So the total degree of the three condition polynomials is $< 21(n+1)T^2 n^2 d^2 \|\mathbf{s}\|_{\infty}$. As $m \geq \frac{\log \frac{1}{\varepsilon} + \log 42 + \log(n+1) + 2 \log(ndT)}{\log q}$, $q^m \geq \frac{42}{\varepsilon} (n+1)T^2 n^2 d^2 \|\mathbf{s}\|_{\infty}$. By Lemma 2.4, if T is the upper bound of all $\#H_i$'s and the interpolation computes the correct polynomials with probability

$$\geq 1 - \frac{21(n+1)T^2 n^2 d^2 \|\mathbf{s}\|_{\infty} - 1}{q^m - 1} \geq 1 - \frac{\varepsilon}{2}.$$

□

We now prove (1) of Theorem 3.2.

Proof. By Lemma 5.2, T_i 's are upper bounds of H_i 's with probability $\geq 1 - \frac{\varepsilon}{2}$. By Lemma 5.3, if each T_i is an upper bound for $\#H_i$, then the interpolation algorithm computes the correct polynomials with probability $1 - \frac{\varepsilon}{2}$. So totally, Algorithm 3.1 returns the correct polynomials with probability $(1 - \frac{\varepsilon}{2})^2 \geq 1 - \varepsilon$. The correctness is proved.

We analyse the complexity. Here T is the upper bound of $\#H_i$'s and $T_G = \sum_{i=1}^{\ell} \#H_i = \#G$

Stage I: In Step 2, randomly choosing a vector \mathbf{s} costs $O(n \log N)$ bit operations. In Step 3, computing $A_{(\mathbf{s}, y)}$, $B_{(\mathbf{s}, y)}$ costs $O^\sim(n(T_A + T_B)(\log d + \log N))$ bit operations. By Theorem 2.10, \mathbf{s} is a suitable vector for A or B with probability $\geq \frac{1}{2}$, so the expected cost is $O^\sim(n(T_A + T_B)(\log d + \log N))$ bit operations. Since $N \in O(\min\{T_A, T_B\})$, the expected cost is $O^\sim(n(T_A + T_B) \log d)$ bit operations

Stage II: In Step 6, we compute $F_1(y, \vec{\sigma}^i)$ and $F_2(y, \vec{\sigma}^i)$. As the partial degree of $F_1(y, \vec{\sigma}^i)$ is $O(d\|\mathbf{s}\|_\infty)$, the complexity is $O^\sim(nT_A \log d \log q^r + TT_A \log q^r)$ bit operations. Plus the cost for $F_2(y, \vec{\sigma}^i)$, the total complexity is $O^\sim(n(T_A + T_B) \log d \log q^r + T(T_A + T_B) \log q^r)$ bit operations. To compute the GCD of $F_1(y, \vec{\sigma}^i)$ and $F_2(y, \vec{\sigma}^i)$, the complexity is $O^\sim(TD\|\mathbf{s}\|_\infty \log q^r)$ bit operations. In Step 8, to test the Hankel matrices, it costs $O^\sim(T \log q^r)$ bit operations. So the complexity is $O^\sim(nT(T_A + T_B) \log d \log q^r + T(T_A + T_B) \log q^r + TD \min\{T_A, T_B\} \log q^r)$ bit operations.

Since q^r is $O(\frac{1}{\varepsilon} d^{2n} n^2 d^2 \|\mathbf{s}\|_\infty)$, the complexity is $O^\sim(n(T_A + T_B) \log D(n \log d + \log \frac{1}{\varepsilon} \log q + T(T_A + T_B)(n \log d + \log \frac{1}{\varepsilon} \log q) + TD \min\{T_A, T_B\}(n + \log \frac{1}{\varepsilon} \log q))$ bit operations.

Stage III: In Step 12, the cost is $O^\sim(n(T_A + T_B) \log d \log q^m)$ bit operations.

Stage IV: In Step 13, we compute $\tilde{F}_1(y, \vec{\alpha}^i)$, $\tilde{F}_2(y, \vec{\alpha}^i)$, $\tilde{F}_1(y, \vec{\alpha}_k^i)$ and $\tilde{F}_2(y, \vec{\alpha}_k^i)$. The complexity is $O^\sim(nT(T_A + T_B) \log d \log q^m)$ bit operations, which is $O^\sim(nT(T_A + T_B)(\log \frac{1}{\varepsilon} + \log d) \log d \log q)$ bit operations. To compute the GCDs, the complexity is $O^\sim(nTD\|\mathbf{s}\|_\infty \log q^m)$ bit operations, which is $O^\sim(nTD \min\{T_A, T_B\} \log \frac{1}{\varepsilon} \log q)$ bit operations.

Stage V: In Step 15, for each interpolation of H_i , as $\deg_{x_j} H_i \leq 2d$ for any $x_j, j = 1, \dots, n$ and $\#H_i \leq T_i$, by Theorem 2.18, the cost is $O^\sim(nT_i \log^2 q^m + nT_i \sqrt{d} \log q)$ bit operations, which is $O^\sim(nT_i \log^2 d \log^2 \frac{1}{\varepsilon} \log^2 q + nT_i \sqrt{d} \log q)$ bit operations. So the total complexity is $O^\sim(nT_G \log^2 d \log^2 \frac{1}{\varepsilon} \log^2 q + nT_G \sqrt{d} \log q)$ bit operations.

Shoup [15] presented an algorithm to construct an irreducible polynomial of degree k over finite field \mathbb{F}_q with an expected number of $O^\sim(k^2 + k \log q)$ operations in \mathbb{F}_q , which is $O^\sim(k^2 \log q + k \log^2 q)$ bit operations. So the complexity for constructing irreducible polynomials of degrees m and r is $O^\sim(n^2 \log^2(d\|\mathbf{s}\|_\infty) \log^2 \frac{1}{\varepsilon} \log q)$ bit operations. Actually, by Lemma 5.1, $nD(T_A + T_B) \geq n^2$. So the total complexity of our algorithm is $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations.

As in Step 1, we always let $q > D$. If $q < D$, we extend \mathbb{F}_q to $\mathbb{F}_{q'}$ with $q' > D$. Finding a new primitive root costs $O(q^{\frac{1}{4}+\epsilon}) = O(D^{\frac{1}{4}+\epsilon})$ bit operations. So if $q \leq D$, we use q' instead of q , the complexity is $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q' + n^2 \log^2 D \log^2 \frac{1}{\varepsilon} \log q')$ bit operations. As $\log q' = \log D$, the cost is $O^\sim(nDT_G(T_A +$

$T_B) \log^2 \frac{1}{\varepsilon}$) bit operations. So the cost is in $O^\sim(nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations. \square

We now prove (2) of Theorem 3.2.

Proof. We consider the worst case. Double the terms bound T , as the bad points for $A_{(s,y)}$ and $B_{(s,y)}$, T becomes $O(d^n)$, the complexity for wrong computing is at most $O^\sim(nDd^n(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$ bit operations. But it happens only with probability $\leq \varepsilon$. So the expected complexity is

$$O^\sim((1 - \varepsilon)nDT_G(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q + \varepsilon nDd^n(T_A + T_B) \log^2 \frac{1}{\varepsilon} \log^2 q)$$

Now we choose $\varepsilon = \frac{1}{nDd^n(T_A + T_B)}$. Then the expected complexity is $O^\sim(n^3DT_G(T_A + T_B) \log^2 q)$ bit operations. \square

6. CONCLUSION

In this paper, we proposed a new method for computing sparse GCDs of multivariate polynomials. Our algorithm works for polynomials over any finite field. We map the multivariate polynomials into univariate ones which keeps the sparse structure. Then recover the target multivariate GCD via a variant of Ben-Or/Tiwari's interpolation algorithm over finite field. We also give the explicit bit complexity for the algorithm, which is better than that of Zippel's algorithm. The algorithm is shown to be 1-3 orders of magnitude faster than the default Maple GCD codes for various benchmarks.

REFERENCES

- [1] Andrew Arnold, *Sparse polynomial interpolation and testing*, Phd Thesis, University of Waterloo, 2016.
- [2] Michael Ben-Or and Prasoorn Tiwari, *A deterministic algorithm for sparse multivariate polynomial interpolation (extended abstract)*, Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA (Janos Simon, ed.), ACM, 1988, pp. 301–309.
- [3] W. Steven Brown, *On euclid's algorithm and the computation of polynomial greatest common divisors*, Journal of the ACM (JACM) **18** (1971), no. 4, 478–504.
- [4] George E. Collins, *Subresultants and reduced polynomial remainder sequences*, J. ACM **14** (1967), no. 1, 128–142.
- [5] Annie Cuyt and Wen-shin Lee, *A new algorithm for sparse interpolation of multivariate polynomials*, Theoretical Computer Science **409** (2008), no. 2, 180–185.
- [6] Mark Giesbrecht and Daniel S Roche, *Diversification improves interpolation*, Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation, 2011, pp. 123–130.
- [7] Jiaxiong Hu and Michael Monagan, *A fast parallel sparse polynomial gcd algorithm*, Proceedings of the ACM on International Symposium on Symbolic and Algebraic Computation, 2016, pp. 271–278.
- [8] ———, *A fast parallel sparse polynomial gcd algorithm*, Journal of Symbolic Computation **105** (2021), 28–63.
- [9] Qiao-Long Huang, *Sparse polynomial interpolation based on diversification*, Science China Mathematics (2021), 1–16.
- [10] Erich Kaltofen, *Greatest common divisors of polynomials given by straight-line programs*, Journal of the ACM (JACM) **35** (1988), no. 1, 231–264.
- [11] Erich Kaltofen and Wen-shin Lee, *Early termination in sparse interpolation algorithms*, Journal of Symbolic Computation **36** (2003), no. 3-4, 365–400.

- [12] Erich Kaltofen and Barry M Trager, *Computing with polynomials given by black boxes for their evaluations: Greatest common divisors, factorization, separation of numerators and denominators*, Journal of Symbolic Computation **9** (1990), no. 3, 301–320.
- [13] Adam R Klivans and Daniel Spielman, *Randomness efficient identity testing of multivariate polynomials*, Proceedings of the thirty-third annual ACM symposium on Theory of computing, 2001, pp. 216–223.
- [14] Joel Moses and David Y. Y. Yun, *The EZ GCD algorithm*, Proceedings of the ACM annual conference, Atlanta, Georgia, USA, August 27-29, 1973 (Irwin E. Perlin and Thomas J. McConnell Jr., eds.), ACM, 1973, pp. 159–166.
- [15] Victor Shoup, *Fast construction of irreducible polynomials over finite fields*, Journal of Symbolic Computation **17** (1994), no. 5, 371–391.
- [16] Min Tang, Bingyu Li, and Zhenbing Zeng, *Computing sparse gcd of multivariate polynomials via polynomial interpolation*, Journal of Systems Science and Complexity **31** (2018), no. 2, 552–568.
- [17] Paul S. Wang, *The EEZ-GCD algorithm*, SIGSAM Bull. **14** (1980), no. 2, 50–60.
- [18] Richard Zippel, *Probabilistic algorithms for sparse polynomials*, Symbolic and Algebraic Computation, EUROSAM '79, An International Symposium on Symbolic and Algebraic Computation, Marseille, France, June 1979, Proceedings (Edward W. Ng, ed.), Lecture Notes in Computer Science, vol. 72, Springer, 1979, pp. 216–226.

SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY
Email address: huangqiaolong@sdu.edu.cn

UCAS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES
Email address: xgao@mmrc.iss.ac.cn