POLYNOMIAL SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we discuss the polynomial solutions of algebraic differential equations. We present an approach to find the polynomial solutions of algebraic differential equations with terms less than or equal to three.

1 Introduction

Constructing elementary function solutions for differential equations may be traced back to the work of Liouville. In a pioneering paper [5], Risch gave an algorithm for finding Liouvillian solutions for integration. In [1], Kovacic presented a method for solving second order linear homogeneous differential equations. In [7], Singer proposed a method for finding Liouvillian solutions of general linear differential equations. In [2], Li and Schwarz gave a method to find rational solutions for a class of partial differential equations. All these work are limited to linear cases. In this paper, we try to give some partial solutions to this problem for non-linear differential equations. We present an approach to find polynomial solutions for algebraic differential equations with terms less than or equal to three. In the two-term case, the method is complete for differential equations with rational function coefficients. In the three-term case, the method is complete only for differential equations with constant coefficients. The approach consists of two phases. First, we determine the upper bound for the degree of the polynomial solutions. Second, we use Wu-Ritt's zero decomposition algorithm to find these solutions. We may also determine whether these solutions are general solutions of the given differential equations. The results reported in this paper are used to define genus for certain differential sets [3].

The rest of the paper is organized as follows. In Section 2, we introduce some notations. In Section 3, we show how to find polynomial solutions with Wu-Ritt's zero decomposition algorithm. In Section 4, we treat the binomial case. In Section 5, we treat the trinomial case.

2 Notations and Terminology

Let $K = \mathbb{C}(x)$ be the field of rational functions over the field of complex numbers $\mathbb{C}$, and $\mathbb{F} = K\{y\}$ the differential polynomial ring over K [6,4], where y is a differential indeterminate. An element $f \in \mathbb{F}$ is called a differential polynomial. The equation $f = 0$ is called an algebraic differential equation if f is a differential polynomial in $\mathbb{F}$ with at least one term involving the derivative of y. We denote by y_i the i-th derivation of y. For a differential polynomial P not in K, let y_o be the highest derivative appear in P. Then o is called the order of P. $S = \frac{\partial P}{\partial y_o}$ is called the *separant* of P. For two d-pols P and Q, let $\mathrm{prem}(P, Q)$ be the pseudo-remainder of P with respect to Q [4,6].

Let E be a universal extension field of K [6]. For a differential polynomial set PS and a differential polynomial D, let

$$\begin{aligned}\mathrm{Zero}(PS) &= \{\eta = (\eta_1, \cdots, \eta_n) \in E^n \mid \forall P \in PS, P(\eta) = 0\}.\\ \mathrm{Zero}(PS/D) &= \mathrm{Zero}(PS) - \mathrm{Zero}(D)\end{aligned}$$

A monomial in $\mathbb{F}$ is always arranged in the following form

$$a \prod_{i=1}^{r} y_{\beta_i}^{\alpha_i}$$

where $a \in \mathbb{C}(x)$, $\alpha_i \in \mathbb{N}^+ (i = 1, \cdots, r)$, $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$. Then an element $f \in \mathbb{F}$ can be written as the following form,

$$\begin{aligned} f &= f_1 + f_2 + \cdots + f_n \\ &= a_1 \textstyle\prod_{i=1}^{r_1} y_{\beta_{1i}}^{\alpha_{1i}} + a_2 \prod_{i=1}^{r_2} y_{\beta_{2i}}^{\alpha_{2i}} + \cdots + a_n \prod_{i=1}^{r_n} y_{\beta_{ni}}^{\alpha_{ni}} = 0 \end{aligned} \tag{2.1}$$

Without loss of generality, we may assume that the a_i's are polynomials of x with degree $d_i, (i = 1, \cdots, n)$ respectively.

In order to determine the upper bound for the degree of the polynomial solutions of the equation (2.1), we introduce the following notions.

Definition 2.1 Given an algebraic differential equation (2.1), the following equations are called the *degree equations*

$$\sum_{i=1}^{r_s} \alpha_{si}(d - \beta_{si}) + d_s = \sum_{i=1}^{r_t} \alpha_{ti}(d - \beta_{ti}) + d_t \quad (1 \leq s < t \leq n) \qquad [s-t]$$

If a degree equation $[s - t]$ $(1 \leq s < t \leq n)$ has a unique positive integer solution d_0, such that $d_0 > \min\{\beta_{s1}, \beta_{t1}\}$, then d is called the *effective solution* of the degree equation $[s - t]$.

Definition 2.2 Given an algebraic differential equation (2.1), if there exist l_j $(1 \le l_1 < l_2 < \cdots < l_m \le n)$ satisfying

$$\sum_{i=1}^{l_1} \alpha_{l_1 i} = \sum_{i=1}^{l_2} \alpha_{l_2 i} = \cdots = \sum_{i=1}^{l_m} \alpha_{l_m i}$$

$$\sum_{i=1}^{l_1} \alpha_{l_1 i}\beta_{l_1 i} - d_{l_1} = \sum_{i=1}^{l_2} \alpha_{l_2 i}\beta_{l_2 i} - d_{l_2} = \cdots = \sum_{i=1}^{l_m} \alpha_{l_m i}\beta_{l_m i} - d_{l_m}$$

then the following differential polynomial is called an *isotropic polynomial* of the equation (2.1).

$$\Lambda_{l_1 l_2 \cdots l_m} = f_{l_1} + f_{l_2} + \cdots + f_{l_m}$$

and the corresponding equation $\Lambda_{l_1 l_2 \cdots l_m} = 0$ is called the *isotropic equation* of equation (2.1).

Definition 2.3 Let $\Lambda_{l_1 l_2 \cdots l_m} = 0$ be one of the isotropic equations of equation (2.1). Then the following equation is called the *characteristic equation* of $\Lambda_{l_1 l_2 \cdots l_m} = 0$

$$c_{l_1} \prod_{i=1}^{r_{l_1}} [d(d-1)\cdots(d-\beta_{l_1 i}+1)]^{\alpha_{l_1 i}} + \cdots + c_{l_m} \prod_{i=1}^{r_{l_m}} [d(d-1)\cdots(d-\beta_{l_m i}+1)]^{\alpha_{l_m i}} = 0 \tag{2.2}$$

Where c_{l_i}'s are the leading coefficients of $a_{l_i} (i = 1, \cdots, m)$ respectively. If d_0 is a positive integer solution for (2.2), such that $d_0 > \min\{\beta_{l_1 1}, \cdots, \beta_{l_m 1}\}$, then d_0 is called an *effective solution* of the characteristic equation (2.2).

3 Polynomial Solutions of Algebraic Differential Equations

In what follows, we assume that $F(y) \in K\{y\}$ is an irreducible differential polynomial with at least one term containing y. If it does not have such term, let β be the least number such that y_β appears in f. Then substituting $y_{\beta+\gamma}$ for $\tilde{y}_\gamma$, we can get the desired form.

We know that

$$\mathrm{sat}(F) = \{P \in K\{y\} | \mathrm{prem}(P, F) = 0\}$$

is a prime ideal and a generic point of $\mathrm{Zero}(\mathrm{sat}(P))$ is called the *general solution* of $P(y) = 0$ [6]. A solution of $F(y) = 0$ is called *singular solution* if the separant of F vanishes at the solutions.

In this paper, we are going to find polynomial solutions for $F(y) = 0$, which can be represented as the following form

$$\eta = \sum_{i=0}^{n} b_i x^i \tag{3.1}$$

where b_i are constants. Let $P(x)$ be a polynomial solution of an algebraic differential equation $f = 0$. If it is the polynomial solution of $f = 0$ which has the highest degree and with arbitrary coefficients, then it is called the *universal polynomial solution* of the equation $f = 0$ and denoted by $[d]$, otherwise, it is called a *sharing polynomial solution* of $f = 0$.

Suppose that we know the solution of $f = 0$ is of form (3.1). Substituting η into $f(y) = 0$ and collecting the coefficients of $f(\eta)$ as a polynomial in x, we get a set of algebraic equations PS for the b_i. Applying Wu-Ritt's zero decomposition theorem [8,9], we have

$$\mathrm{Zero}(PS) = \cup_i \mathrm{Zero}(TS_i/J_i) \tag{3.2}$$

$$= \cup_j \mathrm{Zero}(\mathrm{sat}(AS_j)) \tag{3.3}$$

where AS_j and TS_i are irreducible ascending sets, J_i is the initial product of TS_i, and $\mathrm{sat}(AS_i)$ is the prime ideal with AS_i as a characteristic set [9]. There are subtle distinctions between the two forms. In (3.2), both TS_i and J_i are explicitly given and we may determine $\mathrm{Zero}(TS_i/J_i)$ easily. In (3.3), we have an irredundant decomposition. Therefore, it provides a minimal representation of the solutions of the differential equation.

Theorem 3.1 Let o be the order of the differential equation $f = 0$ in (2.1), and d the dimension of a prime ideal $\mathrm{sat}(AS_i)$ in (3.3). Then $d \le o$ and if $d = o$, $\mathrm{Zero}(\mathrm{sat}(AS_i))$ represents the general solution for $f = 0$.

Proof. Roughly speaking, the dimension of a prime ideal is the maximal number of independent variables which can take arbitrary values, while the value of other variables can be determined from these free parameters. Substituting a generic zero (see [8] for definition) of the prime ideal $\mathrm{Zero}(\mathrm{sat}(AS_i))$ into (2.1), we obtain a solution η' of $f = 0$. This solution contains d independent arbitrary constants. Then d must be less than or equal to the order o of $f = 0$, which represents the maximal number of independent free constants in its solutions. If $d = o$, we will prove that η' is the general solution for $f = 0$. Let ID be the set of differential polynomials in $K\{y\}$ which vanish η'. Then ID is a prime ideal of dimension zero. Let $\{G\}$ be the characteristic set of ID. Since there are d arbitrary constants in η', G must be of order $d = o$. Note that η' is a solution of $f = 0$. Then the pseudo remainder of f with respect to G is zero. But f and G are of the same order and are irreducible. Then f must be the same as G, which implies that η' is a general solution of $f = 0$. $\square$

The method of finding polynomial solutions of $F = 0$ reported in this section is usable only if we can decide the upper bound for the degree of the polynomial solution (3.1). We will give such bounds for two types of special differential equations.

4 Polynomial solutions of binomial algebraic differential equation

Let $f \in \mathbb{F}$ be a differential polynomial with two terms, then we call it a *binomial differential polynomial*. The corresponding equation is called a *binomial algebraic differential equation*. It has the form

$$f = f_1 + f_2 = a_1 \prod_{i=1}^{r} y_{\beta_i}^{\alpha_i} + a_2 \prod_{j=1}^{s} y_{\delta_j}^{\gamma_j} = 0 \tag{4.1}$$

The degree equations of the equation (4.1) has only one equation,

$$\sum_{i=1}^{r} \alpha_i(d-\beta_i) + d_1 = \sum_{j=1}^{s} \gamma_j(d-\delta_j) + d_2 \qquad [1-2]$$

If equation (4.1) has an isotropic polynomial $\Lambda_{l_1 l_2 \cdots l_m}$, then $\Lambda_{l_1 l_2 \cdots l_m} = f$. The characteristic equation is

$$c_1 \prod_{i=1}^{r} [d(d-1)\cdots(d-\beta_i+1)]^{\alpha_i} + c_2 \prod_{j=1}^{s} [d(d-1)\cdots(d-\delta_j+1)]^{\gamma_j} = 0 \tag{4.2}$$

We have the following theorem.

Theorem 4.1 The equation (4.1) has and only has the following polynomial solutions.

(I) The universal polynomial solution.

If $d_u := \min\{\beta_1 - 1, \gamma_1 - 1\} \geq 0$, then $[d_u]$ is the universal polynomial solution of (4.1).

(II) The sharing polynomial solutions.

1. If the degree equation of (4.1) has no positive integer solution, then (4.1) has no sharing polynomial solutions.
2. If the degree equation of (4.1) has an effective solution, let it be d_s. If the degree equation of $f = 0$ has infinite positive integer solutions. Let d_s be

the maximal effective solutions of the characteristic equation (4.2). Let $V(x)=\sum_{i=0}^{d_s} b_i x^{d_o-i}$. If by comparing the coefficients of $f(V(x))=0$, the b_i's exist, then $V(x)$ is a sharing polynomial solutions of (4.1).

Lemma 4.2 If the integers $\alpha_i,\beta_i,\gamma_j,\delta_j (1\le i\le r, 1\le j\le s)$ satisfy the following conditions,

(i) $\delta_1>\delta_2>\cdots>\delta_s\ge 0$, $\beta_1>\beta_2>\cdots>\beta_r\ge 0$;

(ii) $\beta_r\delta_s=0$;

(iii) $\beta_i\ne\delta_j (\forall 1\le i\le s, 1\le j\le r)$;

(iv) $\sum_{i=1}^r \alpha_i=\sum_{i=1}^s\gamma_i$, $\sum_{i=1}^r\alpha_i\beta_i=\sum_{i=1}^s\gamma_i\delta_i$.

then the following equation in indeterminate d cannot be an identical equation.

$$c_1\prod_{i=1}^r[d(d-1)\cdots(d-\beta_i+1)]^{\alpha_i}+c_2\prod_{j=1}^s[d(d-1)\cdots(d-\delta_j+1)]^{\gamma_j}=0$$

where c_1,c_2 are constants.

Proof. Because the two sides of the equation are polynomials, if it is an identical equation, then the two polynomials have the same zeros, hence $r=s,\alpha_1=\gamma_1,\cdots,\beta_1=\delta_1,\cdots$, a contradiction. □

Proof of Theorem 4.1.

Firstly, if $d_u=\min\{\beta_1-1,\gamma_1-1\}\ge 0$, then $P(x)=\sum_{i=0}^{d_u} b_i x^{d_u-i}$ (b_i's are arbitrary constants) is annihilated by f_1 and f_2, henceforth, it is annihilated by f. Then $P(x)=[d_u]$ is the universal polynomial solution of equation (4.1).

Secondly, if $Q(x)=\sum_{i=0}^{d_0} b_i x^{d_0-i}$ is a polynomial solution of (4.1) other than the universal polynomial solution $[d_u]$, then we have $deg(f_1(Q(x))+d_1=deg(f_1(Q(x))+d_2$.

If the degree equation has no positive integer solution, then (4.1) has no sharing polynomial solutions. Otherwise, we need only to determine the upper bound for the degree of the solution. Suppose that the degree equation has an effective solution d_s. Then d_s may be used as the bound. Suppose that the degree equation has infinitely many positive integer solutions. If equation (4.1) has a sharing polynomial solution $Q(x)=\sum_{i=0}^{d_0} b_i x^{d_0-i}$, then the leading coefficient of $f_1(Q(x))$ is the negation of the leading coefficient of $f_2(Q(x))$'s. The leading coefficient of $f_1(Q(x))$ is $c_1 b_0^{\sum_{i=1}^r}\prod_{i=1}^r[d(d-1)\cdots(d-\beta_i+1)]^{\alpha_i}$ and the leading coefficient of $f_2(Q(x))$ is
$c_2 b_0^{\sum_{j=1}^s}\prod_{j=1}^s[d(d-1)\cdots(d-\delta_j+1)]^{\gamma_j}$. By Lemma 4.2, we know that there are finitely many positive integers $d_l (l=1,\cdots,k)$ such that the leading coefficient of $f_1(Q(x))$ is the negation of the $f_2(Q(x))$'s. Then $d_s=\max\{d_1,\cdots,d_k\}$ may be used as a degree bound. □

Example 4.3. Let $F = y_1^2 - 4y$. The degree equation for $F = 0$ is

$$2(d-1) = d$$

which has an effective solution: $d = 2$. Then the polynomial solution for $F = 0$ is of degree two. Substitute $y = b_2x^2 + b_1x + b_0$ into $F = 0$, we obtain the following equations

$$b_2^2 - b_2 = 0, b_1b_2 - b_1 = 0, b_1^2 - 2b_0 = 0.$$

The above equations have two sets of solutions $\{b_2 = b_1 = b_0 = 0\}$, $\{b_2 = 1, b_0 = b_1^2/4\}$. The first set leads to a singular solution $y = 0$ for $F = 0$. The second set leads to the solution $y = (x + b_1/2)^2$, which has one parameter. From Theorem 3.1, $y = (x + b_1/2)^2$ is the general solution for $F = 0$.

5 Polynomial solutions of trinomial algebraic differential equation

Let $f \in \mathbb{F}$ be a differential polynomial with three terms, then we call it a *trinomial differential polynomial.* The corresponding equation is called a *trinomial algebraic differential equation*, *i.e.*

$$f = f_1 + f_2 + f_3 = a_1 \prod_{i=1}^{r_1} y_{\beta_i}^{\alpha_i} + a_2 \prod_{j=1}^{r_2} y_{\delta_j}^{\gamma_j} + a_3 \prod_{k=1}^{r_3} y_{\zeta_k}^{\varepsilon_k} = 0 \qquad (5.1)$$

Here we only consider the case that all of the coefficients are constants. The degree equations of (5.1) become,

$$\sum_{i=1}^{r_1} \alpha_i(d - \beta_i) + d_1 = \sum_{j=1}^{r_2} \gamma_j(d - \delta_j) + d_2 \qquad [1-2]$$

$$\sum_{i=1}^{r_1} \alpha_i(d - \beta_i) + d_1 = \sum_{k=1}^{r_3} \varepsilon_k(d - \zeta_k) + d_3 \qquad [1-3]$$

$$\sum_{j=1}^{r_2} \gamma_j(d - \delta_j) + d_2 = \sum_{k=1}^{r_3} \varepsilon_k(d - \zeta_k) + d_3 \qquad [2-3]$$

We have the following theorem.

Theorem 5.1 The equation (5.1) has and only has the following polynomial solutions:

(I) The universal polynomial solution.

If $d_u := \min\{\beta_1 - 1, \delta_1 - 1, \varepsilon_1 - 1\} \geq 0$, then $[d_u]$ is the universal polynomial solution of (5.1).

(II) The sharing polynomial solutions.

1. If all the three degree equations have no positive integer solution, then (5.1) has no sharing polynomial solution.

2. If one of the degree equations has an effective solution, let it be d_s. If only one of the degree equations $[s-t]$ is an identical equation and the other one has no effective solution, let d_s be the maximal effective solutions of the characteristic equation of the isotropic equation $\Lambda_{st} = f_s + f_t = 0$. If all of the three degree equations are identical equations, let d_s be the maximal effective solutions of the characteristic equation of the isotropic equation $\Lambda_{123} = f = 0$.

   Let $V(x) = \sum_{i=0}^{d_s} b_i x^{d_o - i}$. If by comparing the coefficients of $f(V(x)) = 0$, the b_i's exist, then $V(x)$ is a sharing polynomial solution of (5.1).

Lemma 5.2 If the integers $\alpha_i, \beta_i, \gamma_j, \delta_j, \varepsilon_k, \zeta_k (1 \le i \le r_1, 1 \le j \le r_2, 1 \le k \le r_3)$ satisfy the following conditions,
(i) $\beta_1 > \beta_2 > \cdots > \beta_{r_1} \ge 0$, $\delta_1 > \delta_2 > \cdots > \delta_{r_2} \ge 0$, $\zeta_1 > \zeta_2 \cdots > \zeta_{r_3} \ge 0$;
(ii)$\beta_{r_1} \delta_{r_2} \zeta_{r_3} = 0$;
(iii) there is no $1 \le i \le r_1, 1 \le j \le r_2, 1 \le k \le r_3$, such that $\beta_i = \delta_j = \zeta_k$;
(iv) $\sum_{i=1}^{r_1} \alpha_i = \sum_{j=1}^{r_2} \gamma_j = \sum_{k=1}^{r_3} \varepsilon_k$, $\sum_{i=1}^{r_1} \alpha_i \beta_i = \sum_{j=1}^{r_2} \gamma_j \delta_j = \sum_{k=1}^{r_3} \varepsilon_k \zeta_k$,
then the following equation in indeterminate d cannot be an identical equation.

$$a_1 \prod_{i=1}^{r_1} [d(d-1) \cdots (d - \beta_i + 1)]^{\alpha_i} + a_2 \prod_{j=1}^{r_2} [d(d-1) \cdots (d - \delta_j + 1)]^{\gamma_j}$$

$$+ a_3 \prod_{k=1}^{r_3} [d(d-1) \cdots (d - \zeta_k + 1)]^{\varepsilon_k} = 0 \tag{5.2}$$

Proof. According to the assumption on β_{r_1}, δ_{r_2} and ζ_{r_3}, we divide the proof into two parts.

Part I. Only one of the three integers β_{r_1}, δ_{r_2}, ζ_{r_3} equals zero. Without loss of generality, we assume that $\gamma_{r_3} = 0$. Then the equation (5.2) is

$$a_1 \prod_{i=1}^{r_1} [d(d-1) \cdots (d - \beta_i + 1)]^{\alpha_i} + a_2 \prod_{j=1}^{r_2} [d(d-1) \cdots (d - \delta_j + 1)]^{\gamma_j} =$$

$$-a_3 \prod_{k=1}^{r_3} [d(d-1) \cdots (d - \zeta_k + 1)]^{\varepsilon_k} \tag{5.3}$$

Hence the left side of the equation (5.3) has the factor d with power of $\sum_{i=1}^{r_1} \alpha_i = \sum_{j=1}^{r_2} \gamma_j$, but the right side of the equation (5.3) has the factor d with power of

$\sum_{k=1}^{r_3-1} \varepsilon_k$, which is less than $\sum_{i=1}^{r_1} \alpha_i = \sum_{j=1}^{r_2} \gamma_j$. Hence, the equation cannot be an identical equation in indeterminate d.

Part II. Two of the three integers β_{r_1}, δ_{r_2}, ζ_{r_3} equal zero. Without loss of generality, we assume that $\beta_{r_1} = \delta_{r_2} = 0$. Then the equation (5.2) is

$$a_1 \prod_{i=1}^{r_1} [d(d-1)\cdots(d-\beta_i+1)]^{\alpha_i} + a_2 \prod_{j=1}^{r_2} [d(d-1)\cdots(d-\delta_j+1)]^{\gamma_j} =$$

$$-a_3 \prod_{k=1}^{r_3} [d(d-1)\cdots(d-\zeta_k+1)]^{\varepsilon_k} \tag{5.4}$$

Suppose that $\sum_{i=1}^{r_1} \alpha_i\beta_i = \sum_{j=1}^{r_2} \gamma_j\delta_j = \sum_{k=1}^{r_3} \varepsilon_k\zeta_k = m$ and $\sum_{i=1}^{r_1} \alpha_i = \sum_{j=1}^{r_2} \gamma_j = \sum_{k=1}^{r_3} \varepsilon_k = n$. Then the right side of (5.4) is a polynomial in d with degree m and the left side of (5.4) is the sum of two polynomials which with degrees less that m. Obviously, it cannot be an identical equation in indeterminate d. □

Proof of Theorem 5.1.

Firstly, if $d_u := \min\{\beta_1 - 1, \delta_1 - 1, \varepsilon_1 - 1\} \geq 0$, then $P(x) = \sum_{i=0}^{d_u} b_i x^{d_u - i}$ (b_i's are arbitrary constants) is annihilated by f_1, f_2 and f_3, henceforth, it is annihilated by f. Then $P(x) = [d_u]$ is the universal polynomial solution of the equation (5.1).

Secondly, if $Q(x) = \sum_{i=0}^{d_s} b_i x^{d_0 - i}$ is a polynomial solution of (5.1) other than the universal polynomial solution $[d_u]$, then $d_s > d_u$. We need only to determine d_s .

Case 1. The degree equation has no positive integer solution. Then (5.1) has no sharing polynomial solutions.

Case 2. One of the degree equations has an effective solution. This solution may be used as d_s.

Case 3. One of the three degree equations $[s-t](1 \leq s < t \leq 3)$ has infinite positive integer solutions, and the other two degree equations have no positive integer solution. If $Q(x)$ is a sharing polynomial solution of the equation (5.1), then the leading coefficients of $f_s(Q(x))$ and $f_t(Q(x))$ are opposite. By Lemma 4.2, we know that there are finite positive integers $d_j (j = 1, \cdots, m)$ such that the head coefficients of $f_s(Q(x))$ and $f_t(Q(x))$ are opposite. We may take $d_s = \max\{d_1, \cdots, d_m\}$.

Case 4. All of the three degree equations have infinite positive integer solutions. By Lemma 5.2, the characteristic equation for the isotropic equation $f = 0$ has a finite number of positive integers solutions: $d_j (j = 1, \cdots, m)$. We may take $d_s = \max\{d_1, \cdots, d_m\}$. □

Example 5.3. This example shows that our approach cannot determine the upper bound for the polynomial solutions with the coefficients being polynomials in x. Consider the trinomial algebraic differential equation

$$f = f_1 + f_2 + f_3 = a_1(x)\prod_{i=1}^{r_1} y_{\beta_i}^{\alpha_i} + a_2(x)\prod_{j=1}^{r_2} y_{\delta_j}^{\gamma_j} + a_3(x)\prod_{k=1}^{r_3} y_{\zeta_k}^{\varepsilon_k} = 0$$

where $\sum_{i=1}^{r_1}\alpha_i = \sum_{j=1}^{r_2}\gamma_j = \sum_{k=1}^{r_3}\varepsilon_k$, $\sum_{i=1}^{r_1}\alpha_i\beta_i - d_1 = \sum_{j=1}^{r_2}\gamma_j\delta_j - d_2 = \sum_{k=1}^{r_3}\varepsilon_k\zeta_k - d_3$, $\beta_{r_1} = \delta_{r_2} = 0$, $\alpha_{r_1} = \gamma_{r_2} = 1$. The leading coefficients of $a_1(x), a_2(x), a_3(x)$ are $a, 1-a, 1$ respectively, where

$$a = \frac{\prod_{j=1}^{r_2-1}[(\delta_j-1)!]^{\gamma_j}}{\prod_{j=1}^{r_2-1}[(\delta_j-1)!]^{\gamma_j} - \prod_{i=1}^{r_1-1}[(\beta_i-1)!]^{\alpha_i}}.$$

It is easy to check that our approach cannot determine the upper bound for the polynomial solutions.

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