MACHINE PROOFS IN GEOMETRY

Automated Production of Readable Proofs for Geometry Theorems

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To Wu Wen-Tsün

Foreword

Polya named Descartes' *Rules for the Direction of the Mind* his favorite book on how to think. In the *Rules*, Descartes speculates on the possible existence of a powerful, secret *method* known to the ancients for doing geometry:

> But my opinion is that these writers then with a sort of low cunning, deplorable indeed, suppressed this knowledge. Possibly they acted just as many inventors are known to have done in the case of their discoveries, i.e., they feared that their method being so easy and simple would become cheapened on being divulged, and they preferred to exhibit in its place certain barren truths, deductively demonstrated with show enough of ingenuity, as the results of their art, in order to win from us our admiration for these achievements, rather than to disclose to us that method itself which would have wholly annulled the admiration accorded.

Descartes, that master of both method and geometry, would have been amazed to see the publication of this volume by Chou, Gao, and Zhang, who here present a *method* for proving extremely difficult theorems in geometry, a method so simple and efficient it can be carried out by high school students or computers. In fact, the method has been implemented in a computer program which can prove hundreds of difficult theorems and moreover can produce simple, elegant proofs. In my view, the publication of this book is the single most important event in automated reasoning since Slagle and Moses first implemented programs for symbolic integration.

All too often, research in automated reasoning is concerned with technical questions of marginal interest to the mathematics community at large. Various strategies for efficient substitution and propositional or equational reasoning have dominated the field of automated reasoning, and the mechanical proving of difficult theorems has been a rare event. Mechanical searches for the proofs of difficult theorems are usually guided extensively by the 'user'. Almost never do we find exhibited a computer program that can routinely treat hard problems in any area of mathematics, but in this book we do! The skeptical reader is urged to flip through the 400 difficult theorems in Chapter 6, all mechanically proved, and try his hand. The exceptional simplicity of the mechanically generated proofs presented for many of these theorems illustrate that the authors' method is not some algorithm of mere "in principle," proof-theoretic relevance, such as Tarski's method. No, by applying the simple method formulated by the authors, the reader may also quickly become an expert at geometry proving. The stunning beauty of these proofs is enough to rivet the reader's attention into learning the method by heart.

The key to the method presented here is a collection of powerful, high level theorems, such as the Co-side and Co-angle Theorems. This method can be contrasted with the earlier Wu method, which also proved astonishingly difficult theorems in geometry, but with low-level, mind-numbing polynomial manipulations involving far too many terms to be carried out by the human hand. Instead, using high level theorems, the Chou-Gao-Zhang method employs such extremely simple strategies as the systematic elimination of points in the order introduced to produce proofs of stunning brevity and beauty.

Although theorems are generally regarded as the most important of mathematical results by the mathematics community, it is methods, i.e., constructions and algorithms, that have the most practical significance. For example, the reduction of arithmetical computation to a mechanical activity is the root of the computing industry. Whenever an area of mathematics can be advanced from being an unwieldy body of theorems to a unified method, we can expect serious practical consequences. For example, the reduction of parts of the calculus to tables of integrals and transforms was crucial to the emergence of modern engineering. Although arithmetic calculation and even elementary parts of analysis have reached the point that computers are both faster at them and more trustworthy than people, the impact of the mechanization of geometry has been less palpable. I believe this book will be a milestone in the inevitable endowment of computers with as much geometric as arithmetic prowess.

Yet I greet the publication of this volume with a tinge of regret. Students of artificial intelligence and of automated reasoning often suffer from having their achievements disregarded by society at large precisely because, as Descartes observed, any simple, ingenious invention once revealed, seems first obvious and then negligible. Progress in the automation of mathematics is inherently dependent upon the slow, deep work of first rate mathematicians, and yet this fundamentally important line of work receives negligible societal scientific support in comparison with those who build bigger bombs, longer molecules, or those multi-million-line quagmires called 'systems'. Is it possible that keeping such a work of genius as this book secret would be the better strategy for increased research support, using the text as the secret basis for generating several papers and research proposals around each of the 400 mechanically proved theorems, never revealing the full power of the method?

Robert S. Boyer

December, 1993

Preface

This book reports a recent major advance in automated theorem proving in geometry which should be of interest to both geometry experts and computer scientists. The authors have developed a method and implemented a computer program which, for the first time, produces short, readable, and elegant proofs for *hundreds* of geometry theorems.

Modern computer technology and science make it possible to produce proofs of theorems automatically. However, in practice computer theorem proving is a very difficult task. Historically, geometry theorem proving on computers began in earnest in the fifties with the work of Gelerntner, J. R. Hanson, and D. W. Loveland [103]. This work and most of the subsequent work [129, 144, 83] were *synthetic*, i.e., researchers focused on the automation of the traditional proof method. The main problem of this approach was controlling the search space, or equivalently, guiding the program toward the right deductions. Despite initial success, this approach did not make much progress in proving relatively difficult theorems.

On the other hand, in the 1930s, A. Tarski, introduced a *quantifier elimination method* based on the *algebraic approach* [34] to prove theorems in *elementary geometry*. Tarski's method was later improved and redesigned by A. Seidenberg [147], G. Collins [84] and others. In particular, Collins' cylindrical decomposition algorithm is the first Tarski type algorithm which has been implemented on a computer. Solutions of several nontrivial problems of elementary geometry and algebra have been obtained using the implementation [45, 114].

A breakthrough in the use of the algebraic method came with the work of Wen-Tsün Wu, who introduced an algebraic method which, for the first time, was used to prove hundreds of geometry theorems automatically [164, 36]. Many difficult theorems whose traditional proofs need an enormous amount of human intelligence, such as Feuerbach's theorem, Morley's trisector theorem, etc., can be proved by computer programs based on Wu's method within seconds. In Chou's earlier book [12], there is a collection of 512 geometry theorems proved by a computer program based on Wu's method.

However, if one wishes to look at the proofs produced using Wu's method, he/she will find tedious and formidable computations of polynomials. The polynomials involved in the proofs can contain *hundreds* of terms with more than a dozen variables. Because of this, producing short, readable proofs remains a prominent challenge to researchers in the field of automated theorem proving.

Recently, the authors developed a method which can produce short and readable proofs for hundreds of geometric statements in plane and solid geometries [195, 72, 73, 74]. The starting point of this method is the *mechanization* of the *area method*, one of the oldest and most effective methods in plane geometry. One of the most important theorems in geometry, the Pythagorean theorem, was first proved using the area method. But the area method has generally been considered just some sort of special trick for solving geometry problems. J. Zhang recognized the generality of the method and developed it into a systematic method to solve geometry problems [40, 41, 42]. By a team effort, *we have further developed this systematic method into a mechanical one and implemented a prover*[1] which has been used to produce elegant proofs for hundreds of geometry theorems. This book contains 478 geometry problems solved *entirely automatically* by our prover, including machine proofs of 280 theorems printed in full.

The area method is a combination of the synthetic and algebraic approaches. In the machine proofs, we still use polynomial computation; but we use *geometric invariants* like *areas* and *Pythagoras differences* instead of coordinates of points as the basic quantities, and the geometric meaning for each step of the proof is clear. Another important feature of the area method is that the machine proofs produced by the method/program are generally very short; the formulas in the proofs usually have only *a few terms*, and hence are readable by people.

The method is complete for *constructive geometry theorems*, i.e., those statements whose diagrams can be drawn using a ruler and a pair of compasses only. The area is used to deal with geometry relations like incidence and parallelism. Another basic geometry quantity in our method is the Pythagoras difference, which is used to deal with geometry relations like perpendicular and congruence of line segments. Besides the area and Pythagoras difference, we also use other geometry quantities, such as the full-angle, the volume, the vector, and the complex number. The reason we use more geometry quantities is that for each geometry quantity, there are certain geometry theorems which can be proved easily using this quantity.

Another aspect of *automated geometry theorem proving* relates to the difficulty of learning and teaching geometry. About two thousand years ago, an Egyptian king asked Euclid whether there was an easier way to learn geometry. Euclid's reply was, "There is no royal road to geometry." Of course, the difficulty here was not the basic geometry concepts, such as points, lines, angles, triangles, lengths, areas, etc. The difficulty has been with many other fascinating facts (theorems) and how to use logical reasoning to justify (prove) these theorems based on only a few basic facts (axioms) that are so obvious that they can be taken for granted. One can draw dozens of triangles with three medians and find the fact that the three medians of a triangle intersect at the same points. However, the empirical observation is only a justification of formation of a conjecture whose correctness must be proved by other means – logical reasoning.

[1]The prover is available via ftp at emcity.cs.twsu.edu: pub/geometry.

Unlike algebra, in which most problems can be solved according to some systematic method or algorithm, a human proof requires a different set of tricks for each geometry theorem, making it difficult for students to get started with a proof. In the two thousand years since the time of the Egyptian king, people have continued to wish for an easy way to learn geometry.

In response to this difficulty and in working with middle school students, J. Z. Zhang has established a new geometry axiom system based on the notion of area [40, 41, 42]. Using his new system, Zhang has made a great effort in promoting a new reform in high school geometry education in China. The successful implementation of his method has led to its use in Chinese geometry textbooks for teachers' colleges. In addition, the area method has been used in recent years to train students of Chinese teams for participation in the International Mathematical Olympiads.

One of the goals of this book is to make learning and teaching of geometry easy. The machine proofs generated have a shape that a student of mathematics could learn to design with pencil and paper. By reading the machine produced proofs in this book, many readers might be able to use the mechanical method to prove difficult geometry theorems themselves.

This book consists of six chapters. The first chapter is about the basic concepts of geometry, area and ratio of lengths. Then we introduce the new method, the *area method* for proving geometry theorems. This chapter can stand alone as a supplement to textbooks of high school or college geometry. We present this chapter at an elementary level with many interesting examples solved by the new method, with the intention of attracting many readers at various levels, from high school students to university professors. It is also our hope that people will be able to prove many difficult geometry theorems using the method introduced in this chapter.

Beginning with Chapter 2, we formalize or mechanize the method by describing the method in an algorithmic way. Those who are interested in geometry only will have a clearer idea about this mechanical procedure of proving geometry theorems. Those who wish to write their own computer programs will be able to produce short and readable proofs of difficult geometry theorems. Experts will be able to find further extensions, developments and improvements in this new direction. Only when more and more people are participating in projects of this kind will real advances in geometry education be possible. One of our goals in writing this book is to encourage such research and advances.

The last chapter, Chapter 6, is a collection of 400 theorems proved by our computer program, including machine proofs of 205 theorems printed in full. Most of this chapter was generated mechanically, including machine proofs in TEX typesetting form. This chapter is an integral and important part of the book, because it alone shows the power of our mechanical method and computer program. Even reading the proofs produced by our computer program is enjoyable.

Funding from the National Science Foundation gave us a unique opportunity to form a very strong research team at Wichita State University and to make this book possible. Also the Chinese National Science Foundation provided further support for Gao and Zhang.

The authors wish to thank many people for their support of our research project. Dr. K. Abdali of National Science Foundation constantly encouraged Chou and recommended further funding to invite the third author, J. Zhang, to the U.S. This turned out to be the crucial step for this exciting new development and publication of this book. The encouragements from R. S. Boyer, L. Wos, W. T. Wu, W. W. Bledsoe, and J S. Moore when they heard about our early work were invaluable. They also thank Frank Hwang for his encouragement.

In Wichita, Mary Edgington has also played a key role in promoting the project. The authors also thank Patsy P. Sheffield and L. Yang for their careful reading of the manuscript and helpful suggestions, and thank Tom Wallis, the computer system manager, for providing an excellent computing environment.

Chou used a part of this work in his graduate course "Symbolic and Algebraic Computation;" the manuscript of this book was also read by many students in his class. In particular, we wish to thank Joe G. Moore for his proof-reading and Nirmala Navaneethan for her suggestions.

The support and patience of the authors' families were also very important for the success of this project.

S. C. Chou
X. S. Gao
J. Z. Zhang

December, 1993

Contents

Chapter 1

Geometry Preliminaries

In this chapter, we will provide geometry background for the rest of this book. The presentation is informal and the prerequisite in geometry is minimal. Anyone with a semester of high school geometry can read and understand this chapter. We will present a new systematic proof method, the *area method*, which can be used to solve numerous geometry problems at various levels of difficulty, ranging from problems in high school textbooks to those in mathematics competitions. Those who are mainly interested in machine proofs may skip this chapter and start from Chapter 2 directly.

1.1 Introduction

Geometry, like other sciences, is concerned with the laws in a specific domain. For geometry this domain is space. However, reasoning plays a much more important role in geometry and in other branches of mathematics than in other sciences.

There are two kinds of reasoning: *inductive reasoning* and *deductive reasoning*.

Necessity and curiosity have at all times caused people to investigate phenomena and to find the laws governing the physical universe. Drawing three medians of a triangle, one finds that the three medians intersect at the same point. By making repeated experiments, one can come to the conclusion that the three medians of any triangle always intersect at the same point. This kind of *inductive reasoning*, which is fundamental in experimental sciences, is also very important for observing new facts and laws in mathematics.

However, in geometry or mathematics, inductive reasoning generally cannot serve as the justification of the correctness of the observed fact. To justify the observed fact, we have to give a *proof* of the fact. Here the terminology "proof" has a distinctive meaning. Based on already proved facts (theorems), we need to use *logical or deductive reasoning* to derive this new fact about three medians. Once we prove a fact in this way, we call it a theorem. The truth of the fact is thus beyond doubt. Here we prove a theorem "based on already proved facts." However, the proof of these "already proved facts" is based on other " already proved facts." In the final analysis, we have to choose a few basic facts whose correctness is so evident in our everyday life that we can take them for granted without any proofs. For example, the fact that for any two distinct points there is one and only one line passing through them is so evident that we can use it without proof. Such kinds of facts are very basic and are called *axioms or postulates*.

The selection of axioms is by no means evident. For example, Euclid's fifth postulate, "from any point not on a line, there is one and *only one* line passing through the point and parallel to the given line," is a very evident fact for many people. However, for over two thousand years, mathematicians tried to prove this fact using other basic postulates. The failure of these attempts finally led to the Non-Euclidean geometries and a revolution in mathematics. There is a special branch of mathematics, *foundations of geometry*, which is exclusively concerned with the related topics.

Generally, the traditional proof method in geometry proceeds as follows: first we establish (prove) various basic propositions or lemmas, e.g., the theorems of congruence of two triangles. Then we use these basic propositions to prove new theorems. We can enlarge the set of basic propositions by including some of these newly proved theorems. In the traditional methods used by Euclid and many other geometers since then, theorems of congruence and similarity of triangles are the very basic tools. Although the method based on congruence and similarity leads to elegant proofs for many geometry theorems, it also has weaknesses:

(1) In a diagram of a geometry statement to be proved, rarely do there exist congruent or similar triangles. In order to use the propositions on congruent or similar triangles, one has to construct auxiliary lines. As we know, adding auxiliary lines is one of the most difficult and tricky steps in the proofs of geometry theorems. This also leads to a changeable and uncertain strategy in the effort of finding a proof.

(2) In propositions on congruent or similar triangles, there is asymmetry between the hypothesis and conclusion. For example, in order to prove the congruence of the segments AB and XY, we need to construct (or find) $\triangle ABC$ and $\triangle XYZ$ and to prove the congruence of the two triangles. Generally, this in turn requires us to prove the congruence of three pairs of geometry elements (segments or angles). In order to prove one identity, we need to find three other identities!

As a consequence, it is very difficult to find an effective method for solving geometry problems based on congruence and similarity.

In this book, we will use the area of triangles as the basic tool for solving geometry problems. The traditional area method is one of the oldest and most effective methods in plane geometry. One of the most important theorems in geometry, the Pythagorean theorem, was first proved using the area method. But the area method has generally been considered as a set of special tricks for solving geometry problems. J. Z. Zhang has been studying the area method since 1975. He has recognized its generality and has developed it into a systematic method for solving geometry problems [40, 41, 42].[1] A large number of geometry problems at various levels of difficulty, ranging from basic propositions in high school textbooks to those in mathematics competitions, have been provided with elegant proofs using the area method. This chapter is a summary of the basic facts about the area method together with many geometry theorems solved by this method.

Another feature of the area method is that the deduction in the method is achieved mainly by algebraic computation. This makes the area method ready for mechanization, which is the main theme of this book.

1.2 Directed Line Segments

For most of the book, we are concerned with *plane geometry*. Thus our starting point is a plane, which we sometimes refer to as the *Euclidean plane*. The basic geometry objects on a plane are *points* and *lines*.

We use capital English letters A, B, C, ... to denote points on the Euclidean plane.

For two distinct points A and B, there is one and only one line l that passes through points A and B. We use AB or BA to denote this line. In addition, we can give a line one of the two directions and talk about *directed lines*. Thus directed line AB has the direction from point A to point B, whereas the directed line BA has the opposite direction from point B to A.

Two points A and B on a directed line determine a *directed line segment* whose length $\overline{AB}$ is positive if the direction from A to B is the same as the direction of the line and negative if the direction from A to B is in the opposite direction. Thus

$$\overline{AB} = -\overline{BA} \tag{1.1}$$

and $\overline{AB} = 0$ if and only if $A = B$.

Let A, B, P, and Q be four points on the same line such that $A \neq B$. Then the ratio of $\overline{PQ}$ and $\overline{AB}$ is meaningful; let it be t. We have

$$\frac{\overline{PQ}}{\overline{AB}} = t, or\ \overline{PQ} = t\overline{AB}.$$

If the two directed line segments AB and PQ have the same direction then $t \geq 0$; if they have opposite directions then $t \leq 0$. If P or Q is not on line AB, then we cannot compare

[1]This systematic method has been used to train students of Chinese teams for participating in the International Mathematical Olympiad in solving geometry problems.

or do arithmetic operations between $\overline{AB}$ and $\overline{PQ}$, because the sign of $\overline{AB}$ depends on the direction of line AB which has nothing to do with the direction of line PQ.

If point P is on line AB then $\overline{AB} = \overline{AP} + \overline{PB}$ or

$$\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1. \tag{1.2}$$

We call $\frac{\overline{AP}}{\overline{AB}}$ and $\frac{\overline{PB}}{\overline{AB}}$ the *position ratios* or the *position coordinates* of point P with respect to AB. It is clear that for two real numbers s and t such that $s + t = 1$, there is a unique point P on AB which satisfies

$$\frac{\overline{AP}}{\overline{AB}} = t, \quad \frac{\overline{PB}}{\overline{AB}} = s.$$

In particular, the statement that point O is the midpoint of segment AB means $\frac{\overline{AO}}{\overline{AB}} = \frac{\overline{OB}}{\overline{AB}} = \frac{1}{2}$.

Two distinct points always determine a line. However, three points are generally not on the same line. If they are, we say that the three points are *collinear*. In fact, the most fascinating facts about many elegant geometry theorems discovered over the past two thousand years have been that no matter how you draw a certain geometry figure, the three particular points in the figure are always collinear. Connected with each theorem, there is the collinear line named after the mathematician who discovered it, for example, Pappus' line, Euler's line, Gauss' line, Pascal's line, Simson's line, etc.

Example 1.1 (Pappus' Theorem) *Let points A, B and C be on one line, and A_1, B_1 and C_1 be on another line. Let AB_1 meet A_1B in P, AC_1 meet A_1C in Q, and BC_1 meet B_1C in S. Show that P, Q, and S are collinear.*

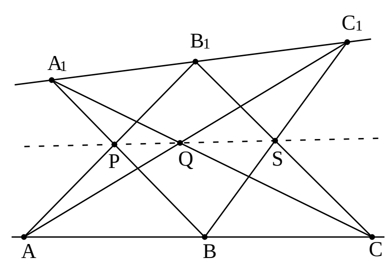


Figure 1-1

Please draw a few diagrams for this geometric configuration on a piece of paper. Note that the three intersection points P, Q and S are always collinear. If you have our software package, you can see this fact more vividly. First, the program helps you to draw one diagram in a few seconds on the computer screen. Then you can use the mouse to move any of the user-chosen points, e.g., point C_1. The diagram is continuously changed on the screen while you can see that the three moving points P, Q, and S are always on the same line. This observation convinces almost everyone that this statement is always true.

However, in order to justify the truth of a fact in mathematics, empirical observations are not enough. We need a proof of the truth of the fact using the mathematical reasoning. The proof of Pappus' theorem (or of many other geometry theorems) is by no means easy, especially to high school students. The main objective of this book is to present a new *systematic* proof method. We believe that the serious reader can learn this method easily.

Once you know this method and work with a few examples, you can prove a difficult theorem in a few minutes. For a proof of Pappus' theorem, see page 14.

Exercises 1.2

1. Let A, B, and C be three collinear points. Show that $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2+2\overline{AB}\cdot\overline{CB}$. (Use (1.1) and (1.2). Do the same for the following exercises.)
2. Let A, B, C, and D be four collinear points. Then $\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}+\overline{AD}\cdot\overline{BC}=0$.
3. Four collinear points A, B, C, and D are called a *harmonic sequence* if $\frac{\overline{AC}}{\overline{BC}}=-\frac{\overline{AD}}{\overline{BD}}$. Show that four collinear points A, B, C, and D form a harmonic sequence if and only if $\frac{\overline{AB}}{\overline{CB}}+\frac{\overline{AB}}{\overline{DB}}=2$.
4. Show that four collinear points A, B, C, and D form a harmonic sequence if and only if $\overline{OC}\cdot\overline{OD}=\overline{OA}^2$ where O is the midpoint of AB.

1.3 Areas and Signed Areas

The next geometric object is the triangle. As we know, three non-collinear points A, B, and C form a triangle which is denoted by $\triangle ABC$. The area of $\triangle ABC$ is denoted by $\triangledown ABC$. The reader can safely assume that $\triangledown ABC$ is $\frac{1}{2}h\cdot BC$ where h is the altitude of the triangle on the side BC. In our area method, however, we do not consider this a basic fact. Instead, we use other simple facts about the area as basic propositions. If A, B, and C are collinear, $\triangle ABC$ is called degenerate and $\triangledown ABC$ is defined to be zero.

For any four points A, B, C, and P in the *same plane*, we can form four triangles $\triangle ABC$, $\triangle PAB$, $\triangle PBC$, and $\triangle PAC$ and the areas of the four triangles satisfy seven different relations, depending on the relative position of the points A, B, C and P.

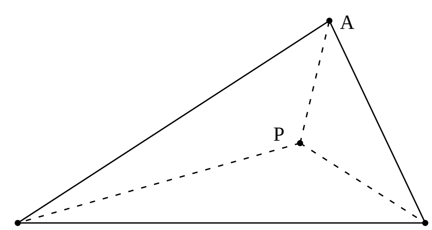


Figure 1-2

If point P is inside the triangle ABC as shown in Figure 1-2, we have

$$\triangledown ABC=\triangledown PAB+\triangledown PBC+\triangledown PCA.$$

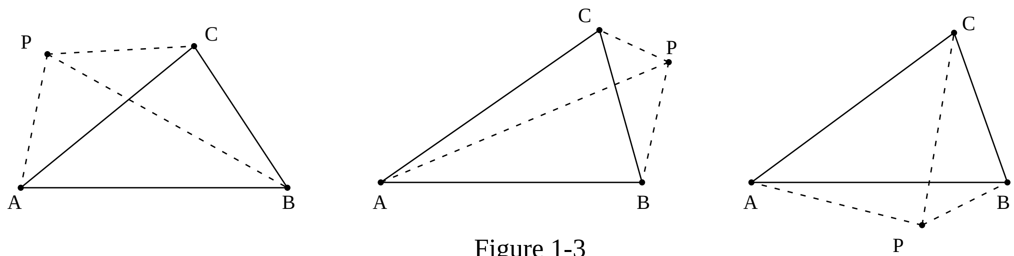


Figure 1-3

For the three cases shown in Figure 1-3, we have

$\bigtriangledown ABC = \bigtriangledown PAB + \bigtriangledown PBC - \bigtriangledown PCA$ If $ABCP$ is convex.
$\bigtriangledown ABC = \bigtriangledown PAB - \bigtriangledown PBC + \bigtriangledown PCA$ If $ABPC$ is convex.
$\bigtriangledown ABC = - \bigtriangledown PAB + \bigtriangledown PBC + \bigtriangledown PCA$ If $APBC$ is convex.

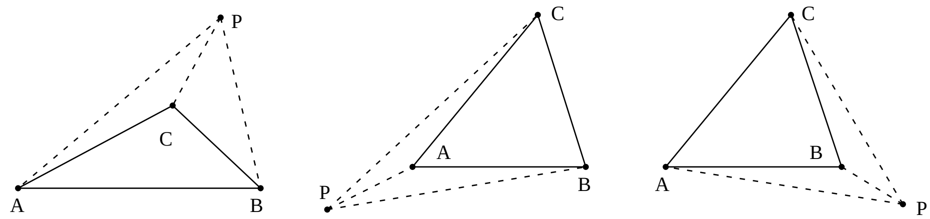


Figure 1-4

For the three cases shown in Figure 1-4, we have

$\bigtriangledown ABC = \bigtriangledown PAB - \bigtriangledown PBC - \bigtriangledown PCA$ If C is in the interior of $\triangle PAB$.
$\bigtriangledown ABC = - \bigtriangledown PAB + \bigtriangledown PBC - \bigtriangledown PCA$ If A is in the interior of $\triangle PBC$.
$\bigtriangledown ABC = - \bigtriangledown PAB - \bigtriangledown PBC + \bigtriangledown PCA$ If B is in the interior of $\triangle PAC$.

We see that the above relations among the areas of triangles formed from the four points are very complicated. However, if we introduce the signed area of an oriented triangle, we can greatly simplify the relations of these areas; the seven relations can be reduced to just one equality.

A triangle ABC has two orientations: if A–B–C is counterclockwise, triangle ABC has the positive orientation; otherwise triangle ABC has the negative orientation. Thus $\triangle ABC$, $\triangle BCA$, and $\triangle CAB$ have the same orientation, whereas $\triangle ACB$, $\triangle CBA$, and $\triangle BAC$ have the opposite orientation.

The *signed area of an oriented triangle ABC*, denoted by S_{ABC}, has the same absolute value as $\bigtriangledown ABC$ and is positive if the orientation of triangle ABC is positive; otherwise S_{ABC} is negative. We thus have

$$S_{ABC} = S_{BCA} = S_{CAB} = -S_{ACB} = -S_{BAC} = -S_{CBA}.$$

Now the seven equations for $\bigtriangledown ABC, \bigtriangledown PAB, \bigtriangledown PBC,$ and $\bigtriangledown PCA$ can be summarized as just one equation: regardless of the position among points A, B, C, and P, we always have

$$S_{ABC} = S_{PBC} + S_{PCA} + S_{PAB} \tag{1.3}$$

Remark for Advanced Readers. Based on analytic (coordinate) geometry, the proof of (1.3) is more strict and straightforward, and we do not have to discuss each of the seven cases separately. Suppose we have a coordinate system on the plane and the coordinates for all four points. For example, the coordinates of the points A, B and C are (a_x, a_y), (b_x, b_y), and (c_x, c_y), respectively. Twice the signed area S_{ABC} of the triangle ABC is the polynomial $(a_x - b_x)(b_y - c_y) - (a_y - b_y)(b_x - c_x)$. Then checking the validity of (1.3) is a straightforward computation.

Furthermore, we do not even need to use the brute force computation of polynomials to prove (1.3). Since (1.3) is valid when P is inside the triangle ABC and (1.3) is equivalent to a polynomial equation, that equation must be true. Since the truth of the equation is independent of the relative position of the four points, (1.3) is valid in all cases. With this argument in mind, the understanding of many other identities in this book or in geometry is much easier. For many geometry statements or theorems of *equality type*, the order relation (inside, between, etc.) is irrelevant. By the above argument based on polynomials, if the statement is true in one case, then it is true in all cases regardless of the relative order position of the points involved.

There are many geometry theorems in which the order relation is essential. Proving of these theorems is beyond the scope of our current computer program based on the area method. One of the important examples is "A triangle with two equal internal angle bisectors is an isosceles triangle." In this chapter we will prove many such theorems using the area method. However, the proofs are informal in the sense that we use some facts other than the basic propositions of the area method used in our computer program. *End of Remark.*

Similarly, we can also define oriented quadrilaterals. Given four points A, B, C, and D, we define an *oriented quadrilateral ABCD* according to the point orientation A–B–C–D. Thus $BCDA$, $CDAB$, and $DABC$ denote the same oriented quadrilateral as $ABCD$ because their point orientation is the same. Four points can form six (4!/4) different orientations, hence six different oriented quadrilaterals: $ABCD$, $ADCB$, $ACBD$, $ADBC$, $ACDB$, and $ABDC$, as shown in Figure 1-5.

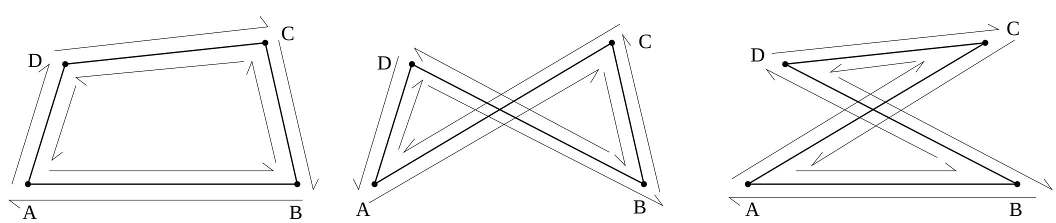


Figure 1-5

Now we can define the area of an oriented quadrilateral $ABCD$ to be

$$S_{ABCD} = S_{ABC} + S_{ACD}.$$

In order to justify that S_{ABCD} is well-defined, we need to prove that for the above definition, we have

$$S_{ABCD} = S_{BCDA} = S_{CDAB} = S_{DABC}$$

which is a direct consequence of (1.3).

The definition of the signed area here can be generalized to an oriented n–polygon with any $n > 4$.

Exercises 1.3

1. Prove the following properties of the areas of quadrilaterals.

   $S_{ABCD} = S_{BCDA} = S_{CDAB} = S_{DABC} = -S_{DCBA} = -S_{CBAD} = -S_{BADC} = -S_{ADCB}$,
   $S_{ABCD} = S_{ABC} - S_{ADC} = S_{BCD} - S_{BAD}$, and
   $S_{ABBC} = S_{ABCC} = S_{AABC} = S_{ABCA} = S_{ABC}$.

2. For any five points A, B, C, P, and Q in the same plane, we have $S_{PAQB} + S_{PBQC} = S_{PAQC}$.

1.4 The Co-side Theorem

The following is the first *basic proposition* of the area method.

Proposition 1.4 *Let A, B, and C be three distinct collinear points, and P a point not on line AB. Then we have $\frac{S_{PBC}}{S_{PAB}} = \frac{\overline{BC}}{\overline{AB}}$, or if $\overline{BC} = t\overline{AB}$ then $S_{PBC} = tS_{PAB}$.*

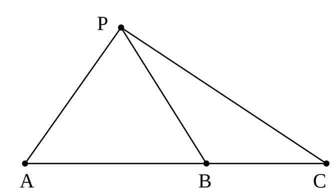


Figure 1-6

Proposition 1.4 is obvious if we use the area formulas: $\triangledown PBC = \frac{1}{2}h \cdot BC$, $\triangledown PAB = \frac{1}{2}h \cdot AB$ where h is the distance from point P to line AB. Notice that the areas involved here are signed areas of oriented triangles.

Two triangles with a common side are said to be a pair of *co-side triangles*. From Proposition 1.4 we can easily infer the co-side theorem, which is the most important proposition of the area method. By using this theorem alone, we can prove many difficult theorems easily.

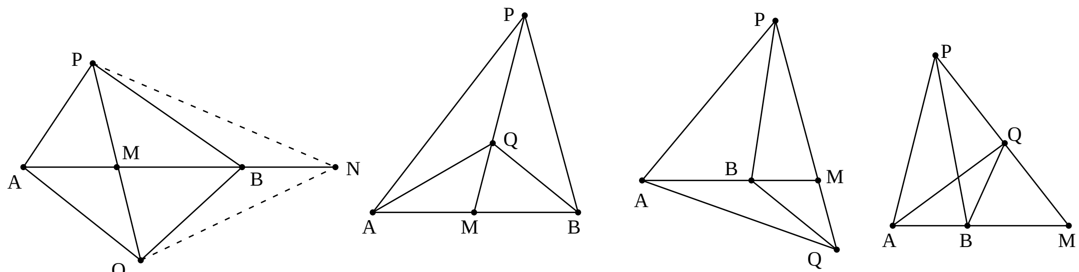


Figure 1-7

Proposition 1.5 (The Co-side Theorem) *Let M be the intersection of the lines AB and PQ and $Q \neq M$. Then we have* $\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}$.

Figure 1-7 shows four possible cases of the co-side theorem. Here we give two proofs of Proposition 1.5, which are valid for all the four cases.

Proof 1. Let N be a point on the line AB such that $\overline{MN} = \overline{AB}$. By Proposition 1.4

$$\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PMN}}{S_{QMN}} = \frac{\overline{PM}}{\overline{QM}}.$$

Proof 2. $\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PAB}}{S_{PAM}} \cdot \frac{S_{PAM}}{S_{QAM}} \cdot \frac{S_{QAM}}{S_{QAB}} = \frac{\overline{AB}}{\overline{AM}} \cdot \frac{\overline{PM}}{\overline{QM}} \cdot \frac{\overline{AM}}{\overline{AB}} = \frac{\overline{PM}}{\overline{QM}}.$

We mentioned early that in the diagrams of geometry theorems, rarely are there similar or congruent triangles. But there are plenty of co-side triangles. For instance, in each of the four diagrams in Figure 1-7 there are 18 pairs of co-side triangles!

Propositions 1.4 and 1.5 are two *basic propositions*. We will expand the set of basic propositions and, based on them, introduce our *area proof method*. Using the area method to prove theorems, no skillful trick is needed: we need only to follow systematic or prescribed steps to reach the completion of a proof. We will use several non-trivial examples to illustrate how to use the basic propositions to prove theorems.

Example 1.6 *Let $\triangle ABC$ be a triangle and P be any point in the plane (inside or outside of the triangle). Let D be the intersection of lines AP and CB, i.e., $D = AP \cap CB$. Also let $E = BP \cap AC$, and $F = CP \cap AB$. Show that* $\frac{\overline{PD}}{\overline{AD}} + \frac{\overline{PE}}{\overline{BE}} + \frac{\overline{PF}}{\overline{CF}} = 1$.

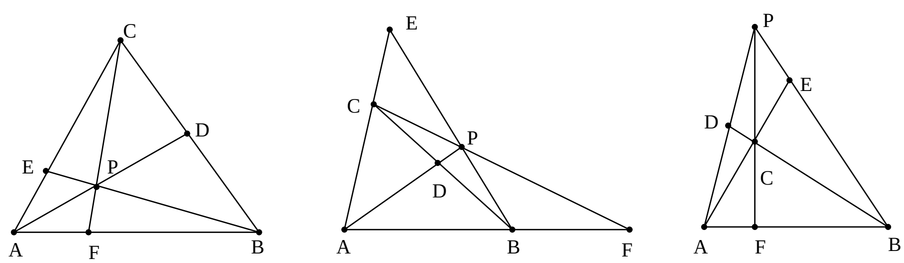


Figure 1-8

The hypotheses of most geometry theorems can be stated in a *constructive way*:

beginning with some arbitrarily chosen points, lines, and circles, we introduce new points by taking arbitrary points on or taking intersections of the lines and circles;

from the constructed points, new lines and circles can be formed;

we now can introduce new points from these new lines and circles, etc;

finally, a figure is formed consisting of points, lines, and circles.

Such kinds of geometry theorems are called *theorems of constructive type.*

Example 1.6 is a theorem of constructive type. Beginning with the arbitrarily chosen (free) points A, B, C and P, we introduce new points D, E, and F by constructing intersections of lines AP and BC, lines BP and AC, and lines CP and AB. Our aim is to *eliminate* the constructed points from the left-hand side of the conclusion

$$\frac{\overline{PD}}{\overline{AD}} + \frac{\overline{PE}}{\overline{BE}} + \frac{\overline{PF}}{\overline{CF}}$$

in the reverse order of the points in which they are introduced until all points in the expression are free points. Then the expression is equal to or is easily proved to be equal to the right-hand side of the conclusion.

Proof. By using the co-side theorem three times we can eliminate points D, E, and F respectively:

$$\frac{\overline{PD}}{\overline{AD}} = \frac{S_{PBC}}{S_{ABC}}, \quad \frac{\overline{PE}}{\overline{BE}} = \frac{S_{PCA}}{S_{ABC}}, \quad \frac{\overline{PF}}{\overline{CF}} = \frac{S_{PAB}}{S_{ABC}}.$$

By (1.3) on page 7,

$$\frac{\overline{PD}}{\overline{AD}} + \frac{\overline{PE}}{\overline{BE}} + \frac{\overline{PF}}{\overline{CF}} = \frac{S_{PBC} + S_{PCA} + S_{PAB}}{S_{ABC}} = \frac{S_{ABC}}{S_{ABC}} = 1. \quad \blacksquare$$

If not using the signed area, we would have to discuss several (seven) cases when P is inside or outside the triangle ABC. Figure 1-8 shows three possible cases of the example. The use of the signed area makes the proof concise and more strict. The reader will see this advantage throughout this book in other examples.

At this point, we need to mention that the *non-degenerate conditions* which are necessary for a geometry statement to be true are not stated explicitly in the example. Here triangle ABC must be non-degenerate, i.e., $S_{ABC} \neq 0$, as we generally implicitly assume in geometry textbooks. But this is not enough. We need each of the intersection points D, E and F to be normal, i.e., there is one and only one intersection point. This imposes conditions on the point P. For detailed discussion of non-degenerate conditions, see Chapter 2.

Example 1.7 (Ceva's Theorem) *The same hypotheses (constructions) as Example 1.6. Show that*

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Proof. Our aim is still to eliminate the constructed points F, E and D from the left-hand side of the conclusion. Using the co-side theorem three times, we can eliminate E, F, and D

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{S_{APC}}{S_{BCP}} \cdot \frac{S_{BPA}}{S_{CAP}} \cdot \frac{S_{CPB}}{S_{ABP}} = 1. \quad \blacksquare$$

The above proof for Ceva's theorem can be extended immediately to prove Ceva's theorem for an arbitrary $(2m+1)$-polygon with $m \geq 1$.

Example 1.8 (Ceva's Theorem for a $(2m+1)$-polygon) *Let a point O and an arbitrary polygon $V_1 \ldots V_{2m+1}$ be given. Let P_i be the intersection of line OV_i and the side $V_{i+m}V_{i+m+1}$. Then*

$$\prod_{i=1}^{2m+1} \frac{\overline{V_{i+k}P_i}}{\overline{P_iV_{i+k+1}}} = 1$$

where the subscripts are understood to be mod $2m+1$.

Proof. By the co-side theorem,

$$\frac{\overline{V_{i+k}P_i}}{\overline{P_iV_{i+m+1}}} = \frac{S_{OV_iV_{i+m}}}{S_{OV_{i+m+1}V_i}}, \quad i = 1, \ldots, 2m+1.$$

Multiplying the above equations together and noticing that the $(i+m)$-th element in the denominator $S_{OV_{(i+m)+m+1}V_{i+m}} = S_{OV_iV_{i+m}}$ is just the i-th element in the numerator, we prove the result. $\blacksquare$

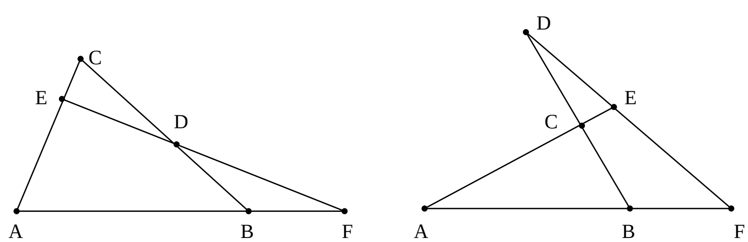


Figure 1-9

Example 1.9 (Menelaus' Theorem) *F, D, and E are three points on sides AB, BC, and CA of a triangle ABC respectively. Show that E, F, and D are collinear if and only if* $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$.

Proof. A *transversal* may meet two sides of a triangle and the third side produced, or all three sides produced (Figure 1-9). The following proof is valid for both cases. If E, F, and D are collinear, by the co-side theorem,

$$\frac{\overline{AF}}{\overline{FB}} = -\frac{S_{AEF}}{S_{BEF}}, \quad \frac{\overline{BD}}{\overline{DC}} = -\frac{S_{BEF}}{S_{CEF}}, \quad \frac{\overline{CE}}{\overline{EA}} = -\frac{S_{CEF}}{S_{AEF}}.$$

Then $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$.

Conversely, let E, F, and D be points on AC, AB, and BC such that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$. Let EF meet BC in H. Then we need only to show $D = H$. By what we just proved $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BH}}{\overline{HC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$. As a consequence $\frac{\overline{BH}}{\overline{HC}} = \frac{\overline{BD}}{\overline{DC}}$, i.e., $D = H$. ▮

We can extend the Menelaus' theorem to m-sided *polygons*.

Example 1.10 (Menelaus' Theorem for an m-polygon) *Let $V_1 \cdots V_m$ be an m-polygon. A line XY meets V_iV_{i+1}, $i = 1, ..., m$ in P_i. Then*

$$\prod_{i=1}^{m} \frac{\overline{V_iP_i}}{\overline{P_iV_{i+1}}} = (-1)^m.$$

Proof. By the co-side theorem,

$$\frac{\overline{V_iP_i}}{\overline{P_iV_{i+1}}} = -\frac{S_{V_iXY}}{S_{V_{i+1}XY}}, \quad i = 1, ..., m.$$

Multiplying the m equations together, we obtain the result. ▮

We can use the area method to solve many geometry problems other than theorem proving.

Example 1.11 *Let ABC be a triangle, D and E be two points on the lines AC and AB such that $\overline{CD} = u\overline{AD}$ and $\overline{AE} = v\overline{BE}$. Let P be the intersection of BD and CE. Express $\frac{\overline{PD}}{\overline{PB}}$ in terms of u and v.*

Solution. We need to eliminate the constructed points P, E, and D successively. By the co-side theorem we have[2]

$$\frac{\overline{PD}}{\overline{PB}} = \frac{S_{DCE}}{S_{BCE}}.$$

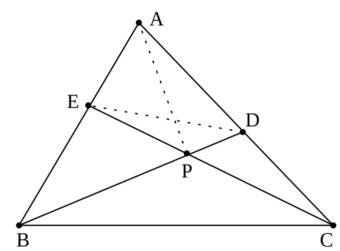


Figure 1-10

[2]Here we mention triangle DCE. The side DE of the triangle is not in the figure. Our method will add those kinds of auxiliary lines automatically.

Now the right-hand side of the above equality is free of the point P. We still need to eliminate the points E and D from it. By the co-side theorem again, we have:

$$\frac{S_{DCE}}{S_{BCE}} = \frac{S_{EDC}}{S_{EAC}} \cdot \frac{S_{EAC}}{S_{EBC}} = \frac{\overline{DC}}{\overline{AC}} \cdot \frac{\overline{EA}}{\overline{EB}} = \frac{-u\overline{AD}}{\overline{AD} - u\overline{AD}} \cdot \frac{v\overline{EB}}{\overline{EB}} = \frac{uv}{(u-1)}. \quad ▮$$

Let us look at the special case when $u = v = -1$, i.e., D and E are the midpoints of the side AC and AB respectively. Then

$$\frac{\overline{PD}}{\overline{PB}} = -\frac{1}{2}.$$

The following theorem follows from this fact immediately.

Example 1.12 (The Centroid Theorem) *The three medians of a triangle meet in a point, and each median is trisected by this point.*

Example 1.13 *The same hypotheses as Example 1.11. Express $\frac{S_{PBC}}{S_{ABC}}$ in terms of u and v.*

Solution. By (1.3) on page 7 and the co-side theorem, we have

$$\frac{S_{ABC}}{S_{PBC}} = \frac{S_{APC}}{S_{PBC}} + \frac{S_{ABP}}{S_{PBC}} + \frac{S_{PBC}}{S_{PBC}} = \frac{\overline{AE}}{\overline{EB}} + \frac{\overline{AD}}{\overline{DC}} + 1 = -v - \frac{1}{u} + 1 = \frac{u - 1 - uv}{u}.$$

Therefore, $\frac{S_{PBC}}{S_{ABC}} = \frac{u}{u-1-uv}$. ▮

Example 1.14 *Take three points D, E, and F on sides AC, AB, and BC of the triangle ABC such that $\frac{\overline{CD}}{\overline{AD}} = u$, $\frac{\overline{AE}}{\overline{BE}} = v$, $\frac{\overline{BF}}{\overline{CF}} = w$. Let $R = AF \cap BD$, $P = BD \cap EC$, and $Q = AF \cap CE$. Express $\frac{S_{PQR}}{S_{ABC}}$ in terms of u, v, and w.*

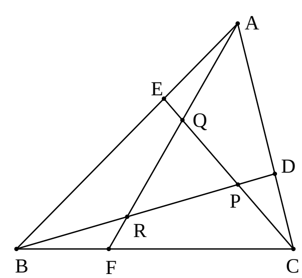


Figure 1-11

Solution. By (1.3) on page 7 and Example 1.13,

$$\begin{aligned} S_{PQR} &= S_{ABC} - S_{PBC} - S_{QCA} - S_{RAB} \\ &= \left(1 - \frac{u}{u-1-uv} - \frac{w}{w-1-wu} - \frac{v}{v-1-wv}\right) S_{ABC} \\ &= \frac{(1+wuv)^2}{(1-u+uv)(1-v+wv)(1-w+wu)} S_{ABC}. \quad ▮ \end{aligned}$$

Remark 1.15

1. *In Example 1.14, if $u = v = w = -\frac{1}{2}$ then we have $\frac{S_{PQR}}{S_{ABC}} = \frac{1}{7}$.*

2. *As a consequence of Example 1.14, we have proved a stronger version of the Ceva theorem: lines AF, BD, and CE are concurrent if and only if $S_{PQR} = 0$, i.e., if and only if $uvw + 1 = 0$ or*

$$\frac{\overline{CD}}{\overline{AD}} \cdot \frac{\overline{AE}}{\overline{BE}} \cdot \frac{\overline{BF}}{\overline{CF}} = -1.$$

Example 1.16 [3] *Let L be the intersection of AB and CD, K the intersection of AD and BC, F the intersection of BD and KL, and G the intersection of AC and KL. Then*

$$\frac{\overline{LF}}{\overline{KF}} = \frac{\overline{LG}}{\overline{GK}}.$$

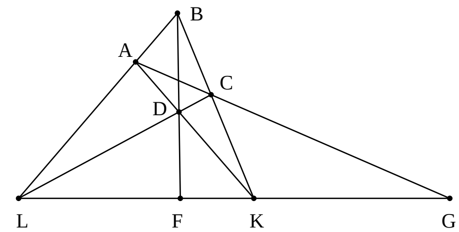


Figure 1-12

Proof 1. By the co-side theorem, we can eliminate point F: $\frac{\overline{LF}}{\overline{KF}} = \frac{S_{LBD}}{S_{KBD}}$. We need further to eliminate points K and L:

$$S_{LBD} = \frac{\overline{LB}}{\overline{AB}} S_{ABD} = \frac{S_{BCD} S_{ABD}}{S_{BCAD}}, S_{KBD} = \frac{\overline{KB}}{\overline{CB}} S_{CBD} = \frac{S_{BAD} S_{CBD}}{S_{BACD}}.$$

Then $\frac{\overline{LF}}{\overline{KF}} = \frac{S_{BCAD}}{S_{BACD}}$. Similarly, we can prove that $\frac{\overline{LG}}{\overline{GK}} = \frac{S_{BCAD}}{S_{BACD}} = \frac{\overline{LF}}{\overline{KF}}$. ∎

Proof 2. The following proof is shorter.

$$\frac{\overline{LF}}{\overline{KF}} = \frac{S_{LBD}}{S_{KBD}} = \frac{S_{LBD}}{S_{KBL}} \frac{S_{KBL}}{S_{KBD}} = \frac{\overline{DA}}{\overline{AK}} \cdot \frac{\overline{LC}}{\overline{DC}} = \frac{S_{DAC}}{S_{AKC}} \frac{S_{LAC}}{S_{DAC}} = \frac{S_{LAC}}{S_{AKC}} = \frac{\overline{LG}}{\overline{GK}}. \quad ∎$$

Example 1.17 (Pappus' Theorem) *Let points A, B and C be on one line, and A_1, B_1 and C_1 be on another line. Let $P = AB_1 \cap A_1B$, $Q = AC_1 \cap A_1C$, and $S = BC_1 \cap B_1C$. Show that P, Q, and S are collinear.*

[3] This theorem is sometimes referred as the basic principle of the projective geometry. It can be used to define the harmonic sequence geometrically. For more details see page 372.

Proof. We generally convert a problem of collinearity into a ratio problem in the following way. Let $Z_1 = PQ \cap BC_1$ and $Z_2 = PQ \cap B_1C$. We need only to prove that $Z_1 = Z_2$. This is equivalent to

$$\frac{\overline{PZ_1}}{\overline{QZ_1}} \cdot \frac{\overline{QZ_2}}{\overline{PZ_2}} = 1. \qquad (1)$$

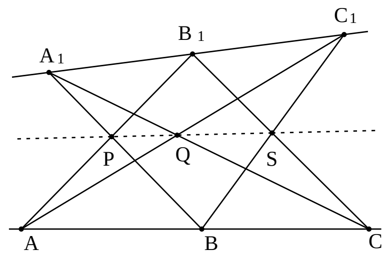


Figure 1-13

We can eliminate points Z_2 and Z_1 from (1) using the co-side theorem.

$$\frac{\overline{PZ_1}}{\overline{QZ_1}} \cdot \frac{\overline{QZ_2}}{\overline{PZ_2}} = \frac{S_{PBC_1}}{S_{QBC_1}} \cdot \frac{S_{QCB_1}}{S_{PCB_1}}. \qquad (2)$$

Now we want to eliminate points Q and P from the right-hand side of (2) using the following identities which can be easily obtained using the co-side theorem.

$$\begin{aligned}
S_{QBC_1} &= \frac{\overline{QC_1}}{\overline{AC_1}} \cdot S_{ABC_1} &&= \frac{S_{A_1CC_1} \cdot S_{ABC_1}}{S_{ACC_1A_1}}\\
S_{QCB_1} &= \frac{\overline{QC}}{\overline{A_1C}} \cdot S_{A_1CB_1} &&= \frac{S_{ACC_1} \cdot S_{A_1CB_1}}{S_{ACC_1A_1}}\\
S_{PBC_1} &= \frac{\overline{PB}}{\overline{A_1B}} \cdot S_{A_1BC_1} &&= \frac{S_{ABB_1} \cdot S_{A_1BC_1}}{S_{ABB_1A_1}}\\
S_{PCB_1} &= \frac{\overline{PB_1}}{\overline{AB_1}} \cdot S_{ACB_1} &&= \frac{S_{A_1BB_1} \cdot S_{ACB_1}}{S_{ABB_1A_1}}.
\end{aligned}$$

Substituting these into (2) and applying Proposition 1.4, we have

$$\begin{aligned}
\frac{S_{PBC_1}}{S_{QBC_1}} \cdot \frac{S_{QCB_1}}{S_{PCB_1}} &= \frac{S_{ABB_1}}{S_{ACB_1}} \cdot \frac{S_{A_1BC_1}}{S_{A_1BB_1}} \cdot \frac{S_{ACC_1}}{S_{ABC_1}} \cdot \frac{S_{A_1CB_1}}{S_{A_1CC_1}}\\
&= \frac{\overline{AB}}{\overline{AC}} \cdot \frac{\overline{A_1C_1}}{\overline{A_1B_1}} \cdot \frac{\overline{AC}}{\overline{AB}} \cdot \frac{\overline{A_1B_1}}{\overline{A_1C_1}} = 1. \quad \blacksquare
\end{aligned}$$

The following extension of the co-side theorem is also very useful.

Proposition 1.18 *Let R be a point on line PQ. Then for any two points A and B*

$$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}} S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}} S_{PAB}.$$

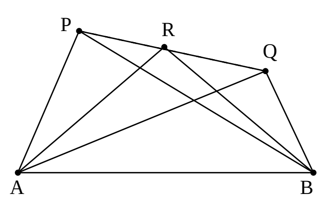


Figure 1-14

Proof. By the co-side theorem $\frac{S_{RAP}}{S_{QAP}} = \frac{\overline{PR}}{\overline{PQ}}, \frac{S_{RPB}}{S_{QPB}} = \frac{\overline{PR}}{\overline{PQ}}$. By (1.3) on page 7,

$$\begin{aligned} S_{RAB} &= S_{PAB} + S_{RAP} + S_{RPB} = S_{PAB} + \frac{\overline{PR}}{\overline{PQ}}(S_{APQ} - S_{BPQ}) \\ &= S_{PAB} + \frac{\overline{PR}}{\overline{PQ}}(S_{APB} + S_{ABQ}) \\ &= (1 - \frac{\overline{PR}}{\overline{PQ}})S_{PAB} + \frac{\overline{PR}}{\overline{PQ}}S_{QAB} \\ &= \frac{\overline{PR}}{\overline{PQ}}S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}S_{PAB}. \quad \blacksquare \end{aligned}$$

Example 1.19 (0.116, 4, 7) *The circumdiameters AP, BQ, CR of a triangle ABC meet the sides BC, CA, AB in the points K, L, M. Show that* $(KP/AK) + (LQ/BL) + (MR/CM) = 1$.

Proof. By the co-side theorem,

$$\frac{\overline{RM}}{\overline{CM}} = \frac{S_{ABR}}{S_{ABC}}, \frac{\overline{QL}}{\overline{BL}} = \frac{-S_{ACQ}}{S_{ABC}}, \frac{\overline{PK}}{\overline{AK}} = \frac{S_{BCP}}{S_{ABC}}.$$

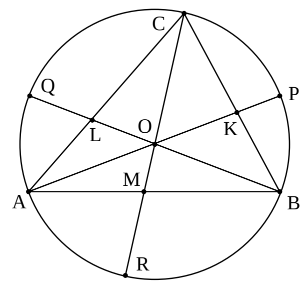


Figure 1-15

Note that O is the midpoint of AP, BQ, and CR. Using Proposition 1.18, we can eliminate points R, Q, and P.

$$S_{ABR} = 2S_{ABO} - S_{ABC}, S_{ACQ} = 2S_{ACO} + S_{ABC}, S_{BCP} = 2S_{BCO} - S_{ABC}.$$

Thus

$$\frac{\overline{RM}}{\overline{MC}} + \frac{\overline{QL}}{\overline{LB}} + \frac{\overline{PK}}{\overline{KA}} = \frac{-(2S_{BCO} + 2S_{AOC} + 2S_{ABO} - 3S_{ABC})}{S_{ABC}} = \frac{S_{ABC}}{S_{ABC}} = 1. \quad \blacksquare$$

Remark 1.20 *In summary, we see that to prove a geometry theorem using the area method systematically, we generally follow three steps: first, formulate the geometry theorem in a constructive way and state the conclusion of the theorem as an expression in areas and ratios of directed line segments; second, based on the basic propositions about areas, try to eliminate the points from the conclusion in the reverse order in which the points are introduced; finally state whether the conclusion is true or not.*

Exercises 1.21

1. There are 48 pairs of co-side triangles in each diagram of Figure 1-8. Try to count how many co-side triangles are there in Figure 1-12.

2. If A, B, C, and D are on the same line, then for points P and Q such that $S_{PCQD} \neq 0$, we have $S_{PAQB}/S_{PCQD} = \overline{AB}/\overline{CD}$.
3. The same hypotheses as Example 1.6. Show that $\frac{\overline{AP}}{\overline{AD}} + \frac{\overline{BP}}{\overline{BE}} + \frac{\overline{CP}}{\overline{CF}} = 2$.
4. Let $ABCD$ be a quadrilateral and O a point. Let E, F, G, and H be the intersections of lines AO, BO, CO, and DO with the corresponding diagonals BD, AC, BD, and AC of the quadrilateral. Show that $\frac{\overline{AH}}{\overline{HC}}\frac{\overline{CF}}{\overline{FA}}\frac{\overline{BE}}{\overline{ED}}\frac{\overline{DG}}{\overline{GB}} = 1$.
5. (Lesening's Theorem) Continuing from Example 1.17, let L_1, L_2, L_3 be the intersections of lines OP and CC_1, lines OQ and BB_1, and lines OS and AA_1. Show that L_1, L_2, L_3 are collinear.

1.5 Parallels

Definition 1.22 *Let AB and CD be two non-degenerate lines. If AB and CD do not have any common point, we say that AB is parallel to CD. We use the notation $AB \parallel CD$ to denote the fact that A, B, C, and D satisfy one of the following conditions: (1) AB and CD are parallel; (2) $A = B$ or $C = D$; or (3) A, B, C and D are on the same line.*

A *parallelogram* is an oriented quadrilateral $ABCD$ such that $AB \parallel CD$, $BC \parallel AD$, and no three vertices of it are on the same line. Let $ABCD$ be a parallelogram. It is clear that line AB and line DC have the same direction. Let $ABCD$ be a parallelogram and P, Q two points on CD. We define

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}}$$

to be the *ratio of two parallel line segments*.

We have the following very useful and simple result which is also basic to our method.

Proposition 1.23 *Let A, B, C, and D be four points. $AB \parallel CD$ if and only if $S_{ABC} = S_{ABD}$ or $S_{ADBC} = 0$.*

If we assume the area for a triangle to be $ah/2$, then Proposition 1.23 is obvious. However, we can prove it using Proposition 1.4 (see page 57 for details). With Proposition 1.23, we can prove many other theorems easily.

Example 1.24 *Let O be the intersection of the two diagonals AC and BD of a parallelogram ABCD. Show that $\overline{AO} = \overline{OC}$, or $\frac{\overline{AO}}{\overline{OC}} = 1$.*

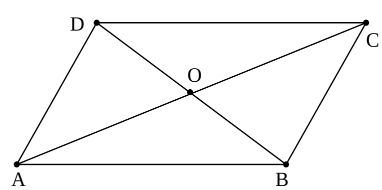


Figure 1-16

The construction of the figure proceeds as follows. First we have three arbitrarily chosen points A, B and C. Then we take a point D such that $\overline{AB} = \overline{DC}$. Finally, we take the intersection of two lines AC and BD to obtain the point O. Thus we need to eliminate O from the expression $\frac{\overline{AO}}{\overline{OC}}$, then eliminate the point D from the new expression. Here is the proof.

Proof.

$$\begin{aligned} \frac{\overline{AO}}{\overline{OC}} &= \frac{S_{ABD}}{S_{BCD}} && \text{by the co-side theorem} \\ &= \frac{S_{ABC}}{S_{BCA}} && (\text{since } AB \parallel CD \text{ and } AD \parallel BC, S_{ABD} = S_{ABC}, S_{BCD} = S_{BCA}.) \\ &= \frac{S_{ABC}}{S_{ABC}} = 1. \end{aligned}$$ ∎

Example 1.25 *Three parallel lines cut two lines at A, B, C, and X, Y, Z respectively. Show that*

$$\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{XY}}{\overline{ZY}}.$$

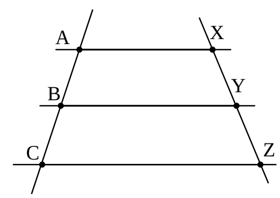


Figure 1-17

This proposition is considered a very basic property of parallel lines in high school geometry. Here we can prove it very elegantly .

Proof. By the co-side theorem and Proposition 1.23, we have

$$\frac{\overline{AB}}{\overline{CB}} = \frac{S_{ABY}}{S_{CBY}} = \frac{S_{XBY}}{S_{ZBY}} = \frac{\overline{XY}}{\overline{ZY}}.$$ ∎

Example 1.26 *Lines AB and CD are parallel. Let $P = AC \cap BD$ and $Q = AD \cap BC$. Line PQ intersects line AB at M. Show that M is the midpoint of AB, i.e., $\overline{AM} = \overline{MB}$.*

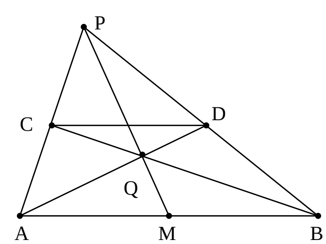


Figure 1-18

Proof 1. By the co-side theorem, we can eliminate M: $\frac{\overline{AM}}{\overline{MB}} = \frac{S_{PAQ}}{S_{PQB}}$. By the co-side theorem again, we can eliminate Q:

$$S_{PAQ} = \frac{\overline{AQ}}{\overline{AD}} S_{PAD} = \frac{S_{ABC}}{S_{ABDC}} S_{PAD}, S_{PQB} = \frac{\overline{BQ}}{\overline{BC}} S_{PCB} = \frac{S_{BDA}}{S_{BDCA}} S_{PCB}.$$

Now by Proposition 1.23, $\frac{\overline{AM}}{\overline{MB}} = \frac{S_{PAD}}{S_{PCB}} = \frac{S_{PCD}+S_{CAD}}{S_{PCD}+S_{CBD}} = 1$ ∎

Proof 2. The following proof is shorter.

$$\frac{\overline{AM}}{\overline{MB}} = \frac{S_{PAQ}}{S_{PQB}} = \frac{S_{PAQ}}{S_{QAB}} \cdot \frac{S_{QAB}}{S_{PQB}} = \frac{\overline{PD}}{\overline{DB}} \cdot \frac{\overline{CA}}{\overline{PC}} = \frac{S_{CPD}}{S_{CDB}} \cdot \frac{S_{CDA}}{S_{CPD}} = \frac{S_{CDA}}{S_{CDB}} = 1.$$

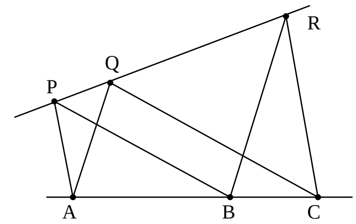


Figure 1-19

Example 1.27 (Pascalian Axiom[4]) *Let A, B and C be three points on one line, and P, Q, and R be three points on another line. If $AQ \parallel RB$ and $BP \parallel QC$ then $AP \parallel RC$.*

Proof. We need to prove $S_{RAP} = S_{CAP}$. Since $AQ \parallel RB$ and $BP \parallel QC$, by Proposition 1.23 we have

$$\begin{aligned} S_{RAP} &= S_{RAQ} + S_{APQ} = S_{BAQ} + S_{APQ} = S_{BAPQ} \\ &= S_{BAP} + S_{BPQ} = S_{BAP} + S_{BPC} = S_{CAP}. \quad \blacksquare \end{aligned}$$

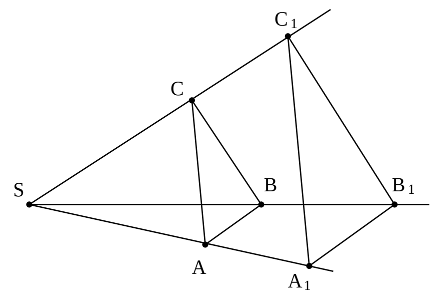


Figure 1-20

Example 1.28 (Desargues' Axiom) *SAA_1, SBB_1, and SCC_1 are three distinct lines. If $AB \parallel A_1B_1$ and $AC \parallel A_1C_1$ then $BC \parallel B_1C_1$.*

Proof. We need to show $S_{B_1BC} = S_{C_1BC}$. Noting that $AB \parallel A_1B_1$ and $AC \parallel A_1C_1$, by the co-side theorem we can eliminate B_1 and C_1

$$S_{BCC_1} = \frac{\overline{CC_1}}{\overline{CS}} S_{SBC} = \frac{\overline{AA_1}}{\overline{AS}} S_{SBC}; \quad S_{BCB_1} = = \frac{\overline{BB_1}}{\overline{BS}} S_{SBC} = \frac{\overline{AA_1}}{\overline{AS}} S_{SBC}.$$

Hence $S_{B_1BC} = S_{C_1BC}$. ▮

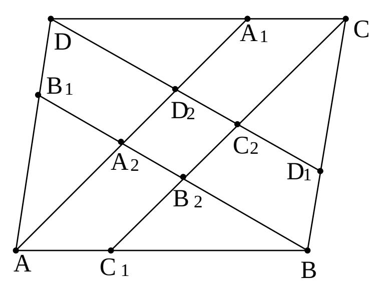


Figure 1-21

Example 1.29 *Let A_1, B_1, C_1, and D_1 be points on the sides CD, DA, AB, and BC of a parallelogram $ABCD$ such that $CA_1/CD = DB_1/DA = AC_1/AB = BD_1/BC = 1/3$. Show that the area of the quadrilateral formed by the lines AA_1 BB_1, CC_1, and DD_1 is one thirteenth of the area of parallelogram $ABCD$.*

[4]This result is a special case of Pappus' theorem. It was referred to as Pascal's theorem by Hilbert in [24] and was used as an axiom in [5].

Proof. By the co-side theorem,

$$\begin{aligned}\frac{S_{ABCD}}{S_{ABA_2}} &= 2\cdot\frac{S_{ABA_2}+S_{A_2BD}+S_{A_2AD}}{S_{ABA_2}}\\ &= 2(1+\frac{\overline{DB_1}}{\overline{B_1A}}+\frac{S_{AA_1D}}{S_{ABA_1}}) = 2(1+\frac{1}{2}+\frac{2}{3}) = \frac{13}{3}\end{aligned}$$

Then

$$\frac{S_{A_2B_2C_2D_2}}{S_{ABCD}} = \frac{S_{ABCD}-S_{ABA_2}-S_{BCB_2}-S_{CDC_2}-S_{DAD_2}}{S_{ABCD}} = 1-\frac{12}{13} = \frac{1}{13}. \quad \blacksquare$$

Proposition 1.30 *Let ABCD be a parallelogram, P and Q be two points. Then* $S_{APQ}+S_{CPQ}=S_{BPQ}+S_{DPQ}$ *or* $S_{PAQB}=S_{PDQC}$.

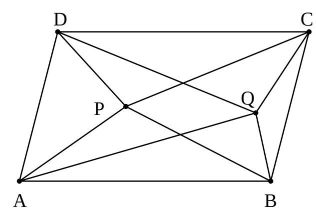


Figure 1-22

Proof. Let O be the intersection of AC and BD. Since O is the midpoint of AC, by Proposition 1.18, $S_{APQ}+S_{CPQ}=2S_{OPQ}$. For the same reason, $S_{BPQ}+S_{DPQ}=2S_{OPQ}$. We have proved the first formula. The second formula is just another form of the first one. $\blacksquare$

Exercises 1.31

1. In Example 1.28, let $AA_1 \parallel BB_1 \parallel CC_1$ are three parallel lines. If $AB \parallel A_1B_1$ and $AC \parallel A_1C_1$ then $BC \parallel B_1C_1$.
2. Let l be a line passing through the vertex of M of a parallelogram $MNPQ$ and intersecting the lines NP, PQ, and NQ at points R, S, and T. Show that $1/MR+1/MS=1/MT$.
3. The diagonals of a parallelogram and those of its inscribed parallelogram are concurrent.
4. Use the same notations as Example 1.29. If $CA_1/CD = DB_1/DA = AC_1/AB = BD_1/BC = r$ then $\frac{S_{A_2B_2C_2D_2}}{S_{ABCD}} = \frac{r^2}{r^2-2r+2}$.
5. Let P, Q be the midpoints of the diagonals of a trapezoid $ABCD$. Then PQ is half of the difference of the two parallel sides of $ABCD$.
6. A line parallel to the base of trapezoid $ABCD$ meets its two sides and two diagonals at H, G, F, and E. Show that $EF=GH$.

1.6 The Co-angle Theorem

In this section, we will discuss more basic theorems, the co-angle theorems, in the area method. However we have not incorporated these theorems into our computer program.

An angle is the figure consisting of a point O and two rays emanating from O. The symbol for angle is $\angle$. We assume the value of the angle is ≥ 0 and $\leq 180°$.

In a triangle ABC, we use $\angle A$, $\angle B$, and $\angle C$ to represent the three inner angles of the triangle at vertices A, B, and C respectively. Then we have the following formula for the area of a triangle ABC (for details see Section 1.8 below.)

$$\nabla ABC = \frac{1}{2}AB \cdot BC \sin(\angle B) = \frac{1}{2}AC \cdot CB \sin(\angle C) = \frac{1}{2}AB \cdot AC \sin(\angle A). \tag{1.4}$$

If $\angle ABC = \angle XYZ$ or $\angle ABC + \angle XYZ = 180°$, we say $\triangle ABC$ and $\triangle XYZ$ are *co-angle triangles*.

Proposition 1.32 (The Co-angle Theorem) *If* $\angle ABC = \angle XYZ$ *or* $\angle ABC + \angle XYZ = 180°$, *we have* $\frac{\nabla ABC}{\nabla XYZ} = \frac{AB \cdot BC}{XY \cdot YZ}$.

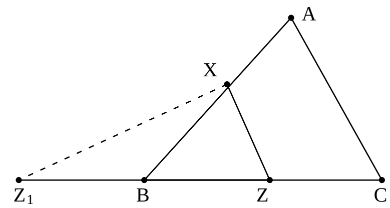


Figure 1-23

Proof 1. This is a consequence of (1.4). ∎

Proof 2. The following proof uses the co-side theorem only. Without loss of generality, we assume that $B = Y$ and Z is on line BC (Figure 1-23). If $\angle ABC = \angle XYZ$, X is on line AB. We have

$$\frac{\nabla ABC}{\nabla XYZ} = \frac{\nabla ABC}{\nabla ABZ} \cdot \frac{\nabla ABZ}{\nabla XYZ} = \frac{BC}{BZ} \cdot \frac{AB}{XY} = \frac{AB \cdot BC}{XY \cdot YZ}.$$

If $\angle ABC + \angle XYZ = 180°$, the result can be proved similarly. ∎

The co-angle theorem, though obvious, can be used to prove nontrivial geometry theorems easily.

Example 1.33 *In triangle* ABC, *if* $\angle B = \angle C$ *then* $AB = AC$.

Proof. By the co-angle theorem, $1 = \frac{\nabla ABC}{\nabla ACB} = \frac{AB \cdot BC}{AC \cdot BC} = \frac{AB}{AC}$. ∎

Example 1.34 *If* $ABCD$ *is a parallelogram, then* $AB = CD$.

Proof. Since $ABCD$ is a parallelogram, we have $\angle CAB = \angle ACD$ and $\nabla ABC = \nabla BCD = \nabla ACD$. By the co-angle theorem, we have

$$1 = \frac{\nabla ABC}{\nabla ACD} = \frac{AC \cdot AB}{AC \cdot CD} = \frac{AB}{CD}. \quad ∎$$

Example 1.35 *F is a point on side BC of $\triangle ABC$ such that AF is the bisector of $\angle BAC$. Then* $\frac{AB}{AC} = \frac{FB}{FC}$.

Proof. By the co-side and co-angle theorems,

$$\frac{FB}{FC} = \frac{\triangledown FAB}{\triangledown FAC} = \frac{FA \cdot AB}{FA \cdot AC} = \frac{AB}{AC}. \quad \blacksquare$$

Example 1.36 *In triangles ABC and XYZ, if $\angle A = \angle X$, $\angle B = \angle Y$ then $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$.*

Proof. From the hypotheses, we also have $\angle C = \angle Z$. By the co-angle theorem,

$$\frac{\triangledown ABC}{\triangledown XYZ} = \frac{AB \cdot AC}{XY \cdot XZ} = \frac{AB \cdot BC}{XY \cdot YZ} = \frac{AC \cdot CB}{XZ \cdot ZY}.$$

The result follows from the above formula immediately. $\blacksquare$

Two triangles are said to be *similar* if their corresponding angles are equal. The above example implies that the corresponding sides of two similar triangles are proportional.

Example 1.37 *In triangle ABC, $AB = AC$, $AB \perp AC$, and M is the midpoint of AB. The perpendicular from A to CM meets BC in P. Show that $PC = 2PB$.*

Proof. Note that $\angle PAC = \angle AMC$ and $\angle PAB = \angle ACM$. By the co-side and co-angle theorems,

$$\begin{aligned} \frac{PC}{PB} &= \frac{\triangledown PAC}{\triangledown PAB} = \frac{\triangledown PAC}{\triangledown MAC} \cdot \frac{\triangledown MAC}{\triangledown PAB} \\ &= \frac{PA \cdot AC}{MA \cdot MC} \cdot \frac{AC \cdot MC}{PA \cdot AB} \\ &= \frac{AC \cdot AC}{MA \cdot AB} = \frac{AB \cdot AB}{MA \cdot AB} = \frac{AB}{MA} = 2. \quad \blacksquare \end{aligned}$$

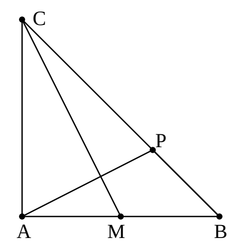


Figure 1-24

Example 1.38 *AM is the median of triangle ABC. D, E are points on AB, AC such that $AD = AE$. DE and AM meet in N. Show that $\frac{DN}{NE} = \frac{AC}{AB}$.*

Proof. From $BM = MC$, we have $\triangledown ABM = \triangledown ACM$. Now by the co-angle theorem,

$$\begin{aligned} \frac{DN}{NE} &= \frac{\triangledown ADN}{\triangledown ANE} = \frac{\triangledown ADN}{\triangledown ABM} \cdot \frac{\triangledown ACM}{\triangledown ANE} \\ &= \frac{AD \cdot AN}{AB \cdot AM} \cdot \frac{AC \cdot AM}{AN \cdot AE} = \frac{AD \cdot AC}{AB \cdot AE} = \frac{AC}{AB}. \quad \blacksquare \end{aligned}$$

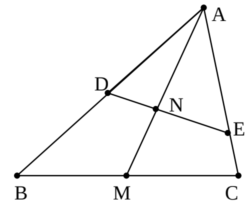


Figure 1-25

Example 1.39 *Four rays passing through a point O meet two lines sequentially in A, B, C, D and P, Q, R, S. Show that* $\frac{AB\cdot CD}{AD\cdot BC} = \frac{PQ\cdot RS}{PS\cdot QR}$.

Proof. By the co-side and co-angle theorem

$$\begin{aligned} & \frac{AB\cdot CD\cdot PS\cdot QR}{AD\cdot BC\cdot PQ\cdot RS} \\ = & \frac{AB}{AD}\cdot\frac{CD}{BC}\cdot\frac{PS}{PQ}\cdot\frac{QR}{RS} \\ = & \frac{\triangledown OAB}{\triangledown OAD}\cdot\frac{\triangledown OCD}{\triangledown OBC}\cdot\frac{\triangledown OPS}{\triangledown OPQ}\cdot\frac{\triangledown OQS}{\triangledown ORS} \\ = & \frac{\triangledown OAB}{\triangledown OPQ}\cdot\frac{\triangledown OCD}{\triangledown ORS}\cdot\frac{\triangledown OPS}{\triangledown OAD}\cdot\frac{\triangledown OQR}{\triangledown OBC} \\ = & \frac{OA\cdot OB\cdot OC\cdot OD\cdot OP\cdot OS\cdot OQ\cdot OR}{OP\cdot OQ\cdot OR\cdot OS\cdot OA\cdot OD\cdot OB\cdot OC} = 1. \end{aligned}$$

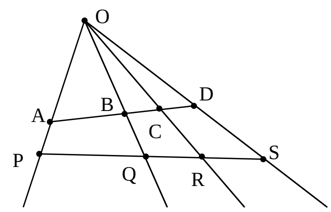


Figure 1-26

Example 1.40 *AM is the median of triangle ABC. A line meets the rays AB, AC, and AM at P, Q, and N respectively. Show that* $\frac{AM}{AN} = \frac{1}{2}(\frac{AC}{AQ} + \frac{AB}{AP})$.

Proof. By the co-angle theorem,

$$\begin{aligned} \frac{AP\cdot AQ}{AB\cdot AC} &= \frac{\triangledown APQ}{\triangledown ABC} = \frac{\triangledown APN}{\triangledown ABC} + \frac{\triangledown AQN}{\triangledown ABC} \\ &= \frac{\triangledown APN}{2\triangledown ABM} + \frac{\triangledown AQN}{2\triangledown ACM} \\ &= \frac{AP\cdot AN}{2AB\cdot AM} + \frac{AQ\cdot AN}{2AC\cdot AM} = \frac{1}{2}(\frac{AP}{AB} + \frac{AQ}{AC})\frac{AN}{AM} \end{aligned}$$

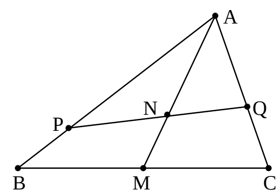


Figure 1-27

Multiplying the two sides of the above formula by $\frac{AM\cdot AB\cdot AC}{AN\cdot AP\cdot AQ}$, we obtain the result. ∎

Example 1.41 *Take two points M, N on the sides AB, AC of triangle ABC such that $\angle MCB = \angle NBC = \angle A/2$. Then $BM = CN$.*

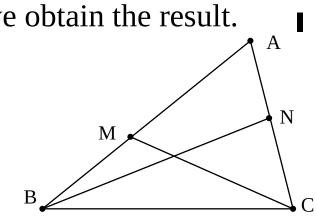


Figure 1-28

Proof. Since $\angle BMC = \angle A + \angle ACB - \angle A/2$ and $\angle CNB = \angle A + \angle ABC - \angle A/2$, we have $\angle BMC + \angle CNB = \angle A + \angle ACB + \angle ABC = 180°$. Hence in triangles BMC and CNB, we have $\angle MCB = \angle NBC$, $\angle BMC + \angle CNB = 180°$. By the co-angle theorem,

$$\frac{MC\cdot BC}{NB\cdot BC} = \frac{\triangledown BMC}{\triangledown CNB} = \frac{BM\cdot MC}{CN\cdot NB}$$

i.e., $1 = BM/CN$. ∎

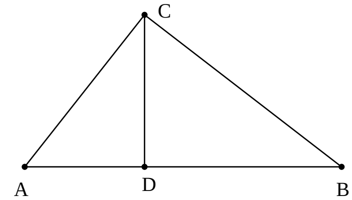


Figure 1-29

Example 1.42 (Pythagorean Theorem[5]) *In triangle ABC, let $BC = a, CA = b, AB = c$, and $\angle ACB = 90^\circ$. Show that $a^2 + b^2 = c^2$.*

Proof. Let the altitude on side AB be $CD = h$. Then $\triangledown ACD + \triangledown BCD = \triangledown ABC$, i.e., $\frac{\triangledown ACD}{\triangledown ABC} + \frac{\triangledown BCD}{\triangledown ABC} = 1$. Since $\angle ACD = \angle ABC, \angle BCD = \angle CAB$, by the co-angle theorem we have

$$bh/(ac) + ah/(bc) = 1.$$

Also note that $ab = 2 \triangledown ABC = ch$, we have $h = ab/c$. Substituting this into the above equation, we prove the result. ∎

Proposition 1.43 (The Co-angle Inequality) *If $\angle ABC > \angle XYZ$ and $\angle ABC + \angle XYZ < 180^\circ$ then $\frac{\triangledown ABC}{\triangledown XYZ} > \frac{AB \cdot BC}{XY \cdot YZ}$.*

Proof 1. This is a consequence of (1.4) on page 21 and the property of the sine function. ∎

Proof 2. The following proof does not use the trigonometric functions. Draw an isosceles triangle PQR such that $QP = PR$ and $\angle QPR = \angle ABC - \angle XYZ$. Produce QR to S such that $\angle RPS = \angle XYZ$. Then we have $\angle QPS = \angle ABC$ and $\triangledown QPS > \triangledown RPS$. By the co-angle theorem

$$\begin{aligned}
\frac{\triangledown ABC}{\triangledown XYZ} &= \frac{\triangledown ABC}{\triangledown QPS}\frac{\triangledown QPS}{\triangledown RPS}\frac{\triangledown RPS}{\triangledown XYZ} \\
&= \frac{AB \cdot BC}{QP \cdot PS}\frac{\triangledown QPS}{\triangledown RPS}\frac{RP \cdot PS}{XY \cdot YZ} \\
&= \frac{AB \cdot BC}{XY \cdot YZ}\frac{\triangledown QPS}{\triangledown RPS} \quad (QP = RP) \\
&> \frac{AB \cdot BC}{XY \cdot YZ}.
\end{aligned}$$
∎

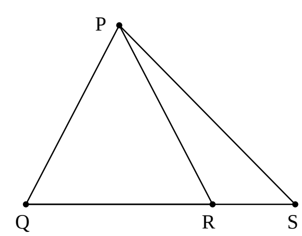


Figure 1-30

Corollary 1.44

1. *If $\angle ABC > \angle XYZ$ and $\angle ABC + \angle XYZ > 180^\circ$ then $\frac{\triangledown ABC}{\triangledown XYZ} < \frac{AB \cdot BC}{XY \cdot YZ}$.*

[5]In Chinese literature, this theorem is called the Gou-Gu theorem and is attributed to Shang-Gao (1100 B.C.). This celebrated theorem is one of the most important theorems in the whole realm of geometry. There are about 370 proofs for this theorem in [28].

2. (The Converse of the Co-angle Theorem) *If* $\frac{\triangledown ABC}{\triangledown XYZ} = \frac{AB \cdot BC}{XY \cdot YZ}$ *then* $\angle ABC = \angle XYZ$ *or* $\angle ABC + \angle XYZ = 180°$.

Example 1.45 *In triangle* ABC, *if* $\angle B > \angle C$ *then* $AC > AB$.

Proof. By the co-angle inequality, $1 = \frac{\triangledown ABC}{\triangledown ACB} > \frac{AB \cdot BC}{AC \cdot BC} = \frac{AB}{AC}$. ▮

Corollary 1.46

1. *Among the segments from a point to any point on a line, the perpendicular is the shortest.* ▮

2. *The area of any quadrilateral is less than or equal to half of the products of its two diagonals.*

Example 1.47 *If* $AB \geq AC$ *and* P *is a point between* B *and* C *then* $AB > AP$.

Proof. Since $AB \geq AC$, we have $\angle ACB \geq \angle ABC$. Then $\angle APB = \angle ACB + \angle CAP > \angle ABC = \angle ABP$. By the co-angle inequality, we have $AB > AP$. ▮

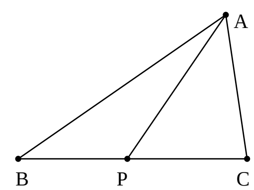


Figure 1-31

Example 1.48 *The sum of any two sides of a triangle is larger than the third one.*

Proof. Produce BC to D such that $CD = AC$. Then $\angle BDA = \angle CDA = \angle CAD = \angle BAD - \angle BAC < \angle BAD$. In triangle ABD, by Example 1.45 we have $AB < BD = BC + CD = BC + AC$. ▮

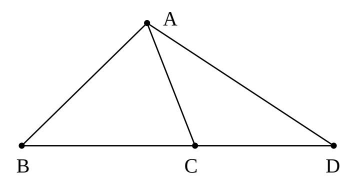


Figure 1-32

Example 1.49 [6] *Take three points* M, K, *and* L *on the three sides* AB, BC, *and* CA *of triangle* ABC *respectively. Show that at least one of* $\triangledown AML$, $\triangledown BMK$, *and* $\triangledown CKL$ *is less that* $\frac{1}{4}\triangledown ABC$.

Proof. Let $AM = r \cdot AB$, $BK = s \cdot BC$, and $CL = t \cdot AC$. Then r, s, and t are positive numbers less that 1. Also $BM = (1-r)AB$, $CK = (1-s)BC$, and $AL = (1-t)AC$. By the co-angle theorem,

[6] This is a problem from the 1966 International Mathematical Olympiad.

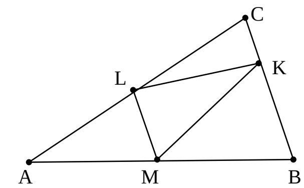


Figure 1-33

$$\begin{aligned}
& \frac{\nabla AML}{\nabla ABC}\cdot\frac{\nabla BMK}{\nabla ABC}\cdot\frac{\nabla CKL}{\nabla ABC}\\
= & \frac{AM}{AB}\cdot\frac{AL}{AC}\cdot\frac{BM}{AB}\cdot\frac{BK}{BC}\cdot\frac{CK}{BC}\cdot\frac{CL}{AC}\\
= & r\cdot(1-t)\cdot(1-r)\cdot s\cdot(1-s)\cdot t\\
= & r(1-r)\cdot s(1-s)\cdot t(1-t)\le(1/4)\cdot(1/4)\cdot(1/4).
\end{aligned}$$

As a consequence, one of $\frac{\nabla AML}{\nabla ABC}$, $\frac{\nabla BMK}{\nabla ABC}$, and $\frac{\nabla CKL}{\nabla ABC}$ must be less than 1/4. ▮

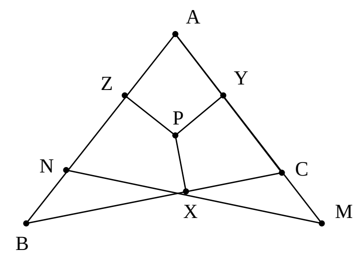


Figure 1-34

Example 1.50 (**Erdös' Inequality**) *P is a point inside the triangle ABC. Let the distances between the point P and the lines BC, CA, and AB be x, y, and z respectively. Show that $PA+PB+PC\ge 2(x+y+z)$.*

Proof. Take points N and M on AB and AC such that $AM=AB$, $AN=AC$. Then $MN=BC$ and

$$z\cdot AC+y\cdot AB=z\cdot AN+y\cdot AM=2(\nabla APN+\nabla APM)\le PA\cdot MN=PA\cdot BC$$

i.e., $z\cdot(AC/BC)+y\cdot(AB/BC)\le PA$. Similarly

$$x\cdot(AB/AC)+z\cdot(BC/AC)\le PB;\quad x\cdot(AC/AB)+y\cdot(BC/AB)\le PC.$$

Adding the three equations together, we have

$$PA+PB+PC\ge x\cdot(\frac{AB}{AC}+\frac{AC}{AB})+y(\frac{BC}{AB}+\frac{AB}{BC})+z\cdot(\frac{BC}{AC}+\frac{AC}{BC})\ge 2(x+y+z).$$ ▮

Example 1.51 (The Steiner-Lehmus Theorem[7]) *In triangle ABC, if the bisectors for angles B and C are equal then $AB=AC$.*

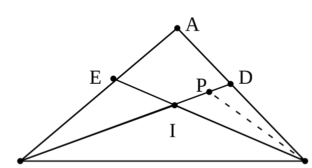


Figure 1-35

Proof. Without loss of generality, we assume $AB\ge AC$. Then $\angle ACB\ge ABC$. Let I be the intersection of BD and CE. Then $\angle DCI\ge EBI$. Take a point P between DI such that $\angle PCI=\angle EBI$. We need only to show that $P=D$. In triangles PCI and EBI, $\angle PCI=\angle EBI$, $\angle PIC=\angle EIB$. Then $\angle CPI=\angle BEI$.

In triangles PBC and EBC, $\angle CPB=\angle BEC$, $\angle PCB\ge\angle EBC$. By the co-angle theorem and the co-angle inequality,

$$\frac{PC\cdot PB}{BE\cdot CE}=\frac{\nabla PBC}{\nabla EBC}\ge\frac{PC\cdot BC}{BE\cdot BC},$$

i.e., $\frac{PB}{CE} \geq 1$. Therefore $PB \geq CE = DB$. Since P is on DB, we have $P = D$. ∎

Exercises 1.52

1. We mentioned earlier that rarely are there congruent and similar triangles in the diagram of a geometry theorem, but there are many co-side triangles. There are also many co-angle triangles in any figure. Try to count the co-angle triangles in Figures 1-23, 1-24, and 1-25. This is why the co-angle theorem works well for many geometry theorems.
2. On the two sides AB and AC two squares $ABDE$ and $ACFG$ are erected externally. Show that $\triangledown ABC = \triangledown AEG$.
3. In triangle ABC, the bisector of the external angle (at vertex A) meets BC in D. Show that $\frac{AB}{AC} = \frac{BD}{CD}$.
4. In triangle ABC, the bisectors of the inner and external angles (at vertex A) meet BC in D and E. Show that $\frac{BD}{CD} = \frac{BE}{CE}$.

1.7 Pythagoras Differences

To solve geometry problems involving perpendiculars and congruence of line segments, we need to introduce a new geometry quantity: the *Pythagoras difference*. To do that, we first introduce the concept of co-area of triangles.

On side AB of a triangle ABC, a square $ABPQ$ is erected such that S_{ABC} and S_{ABPQ} have the same sign (Figure 1-36). The *co-area* C_{BAC} is a real number such that

$$C_{BAC} = \begin{cases} \triangledown ACQ & \text{if } \angle A \leq 90^\circ; \\ -\triangledown ACQ & \text{if } \angle A > 90^\circ. \end{cases}$$

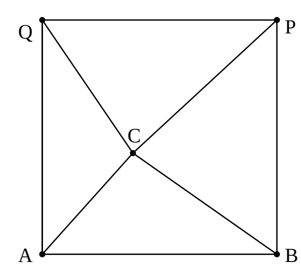


Figure 1-36

Similarly C_{ABC} is equal to $\triangledown BPC$ or $-\triangledown BPC$ according to whether $\angle B$ is acute or obtuse. Generally speaking, C_{BAC}, C_{ABC}, and C_{ACB} are different. But we have $C_{BAC} = C_{CAB}, C_{ABC} = C_{CBA}$, and $C_{ACB} = C_{BCA}$.

Proposition 1.53 *For triangle ABC, we have $C_{ABC} + C_{BAC} = AB^2/2$.*

Proof. As in Figure 1-36, if both $\angle A$ and $\angle B$ are acute then

$$C_{ABC} + C_{BAC} = \triangledown BPC + \triangledown ACQ = \triangledown ABPQ/2 = AB^2/2.$$

[7]For interesting extensions of this theorem, see [37].

If $\angle A$ is obtuse and $\angle B$ is acute then

$$C_{ABC} + C_{BAC} = \triangledown BPC - \triangledown ACQ = \triangledown ABPQ/2 = AB^2/2.$$

If $\angle A$ is acute and $\angle B$ is obtuse then

$$C_{ABC} + C_{BAC} = -\triangledown BPC + \triangledown ACQ = \triangledown ABPQ/2 = AB^2/2. \blacksquare$$

By Proposition 1.53,

$$C_{ABC} + C_{BAC} = \frac{AB^2}{2}, C_{BCA} + C_{ABC} = \frac{BC^2}{2}, C_{BAC} + C_{BCA} = \frac{CA^2}{2}.$$

From the above equations, we have

$$\begin{aligned} C_{ABC} &= (AB^2 + BC^2 - AC^2)/4, \\ C_{BAC} &= (AB^2 + AC^2 - BC^2)/4, \\ C_{ACB} &= (AC^2 + BC^2 - AB^2)/4. \end{aligned}$$

Definition 1.54 *We call $AB^2 + BC^2 - AC^2$ the Pythagoras difference of triangle ABC with regard to B, and denote it by P_{ABC}, i.e.,*

$$P_{ABC} = 4C_{ABC} = AB^2 + BC^2 - AC^2.$$

Proposition 1.55 (Pythagorean Theorem)

1. $P_{ABC} = 0$ *if and only if* $\angle ABC = 90^\circ$.
2. $P_{ABC} > 0$ *if and only if* $\angle ABC < 90^\circ$.
3. $P_{ABC} < 0$ *if and only if* $\angle ABC > 90^\circ$.

Proof. As shown in Figure 1-36, it is clear that $\triangledown BPC = 0$ if and only if $\angle ABC = 90^\circ$. The second and third cases come from the definition of the co-areas directly. ▮

Proposition 1.56 (The Pythagoras Difference Theorem) *If $\angle ABC \neq 90^\circ$, we have*

1. $\angle ABC = \angle XYZ$ *if and only if* $\frac{P_{ABC}}{P_{XYZ}} = \frac{AB \cdot BC}{XY \cdot YZ}$;
2. $\angle ABC + \angle XYZ = 180^\circ$ *if and only if* $\frac{P_{ABC}}{P_{XYZ}} = -\frac{AB \cdot BC}{XY \cdot YZ}$.

Proof. This proposition is a consequence of the definition of Pythagoras difference and the converse of the co-angle theorem on page 25. ▮

Example 1.57 *In triangle ABC, let* $BC = a, CA = b$, *and* $AB = c$. *Express the length m of the median CM in terms of a, b, and c.*

Solution. Without loss of generality, let us assume that $\angle A$ is not a right angle. Applying Proposition 1.56 to triangles BAC and MAC, we have

$$\frac{P_{MAC}}{P_{BAC}} = \frac{MA \cdot CA}{AB \cdot CA} = \frac{1}{2} \qquad (AB = 2MA).$$

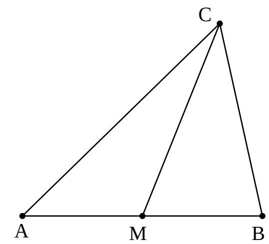


Figure 1-37

Hence $P_{BAC} = 2P_{MAC}$. i.e., $b^2 + c^2 - a^2 = 2(b^2 + c^2/4 - m^2)$. Now it is clear that

$$m^2 = (a^2 + b^2 - c^2/2)/2. \quad \blacksquare$$

Example 1.58 *In triangle ABC, let* $BC = a, CA = b$, *and* $AB = c$. *Express the length h of the altitude CD in terms of a, b, and c.*

Solution. Without loss of generality, we assume that $\angle A$ is acute. Applying Proposition 1.56 to triangles DAC and BAC, we have

$$P_{DAC}/P_{BAC} = (AD \cdot b)/(b \cdot c) = AD/c.$$

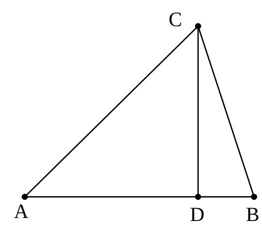


Figure 1-38

Let $x = AD$. By the Pythagorean theorem, $b^2 - h^2 = x^2$. Then $P_{DAC} = x^2 + b^2 - h^2 = 2x^2$. Thus

$$x/c = P_{DAC}/P_{BAC} = 2x^2/(b^2 + c^2 - a^2),$$

i.e., $x = (b^2 + c^2 - a^2)/(2c)$. Since $x^2 = b^2 - h^2$, we obtain the final result:

$$h^2 = b^2 - (b^2 + c^2 - a^2)^2/(4c^2). \quad \blacksquare$$

Example 1.59 *In triangle ABC, let* $BC = a, CA = b$, *and* $AB = c$. *Express the length of the bisector CF in terms of a, b, and c.*

Solution. We assume that $\angle A \neq 90^\circ$. Applying Proposition 1.56 to triangles BAC and FAC

$$P_{FAC}/P_{BAC} = (AF \cdot b)/b \cdot c = AF/c.$$

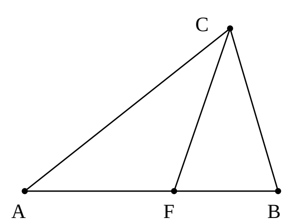


Figure 1-39

By Example 1.36, $AF/BF = b/a$; hence $AF = (bc)/(a+b)$. Let $x = CF$. We have

$$AF/c = P_{FAC}/P_{BAC} = (AF^2 + b^2 - x^2)/(b^2 + c^2 - a^2).$$

Substituting $AF = bc/(a+b)$ into the above equation, we obtain

$$x^2 = ab((a+b)^2 - c^2)/(a+b)^2. \quad \blacksquare$$

We define the *Pythagoras difference for an oriented quadrilateral ABCD* as follows.

$$P_{ABCD} = AB^2 - BC^2 + CD^2 - DA^2.$$

It is easy to derive the following properties for the Pythagoras differences of quadrilaterals.

1. $P_{ABCD} = P_{CDAB} = P_{BADC} = P_{DCBA}$.
2. $P_{ABCD} = -P_{BCDA} = -P_{DABC} = -P_{ADCB} = -P_{CBAD}$.
3. $P_{ABCD} = P_{BAC} - P_{DAC} = P_{ABD} - P_{CBD} = P_{DCA} - P_{BCA} = P_{CDB} - P_{ADB}$.
4. $P_{ABBC} = P_{ABC}, P_{AABC} = -P_{BAC}, P_{ABCC} = -P_{ACB}, P_{ABCA} = P_{BAC}$.
5. $P_{ABAC} = 0, P_{ABCB} = 0$.
6. $P_{APBQ} + P_{BPCQ} = P_{APCQ}$. (The additivity of the Pythagoras difference.)

Proposition 1.60 *If A, B, and C are collinear, we have $P_{ABC} = 2\overline{BA}\cdot\overline{BC}$.*

Proof. Since $\overline{AC} = \overline{AB} - \overline{CB}$, we have

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 = \overline{AB}^2 + \overline{CB}^2 - (\overline{AB} - \overline{CB})^2 = 2\overline{BA}\cdot\overline{BC}. \quad \blacksquare$$

Definition 1.61 *We use the notation $AB\perp CD$ to denote that four points A, B, C, and D satisfy one of the following conditions: line AB is perpendicular to line CD; $A = B$; or $C = D$.*

The following result gives a criterion for $AB\perp CD$ using the Pythagoras difference.

Proposition 1.62 *$AC\perp BD$ if and only if $P_{ABCD} = P_{ABD} - P_{CBD} = 0$.*

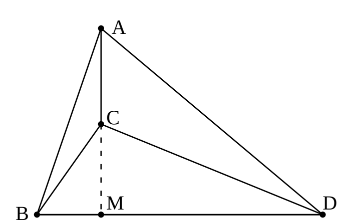


Figure 1-40

Proof. Let M and N be the feet of the perpendiculars from points A and C to line BD. Then by the Pythagorean theorem,

$$\begin{aligned}P_{ABD} &= \overline{AB}^2 + \overline{BD}^2 - \overline{AD}^2\\ &= \overline{AM}^2 + \overline{BM}^2 + \overline{BD}^2 - \overline{AM}^2 - \overline{MD}^2\\ &= \overline{BM}^2 + \overline{BD}^2 - (\overline{BD} - \overline{BM})^2\\ &= 2\overline{BM}\cdot\overline{BD}.\end{aligned}$$

Similarly $P_{CBD} = 2\overline{BN}\cdot\overline{BD}$. We have $P_{ABD} = P_{CBD}$ if and only if $\overline{MB} = \overline{NB}$, i.e., $M = N$ which is equivalent to $AC \perp BD$. ∎

Proposition 1.63 *Let P and Q be the feet of the perpendiculars from points A and C to BD. Then $P_{ABCD} = 2\overline{QP}\cdot\overline{BD}$.*

Proof. By Propositions 1.62 and 1.60,

$$\begin{aligned}P_{ABCD} &= P_{ABD} - P_{CBD} = P_{PBD} - P_{QBD}\\ &= 2\overline{BP}\cdot\overline{BD} - 2\overline{BQ}\cdot\overline{BD} = 2\overline{QP}\cdot\overline{BD}.\end{aligned}$$ ∎

Proposition 1.64 *Let $ABCD$ be a parallelogram. Then for two points P and Q, we have $P_{PAQB} = P_{PDQC}$.*

Proof. This is a consequence of Proposition 1.63. ∎

Proposition 1.65 *Let D be the foot drawn from point P to a line AB ($A \neq B$). Then we have*

$$\frac{\overline{AD}}{\overline{DB}} = \frac{P_{PAB}}{P_{PBA}},\quad \frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2},\quad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$

Proof. By Proposition 1.62,

$$P_{PAB} = P_{DAB} = 2\overline{AB}\cdot\overline{AD},\quad P_{PBA} = P_{DBA} = 2\overline{BA}\cdot\overline{BD}.$$

The result is clear now. ∎

Proposition 1.66 *Let AB and PQ be two nonperpendicular lines and Y be the intersection of line PQ and the line passing through A and perpendicular to AB (Figure 1-41). Then*

$$\frac{\overline{PY}}{\overline{QY}} = \frac{P_{PAB}}{P_{QAB}},\quad \frac{\overline{PY}}{\overline{PQ}} = \frac{P_{PAB}}{P_{PAQB}},\quad \frac{\overline{QY}}{\overline{PQ}} = \frac{P_{QAB}}{P_{PAQB}}.$$

Proof. We need only to show the first equation. Let P_1 and Q_1 be the orthogonal projections from P and Q to line AB respectively. By Proposition 1.62, $\frac{P_{PAB}}{P_{QAB}} = \frac{P_{P_1AB}}{P_{Q_1AB}} = \frac{\overline{AP_1}\cdot\overline{AB}}{\overline{AQ_1}\cdot\overline{AB}} = \frac{\overline{AP_1}}{\overline{AQ_1}} = \frac{\overline{PY}}{\overline{QY}}$. ▮

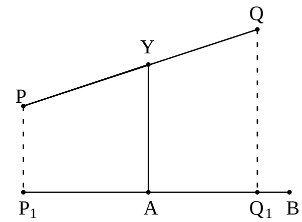


Figure 1-41

Example 1.67 (The Orthocenter Theorem) *Show that the three altitudes of a triangle are concurrent.*

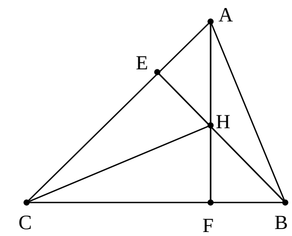


Figure 1-42

Proof. Let the two altitudes AF and BE of triangle ABC meet in H. We need only to prove $CH \perp AB$, i.e., $P_{ACH} = P_{BCH}$. By Proposition 1.62, $P_{ACH} = P_{ACB} = P_{BCA} = P_{BCH}$. ▮

Example 1.68 (Orthocenter-dual) *Let $ABCO$ be a quadrilateral. From point O perpendiculars to OA, OB, and OC are drawn which meet BC, CA, and AB in D, E, and F respectively. Show that D, E, and F are collinear.*

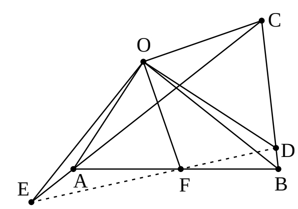


Figure 1-43

Proof. Let DE and AB meet in Z. We need only to show $Z = F$, i.e., $\frac{\overline{AF}}{\overline{BF}} = \frac{\overline{AZ}}{\overline{BZ}}$. By Proposition 1.66, we can eliminate F: $\frac{\overline{AF}}{\overline{BF}} = \frac{P_{AOC}}{P_{BOC}}$. By the co-side theorem, we can eliminate Z: $\frac{\overline{BZ}}{\overline{AZ}} = \frac{S_{BDE}}{S_{ADE}}$. To eliminate E, by Proposition 1.66 we have

$$S_{BDE} = \frac{\overline{EC}}{\overline{AC}} S_{BDA} = \frac{P_{COB}}{P_{COAB}} S_{BDA}, \quad S_{ADE} = \frac{\overline{EA}}{\overline{AC}} S_{ACD} = \frac{P_{AOB}}{P_{COAB}} S_{ACD}.$$

Then

$$\begin{aligned}\frac{\overline{AF}}{\overline{BF}} \cdot \frac{\overline{BZ}}{\overline{AZ}} &= \frac{P_{AOC}}{P_{BOC}} \cdot \frac{S_{BDA} P_{COB}}{P_{COAB}} \cdot \frac{P_{COAB}}{P_{AOB} S_{ACD}} \\ &= \frac{P_{AOC}}{P_{AOB}} \cdot \frac{S_{BDA}}{S_{ACD}} = \frac{P_{AOC}}{P_{AOB}} \cdot \frac{\overline{BD}}{\overline{CD}} = \frac{P_{AOC}}{P_{AOB}} \cdot \frac{P_{AOB}}{P_{AOC}} = 1. \quad ▮\end{aligned}$$

Example 1.69 *On the two sides AB and AC of triangle ABC, two squares $ABDE$ and $ACFG$ are drawn externally. Show that $BG \perp CE$.*

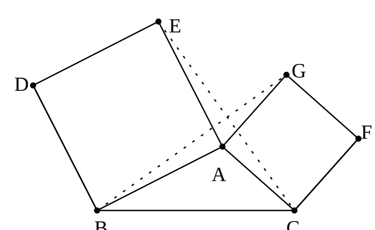


Figure 1-44

Proof.

$$\begin{aligned}
&P_{BCGE} = P_{BCGA} + P_{BAGE} && \text{(Additivity)}\\
&= P_{BCAA} + P_{ACGA} + P_{BAAE} + P_{AAGE} && \text{(Additivity)}\\
&= -P_{BAC} - P_{GAE} && (AG \perp AC, AB \perp AE)\\
&= 0 && (\angle BAC + \angle GAE = 180^\circ) \blacksquare
\end{aligned}$$

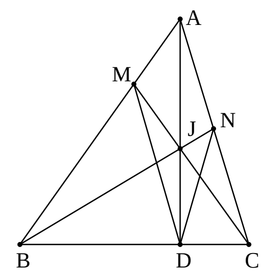


Figure 1-45

Example 1.70 *In triangle ABC, take a point J on the altitudes AD. Lines BJ and CJ meet AC and AB in N and M respectively. Show that $\angle MDA = \angle ADN$.*

Proof. To prove $\angle MDA = \angle ADN$, by the co-angle theorem and Proposition 1.56 we need only to show $P_{MDA}/S_{MDA} = P_{ADN}/S_{ADN}$.

$$\begin{aligned}
&S_{MDA} = \frac{\overline{AM}}{\overline{AB}} S_{BDA} = \frac{S_{AJC}}{S_{AJBC}} S_{BDA} && \text{the co-side theorem;}\\
&S_{ADN} = \frac{\overline{AN}}{\overline{AC}} S_{ADC} = \frac{S_{ABJ}}{S_{ABCJ}} S_{ADC} && \text{the co-side theorem;}\\
&P_{ADN} = \frac{\overline{NC}}{\overline{AC}} P_{ADA} = \frac{S_{BCJ}}{S_{ABCJ}} P_{ADA} && \text{Proposition 1.65;}\\
&P_{MDA} = \frac{\overline{MB}}{\overline{AB}} P_{ADA} = \frac{S_{BCJ}}{S_{AJBC}} P_{ADA} && \text{Proposition 1.65;}
\end{aligned}$$

Then

$$\frac{P_{ADN}}{P_{MDA}} \frac{S_{MDA}}{S_{ADN}} = \frac{S_{BDA}}{S_{ADC}} \frac{S_{AJC}}{S_{ABJ}} = \frac{\overline{BD}}{\overline{DC}} \frac{\overline{DC}}{\overline{BD}} = 1. \quad \blacksquare$$

Exercises 1.71

1. In Example 1.69, show that $BG = CE$.
2. In Example 1.69, let M be the midpoint of BC. Show that $AM \perp EG$ and $EG = 2AM$.
3. The sum of the squares of the diagonals of a parallelogram is equal to sum of the squares of the four sides of the given parallelogram. (Use Example 1.57.)
4. In the quadrilateral $ABCD$, if $\angle ABC = \angle CDA = 90^\circ$ and P is the intersection of AC and BD then $\frac{P_{BAD}}{P_{BCD}} = \frac{AP}{CP}$.

1.8 Trigonometric Functions

We have discussed two major geometric quantities: the area and the Pythagoras difference. Area is a well-known concept and has been used since the time of Euclid. On the other hand, the Pythagoras difference is unfamiliar to most readers. In this section, we will introduce the trigonometric functions and use them to represent areas and Pythagoras differences.

The sine and cosine functions can be defined in the usual way. Let ABC be a triangle with $\angle B = 90°$. Then

$$\sin(\angle A) = \frac{BC}{AC}, \quad \cos(\angle A) = \frac{AB}{AC}.$$

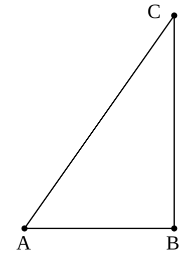


Figure 1-46

We can also use the area to define trigonometric functions.

Definition 1.72 $\sin(\angle A)$ *is twice the signed area of a triangle* ABC *such that* $AB = AC = 1$.

It is easy to see that the two definitions are consistent. The following properties of the trigonometric functions can be derived from the definition directly.

1. $\sin(0) = \sin(180°) = 0$; $\sin(90°) = 1$.
2. $\sin(\angle X) = \sin(180° - \angle X)$.
3. $\sin(\angle A) = \sin(\angle B)$ if and only if $\angle A = \angle B$ or $\angle A + \angle B = 180°$.
4. Let us assume that $\angle A + \angle B \le 180°$. Then $A < B$ implies $\sin(\angle A) < \sin(\angle B)$, and vise versa. (This is a consequence of Proposition 1.43.)

Proposition 1.73 $S_{ABC} = \frac{1}{2}AB \cdot AC \cdot \sin(\angle A)$.

Proof. Take two points X and Y on AB and AC respectively such that $AX = AY = 1$. By the co-angle theorem

$$\triangledown ABC = \triangledown AXY \frac{AB \cdot AC}{AX \cdot AY} = \triangledown AXY \cdot AB \cdot AC = \frac{1}{2} AB \cdot AC \cdot \sin(\angle A). \quad ▮$$

Applying the above proposition to the three angles of triangle ABC respectively, we have

$$\triangledown ABC = \frac{1}{2} bc \sin(\angle A) = \frac{1}{2} ac \sin(\angle B) = \frac{1}{2} ab \sin(\angle C).$$

As a direct consequence, we have

Proposition 1.74 (The Sine Law) *In triangle* ABC*, let* $BC = a, CA = b$*, and* $AB = c$*. Then*

$$\sin(\angle A)/a = \sin(\angle B)/b = \sin(\angle C)/c.$$

The following is a very useful property of trigonometric functions.

Proposition 1.75 (The Visual Angle Theorem)
Emanating from P, there are three rays PA, PB, and PC such that $\angle APC = \alpha$, $\angle CPB = \beta$, and $\angle APB = \gamma = \alpha + \beta < 180^\circ$. Then A, B, and C are collinear if and only if

$$\sin(\angle\gamma)/PC = \sin(\angle\alpha)/PB + \sin(\angle\beta)/PA.$$

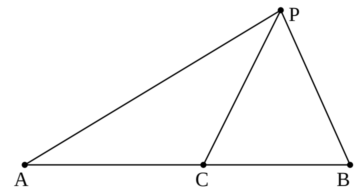


Figure 1-47

Proof. If points A, B, and C are collinear then we have $\triangledown PAB = \triangledown PAC + \triangledown PCB$, i.e.,

$$PA \cdot PB \cdot \sin(\angle\gamma) = PA \cdot PC \cdot \sin(\angle\alpha) + PB \cdot PC \cdot \sin(\angle\beta).$$

We obtain the result by dividing $PA \cdot PB \cdot PC$ from both sides of the above formula. Conversely, from the above formula, we have $\triangledown PAB = \triangledown PAC + \triangledown PCB$ which implies $\triangledown ABC = |\triangledown PAB - \triangledown PAC - \triangledown PCB| = 0$, i.e., A, B, and C are collinear. ∎

Proposition 1.75 has many applications.

Example 1.76 *In triangle ABC, $\angle ACB = 120^\circ$; CE is the bisector of angle C. Show that $1/CE = 1/CA + 1/CB$.*

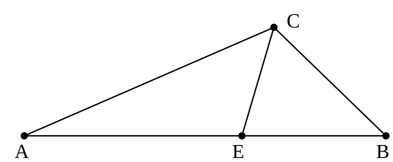


Figure 1-48

Proof. By Proposition 1.75,

$$\sin(120^\circ)/CE = \sin(60^\circ)/CA + \sin(60^\circ)/CB.$$

Since $\sin(120^\circ) = \sin(180^\circ - 120^\circ) = \sin(60^\circ)$, we obtain the result by dividing $\sin(60^\circ)$ from both sides of the above equation. ∎

Example 1.77 *In the right triangle ABC, let $BC = a, AC = b$, and $CD = h$ be the altitude on the hypotenuse. Show that $1/h^2 = 1/a^2 + 1/b^2$.*

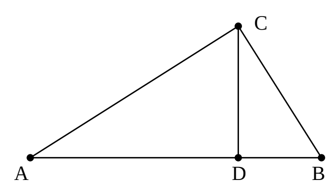


Figure 1-49

Proof. Let $\alpha = \angle ACD, \beta = \angle BCD$. By Proposition 1.75

$$\sin(\alpha + \beta)/h = \sin(\alpha)/a + \sin(\beta)/b.$$

We have $\sin(\alpha + \beta) = \sin(\angle ACB) = 1, \sin(\alpha) = \sin(\angle CBD) = h/a, \sin(\beta) = \sin(\angle CAD) = h/b$. Substituting these into the above equation, we obtain the result. ∎

Definition 1.78 *Let $\angle A$ be an angle. The cosine of $\angle A$ is defined as follows.*

$$\cos(\angle A) = \begin{cases} \sin(90^\circ - \angle A) & \text{if } \angle A \le 90^\circ; \\ -\sin(\angle A - 90^\circ) & \text{if } \angle A > 90^\circ. \end{cases}$$

Proposition 1.79 (The Cosine Law) *In triangle ABC, we have*

$$P_{ABC} = 2AB \cdot CB \cdot \cos(\angle B).$$

Proof. In Figure 1-36, if $\angle B \le 90^\circ$

$$C_{ABC} = \frac{1}{2}AB \cdot BC \cdot \sin(90^\circ - \angle B) = \frac{1}{2}AB \cdot BC \cdot \cos(\angle B).$$

Thus $P_{ABC} = 4C_{ABC} = 2AB \cdot BC \cdot \cos(\angle B)$. Other cases can be proved similarly. ∎

Example 1.80 *Let α and β both be acute angles. Show that*

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

Proof. In Proposition 1.75, let us assume that $PC \perp AB$, $PC = h$, $PA = b$, and $PB = a$. Then

$$\sin(\alpha + \beta) = (h/a)\sin(\alpha) + (h/b)\sin(\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha). \quad ∎$$

It is easy to show that the angles in the preceding example are not necessarily acute. In Example 1.80, let $\alpha + \beta = 90^\circ$. We have the well-known formula:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1.$$

Example 1.81 $\sin(30^\circ) = \frac{1}{2}$; $\sin(45^\circ) = \sqrt{2}/2$; $\sin(60^\circ) = \sqrt{3}/2$.

Proof. In Proposition 1.80, setting $\alpha = \beta = 30^\circ$, we have

$$\sin(60^\circ) = 2\sin(30^\circ)\cos(30^\circ).$$

Since $\cos(30^\circ) = \sin(90^\circ - 30^\circ) = \sin(60^\circ)$, we have $\sin(30^\circ) = \frac{1}{2}$. Similarly setting $\alpha = 30^\circ, \beta = 60^\circ$, we have $\sin(60^\circ) = \sqrt{3}/2$; setting $\alpha = \beta = 45^\circ$, we have $\sin(45^\circ) = \sqrt{2}/2$. ∎

So far, the angles are always positive. To represent the signed areas using trigonometric functions, we need to introduce the *oriented angle*, which is also denoted by $\angle$.

Definition 1.82 *The oriented angle $\measuredangle ABC$ is a real number such that (1) the absolute value of $\measuredangle ABC$ is the same as the ordinary angle $\angle ABC$, and (2) $\measuredangle ABC$ has the same sign with S_{ABC}.*

It is clear that for any angle α, we have $-180^\circ < \alpha \leq 180^\circ$. The arithmetic for oriented angles is understood to be mod 360° and between -180° and 180°,

Definition 1.83 *We extend the definition of the sine and cosine functions to the oriented angles as follows. Let* $\measuredangle\alpha$ *be a negative angle. Then*

$$\sin(\measuredangle\alpha) = -\sin(-\measuredangle\alpha), \quad \cos(\measuredangle\alpha) = \cos(-\measuredangle\alpha).$$

With the above definition, the properties for these two functions proved before are still valid. In particular, we have

Proposition 1.84 $S_{ABC} = \frac{1}{2}AB \cdot BC \cdot \sin(\measuredangle ABC), P_{ABC} = 2AB \cdot BC \cdot \cos(\measuredangle ABC)$.

Example 1.85 (The Herron-Qin Formula[8]) *In triangle* ABC*,*

$$16S_{ABC}^2 = 4\overline{AB}^2 \cdot \overline{CB}^2 - P_{ABC}^2.$$

Proof. By Proposition 1.84, $\sin(\measuredangle ABC) = \frac{2S_{ABC}}{AB \cdot BC}$, $\cos(\measuredangle ABC) = \frac{P_{ABC}}{2AB \cdot BC}$. Since $\sin(\measuredangle ABC)^2 + \cos(\measuredangle ABC)^2 = 1$, we have

$$\frac{4S_{ABC}^2}{AB^2 \cdot BC^2} + \frac{P_{ABC}^2}{4AB^2 \cdot BC^2} = 1.$$

Thus $16S_{ABC}^2 = 4\overline{AB}^2 \cdot \overline{CB}^2 - P_{ABC}^2$. ∎

Definition 1.86 *The oriented angle between two directed lines* PQ *and* AB*, denoted by* $\measuredangle(PQ, AB)$*, is defined as follows. Take points* O, X*, and* Y *such that* $OYQP$ *and* $OXBA$ *are parallelograms. Then we define* $\measuredangle(PQ, AB) = \measuredangle XOY$.

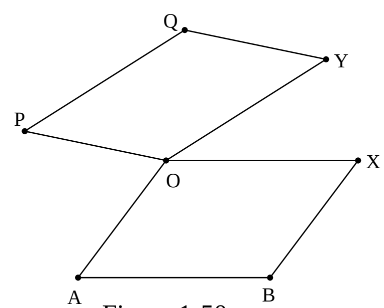


Figure 1-50

With the above definition, it is easy to check the following properties for the oriented angles.

1. $\measuredangle(PQ, AB) = -\measuredangle(AB, PQ)$.
2. $\measuredangle(PQ, AB) = 180^\circ + \measuredangle(QP, AB) = 180^\circ + \measuredangle(PQ, BA)$.

[8]Qin (1247 A.D.) is an ancient Chinese mathematician who discovered the area formula for triangles in exact form of this proposition [171].

3. $\measuredangle(PQ, AB) = \measuredangle(QP, BA)$.

4. $\measuredangle(PQ, AB) + \measuredangle(AB, CD) = \measuredangle(PQ, CD)$.

We can now represent the signed area and the Pythagoras difference of quadrilaterals using trigonometric functions.

Proposition 1.87 $S_{ABCD} = \frac{1}{2} AC \cdot BD \cdot \sin(\measuredangle(AC, BD))$.

Proof. Take a point X such that $AXDB$ is a parallelogram. Then $\overline{AX} = \overline{BD}$. By Proposition 1.30, $S_{ABCD} = S_{AACX} = S_{XAC} = \frac{1}{2} XA \cdot AC \cdot \sin(\measuredangle XAC) = \frac{1}{2} AC \cdot BD \cdot \sin(\measuredangle(AC, BD))$. ∎

Proposition 1.88 $P_{ABCD} = 2AC \cdot BD \cdot \cos(\measuredangle(AC, DB))$.

Proof. Take a point X such that $CXDB$ is a parallelogram. Then $\overline{CX} = \overline{BD}$. By Proposition 1.64, $P_{ABCD} = -P_{CBAD} = -P_{CCAX} = P_{XCA} = 2XA \cdot AC \cdot \cos(\measuredangle XCA) = 2AC \cdot BD \cdot \cos(\measuredangle(AC, DB))$. ∎

Example 1.89 (The Herron-Qin Formula for Quadrilaterals) *For a quadrilateral ABCD, we have* $16S^2_{ABCD} = 4\overline{AC}^2 \cdot \overline{BD}^2 - P^2_{ABCD}$.

Proof. By Propositions 1.87 and 1.88, $\sin(\measuredangle(AC, BD)) = \frac{2S_{ABCD}}{AC \cdot BD}$, $\cos(\measuredangle(AC, DB)) = \frac{P_{ABCD}}{2AC \cdot BD}$. Since $\sin(\measuredangle(AC, BD))^2 + \cos(\measuredangle(AC, DB))^2 = 1$, we have

$$\frac{4S^2_{ABCD}}{AC^2 \cdot BD^2} + \frac{P^2_{ABCD}}{4AC^2 \cdot BD^2} = 1.$$

Thus $16S^2_{ABCD} = 4\overline{AC}^2 \cdot \overline{BD}^2 - P^2_{ABCD}$. ∎

Exercises 1.90

1. Use Example 1.80 to prove the following formulas

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha + \beta) &= \sin(\alpha)\sin(\beta) - \cos(\alpha)\cos(\beta) \\ \cos(\alpha - \beta) &= \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{aligned}$$

2. Let $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$. Show that

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}, \quad \tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}.$$

3. Prove the following version of the Herron-Qin formula for quadrilaterals.

$$S^2_{ABCD} = (p-a)(p-b)(p-c)(p-d) - a\cdot b\cdot c\cdot d\cdot \cos(\frac{\angle B+\angle D}{2})^2$$

where $a = AB, b = BC, c = CD, d = DA$, and $P = (a+b+c+d)/2$. (Hint. We have $S^2_{ABCD} = (S_{ABC} + S_{ACD})^2$. Then use Proposition 1.84.)

4. Continue from the above exercise. If $ABCD$ is convex and cyclic then $S^2_{ABCD} = (p-a)(p-b)(p-c)(p-d)$.

5. Continue from the above exercise. If $ABCD$ also has an inscribed circle then $S^2_{ABCD} = abcd$.

1.9 Circles

In this section, we will discuss another important geometric object: the circle. Points on the same circle are called *co-circular* or *cyclic*.

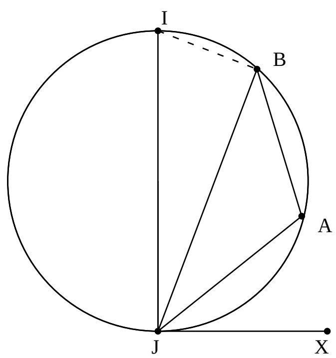


Figure 1-51

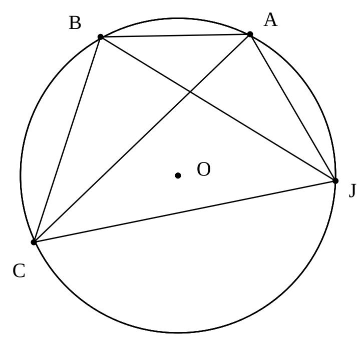


Figure 1-52

Let J be a fixed reference point on the circle, I the antipodal of J, and X a point on the tangent of the circle at point J such that $S_{IJX} > 0$ (Figure 1-51). For any point A on the circle, $\measuredangle A$ is defined to be the oriented angle $\measuredangle AJX$, i.e., the (oriented) inscribed angle corresponding to the arc JA. Then for two points A and B on the circle, $\measuredangle B - \measuredangle A = \measuredangle BJA$ (Figure 1-51).

We now define the *oriented chord* on the circle. The absolute value of the oriented chord $\widetilde{AB}$ is AB while its sign is the same as the sign of the oriented angle $\measuredangle BJA$, or equivalently the same as the sign of S_{BJA}. In Figure 1-51, since $S_{BJA} > 0$, we have $\widetilde{AB} > 0$. The oriented chord $\widetilde{JA}$ is always positive.

Proposition 1.91 *With the above notations, we have*

$$\widetilde{AB} = d\sin(\measuredangle BJA) = d\sin(\measuredangle B - \measuredangle A)$$

$$\widetilde{JA} = d\sin(\measuredangle AJX)$$

where d is the diameter of the circle.

Proof. The signs of the two sides of the equation are equal. Thus we need only to check the absolute values of both sides of the equation. Since $\angle BAJ = \angle BIJ$ or $\angle BAJ + \angle BIJ = 180°$, we have

$$AB = \sin(\angle BJA)\frac{BJ}{\sin\angle BAJ} = \sin(\angle BJA)\frac{BJ}{\sin\angle BIJ} = \sin(\angle BJA)\cdot IJ. \quad \blacksquare$$

Proposition 1.92 *Let the diameter of the circumcircle of triangle ABC be d. Then* $S_{ABC} = \widetilde{AB}\cdot\widetilde{BC}\cdot\widetilde{AC}/(2d)$.

Proof. As shown in Figure 1-52, we have $\angle ABC = \angle CJA$ or $\angle ABC = 180° - \angle CJA$. By Propositions 1.73 and 1.91,

$$\triangledown ABC = \frac{1}{2}AB\cdot BC|\sin(\angle ABC)| = \frac{1}{2}AB\cdot BC\cdot|\sin(\angle CJA)| = \frac{1}{2d}AB\cdot BC\cdot CA.$$

We still need to check whether the signs of both sides of the formula are the same. At first, it is easy to see that when interchanging two vertices of the triangle, the signs of both sides of the equation will change. Therefore, we need only to check a particular position for the three vertices A, B, and C, e.g., the case shown in Figure 1-52. In this case, we have $S_{ABC} \geq 0, S_{CJB} \geq 0, S_{BJA} \geq 0$, and $S_{CJA} \geq 0$. Hence $\widetilde{AB} \geq 0$, $\widetilde{BC} \geq 0$, and $\widetilde{AC} \geq 0$. $\blacksquare$

Proposition 1.93 (The Co-circle Theorem) *If the circumcircles of triangles ABC and XYZ are the same or equal then* $\frac{S_{ABC}}{S_{XYZ}} = \frac{\widetilde{AB}\cdot\widetilde{BC}\cdot\widetilde{CA}}{\widetilde{XY}\cdot\widetilde{YZ}\cdot\widetilde{ZX}}$.

Proof. This is a direct consequence of Proposition 1.92. $\blacksquare$

Example 1.94 (Ptolemy's Theorem) *Let A, B, C, and D be four points on the same circle. Then* $\widetilde{AB}\cdot\widetilde{CD} + \widetilde{AD}\cdot\widetilde{BC} = \widetilde{AC}\cdot\widetilde{BD}$.

Proof. We chose A to be the reference point. By Proposition 1.91,

$$\begin{aligned} & \widetilde{AB}\cdot\widetilde{CD} + \widetilde{AD}\cdot\widetilde{BC} - \widetilde{AC}\cdot\widetilde{BD} \\ = \;& d^2(\sin(\measuredangle B)\sin(\measuredangle D - \measuredangle C) + \sin(\measuredangle D)\sin(\measuredangle C - \measuredangle B) - \sin(\measuredangle C)\sin(\measuredangle D - \measuredangle B)) \\ = \;& 0. \quad \blacksquare \end{aligned}$$

Example 1.95 (Brahmagupta's Formula) *Let A, B, C, and D be four points on the same circle. Then*

$$S^2_{ABCD} = (p - \widetilde{AB})(p - \widetilde{BC})(p - \widetilde{CD})(p - \widetilde{AD})$$

where $p = \frac{1}{2}(\widetilde{AB} + \widetilde{BC} + \widetilde{CD} + \widetilde{AD})$.

Proof. By Example 1.89 and Ptolemy's theorem

$$\begin{aligned}16S^2_{ABCD} &= 4\widetilde{AC}^2\cdot\widetilde{BD}^2 - P^2_{ABCD}\\ &= 4(\widetilde{AB}\cdot\widetilde{CD}+\widetilde{AD}\cdot\widetilde{BC})^2 - P^2_{ABCD}\\ &= ((\widetilde{AB}+\widetilde{CD})^2-(\widetilde{AD}-\widetilde{BC})^2)((\widetilde{AD}+\widetilde{BC})^2-(\widetilde{AB}-\widetilde{CD})^2)\\ &= 16(p-\widetilde{AB})(p-\widetilde{BC})(p-\widetilde{CD})(p-\widetilde{AD}).\end{aligned}$$ ▮

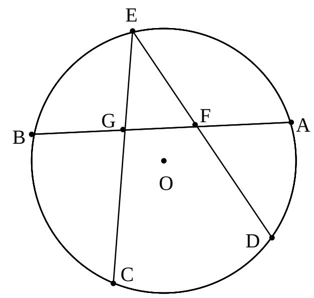


Figure 1-53

Example 1.96 *$A, B, C,$ and D are four co-circle points. For any point E on the same circle, lines DE and CE meet AB in F and G. Show that $\frac{\overline{AF}}{\overline{BF}}\cdot\frac{\overline{BG}}{\overline{AG}}$ is independent of E.*

Proof. By the co-side theorem and the co-circle theorem,

$$\begin{aligned}\frac{\overline{AF}}{\overline{BF}}\cdot\frac{\overline{BG}}{\overline{AG}} &= \frac{S_{ADE}S_{BCE}}{S_{BDE}S_{ACE}}\\ &= \frac{\widetilde{AD}\cdot\widetilde{DE}\cdot\widetilde{EA}\cdot\widetilde{BC}\cdot\widetilde{CE}\cdot\widetilde{EB}}{\widetilde{BD}\cdot\widetilde{DE}\cdot\widetilde{EB}\cdot\widetilde{AC}\cdot\widetilde{CE}\cdot\widetilde{EA}} = \frac{\widetilde{AD}\cdot\widetilde{BC}}{\widetilde{BD}\cdot\widetilde{AC}}\end{aligned}$$

which is independent of E. ▮

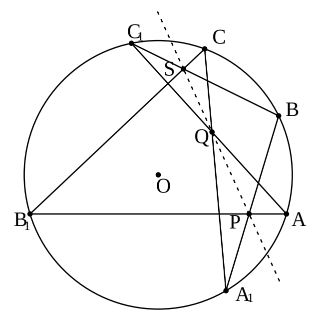


Figure 1-54

Example 1.97 (Pascal's Theorem) *Let A, B, C, A_1, B_1, and C_1 be six points on a circle. Let $P = AB_1 \cap A_1B$, $Q = AC_1 \cap A_1C$, and $S = BC_1 \cap B_1C$. Show that P, Q, and S are collinear.*

Proof. Note that the points $P, Q,$ and R in this example are constructed in the same way as in Example 1.17 on page 14. Let $Z_1 = PQ \cap BC_1$ and $Z_2 = PQ \cap B_1C$. We need only to show

$$G = \frac{\overline{PZ_1}}{\overline{QZ_1}}\cdot\frac{\overline{QZ_2}}{\overline{PZ_2}} = 1.$$

By Example 1.17, we have $G = \frac{S_{ABB_1}S_{A_1BC_1}S_{ACC_1}S_{A_1CB_1}}{S_{ACB_1}S_{A_1BB_1}S_{ABC_1}S_{A_1CC_1}}$. Now $G = 1$ follows from the co-circle theorem immediately. ▮

Example 1.98 (The General Butterfly Theorem) *$A, B, C, D, E,$ and F are six co-circle points. Lines CD and EF meet AB in M and N. Lines CF and DE meet AB in G and H. Show that $\frac{\overline{MG}}{\overline{AG}}\cdot\frac{\overline{BH}}{\overline{NH}} = \frac{\overline{BM}}{\overline{AN}}$. (Figure 1-55)*

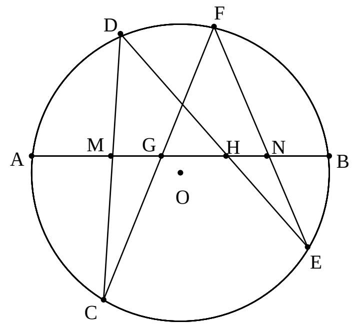


Figure 1-55

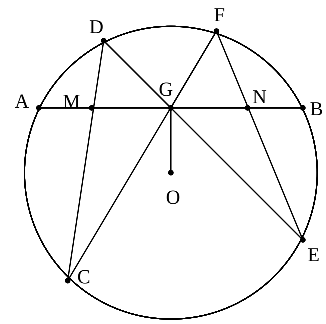


Figure 1-56

Proof. By the co-side theorem

$$G = \frac{\overline{MG}}{\overline{AG}} \cdot \frac{\overline{BH}}{\overline{NH}} \cdot \frac{\overline{AN}}{\overline{BM}} = \frac{S_{MCF}}{S_{ACF}} \frac{S_{BDE}}{S_{NDE}} \frac{\overline{AN}}{\overline{AB}} \cdot \frac{\overline{AB}}{\overline{BM}} = \frac{S_{MCF}}{S_{ACF}} \frac{S_{BDE}}{S_{NDE}} \frac{S_{AFE}}{S_{AFBE}} \frac{S_{ADBC}}{S_{BDC}}.$$

By the co-side theorem again,

$$\frac{S_{MCF}}{S_{DCF}} = \frac{\overline{MC}}{\overline{DC}} = \frac{S_{ABC}}{S_{ADBC}}, \frac{S_{FDE}}{S_{NDE}} = \frac{\overline{FE}}{\overline{NE}} = \frac{S_{AFBE}}{S_{ABE}}.$$

Then $G = \frac{S_{DCF} S_{ABC} S_{BDE} S_{AFE}}{S_{ACF} S_{ABE} S_{FDE} S_{BDC}}$. Now $G = 1$ follows from the co-circle theorem immediately. ∎

In Example 1.98, when $G = H$ becomes the midpoint of AB, we obtain the ordinary butterfly theorem:

Example 1.99 (The Butterfly Theorem) *C, D, E, and F are four points on circle O (Figure 1-56). G is the intersection of DE and CF. Through G draw a line perpendicular to OG, meeting CD in M and EF in N. Show that G is the midpoint of MN.*

In the above examples, no Pythagoras differences occur in the proof. To deal with Pythagoras differences, we need to develop some new tools.

We still assume that there is a fixed reference point J on the circle whose diameter is d.

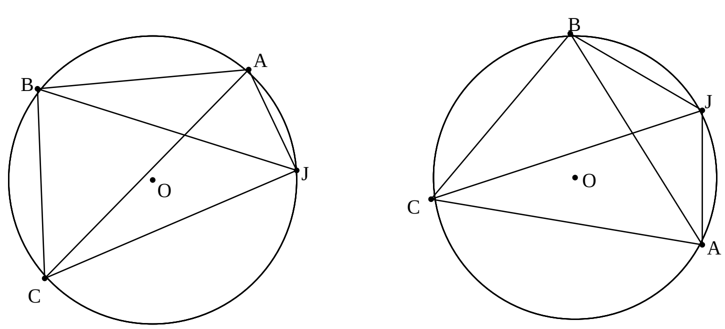


Figure 1-57

Proposition 1.100 *Let d be the diameter of the circumcircle of triangle ABC. Then* $P_{ABC} = 2\widetilde{AB}\cdot\widetilde{CB}\cos(\measuredangle CJA)$.

Proof. By the cosine theorem for oriented angles, $|P_{ABC}| = 2AB\cdot BC|\cos(\angle ABC)|$. As shown in Figure 1-57, we have $\angle ABC = \angle CJA$ or $\angle ABC = 180^\circ - \angle CJA$. Then

$$|P_{ABC}| = 2AB\cdot BC\cdot|\cos(\angle CJA)|.$$

We still need to check whether the signs of both sides of the equation are equal. At first, when interchanging the position of A and C, the signs of both sides of the equation will not change. Therefore, we need only to consider the following two cases: J is on the arc AC or on the arc AB. In the first case, we have $\widetilde{AB}\geq 0, \widetilde{CB}\leq 0$. Since $\measuredangle ABC + \measuredangle CJA = 180^\circ$, P_{ABC} and $\cos(\measuredangle CJA)$ always have opposite signs. Therefore the proposition is true in this case. In the second case, we have $\widetilde{AB}\leq 0, \widetilde{CB}\leq 0$. By the inscribe angle theorem, $\measuredangle CJA = \measuredangle CBA$. Thus P_{ABC} and $\cos(\measuredangle CJA)$ always have the same sign. ∎

Proposition 1.101 (Co-circle Theorem for Pythagoras Differences) *If the circumcircles of triangles ABC and XYZ are the same and $P_{XYZ}\neq 0$, then*

$$\frac{P_{ABC}}{P_{XYZ}} = \frac{\widetilde{AB}\cdot\widetilde{BC}\cos(\measuredangle AJC)}{\widetilde{XY}\cdot\widetilde{YZ}\cos(\measuredangle XJZ)}.$$

Proof. This is a direct consequence of Proposition 1.100. ∎

Example 1.102 (Simson's Theorem) *Let D be a point on the circumcircle of triangle ABC. From D three perpendiculars are drawn to the three sides BC, AC, and AB of triangle ABC. Let E, F, and G be the three feet respectively. Show that E, F and G are collinear.*

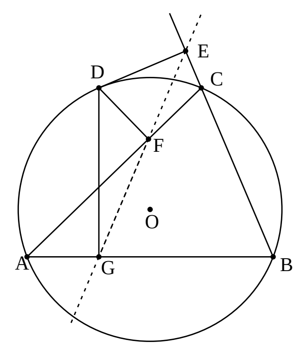


Figure 1-58

Proof. By Menelaus' theorem (Example 1.9 on page 11), we need only to show

$$G = \frac{\overline{AG}}{\overline{GB}}\cdot\frac{\overline{BE}}{\overline{EC}}\cdot\frac{\overline{CF}}{\overline{FA}} = -1.$$

By Propositions 1.65 and 1.101

$$\begin{aligned} G &= \frac{P_{BAD}P_{ACD}P_{DBC}}{P_{ABD}P_{DAC}P_{DCB}} \\ &= \frac{\widetilde{BA}\cdot\widetilde{AD}\cos(\measuredangle BJD)\widetilde{AC}\cdot\widetilde{CD}\cos(\measuredangle AJD)\widetilde{DB}\cdot\widetilde{BC}\cos(\measuredangle DJC)}{\widetilde{AB}\cdot\widetilde{BD}\cos(\measuredangle AJD)\widetilde{DA}\cdot\widetilde{AC}\cos(\measuredangle DJC)\widetilde{DC}\cdot\widetilde{CB}\cos(\measuredangle DJB)} \\ &= -1 \end{aligned}$$

Example 1.103 *Let $V_1, \cdots, V_m$, and P be $m+1$ co-circle points, P_i the orthogonal projections from P to V_iV_{i+1}, $i = 1, ..., m$. Show that*

$$\prod_{i=1}^{m} \frac{\overline{V_iP_i}}{\overline{P_iV_{i+1}}} = (-1)^m$$

where the subscripts are understood to be mod m.

Proof. By Propositions 1.65 and 1.101

$$\frac{\overline{V_iP_i}}{\overline{P_iV_{i+1}}} = \frac{P_{PV_iV_{i+1}}}{P_{PV_{i+1}V_i}} = \frac{\widetilde{PV_i} \cdot \widetilde{V_iV_{i+1}} \cos(\measuredangle PJV_{i+1})}{\widetilde{PV_{i+1}} \cdot \widetilde{V_{i+1}V_i} \cos(\measuredangle PJV_i)}, \quad i = 1, ..., m.$$

Multiplying the m equations together, we obtain the result. ∎

Exercises 1.104

1. A, B, C, and D are four co-circle points. AB and CD meet in P. Show that $\widetilde{PA} \cdot \widetilde{PB} = \widetilde{PC} \cdot \widetilde{PD}$.
2. Show that the diameter of the circumcircle of triangle ABC is equal to the product of two sides dividing the altitude on the third side of the given triangle.
3. Prove Example 1.99 directly.
4. In Example 1.102, if D is an arbitrary point. Show that $DO^2 = AO^2(1 - \frac{4S_{EFG}}{S_{ABC}})$.
5. Continue from the above exercise. Show that D is on a fixed circle with O as its center if and only if S_{EFG} is a constant.

1.10 Full-Angles

To define the concept of angles, we need the concept of *rays*. As a consequence, we need to compare the order of points on a line. In algebraic language, this means that we need to use inequalities to describe angles. In this section, we introduce a new kind of angle, the description of which does not require inequalities. Angles of this kind will be used to simplifying the proofs of many geometry theorems.

Definition 1.105 *A full-angle consists of an ordered pair of lines l and m and is denoted by $\angle[l, m]$. Two full-angles $\angle[l, m]$ and $\angle[u, v]$ are equal if there exists a rotation K such that $K(l) \parallel u$ and $K(m) \parallel v$.*

If A, B and C, D are distinct points on l and m respectively, then $\angle[l, m]$ is also denoted by $\angle[AB, CD]$, $\angle[BA, CD]$, $\angle[AB, DC]$, $\angle[AB, m]$, and $\angle[l, DC]$. For three points A, B, and C, let $\angle[ABC] = \angle[AB, BC]$.

Definition 1.106 *If* $l \perp m$, $\angle[l, m]$ *is said to be a right full-angle and is denoted by* $\angle[1]$. *If* $l \parallel m$, $\angle[l, m]$ *is said to be a flat full-angle and is denoted by* $\angle[0]$.

To give a criterion for the equality of two full-angles, we need to introduce the *tangent function* for full-angles.

Definition 1.107 *The tangent function for the full-angle* $\angle[PQ, AB]$ *is defined to be*

$$\tan(\angle[PQ, AB]) = \frac{\sin(\measuredangle(PQ, AB))}{\cos(\measuredangle(PQ, AB))}.$$

We need to check that this definition is well-defined. That is when interchanging P, Q or A, B, $\frac{\sin(\measuredangle(PQ,AB))}{\cos(\measuredangle(PQ,AB))}$ does not change. This comes from the following equations

$$\begin{aligned}
&\sin(\measuredangle(PQ, AB)) = -\sin(\measuredangle(PQ, BA)) = -\sin(\measuredangle(QP, AB)),\\
&\cos(\measuredangle(PQ, AB)) = -\cos(\measuredangle(PQ, BA)) = -\cos(\measuredangle(QP, AB)).
\end{aligned}$$

As a consequence, we see that the sine and cosine functions for a full-angle are meaningless.

Proposition 1.108 $\angle[AB, PQ] = \angle[XY, UV]$ *if and only if* $\measuredangle(AB, PQ) = \measuredangle(XY, UV)$ *or* $\measuredangle(AB, PQ) - \measuredangle(XY, UV) = 180^\circ$.

Proof. Without loss of generality, we may assume $AB \parallel XY$. Then $\angle[AB, PQ] = \angle[XY, UV]$ if and only if $PQ \parallel UV$, i.e., if and only if $\measuredangle(AB, PQ) = \measuredangle(XY, UV)$ or $\measuredangle(AB, PQ) - \measuredangle(XY, UV) = 180^\circ$. ∎

Proposition 1.109 $\tan(\angle[AB, PQ]) = \frac{4S_{APBQ}}{P_{AQBP}}$.

Proof. This is a consequence of Propositions 1.87 and 1.88. ∎

Proposition 1.110
$\angle[AB, PQ] = \angle[XY, UV]$ *if and only if* $\tan(\angle[AB, PQ]) = \tan(\angle[XY, UV])$.

Proof. If $\angle[AB, PQ] = \angle[XY, UV]$, by Proposition 1.108 we have $\measuredangle(AB, PQ) = \measuredangle(XY, UV)$ or $\measuredangle(AB, PQ) - \measuredangle(XY, UV) = 180^\circ$. It is clear that $\tan(\angle[AB, PQ]) = \tan(\angle[XY, UV])$ for both cases. If $\tan(\angle[AB, PQ]) = \tan(\angle[XY, UV])$ then we have

$$\frac{\sin(\measuredangle(AB, PQ))}{\cos(\measuredangle(AB, PQ))} = \frac{\sin(\measuredangle(XY, UV))}{\cos(\measuredangle(XY, UV))}.$$

The above equation holds if and only if $\measuredangle(AB, PQ) = \measuredangle(XY, UV)$ or $\measuredangle(AB, PQ) - \measuredangle(XY, UV) = 180^\circ$, i.e., $\angle[AB, PQ] = \angle[XY, UV]$ by Proposition 1.108. ∎

Example 1.111 *Let ABC be a triangle such that $AB = AC$. Then $\angle[ABC] = \angle[BCA]$. Conversely, if $\angle[ABC] = \angle[BCA]$ and A, B, and C are not collinear then $AB = AC$.*

Proof. If $AB = AC$, we have

$$\tan(\angle[ABC]) = \frac{4S_{ABC}}{P_{ABC}} = \frac{4S_{BCA}}{BC^2} = \frac{4S_{BCA}}{P_{BCA}} = \tan(\angle[BCA]).$$

Then $\angle[ABC] = \angle[BCA]$. Conversely, if $\angle[ABC] = \angle[BCA]$ and $S_{ABC} \neq 0$, by the definition of the tangent function we have $P_{ABC} = P_{BCA}$, i.e., $AB^2 = AC^2$. $\blacksquare$

In the above example, we do not need to say that the corresponding "inner" angles of an isosceles triangles are equal. To describe the "inner" angle, we need inequalities.

Example 1.112 *Let l, m, and t be three lines. Then $l \parallel m$ if and only if $\angle[l, t] = \angle[m, t]$.*

Notice that in the above criterion for parallel, we do not need to mention that the angles should be the "corresponding angles," the exact description of which needs inequalities.

Example 1.113 (The Inscribed Angle Theorem) *If A, B, C, and D are cyclic points then $\angle[AB, BC] = \angle[AD, DC]$.*

If using angle in the usual sense, we need two conditions: $\angle ABC = \measuredangle ADC$ or $\angle ABC + \angle ADC = 180°$ and to distinguish these two cases, we need inequalities.

The proofs for the above two examples will be given later. From these examples, we see that the concept of full-angles makes many geometry relations concise. We will see later that this will lead to short proofs for many geometry theorems.

Definition 1.114 *Let l, m, u, and v be four lines. Let K be a rotation such that $K(l) \parallel v$. We define $\angle[u, v] + \angle[l, m] = \angle[u, K(m)]$.*

It is easy to check that the addition of full-angles is associative and commutative. The main properties of the full-angles are summarized as follows.

Q1 $\angle[u, v] = \angle[0]$ if and only if $u \parallel v$.

Q2 $\angle[u, v] = \angle[1]$ if and only if $u \perp v$.

Q3 $\angle[1] + \angle[1] = \angle[0]$.

Q4 $\angle[u, v] + \angle[0] = \angle[u, v]$.

Q5 $\angle[u, v] + \angle[l, m] = \angle[l, m] + \angle[u, v]$.

Q6 $\angle[u,v] + (\angle[l,m] + \angle[s,t]) = (\angle[u,v] + \angle[l,m]) + \angle[s,t]$.

Q7 $\angle[u,s] + \angle[s,v] = \angle[u,v]$.

Q8 if $\angle[u,v] = \angle[0]$ then for any line t we have $\angle[u,t] = \angle[v,t]$. Conversely, if for a line t we have $\angle[u,t] = \angle[v,t]$ then $\angle[u,v] = 0$.

Q8 is Example 1.112 which can be derived from Q1–Q7 as follows. If $\angle[u,v] = 0$, we have

$$\begin{aligned}
\angle[u,t] &= \angle[u,v] + \angle[v,t] && \text{(Q7)}\\
&= \angle[0] + \angle[v,t] && \text{the hypothesis}\\
&= \angle[v,t] + \angle[0] && \text{(Q5)}\\
&= \angle[v,t] && \text{(Q4).}
\end{aligned}$$

Conversely, if $\angle[u,t] = \angle[v,t]$

$$\begin{aligned}
\angle[u,v] &= \angle[u,t] + \angle[t,v] && \text{(Q7)}\\
&= \angle[v,t] + \angle[t,v] && \text{the hypothesis}\\
&= \angle[v,v] && \text{(Q7)}\\
&= \angle[0] && \text{(Q1).}
\end{aligned}$$

The following properties of the full-angles are also often used.

Q9 If $AB = AC$, we have $\angle[AB,BC] = \angle[BC,AC]$. Conversely, if $\angle[AB,BC] = \angle[BC,AC]$ then $AB = AC$ or A, B, and C are collinear.

Q10 Points A, B, C, and D are on the same circle or on same line if and only if $\angle[AB,BC] = \angle[AD,DC]$.

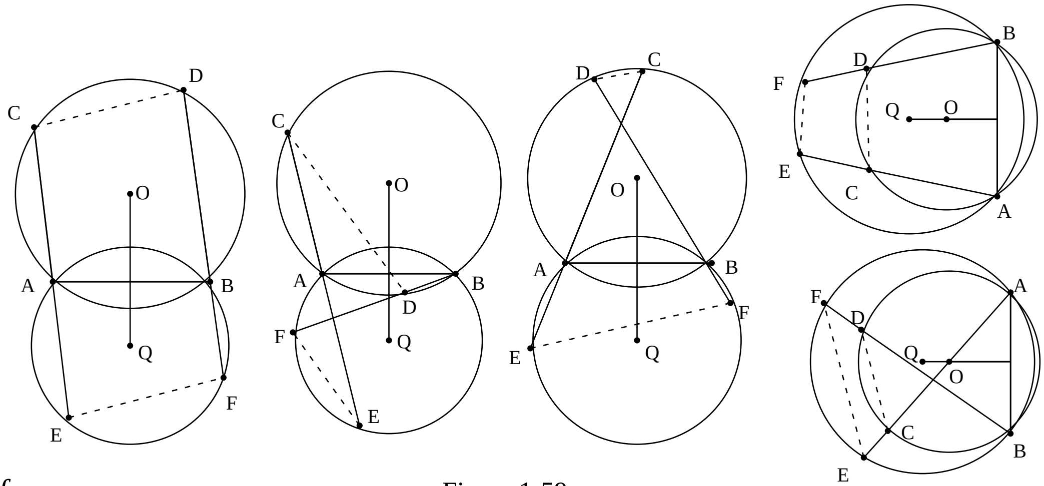
Q11 If AB is the diameter of the circumcircle of triangle ABC then $\angle[AC,BC] = \angle[1]$

Q12 If O is the circumcenter of triangle ABC then $\angle[BO,OC] = 2\angle[AB,AC]$.

Q9 is Example 1.111. For the proofs of Q10, Q11, and Q112, see Example 3.52 on page 131.

Example 1.115 *Two circles O and Q meet in two points A and B. A line passing through A meets circles O and Q in C and E. A line passing through B meets circles O and Q in D and F. Show that $CD \parallel EF$.*

Figure 1-59 shows five possible cases for this example. The following proof based on full-angles is valid for all cases. If we do not use full-angles, we must give different proofs for different figures.



Proof.

Figure 1-59

$$
\begin{aligned}
\angle[DC, FE] &= \angle[DC, DB] + \angle[DB, FE] \qquad \text{(by Q7)}\\
&= \angle[AC, AB] + \angle[FB, FE]\\
&\quad \text{(by Q10 and Q8 since } A, B, C, D \text{ are cyclic and } D \in BF)\\
&= \angle[AE, AB] + \angle[AB, AE]\\
&\quad \text{(by Q8 and Q10 since } E \in AC \text{ and } A, B, F, E \text{ are cyclic)}\\
&= \angle[AE, AE] \qquad \text{(by Q7)}\\
&= \angle[0]. \quad \text{(by Q2)}
\end{aligned}
$$

▮

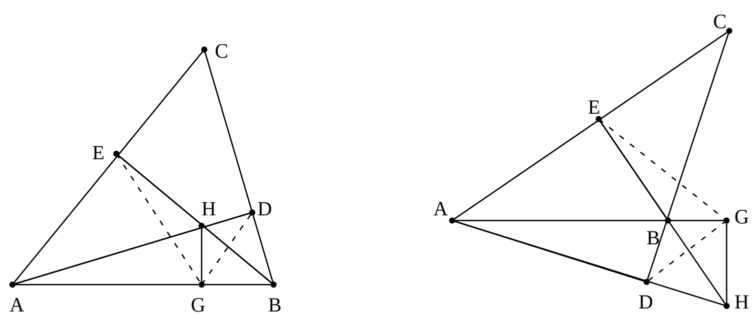


Figure 1-60

Example 1.116 *In triangle ABC, two altitudes AD and BE meet in H. G is the foot of the perpendicular from point H to AB. Show that* $\angle[DG, GH] = \angle[HG, GE]$.

Proof.

$$
\begin{aligned}
&\angle[DG, GH] + \angle[GE, HG]\\
= {} &\angle[DB, BH] + \angle[AE, HA] && (\text{Q10}; B,D,G,H; A,E,G,H \text{ cyclic.})\\
= {} &\angle[BC, BE] + \angle[AC, AD] && (\text{Q8}; D\in BC; B\in EH; E\in AC; H\in AD.)\\
= {} &\angle[BC, AC] + \angle[AC, BE] + \angle[AC, BC] + \angle[BC, AD] && (\text{Q7})\\
= {} &\angle[BC, AC] + \angle[1] + \angle[AC, BC] + \angle[1] && (AC\perp BE \text{ and } BA\perp AD)\\
= {} &\angle[BC, BC] + \angle[0] && (\text{Q7,Q3})\\
= {} &\angle[0].
\end{aligned}
$$

▮

As in Figure 1-60, if both $\angle BAC$ and $\angle ABC$ are acute angles, we have $\angle DGH = \angle EGH$; if one of them is obtuse, then $\angle DGH + \angle EGH = 180°$. Thus, if we do not use full-angles, we must give two proofs for the two cases.

Example 1.117 (The Nine Point Circle) *Let the midpoints of the sides AB, BC, and CA of $\triangle ABC$ be L, M, and N, and AD the altitude on BC. Show that L, M, N, and D are on the same circle.*

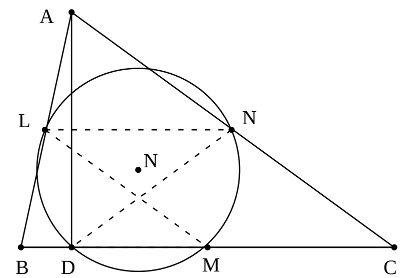


Figure 1-61

Proof. We need to show $\angle[LM, MD] = \angle[LN, ND]$ or equivalently, $\angle[LM, MD] + \angle[ND, LN] = \angle[0]$.

$$\begin{aligned}
&\angle[LM, MD] + \angle[ND, LN] \\
&= \angle[AC, BC] + \angle[ND, LN] && \text{(Q8; } LM \parallel AC\text{; } M, D, B, C \text{ are collinear)} \\
&= \angle[AC, BC] + \angle[ND, BC] + \angle[BC, LN] && \text{(Q7)} \\
&= \angle[AC, BC] + \angle[BC, AC] + \angle[0] && \text{(Q9,Q1; } ND = NC\text{; } BC \parallel LN) \\
&= \angle[AC, AC] = \angle[0]. \quad \blacksquare
\end{aligned}$$

Example 1.118 *The circumcenter of triangle ABC is O. AD is the altitude on side BC. Show that $\angle OAD = |\angle C - \angle B|$.*

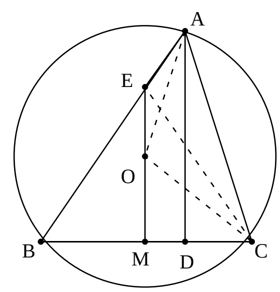


Figure 1-62

Proof. Let M be the midpoint of BC, MO and AB meet in E. We need only to show that $\angle[AD, AO] = \angle[CE, AC]$.

$$\begin{aligned}
&\angle[AD, AO] + \angle[AC, CE] \\
&= \angle[AD, AC] + \angle[AC, AO] + \angle[AC, BC] + \angle[BC, CE] && \text{(Q7)} \\
&= \angle[AD, BC] + \angle[CO, AC] + \angle[BE, BC] && \text{(Q7,Q9 (} BE = CE)) \\
&= \angle[1] + \angle[CO, MO] + \angle[MO, AC] + \angle[BA, BC] && \text{(Q7; } E \in BA; AD \perp BC) \\
&= \angle[1] + \angle[AC, BA] + \angle[MO, AC] + \angle[BA, BC] && \text{(Q12)} \\
&= \angle[1] + \angle[MO, BC] && \text{(Q7)} \\
&= \angle[1] + \angle[1] = \angle[0]. \quad \blacksquare
\end{aligned}$$

Summary of Chapter 1

- Signed areas and Pythagoras differences are used to describe some basic geometry relations: collinearity, parallelism, perpendicularity, congruence of line segments, and congruence of full-angles.

  1. Three points A, B, and C are collinear if and only if $S_{ABC} = 0$.
  2. $PQ \parallel AB$ if and only if $S_{PAQB} = S_{PAB} - S_{QAB} = S_{BPQ} - S_{APQ} = 0$.
  3. $PQ \perp AB$ if and only if $P_{PAQB} = P_{PAB} - P_{QAB} = P_{BPQ} - P_{APQ} = 0$.
  4. $\angle[AB, PQ] = \angle[XY, UV]$ if and only if $\frac{S_{APBQ}}{P_{APBQ}} = \frac{S_{XUYV}}{P_{XUYV}}$.

- The following results are powerful tools for solving difficult geometry problems.

  1. (The Co-side Theorem) Let M be the intersection of the lines AB and PQ and $Q \neq M$. Then we have $\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}$.
  2. (The Co-angle Theorem) Let X, Y and Z be three non-collinear points. Then $\angle ABC = \angle XYZ$ or $\angle ABC + \angle XYZ = 180°$ if and only if $\frac{\triangledown ABC}{\triangledown XYZ} = \frac{AB \cdot BC}{XY \cdot YZ}$.
  3. (The Pythagoras Difference Theorem) Let $\angle ABC \neq 90°$. Then $\angle ABC = \angle XYZ$ if and only if $\frac{P_{ABC}}{P_{XYZ}} = \frac{AB \cdot BC}{XY \cdot YZ}$; $\angle ABC + \angle XYZ = 180°$ if and only if $\frac{P_{ABC}}{P_{XYZ}} = -\frac{AB \cdot BC}{XY \cdot YZ}$.
  4. (The Visual Angle Theorem) Emanating from P, there are three rays PA, PB, and PC such that $\angle APC = \alpha$, $\angle CPB = \beta$, and $\angle APB = \gamma = \alpha + \beta < 180°$. Then A, B, and C are collinear if and only if $\sin(\angle\gamma)/PC = \sin(\angle\alpha)/PB + \sin(\angle\beta)/PA$.

- The concept of oriented angles is introduced to represent the signed areas and Pythagoras differences.

$$\begin{aligned}
S_{ABC} &= \frac{1}{2} \cdot AB \cdot BC \cdot \sin(\measuredangle ABC) \\
P_{ABC} &= 2 \cdot AB \cdot BC \cdot \cos(\measuredangle ABC) \\
S_{ABCD} &= \frac{1}{2} \cdot AC \cdot BD \cdot \sin(\measuredangle(AC, BD)) \\
P_{ABCD} &= 2 \cdot AC \cdot BD \cdot \cos(\measuredangle(AC, DB))
\end{aligned}$$

- The Herron-Qin formulas give connections between the signed area and the Pythagoras difference.

$$\begin{aligned}
16S_{ABC}^2 &= 4\overline{AB}^2 \cdot \overline{CB}^2 - P_{ABC}^2 \\
16S_{ABCD}^2 &= 4\overline{AC}^2 \cdot \overline{BD}^2 - P_{ABCD}^2.
\end{aligned}$$

Chapter 2

The Area Method

In this chapter, we will present the area method in the narrow sense, that is, as a method of mechanical theorem proving for constructive geometry statements only involving two geometry relations: collinearity and parallel. In other words, we are dealing with constructive geometry statements in affine geometry.

2.1 Traditional Proofs Versus Machine Proofs

We start this section with a comment on the traditional Euclidean proof method from [2].

> One of the main defects in the traditional Euclidean proof is its almost complete disregard of such notions as the *two sides of a line* and the *interior of an angle*. Without clarification of these ideas, absurd consequences result.

The following example shows that this defect may occur even in very simple proofs.

Example 2.1 *Let $ABCD$ be a parallelogram (i.e, $AB \parallel CD$, $BC \parallel AD$), E be the intersection of the diagonals AC and BD. Show that $AE = CE$.*

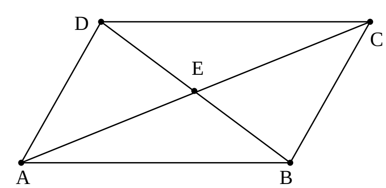


Figure 2-1

The traditional proof of this theorem is first to prove $\triangle ACB \cong \triangle CAD$ (hence $AB = CD$), then to prove $\triangle AEB \cong \triangle CED$ (hence $AE = CE$). In proving the congruence of these triangles, we have repeatedly used the fact $\angle CAB = \angle ACD$. This fact is quite evident because the two angles are the alternative angles with respect to parallels AB and CD. However, here we have implicitly assumed the "trivial fact" that points D and B are on opposite sides of line AC. The last fact is harder to prove than the original statement. (Please try it!)

This extremely simple example reveals the difficulty in implementing a powerful and sound geometry theorem *prover* based on congruence and similarity of triangles. Of course, one may develop an interactive prover so that the user can input some trivial facts such as the one in the previous paragraph. These facts can be stored by the program in a data base. Then one would face a much more severe problem of the *consistency of proofs*.

Example 2.2 *Every triangle is isosceles. Let ABC be a triangle as shown in Figure 2-2. We want to prove $CA = CB$.*

Proof. Let D be the intersection of the perpendicular bisector of AB and the internal bisector of angle ACB. Let $DE \perp AC$ and $DF \perp CB$. It is easy to see that $\triangle CDE \cong \triangle CDF$ and $\triangle ADE \cong \triangle BDF$. Hence $CE + EA = CF + FB$, i.e., $CA = CB$. ∎

Try to solve this *paradox*.

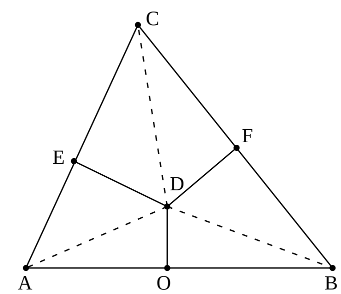


Figure 2-2

Another defect in the traditional Euclidean proofs is the lack of *non-degenerate conditions*. Each geometry theorem is valid only under some auxiliary conditions which are not stated explicitly in the theorem. For instance, in Example 2.1 we need to assume that A, B, and C are not collinear. We call such kinds of conditions the non-degenerate conditions of the theorem, which may become complicated for difficult theorems. Without explicitly stated non-degenerate conditions, the traditional proofs of geometry theorems are generally not strict. First, in each step of the proof, we use some lines or triangles which are implicitly assumed to be in normal positions, i.e., each line is uniquely determined and each triangle does not degenerate to a line. However, in a machine proof these conditions should be explicitly stated or justified. Second, during a proof, we need to use other known theorems; but in the statement of the cited theorems the nondegenerate conditions are usually not given explicitly. Hence the correctness of the use of these known theorems is not fully justified.

Partially due to these defects, it is very difficult to incorporate the traditional Euclidean proof methods into a computer program so that skillful proofs can be automatically produced by computers. Researchers have been studying automated generation of traditional proofs using computer programs since the work by H. Gelernter, J. R. Hanson, and D. W. Loveland [103] in the early 60s. In spite of the enormous amount of research and great improvements [43, 83, 106, 141, 129, 144], the successes in this direction have been limited in the sense that no program has been developed which can prove non–trivial geometry theorems efficiently.

On the other hand, A. Tarski, introduced a decision procedure for what he called *elementary geometry*, based on the algebraic method in the 1930s [34]. Tarski's quantifier elimination method was later improved and redesigned by A. Seidenberg [147], G. Collins [84] and others. In particular, Collins' *cylindrical decomposition algorithm* is the first Tarski type algorithm which has been implemented on a computer. Solutions of several nontrivial problems of elementary geometry and algebra have been obtained using the implementation [45, 46, 114].

Meanwhile, Wen–Tsün Wu introduced a highly successful algebraic method of mechanical geometry theorem proving [164]. Inspired by Wu's work, many researchers have developed efficient computer programs for proving geometry theorems [36, 12, 55, 92, 120, 126, 133, **?**, 156]. Nearly *one thousand* theorems from Euclidean geometry, non-Euclidean geometry, differential geometry, and mechanics have been proved by various provers based on Wu's method and its variants [12, 69, 70, 135, 157]. Many hard theorems whose traditional proofs need an enormous amount of human intelligence, such as Feuerbach's theorem, Morley's trisector theorem etc., can be proved by computer programs based on algebraic methods within seconds. In addition, Wu first recognized the importance of the non-degenerate conditions in mechanical geometry theorem proving.

However, algebraic methods can only tell whether a statement is true or not. If we want to know the proofs, we usually have to look at tedious computations of polynomials in the coordinates of the related points. So the goal of automatically producing *readable* proofs for geometry theorems has not been achieved.

The goal of this book is to present a method which can produce short and readable proofs for geometry statements efficiently. The starting point of this book is the mechanization of a special case of the *area method* discussed in Chapter 1. Our machine proof method has the following advantages.

- Auxiliary points and lines will be added automatically if they are needed.
- Sufficient non-degenerate conditions can be generated automatically.
- The proof produced according to the method is independent of the diagram.

A key fact behind the success of our method (or Wu's algebraic method) is that the validity of most elementary geometry theorems involving equalities only is independent of the relative order positions of the points involved. Such geometry theorems belong to *unordered geometry*. In unordered geometry, the proofs of these theorems can be very simple. However, the ordinary proofs of these theorems involve the order relation (among points and lines), hence are not only complicated, but also not strict, see for example, Example 2.1.

Here we list several other examples. In the statement of Ceva's theorem (Example 1.7 on page 11) we usually assume that P is inside the triangle ABC. But this restriction on P is not necessary: the statement is true regardless of whether P is inside or outside triangle ABC. The proof produced by the area method is valid for all cases. Also see Examples 1.9 on page 11, 1.115 on page 47, and 1.116 on page 48.

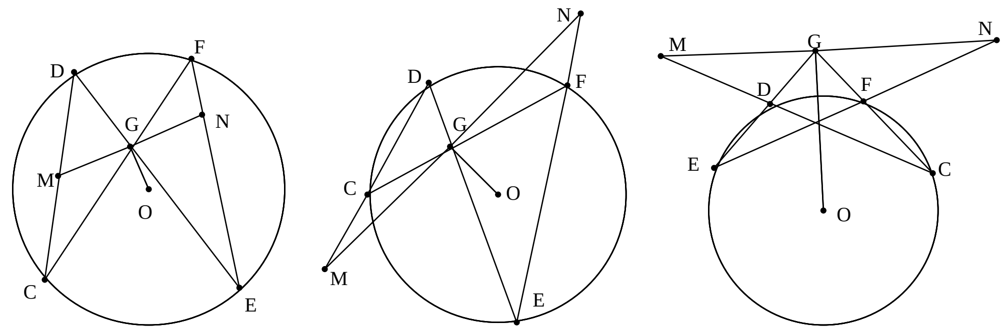


Figure 2-3

For the Butterfly theorem (Example 1.99 on page 42), as shown in Figure 2-3, we have three different diagrams, which are often treated as different theorems in geometry textbooks. The proof for this theorem based on the area method is valid for all three cases.

Remark 2.3

1. For a strict proof of Example 2.1 using the area, see Example 1.24 on page 17. To prove it using congruence triangles, you may start from points A, B, and C to construct point D as follows: E is the midpoint of AC and D is the symmetry of B with respect to E.
2. The problem in the "proof" of Example 2.2 is that we use a wrong diagram. The correct diagram for Example 2.2 is Figure 2-4.

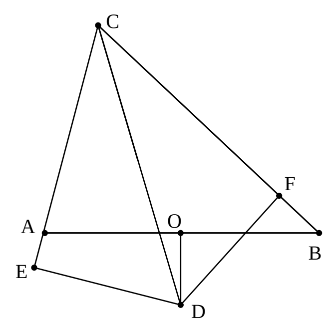


Figure 2-4

2.2 Signed Areas of Oriented Triangles

We will formally define two geometry quantities: the *ratio of the signed lengths of directed parallel line segments* and the *signed areas of oriented triangles*. Properties of these two quantities will serve as the basis of our area method. Those who are mainly interested in machine proofs may skip the next subsection and read Subsection 2.2.2 directly.

2.2.1 The Axioms

We use capital English letters with or without subscripts to denote points. The only basic geometry relation is a trinary relation *collinear*, i.e., three points A, B, and C are collinear. The exact meaning of collinear is given by Axioms A.1–A.6.

Let **R** be the field of the real numbers.

Axiom A.1 *For three collinear points P, A, and B such that $A \neq B$, $\frac{\overline{AP}}{\overline{AB}}$ is an element in* **R** *and satisfies*

$$\frac{\overline{AP}}{\overline{AB}} = -\frac{\overline{PA}}{\overline{AB}} = \frac{\overline{PA}}{\overline{BA}} = -\frac{\overline{AP}}{\overline{BA}}$$

and $\frac{\overline{PA}}{\overline{AB}} = 0$ iff (abbr. if and only if) $P = A$.

$\frac{\overline{AP}}{\overline{AB}}$ is called the *ratio of the directed segments AP* and AB. Let $r = \frac{\overline{AP}}{\overline{AB}}$. We sometimes also write $\overline{AP} = r\overline{AB}$.

Axiom A.2 *Let A and B be two distinct points. For $r \in$* **R**, *there exists a unique point P which is collinear with A and B and satisfies (1) $\frac{\overline{AP}}{\overline{AB}} = r$ and (2) $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.*

Three points A, B, and C determine an *oriented triangle ABC*. We use the order of the vertices of a triangle to represent its orientation. Thus triangles $\triangle ABC$, $\triangle BCA$, and $\triangle CAB$ have the opposite orientation, whereas $\triangle ACB$, $\triangle CBA$, and $\triangle BAC$ have the same orientation, i.e., a triangle has two orientations.

The *signed area* of an oriented triangle ABC, denoted by S_{ABC}, is an element in **R** which satisfies the following four basic properties.

Axiom A.3 $S_{ABC} = S_{CAB} = S_{BCA} = -S_{BAC} = -S_{CBA} = -S_{ACB}$. *If A, B, and C are three non-collinear points, we have $S_{ABC} \neq 0$.*

Axiom A.4 *There exist at least three points A, B, and C such that $S_{ABC} \neq 0$.*

Axiom A.5 *For any four points A, B, C, and D, we have*

$$S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}.$$

Axioms A.4 and A.5 are called the *dimension axioms*. Axiom A.4 ensures that not all the points are collinear. Axiom A.5 ensures that all the points are in one plane.

As a consequence of Axiom A.5, we can define the signed area of oriented quadrilaterals. The *signed area of an oriented quadrilateral $ABCD$* is defined to be

$$S_{ABCD} = S_{ABC} + S_{ACD}.$$

By Axioms A.3 and A.5, it is clear that

$S_{ABCD} = S_{ABD} - S_{CBD}$;
$S_{ABCD} = S_{BCDA} = S_{CDAB} = S_{DABC}$;
$S_{ABCD} = -S_{ADCB} = -S_{DCBA} = -S_{CBAD} = -S_{BADC}$.

Axiom A.6 *Let A, B, and C be three collinear points such that $\overline{AB} = \lambda\overline{AC}$. Then for any point P, we have $S_{PAB} = \lambda S_{PAC}$.*

Axiom A.6 is the most important property of the area. We will see in the next section that most of the interesting and nontrivial properties of area come from this axiom.

It is convenient to extend the notion of collinearity to be a geometry relation among any set of points: one or two points are always collinear, and a set of points are collinear if any three points in it are collinear. We can thus introduce a new geometry object: the line.

Definition 2.4 *A line is a maximal set of collinear points. Let l be a line and $P \in l$. Then we say that P is on line l (instead of in line l).*

Proposition 2.5 *Three points A, B, and C are collinear iff $S_{ABC} = 0$.*

Proof. If $S_{ABC} = 0$, then by Axiom A.3 A, B, and C are collinear. Now let us assume that A, B, and C are collinear. If $A = C$, since $\overline{CC} = 2\overline{CC} = 0$, by Axiom A.6 we have $S_{ACC} = 2S_{ACC} = 0$. If $A \neq C$ and $\lambda = \frac{\overline{AB}}{\overline{AC}}$, by Axiom A.6, $S_{ABC} = \lambda S_{ACC} = 0$. ∎

Corollary 2.6 *Two distinct points A and B determine a unique line AB which is the set of all points P satisfying $S_{ABP} = 0$.*

Proof. Let C, D, and E be three distinct points on line AB. We need to show that C, D, and E are collinear, i.e., $S_{CDE} = 0$. By Proposition 2.5

$$S_{ADE} = \frac{\overline{AD}}{\overline{AB}} \cdot S_{ABE} = 0.$$

Then A, D, and E are collinear. Similarly C, D, and A are collinear. By Proposition 2.5

$$S_{CDE} = \frac{\overline{DE}}{\overline{DA}} \cdot S_{CDA} = 0. \quad ∎$$

In what follows, when speaking about a line AB, we always assume that $A \neq B$. A point P on line AB is determined uniquely by $\frac{\overline{AP}}{\overline{AB}}$ or $\frac{\overline{PB}}{\overline{AB}}$. We thus call

$$x_P = \frac{\overline{AP}}{\overline{AB}}, \quad y_P = \frac{\overline{PB}}{\overline{AB}}$$

the *position ratio* or *position coordinates* of the point P with respect to AB. It is clear that $x_P + y_P = 1$.

2.2.2 Basic Propositions

The basic propositions presented in this section are the basis of our area method. We first extend Axiom A.6 to the following convenient form.

Proposition 2.7 *If points C and D are on line AB and P is any point not on line AB (Figure 2-5), then*

$$\frac{S_{PCD}}{S_{PAB}} = \frac{\overline{CD}}{\overline{AB}}.$$

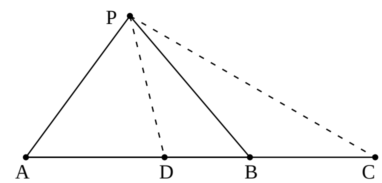


Figure 2-5

Proof. Without loss of generality, let us assume $C \neq A$. Then

$$\frac{S_{PCD}}{S_{PAB}} = \frac{S_{PCD}}{S_{PCA}} \cdot \frac{S_{PCA}}{S_{PAB}} = \frac{\overline{CD}}{\overline{CA}} \cdot \frac{\overline{CA}}{\overline{AB}} = \frac{\overline{CD}}{\overline{AB}}. \quad \blacksquare$$

Proposition 2.8 (The Co-side Theorem) *Let M be the intersection of two lines AB and PQ. Then*

$$\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}; \quad \frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}}; \quad \frac{\overline{QM}}{\overline{PQ}} = \frac{S_{QAB}}{S_{PAQB}}.$$

Proof. See Proposition 1.5 on page 9. ▮

Proposition 2.9 *Let R be a point on line PQ. Then for any two points A and B*

$$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}} S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}} S_{PAB}.$$

Proof. See Proposition 1.18 on page 15. ▮

Similar to Chapter 1, we use the notation $AB \parallel PQ$ to denote the fact that A, B, P, and Q satisfy one of the following conditions: (1) $A = B$ or $P = Q$; (2) A, B, P and Q are on the same line; or (3) line AB and line PQ do not have a common point.

Proposition 2.10 *$PQ \parallel AB$ iff $S_{PAB} = S_{QAB}$, i.e., iff $S_{PAQB} = 0$.*

Proof. If $S_{PAB} \neq S_{QAB}$, it is clear that $A \neq B$, $P \neq Q$, and A, B, P, and Q are not collinear. Let O be a point on line PQ such that $\frac{\overline{PO}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}}$. Thus $\frac{\overline{OQ}}{\overline{PQ}} = -\frac{S_{QAB}}{S_{PAQB}}$. By Proposition 2.9, $S_{OAB} = \frac{\overline{PO}}{\overline{PQ}} S_{QAB} + \frac{\overline{OQ}}{\overline{PQ}} S_{PAB} = 0$. By Proposition 2.5, point O is also on line AB, i.e., AB is not parallel to line PQ. Conversely, if $PQ \nparallel AB$ then $A \neq B$, $P \neq Q$ and lines AB and PQ

intersection at a unique point O. By Proposition 2.8, $\frac{\overline{OP}}{\overline{OQ}} = \frac{S_{PAB}}{S_{QAB}} = 1$. Thus $P = Q$ which is a contradiction. ▮

A *parallelogram* is a quadrilateral $ABCD$ such that $AB \parallel CD$, $BC \parallel AD$, and no three vertices of it are on the same line. Let $ABCD$ be a parallelogram and P, Q be two points on CD. We define the *ratio of two parallel line segments* as follows

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}}.$$

In our machine proofs, *auxiliary parallelograms* are often added automatically and the following two propositions are used frequently.

Proposition 2.11 *Let $ABCD$ be a parallelogram. Then for two points P and Q, we have*

$$S_{APQ} + S_{CPQ} = S_{BPQ} + S_{DPQ} \text{ or } S_{PAQB} = S_{PDQC}.$$

Proof. See Proposition 1.30 on page 20.▮

Proposition 2.12 *Let $ABCD$ be a parallelogram and P be any point. Then*

$$S_{PAB} = S_{PDC} - S_{ADC} = S_{PDAC}.$$

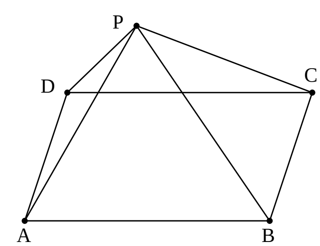


Figure 2-6

Proof. By Proposition 2.11,

$$S_{PAB} = S_{PDB} - S_{PCB} = S_{DBCP} = S_{DBC} + S_{DCP} = S_{PDC} - S_{ADC}. \quad ▮$$

So far, we do not mention the existence of parallel lines. The following statement shows that Euclid's parallel postulate is a consequence of our axioms.

Example 2.13 (Euclid's Parallel Axiom) *Passing through a point not on a line l, there exists a unique line which is parallel to l.*

Proof. Let P be a point not on the line AB. By Axiom A.2, we can take points O and Q such that O is the midpoint of PA and Q is the symmetric point of B with O, i.e., $\overline{QO} = \overline{OB}$. By the co-side theorem $S_{QAB} = 2S_{OAB} = S_{PAB}$. By Proposition 2.10, $PQ \parallel AB$. To show the uniqueness, let T be another point such that $TP \parallel AB$. By Proposition 2.12, $S_{TPQ} = S_{TAB} - S_{PAB} = 0$, i.e., T is on line PQ. ▮

Example 2.14 *If $PR \parallel AC$ and $QS \parallel BD$ then* $\frac{S_{PQRS}}{S_{ABCD}} = \frac{\overline{PR}}{\overline{AC}} \cdot \frac{\overline{QS}}{\overline{BD}}$.

Proof. Let X, Y be points such that $\overline{PR} = \overline{AX}$, $\overline{QS} = \overline{BY}$. By Proposition 2.11,

$$S_{PQRS} = S_{PBRY} = S_{PBY} - S_{RBY} = \frac{\overline{BY}}{\overline{BD}}(S_{PBD} - S_{RBD}) = \frac{\overline{QS}}{\overline{BD}} S_{PBRD}.$$

Similarly, we have $S_{PBRD} = \frac{\overline{PR}}{\overline{AC}} S_{ABCD}$. ∎

Example 2.15 *If line PQ is parallel to line AB then* $\frac{\overline{AB}}{\overline{PQ}} = \frac{S_{PAB}}{S_{AQP}}$

Proof. Let R be a point such that $\overline{AR} = \overline{PQ}$. By Propositions 2.7 and 2.11,

$$\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{AB}}{\overline{AR}} = \frac{S_{PAB}}{S_{PAR}} = \frac{S_{PAB}}{S_{PAQ}}. \quad ∎$$

2.3 The Hilbert Intersection Point Statements

In Chapter VI of Hilbert's classic book, "Foundations of Geometry", he introduced a class of geometry statements which he called the "pure point of intersection theorems." According to Hilbert, every "*pure point of intersection theorem*" can be put in the following form:

> Choose an arbitrary set of a finite number of points and lines. Then draw in a prescribed manner any parallels to some of these lines. Choose any points on some of the lines and draw any lines through some of these points. Then, if connecting lines, points of intersection and parallels are constructed through the points existing already in the prescribed manner, a definite set of finitely many lines is eventually reached, about which the theorem asserts that they either pass through the same point or are parallel.

Hilbert also gave a mechanical proving method for statements of this type. His result can be summarized as follows.

> Theorem 62 in [24]. Every pure point of intersection theorem that holds in affine geometry takes, through the construction of suitable auxiliary points and lines, the form of a combination of finite number of *Pascal's configuration*s.

By Pascal's configuration, he meant the diagram of Example 1.27 on page 19.

The above result called *Hilbert's Mechanization theorem* by Wu [166], is the first mechanical theorem proving method for a class of geometry statements, i.e., class $\mathbf{C}_{\mathrm{H}}$ in this book.

Hilbert's mechanization theorem works as follows. First, we prove a theorem using algebraic calculations (see [166] and Chapter 3 of [36]). Since each arithmetic operation of numbers, e.g. $a + b = b + a, a * b = b * a$, can be represented by Pascal configurations, the algebraic proof can thus be converted into a series of Pascal configurations. But the geometric proofs produced in this way are expected to be very long and cumbersome, and as far as we know no single theorem has been proved in this way. The aim of this chapter is to provide an efficient method of producing short and readable proofs for the Hilbert pure point of intersection statements.

2.3.1 Description of the Statements

To describe the *Hilbert intersection point statements* precisely, we need the concepts of geometry quantities and constructions.

In this chapter, by a *geometric quantity* we mean

- the ratio of two oriented segments on one line or on two parallel lines, or
- the signed area of an oriented triangle or an oriented quadrilateral.

A *construction* is used to introduce a new point from some known points. We need the following constructions.

C1 Take arbitrary points $Y_1, \cdots, Y_m$ on the plane. Y_i are free points, i.e., Y_i can move freely on the plane.

C2 Take a point Y on line PQ. Point Y is a semi-free point, i.e. point Y can move freely on the line PQ.

To make sure that point Y can be taken properly, we will introduce a nondegenerate (abbr. ndg) condition $P \neq Q$, i.e., the line PQ is well defined.

C3 Take a point Y on line PQ such that $\overline{PY} = \lambda\overline{PQ}$ where λ can be a rational number, a rational expression in geometry quantities, or a variable. Notice that λ is the position ratio of point Y with regard to PQ.

If λ is a fixed quantity then Y is a fixed point; if λ is a variable then Y is a semifree point. The ndg conditions for this construction are $P \neq Q$ and λ is meaningful, i.e., its denominator does not vanish.

C4 Take the intersection Y of line PQ and line UV. Point Y is a fixed point. The ndg condition is that $P \neq Q$, $U \neq V$, and the lines PQ and UV have one and only one common point, i.e., $PQ \nparallel UV$.

C5 Take a point Y on the line passing through point R and parallel to line PQ. Here Y is a semi-free point. The ndg condition is $P \neq Q$.

C6 Take a point Y on the line passing through R and parallel to line PQ such that $\overline{RY} = \lambda\overline{PQ}$, where λ can be a rational number, a rational expression in geometry quantities, or a variable.

If λ is a fixed quantity then Y is a fixed point; if λ is a variable then Y is a semifree point. The ndg conditions are the same as those of C3.

C7 Take the intersection Y of line UV and the line passing through R and parallel to line PQ. Point Y is a fixed point. The ndg condition is that $PQ \nparallel UV$.

C8 Take the intersection Y of the line passing through point R and parallel to PQ and the line passing through point W and parallel to line UV. Point Y is a fixed point. The ndg condition is that $PQ \nparallel UV$.

Point Y in each of the above constructions is said to be introduced by that construction.

We need to show that the above constructions are always possible. That is the introduced points do exist. C1 and C2 are trivial. The existence of the point Y in C3 comes from Axiom A.2. C5 and C6 come from Example 2.13 and C3. By Example 2.13, C7 and C8 can be reduced to C4. For C4, since $PQ \nparallel UV$, line PQ and UV have a unique common point.

Definition 2.16 *A Hilbert intersection point statement can be represented by a list*

$$S = (C_1, C_2, \ldots, C_k, G)$$

where

1. *Each construction C_i introduces a new point from the points which are introduced by the previous C_j, $j = 1, \cdots i-1$; and*
2. *$G = (E_1, E_2)$ where E_1 and E_2 are polynomials in some geometric quantities about the points introduced by the constructions C_i and $E_1 = E_2$ is the conclusion of S.*

The ndg condition of S is the set of ndg conditions of C_i and the condition that the denominators of the length ratios in E_1 and E_2 are not zero. The set of all the Hilbert intersection point statements is denoted by $\mathbf{C}_{\mathrm{H}}$.

As indicated by the definition, the ndg conditions of a statement in $\mathbf{C}_{\mathrm{H}}$ can be generated automatically. Take Ceva's theorem (Example 1.7 on page 11) as an example.

Example 2.17 (Ceva's Theorem) *We describe the statement in the following constructive way.*

Take four arbitrary points A, B, C, and P.
Take the intersection D of BC and AP.

Take the intersection E of AC and BP.
Take the intersection F of AB and CP.
Show that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$

According to the definition, the ndg conditions for Ceva's Theorem are

$$BC \nparallel AP; AC \nparallel BP; AB \nparallel CP; F \neq B; D \neq C; E \neq A,$$

i.e., point P can not be on the three sides of $\triangle ABC$ and the three dotted lines in Figure 2-7.

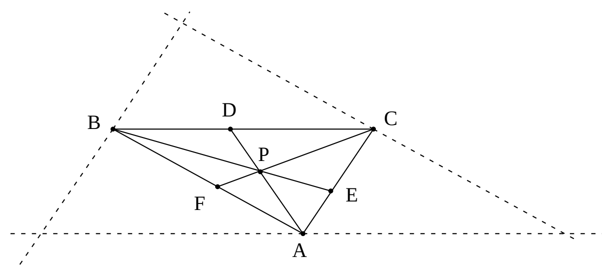


Figure 2-7

You may wonder that the condition "A, B, and C not collinear" is not in the ndg conditions. Indeed, when A, B, and C are three different points (this comes from the ndg condition) on the same line, Ceva's theorem is still true (now $F = C, D = A$, and $E = B$) and the proofs based on the area method is still valid in this case. The ndg conditions produced according to our method guarantee that we can produce a proof for the statement. Certainly, we can avoid this seemingly unpleasant fact by introducing a new construction, TRIANGLE, which introduces three noncollinear points. But theoretically, this is not necessary.

Also the ndg conditions are not unique for a geometry statement: they depend on the constructive description of the statements. For instance, Ceva's theorem can be described as follows.

Take three arbitrary points A, B, C.
Take a point E on line AC.
Take a point F on line AB.
Take the intersection P of BE and CF.
Take the intersection D of BC and AP.
Show that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$

Now the ndg conditions of Ceva's theorem are $A \neq C, A \neq B, BE \nparallel CG, BC \nparallel AP, B \neq F, C \neq D$, and $A \neq P$.

The ndg conditions generated according to Definition 2.16 are sufficient, i.e., if a geometric statement is true in the usual sense then it must be true strictly under these ndg conditions. For more details, see Algorithm 2.32 on page 70.

2.3.2 The Predicate Form

A Hilbert intersection point statement can be transformed into predicate form. We first introduce some basic *predicates*.

1. (POINT P): P is a point in the plane.
2. (COLL P_1, P_2, P_3): points P_1, P_2, and P_3 are on the same line. It is equivalent to $S_{P_1P_2P_3} = 0$.
3. (PARA P_1, P_2, P_3, P_4): $P_1P_2 \parallel P_3P_4$. It is equivalent to $S_{P_1P_3P_2P_4} = 0$.

Each construction is equivalent to the conjunction of several predicates.

C2 Take a point Y on line PQ. The predicate form is (COLL Y P Q) and $P \neq Q$.

C3 Take a point Y on line PQ such that $\overline{PA} = \lambda\overline{PQ}$. The predicate form is (COLL Y P Q), $\lambda = \frac{\overline{PY}}{\overline{PQ}}$, and $P \neq Q$.

C4 Take the intersection Y of line PQ and line UV. The predicate form is (COLL Y P Q), (COLL Y U V), and $\neg$(PARA U V P Q).

C5 Take a point on the line passing through point R and parallel to line PQ. The predicate form is (PARA Y R P Q) and $P \neq Q$.

C6 Take a point Y on the line passing through R and parallel to line PQ such that $\overline{RY} = \lambda\overline{PQ}$. The predicate form is (PARA Y R P Q), $\lambda = \frac{\overline{RY}}{\overline{PQ}}$, and $P \neq Q$.

C7 Take the intersection Y of line UV and the line passing through R and parallel to line PQ. The predicate form is (COLL Y U V), (PARA Y R P Q), and $\neg$(PARA P Q U V).

C8 Take the intersection Y of the line passing through point R and parallel to PQ and the line passing through point W and parallel to line UV. The predicate form is (PARA Y R P Q), (PARA Y W U V), and $\neg$(PARA P Q U V).

The predicate form of each construction C has two parts: the equation part $E(C)$ and the ndg condition $\neg D(C)$.

Now a constructive statement $S = (C_1, \cdots, C_r, (E, F))$ can be transformed into the following *predicate form*

$$\forall P_i[(E(C_1) \wedge \cdots \wedge E(C_r) \wedge \neg D(C_1) \wedge \cdots \wedge \neg D(C_r)) \Rightarrow (E = F)]$$

where P_i is the point introduced by C_i.

It is clear that the predicate form of a statement depends on how we describe the statement constructively. For the first constructive description of Ceva's theorem (Example 2.17 on page 61), its predicate form is

$$\forall A, B, C, P, E, F, D(HYP \Rightarrow CONC)$$

where

$$\begin{aligned} HYP = \quad & (COLL\ D\ B\ C) \wedge (COLL\ D\ A\ O) \wedge \neg(PARA\ B\ C\ A\ O) \wedge \\ & (COLL\ E\ A\ C) \wedge (COLL\ E\ B\ O) \wedge \neg(PARA\ A\ C\ B\ O) \wedge \\ & (COLL\ F\ A\ B) \wedge (COLL\ F\ C\ D) \wedge \neg(PARA\ A\ B\ C\ O) \wedge \\ & B \neq F \wedge D \neq C \wedge A \neq E \\ CONC = \quad & (\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1). \end{aligned}$$

Exercises 2.18

1. We define a new predicate (CONC P_1 P_2 P_3 P_4 P_5 P_6) which means that the lines P_1P_2, P_3P_4, and P_5P_6 are concurrent. Use an equation in areas to represent this predicate.
2. Construction C3 is to take a point on a line with position ratio λ. Show that if point Y is introduced by one of the eight constructions then Y can also be introduced by constructions C1 and C3. The reason we use more constructions is that we want to describe geometry statements using fewer constructions, and as a consequence to obtain short proofs for the statements.

2.4 The Area Method

Before presenting the method, let us re-examine the proof of Ceva's theorem (Example 1.7 on page 11). By describing Ceva's theorem constructively (Example 2.17 on page 61), we can introduce an order among the points naturally: A, B, C, P, D, E, and F, i.e., the order according to which the points are introduced. The proof is actually to eliminate the points from the conclusions according to the reverse order: F, E, D, P, C, B, and A. We thus have the proof:

$$\begin{array}{ll} \frac{\overline{AF}}{\overline{FB}} = -\frac{S_{ACP}}{S_{BCP}} & \text{Eliminate point } F. \\ \frac{\overline{CE}}{\overline{EA}} = \frac{S_{BCP}}{S_{ABP}} & \text{Eliminate point } E. \\ \frac{\overline{BD}}{\overline{DC}} = -\frac{S_{ABP}}{S_{ACP}} & \text{Eliminate point } D. \end{array}$$

Then

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{S_{ACP}S_{BCP}S_{ABP}}{S_{BCP}S_{ACP}S_{ABP}} = 1.$$

Thus the key step of the method is to *eliminate points from geometry quantities*. We will show how this is done in the following three subsections.

2.4.1 Eliminating Points from Areas

We first show that construction C2 is a special case of construction C3. This is because taking an arbitrary point Y on line UV is equivalent to taking a point Y on UV such that $\overline{UA} = \lambda\overline{UV}$ for an indeterminate λ. Similarly, construction C5 is a special case of construction C6: taking an arbitrary point on the line passing through point W and parallel to UV is equivalent to taking a point Y such that $\overline{WA} = \lambda\overline{UV}$ for an indeterminate λ.

We will discuss C1 in Section 2.4.3. Thus we need only to consider five constructions C3, C4, C6, C7, and C8.

Lemma 2.19 *Point Y is introduced by construction C3, i.e., Y satisfies $\overline{PY} = \lambda\overline{PQ}$. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \lambda S_{ABQ} + (1-\lambda)S_{ABP}.$$

Proof. This is Proposition 2.9. ∎

Lemma 2.20 *Point Y is introduced by construction C4, i.e., $Y = PQ \cap UV$. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \frac{1}{S_{PUQV}}(S_{PUV}S_{ABQ} + S_{QVU}S_{ABP}).$$

Proof. By Proposition 2.9, we have

$$S_{ABY} = \frac{\overline{PY}}{\overline{PQ}}S_{ABQ} + \frac{\overline{YQ}}{\overline{PQ}}S_{ABP}.$$

By the co-side theorem, we have $\frac{\overline{PY}}{\overline{PQ}} = \frac{S_{PUV}}{S_{PUQV}}$, $\frac{\overline{YQ}}{\overline{PQ}} = \frac{S_{QVU}}{S_{PUQV}}$. Substituting this into the previous equation, we obtain the conclusion. Since $PQ \nparallel UV$, we have $S_{PUQV} \neq 0$. ∎

Lemma 2.21 *Point Y is introduced by construction C6, i.e., Y satisfies $\overline{RY} = \lambda\overline{PQ}$. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = S_{ABR} + \lambda S_{APBQ}$$

Proof. We take a point S such that $\overline{RS} = \overline{PQ}$. By Lemma 2.19,

$$S_{ABY} = \lambda S_{ABS} + (1-\lambda)S_{ABR}.$$

By Proposition 2.11, we have

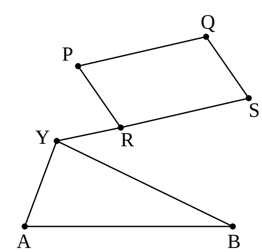


Figure 2-8

$$S_{ABS} = S_{ABR} + S_{ABQ} - S_{ABP} = S_{ABR} + S_{APBQ}.$$

Substituting this into the previous formula, we obtain the conclusion. ∎

Lemma 2.22 *Point Y is introduced by construction C7, i.e., Y is the intersection of line UV and the line passing through R and parallel to line PQ. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \frac{1}{S_{PUQV}}(S_{PUQR}S_{ABV} - S_{PVQR}S_{ABU}).$$

Proof. Take a point S such that $\overline{RS} = \overline{PQ}$. By Lemma 2.20, we have

$$S_{ABY} = \frac{1}{S_{RUSV}} \cdot (S_{USR}S_{ABV} + S_{VRS}S_{ABU}). \quad (1)$$

We also have

$S_{RUSV} = S_{PUQV}$ by Proposition 2.11.
$S_{USR} = S_{UQP} - S_{RQP} = S_{PUQR}$ by Proposition 2.12.
$S_{VSR} = S_{VQP} - S_{RPQ} = S_{PRQV}$ by Proposition 2.12.

Substituting these into (1), we obtain the conclusion. ∎

Lemma 2.23 *Let point Y be introduced by construction C8. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \frac{S_{PWQR}}{S_{PUQV}} \cdot S_{AUBV} + S_{ABW}.$$

Proof. By Lemma 2.21, $S_{ABY} = S_{ABW} + \frac{\overline{WY}}{\overline{UV}}S_{AUBV}$. Now the lemma comes from Lemma 2.26 below. ∎

2.4.2 Eliminating Points from Length Ratios

Lemma 2.24 *Let point Y be introduced by construction C3. To eliminate Y from $\frac{\overline{AY}}{\overline{CD}}$, we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \dfrac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \textit{if } A \in PQ; \\ \dfrac{S_{APQ}}{S_{CPDQ}} & \textit{otherwise.} \end{cases}$$

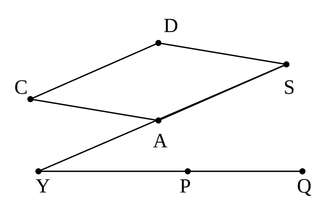


Figure 2-9

Proof. If $A \in PQ$ then $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AP}+\overline{PY}}{\overline{CD}} = \frac{\frac{\overline{AP}}{\overline{PQ}}+\frac{\overline{PY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}} = \frac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}}$. Otherwise, take a point S such that $\overline{AS} = \overline{CD}$. Then Y is the intersection of PQ and AS and $AS \parallel CD$. By Propositions 2.8 and 2.11,

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{S_{APQ}}{S_{APSQ}} = \frac{S_{APQ}}{S_{CPDQ}}. \quad ∎$$

Lemma 2.25 *Let point Y be introduced by construction C4. To eliminate point Y from $\frac{\overline{AY}}{\overline{CD}}$, we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{AUV}}{S_{CUDV}} & \text{if } A \text{ is not on } UV \\ \frac{S_{APQ}}{S_{CPDQ}} & \text{otherwise} \end{cases}$$

Proof. The proof is the same as the second case of Lemma 2.24. ▮

Lemma 2.26 *Let point Y be introduced by construction C6. Then we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in RY. \\ \frac{S_{APRQ}}{S_{CPDQ}} & \text{if } A \notin RY. \end{cases}$$

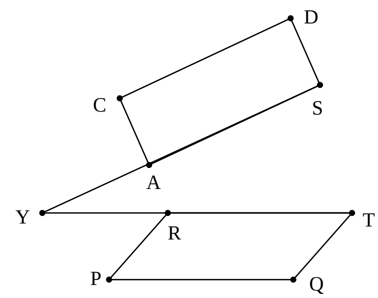


Figure 2-10

Proof. The first case is obvious. For the second case, take points T and S such that $\frac{\overline{RT}}{\overline{PQ}} = 1$ and $\frac{\overline{AS}}{\overline{CD}} = 1$. By the co-side theorem,

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{S_{ART}}{S_{ARST}} = \frac{S_{APRQ}}{S_{CPDQ}}. \quad ▮$$

Lemma 2.27 *Let Y be introduced by construction C7. To eliminate point Y from $G = \frac{\overline{AY}}{\overline{CD}}$, we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{AUV}}{S_{CUDV}} & \text{if } A \text{ is not on } UV \\ \frac{S_{APRQ}}{S_{CPDQ}} & \text{if } A \text{ is on } UV \end{cases}$$

Proof. If A is not on UV then the proof is the same as the second case of Lemma 2.24. If A is on UV, then the proof is the same as the second case of Lemma 2.26. Since $PQ \nparallel UV$, we have $S_{CPDQ} \neq 0$. ▮

Lemma 2.28 *Let point Y be introduced by construction C8. To eliminate point Y from $G = \frac{\overline{AY}}{\overline{CD}}$, we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{APRQ}}{S_{CPDQ}} & \text{if } AY \text{ is not parallel to } PQ \\ \frac{S_{AUWV}}{S_{CUDV}} & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as the proof of the second case of Lemma 2.26. ▮

2.4.3 Free Points and Area Coordinates

In Subsections 2.4.1 and 2.4.2, we present methods of eliminating fixed or semi-free points from geometry quantities. For a geometry statement $S = (C_1, C_2, \ldots, C_k, (E, F))$, we can use these lemmas to eliminate all the nonfree points introduced by C_i. Now the new E and F are rational expressions in indeterminates, areas and Pythagoras differences of *free points*. These geometric quantities are generally not independent, e.g. for any four points A, B, C, D we have

$$S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}.$$

In order to reduce E and F to expressions of independent variables, we introduce the concept of *area coordinates*.

Definition 2.29 *Let A, O, U, and V be four points such that O, U, and V are not collinear. The area coordinates of A with respect to OUV are*

$$x_A = \frac{S_{OUA}}{S_{OUV}}, \quad y_A = \frac{S_{OAV}}{S_{OUV}}, \quad z_A = \frac{S_{AUV}}{S_{OUV}}.$$

It is clear that $x_A + y_A + z_A = 1$. Since x_A, y_A, and z_A are not independent, we also call x_A, y_A the area coordinates of Q with respect to OUV.

Proposition 2.30 *The points in the plane are in a one to one correspondence with the triples (x, y, z) satisfying $x + y + z = 1$.*

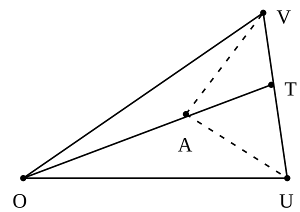


Figure 2-11

Proof. Let O, U, and V be three non-collinear points. Then for each point A, its area coordinates satisfy $x_A + y_A + z_A = 1$. Conversely, for any x, y, and z such that $x + y + z = 1$ we will find a point A whose area coordinates are x, y, and z. If $z = 1$, take a point A such that $\frac{\overline{OA}}{\overline{UV}} = x$. Then by Lemma 2.21, $x_A = \frac{S_{OUA}}{S_{OUV}} = x$, $y_A = -x = y$, and $z_A = 1$. If $z \neq 1$, take a point T on UV such that $\frac{\overline{UT}}{\overline{UV}} = \frac{x}{1-z}$; take a point A on OT such that $\frac{\overline{AT}}{\overline{OT}} = z$. By the co-side theorem, $z_A = \frac{S_{AUV}}{S_{OUV}} = \frac{\overline{AT}}{\overline{OT}} = z$. By the co-side theorem again, we have

$$x_A = \frac{S_{OUA}}{S_{OUV}} = (1 - z)\frac{S_{OUT}}{S_{OUV}} = x\frac{S_{OUV}}{S_{OUV}} = x.$$

Similarly, $y_A = y$. ∎

The following lemma reduces any area to an expression of area coordinates with respect to three given reference points.

Lemma 2.31 *Let O, U, and V be three noncollinear points. Then for points A, B, and Y, we have*

$$S_{ABY} = \frac{1}{S_{OUV}} \begin{vmatrix} S_{OUA} & S_{OVA} & 1 \\ S_{OUB} & S_{OVB} & 1 \\ S_{OUY} & S_{OVY} & 1 \end{vmatrix}.$$

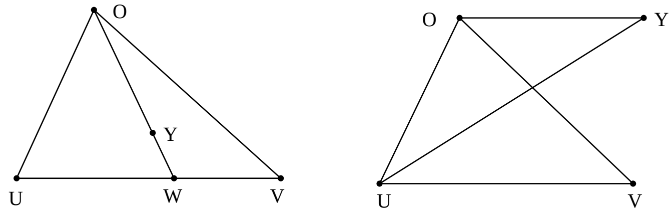


Figure 2-12

Proof. Since by Axiom A.5 $S_{ABY} = S_{OAB} + S_{OBY} - S_{OAY}$, we need only to compute S_{OBY} and S_{OAY}. Let W be the intersection of UV and OY. Then by Lemma 2.20 we have

$$S_{OBW} = \frac{1}{S_{OUYV}}(S_{OBV} \cdot S_{OUY} + S_{OBU} \cdot S_{OYV}).$$

By Proposition 2.8, we have $\frac{S_{OBY}}{S_{OBW}} = \frac{S_{OUYV}}{S_{OUV}}$. Thus

$$S_{OBY} = \frac{1}{S_{OUV}} \cdot (S_{OBV} \cdot S_{OUY} + S_{OBU} \cdot S_{OYV}). \tag{1}$$

If $OY \parallel UV$, (1) can be proved as follows. By Example 2.15, $\frac{\overline{OY}}{\overline{UV}} = \frac{S_{OUY}}{S_{OUV}}$. By Lemma 2.21

$$S_{OBY} = \frac{\overline{OY}}{\overline{UV}} \cdot S_{OUBV} = \frac{S_{OUY}}{S_{OUV}}(S_{OBV} + S_{OUB}) = \frac{S_{OBV} \cdot S_{OUY} + S_{OBU} \cdot S_{OYV}}{S_{OUV}}.$$

Now we have proved that (1) is true under the condition that O, U, V are not collinear. Similarly, we have

$$\begin{aligned} S_{OAY} &= \frac{1}{S_{OUV}} \cdot (S_{OAV} \cdot S_{OUY} + S_{OAU} \cdot S_{OYV}); \\ S_{OAB} &= \frac{1}{S_{OUV}} \cdot (S_{OAV} \cdot S_{OUB} + S_{OAU} \cdot S_{OBV}). \end{aligned}$$

Substituting (1) and the above formulas into $S_{ABY} = S_{OAB} + S_{OBY} - S_{OAY}$, we obtain the conclusion. ∎

Use the same notations as in Lemma 2.31, let $x_A = \frac{S_{OUA}}{S_{OUV}}, y_A = \frac{S_{OVA}}{S_{OUV}}$; $x_B = \frac{S_{OUB}}{S_{OUV}}, y_B = \frac{S_{OVB}}{S_{OUV}}$; $x_Y = \frac{S_{OUY}}{S_{OUV}}, y_Y = \frac{S_{OVY}}{S_{OUV}}$. Then the formula in Lemma 2.31 becomes

$$S_{ABY} = S_{OUV} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_Y & y_Y & 1 \end{vmatrix}$$

which is quite similar to the formula of the area of a triangle in terms of the Cartesian coordinates of its three vertices.

Algorithm 2.32 (AFFINE)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a statement in $\mathbf{C}_{\mathrm{H}}$.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \cdots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the ndg conditions of C_i are satisfied. The ndg conditions of a statement have two forms: $A \neq B$ and $PQ \nparallel UV$. For the first case, we check whether $\frac{\overline{AB}}{\overline{XY}} = 0$ where X, Y are two arbitrary points on AB. For the second case, we check whether $S_{PUV} = S_{QUV}$. If the ndg condition of a geometry statement is not satisfied, the statement is *trivially true*. The algorithm terminates.

S3. Let $G_1, \cdots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \cdots, s$ do S4.

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using Lemmas 2.24–2.31 and replace G_j by H_j in E and F to obtain the new E and F.

S5. Finally, E and F are expressions of free parameters. If E is identical to F, S is true under the ndg conditions. Otherwise S is false in the Euclidean plane geometry.

Proof of the correctness. If $E = F$ then it is clear that S is true. Notice that all the elimination lemmas in this section have the property that after applying a lemma to a geometry quantity that has geometric meaning (i.e., its denominator is not zero), the expression obtained also has geometric meaning under the ndg conditions of this statement. Therefore all the geometric quantities occurring in the proof have geometric meaning.

The geometric quantities in E and F are all free parameters, i.e., in the geometric configuration of S they can take arbitrary values. Since $E \neq F$, by Proposition 2.33 below we can take some concrete values for these quantities such that when replacing these quantities by the corresponding values in E and F, we obtain two different numbers. In other words, we obtain a counter example for S. ∎

Proposition 2.33 *Let P be a nonzero polynomial of indeterminates $x_1, \cdots, x_n$ with real numbers as coefficients. Show that we can find rational numbers $e_1, \cdots, e_n$ such that $P(e_1, \cdots, e_n) \neq 0$.*

The proof is left as an exercise.

For the complexity of the algorithm, let m and n be the number of free and non-free points in a statement respectively. To eliminate each non-free point, we need to apply the lemmas in Subsections 2.4.1 and 2.4.2 to each geometry quantity involving this point once. Also note that each lemma will replace a geometric quantity by a rational expression with degree less than or equal to two. Then if the conclusion of the geometry statement is of degree d, the expression obtained after eliminating the n non-free points is at most degree $2^n d$. Also note that after eliminating the m free points using Lemma 2.31, each quantity will be replaced by an expression of degree two. Then the final result is at most degree $2d2^n = d2^{n+1}$.

This simple exponential complexity of the algorithm seems discouraging. But we will see that on the contrary this method can produce short proofs for almost all statements in $\mathbf{C}_{\mathrm{H}}$ very efficiently. One reason is that during the proof, the common factors of E and F can be removed. This simple trick alone usually reduces the sizes of the polynomials occurring in a proof drastically. Also the algorithm still has much room for improvement in order to obtain short proofs, as shown in Section 2.5 below.

Exercises 2.34

1. Let O, U, and V be three noncollinear points. Then for points A, B, and Y, we have

$$S_{ABY} = \frac{1}{S_{OUV}} \begin{vmatrix} S_{OUA} & S_{OVA} & S_{UVA} \\ S_{OUB} & S_{OVB} & S_{UVB} \\ S_{OUY} & S_{OVY} & S_{UVY} \end{vmatrix}.$$

2. Show that each polynomial $P(x) \in R[x]$ of degree d has at most d different roots. Use this result to prove Proposition 2.33.
3. Let $P(x) = x^d + a_{d-1}x^{d-1} + ... + a_0$ be a polynomial, and $m = max(1, \sum_{i=1}^{d} |a_i|)$. Then for any $r > m$ we have $P(r) \neq 0$. Use this result to prove Proposition 2.33.
4. Prove the Examples in Sections 1.2 and 1.3 using Algorithm 2.32.

2.4.4 Working Examples

Before going further, we want to explain a little bit about the meaning of the axioms, propositions, lemmas, and algorithms in this book. Since the goal of this book is to provide a method of proving theorems, the algorithms are our final goals. The input to an algorithm is a geometry statement. The output of the algorithm is a proof or a disproof of the statement. The algorithms use the lemmas to eliminate points from geometry quantities. In the proofs of the lemmas, only the basic propositions are used. Finally, the basic propositions are consequences of the axioms.

We can thus divide the proofs produced by Algorithm 2.32 into three levels:

1. *a proof is at the first level or lemma level* if in that proof we only use the lemmas;

2. *a proof is at the second level or proposition level* if in that proof we not only give the result obtained by applying the lemmas but also the process of how the results are obtained by using the basic propositions;

3. *a proof is at the third level or axiom level* if in that proof we only use the axioms.

Theoretically, proofs at all levels can be produced automatically. But only proofs at the first or second level are relatively short. If we limit ourselves to the six axioms only, the proofs produced according to our algorithms are generally very long. Also, it is not reasonable to limit oneself to axioms only. The proofs of the geometry statements in Chapter 1 are all at the proposition level and most of the proofs given in this chapter are at the lemma level.

Algorithm 2.32 has been implemented as a prover on a computer. At the present time, this prover can only produce proofs at the lemma level. In what follows, when speaking about *a machine proof*, we mean the proof (in LaTeX form) produced by this prover. For instance, the machine proof for Ceva's theorem (Example 1.7) is as follows.

The machine proof

$$-\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}\cdot\frac{\overline{AF}}{\overline{BF}}$$
$$\stackrel{F}{=} \frac{-(-S_{ACP})}{-S_{BCP}}\cdot\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\stackrel{E}{=} \frac{-S_{BCP}\cdot S_{ACP}}{S_{BCP}\cdot(-S_{ABP})}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\stackrel{simplify}{=} \frac{S_{ACP}}{S_{ABP}}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\stackrel{D}{=} \frac{S_{ABP}\cdot S_{ACP}}{S_{ABP}\cdot S_{ACP}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}}\stackrel{F}{=}\frac{S_{ACP}}{S_{BCP}}$$
$$\frac{\overline{CE}}{\overline{AE}}\stackrel{E}{=}\frac{S_{BCP}}{-S_{ABP}}$$
$$\frac{\overline{BD}}{\overline{CD}}\stackrel{D}{=}\frac{S_{ABP}}{S_{ACP}}$$

In the proof, $a \stackrel{P}{=} b$ means that b is the result obtained by eliminating point P from a; $a \stackrel{simplify}{=} b$ means that b is obtained by canceling some common factors from the denominator and numerator of a; "eliminants" are the results obtained by eliminating points from separate geometry quantities. The prover can also give the ndg conditions and the predicate form of the geometry statement.

We use a sequence of consecutive equations to represent a proof. Some might argue that this proof looks different from the usual form of proofs. It is actually very easy to rewrite a proof in consecutive equations as the usual form. For instance, the above machine proof of Ceva's theorem is essentially the same as the proof of Ceva's theorem on page 11.

It is clear that the proofs produced according to Algorithm 2.32 depend on how we describe a geometry statement constructively. For the same statement, some descriptions will lead to long proofs while other descriptions will lead to short ones. Both the way of introducing points and the way of formulating the conclusions will affect the output. We use some examples to show some heuristic rules in specifying the statement in constructive form which may lead to short proofs.

Example 2.35 (Menelaus' Theorem) *A transversal meets the sides AB, BC, and CA of a triangle ABC in F, D, and E. Show that* $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$.

First describe the statement in the constructive way.

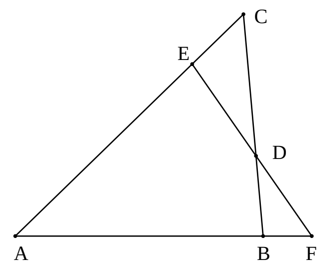


Figure 2-13

Take three arbitrary points A, B, C.
Take a point D on line BC.
Take a point E on line AC.
Take the intersection F of line DE and line AB.
Show that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$.

The ndg conditions are $C \neq B$, $A \neq C$, $DE \nparallel BA$, $B \neq F$, $C \neq D$, and $A \neq E$.

The machine proof

$$\frac{\overline{CE}}{\overline{AE}} \cdot \frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{AF}}{\overline{BF}}$$

$$\stackrel{F}{=} \frac{-S_{ADE}}{-S_{BDE}} \cdot \frac{\overline{CE}}{\overline{AE}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{E}{=} \frac{(\frac{\overline{AE}}{\overline{AC}}-1)\cdot(-S_{ACD})\cdot\frac{\overline{AE}}{\overline{AC}}}{(-S_{ABD}\cdot\frac{\overline{AE}}{\overline{AC}}+S_{ABD})\cdot\frac{\overline{AE}}{\overline{AC}}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{simplify}{=} \frac{S_{ACD}}{S_{ABD}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{D}{=} \frac{\frac{\overline{BD}}{\overline{BC}}\cdot(S_{ABC}\cdot\frac{\overline{BD}}{\overline{BC}}-S_{ABC})}{S_{ABC}\cdot\frac{\overline{BD}}{\overline{BC}}\cdot(\frac{\overline{BD}}{\overline{BC}}-1)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}} \stackrel{F}{=} \frac{S_{ADE}}{S_{BDE}}$$

$$S_{BDE} \stackrel{E}{=} -\left((\frac{\overline{AE}}{\overline{AC}}-1)\cdot S_{ABD}\right)$$

$$S_{ADE} \stackrel{E}{=} -\left(S_{ACD}\cdot\frac{\overline{AE}}{\overline{AC}}\right)$$

$$\frac{\overline{CE}}{\overline{AE}} \stackrel{E}{=} \frac{\frac{\overline{AE}}{\overline{AC}}-1}{\frac{\overline{AE}}{\overline{AC}}}$$

$$S_{ABD} \stackrel{D}{=} S_{ABC}\cdot\frac{\overline{BD}}{\overline{BC}}$$

$$S_{ACD} \stackrel{D}{=} (\frac{\overline{BD}}{\overline{BC}}-1)\cdot S_{ABC}$$

$$\frac{\overline{BD}}{\overline{CD}} \stackrel{D}{=} \frac{\frac{\overline{BD}}{\overline{BC}}}{\frac{\overline{BD}}{\overline{BC}}-1}$$

The above proof produced according to our algorithm is not the simplest one. By describing the example constructively as follows, we can obtain a much shorter proof.

Take arbitrary points A, B, C, X, Y.
D is the intersection of BC and XY.
E is the intersection of AC and XY.
F is the intersection of AB and XY.
Show that $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1$.

The machine proof

$$\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}\cdot\frac{\overline{AF}}{\overline{BF}}$$
$$\overset{F}{=} \frac{S_{AXY}}{S_{BXY}}\cdot\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\overset{E}{=} \frac{S_{CXY}\cdot S_{AXY}}{S_{BXY}\cdot S_{AXY}}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\overset{simplify}{=} \frac{S_{CXY}}{S_{BXY}}\cdot\frac{\overline{BD}}{\overline{CD}}$$
$$\overset{D}{=} \frac{S_{CXY}\cdot S_{BXY}}{S_{BXY}\cdot S_{CXY}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}}\overset{F}{=}\frac{S_{AXY}}{S_{BXY}}$$
$$\frac{\overline{CE}}{\overline{AE}}\overset{E}{=}\frac{S_{CXY}}{S_{AXY}}$$
$$\frac{\overline{BD}}{\overline{CD}}\overset{D}{=}\frac{S_{BXY}}{S_{CXY}}$$

Example 2.36 (Gauss-line Theorem) *Let A_0, A_1, A_2, and A_3 be four points on a plane, X be the intersection of A_1A_2 and A_0A_3, and Y be the intersection of A_0A_1 and A_2A_3. Let M_1, M_2, and M_3 be the midpoints of A_1A_3, A_0A_2, and XY respectively. Then M_1, M_2, and M_3 are collinear.*

The constructive description
Take arbitrary points A_0, A_1, A_2, and A_3.
$X = A_0A_3 \cap A_1A_2$.
$Y = A_2A_3 \cap A_1A_0$.
M_1 is the midpoint of A_1A_3.
M_2 is the midpoint of A_0A_2.
M_3 is the midpoint of XY.
Show that $S_{M_1M_2M_3} = 0$

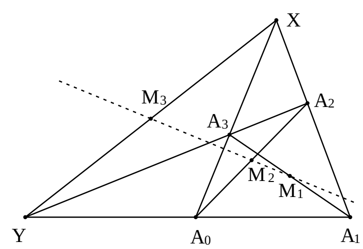


Figure 2-14

Here is the machine proof.

$$S_{M_1M_2M_3}$$
$$\overset{n}{=} \tfrac{1}{2}S_{YM_1M_2}+\tfrac{1}{2}S_{XM_1M_2}$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(\tfrac{1}{2}S_{A_2YM_1}+\tfrac{1}{2}S_{A_2XM_1}+\tfrac{1}{2}S_{A_0YM_1}+\tfrac{1}{2}S_{A_0XM_1})$$
$$\overset{n}{=} (\tfrac{1}{4})\cdot(-\tfrac{1}{2}S_{A_2A_3X}+\tfrac{1}{2}S_{A_1A_2Y}-\tfrac{1}{2}S_{A_0A_3Y}-\tfrac{1}{2}S_{A_0A_1X})$$
$$\overset{n}{=} (-\tfrac{1}{8})(S^2_{A_0A_2A_1A_3}S_{A_2A_3X}+S^2_{A_0A_2A_1A_3}S_{A_0A_1X}+S_{A_0A_2A_1A_3}S_{A_1A_2A_3}S_{A_0A_1A_2} - S_{A_0A_2A_1A_3}S_{A_0A_2A_3}S_{A_0A_1A_3})$$
$$\overset{simplify}{=} (-\tfrac{1}{8})\cdot(S_{A_0A_2A_1A_3}\cdot S_{A_2A_3X}+S_{A_0A_2A_1A_3}\cdot S_{A_0A_1X}+S_{A_1A_2A_3}\cdot S_{A_0A_1A_2}-S_{A_0A_2A_3}\cdot S_{A_0A_1A_3})$$
$$\overset{n}{=} (-\tfrac{1}{8})\cdot(-S_{A_0A_2A_1A_3}\cdot S_{A_0A_1A_3A_2}\cdot S_{A_1A_2A_3}\cdot S_{A_0A_2A_3}+S_{A_0A_2A_1A_3}\cdot S_{A_0A_1A_3A_2}\cdot S_{A_0A_1A_3}\cdot S_{A_0A_1A_2}+ S^2_{A_0A_1A_3A_2}\cdot S_{A_1A_2A_3}\cdot S_{A_0A_1A_2}-S^2_{A_0A_1A_3A_2}\cdot S_{A_0A_2A_3}\cdot S_{A_0A_1A_3})$$
$$\overset{simplify}{=} (\tfrac{1}{8})\cdot(S_{A_0A_2A_1A_3}\cdot S_{A_1A_2A_3}\cdot S_{A_0A_2A_3}-S_{A_0A_2A_1A_3}\cdot S_{A_0A_1A_3}\cdot S_{A_0A_1A_2}- S_{A_0A_1A_3A_2}\cdot S_{A_1A_2A_3}\cdot S_{A_0A_1A_2}+S_{A_0A_1A_3A_2}\cdot S_{A_0A_2A_3}\cdot S_{A_0A_1A_3})$$
$$\overset{n}{=} (\tfrac{1}{8})\cdot(0) \overset{simplify}{=} 0$$

Here $a \overset{n}{=} b$ means b is the numerator of a. To show $a = 0$, we need only to show its numerator is zero.

If we describe the statement as follows, then we can obtain a shorter proof.

> Take arbitrary points A_0, A_1, A_2, A_3.
> $X = A_0A_3 \cap A_1A_2$. $Y = A_2A_3 \cap A_1A_0$.
> M_1 is the midpoint of A_1A_3. M_2 is the midpoint of A_0A_2.
> M_3 is the midpoint of XY. $Z = M_2M_1 \cap XY$.
> Show that $\frac{\overline{XM_3}}{\overline{YM_3}} = \frac{\overline{XZ}}{\overline{YZ}}$.

The machine proof

$$\left(\frac{\overline{XM_3}}{\overline{YM_3}}\right)/\left(\frac{\overline{XZ_{M_3}}}{\overline{YZ_{M_3}}}\right)$$

$$\overset{Z_{M_3}}{=} \frac{-S_{YM_1M_2}}{-S_{XM_1M_2}} \cdot \frac{\overline{XM_3}}{\overline{YM_3}}$$

$$\overset{M_3}{=} \frac{(\frac{1}{2})\cdot S_{YM_1M_2}}{S_{XM_1M_2}\cdot(-\frac{1}{2})}$$

$$\overset{M_2}{=} \frac{-(\frac{1}{2}S_{A_2YM_1}+\frac{1}{2}S_{A_0YM_1})}{\frac{1}{2}S_{A_2XM_1}+\frac{1}{2}S_{A_0XM_1}}$$

$$\overset{M_1}{=} \frac{-(\frac{1}{2}S_{A_1A_2Y}-\frac{1}{2}S_{A_0A_3Y})}{-\frac{1}{2}S_{A_2A_3X}-\frac{1}{2}S_{A_0A_1X}}$$

$$\overset{Y}{=} \frac{S_{A_0A_1A_2A_3}}{S_{A_0A_1A_2A_3}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{XZ_{M_3}}}{\overline{YZ_{M_3}}} \overset{Z_{M_3}}{=} \frac{S_{XM_1M_2}}{S_{YM_1M_2}}$$

$$\frac{\overline{XM_3}}{\overline{YM_3}} \overset{M_3}{=} -(1)$$

$$S_{XM_1M_2} \overset{M_2}{=} \frac{1}{2}(S_{A_2XM_1}+S_{A_0XM_1})$$

$$S_{YM_1M_2} \overset{M_2}{=} \frac{1}{2}(S_{A_2YM_1}+S_{A_0YM_1})$$

$$S_{A_0XM_1} \overset{M_1}{=} -\frac{1}{2}(S_{A_0A_1X})$$

$$S_{A_2XM_1} \overset{M_1}{=} -\frac{1}{2}(S_{A_2A_3X})$$

$$S_{A_0YM_1} \overset{M_1}{=} -\frac{1}{2}(S_{A_0A_3Y})$$

$$S_{A_2YM_1} \overset{M_1}{=} \frac{1}{2}(S_{A_1A_2Y})$$

$$S_{A_2A_3X}+S_{A_0A_1X}=S_{A_0A_1A_2A_3}$$

So for a "nice" expression of the conclusion of a statement, we can obtain a short proof. The idea is to express the conclusion as $a = b$ such that a and b are symmetric in some sense. For example, if we want to prove three points P, Q, and R are collinear, and if one of the three points, say R, is on (LINE E F) where E and F are points in the statement, then we usually introduce a new point N by (INTER N (LINE P Q) (LINE E F)) and prove the equivalent conclusion $N = R$ or $\frac{\overline{ER}}{\overline{FR}} = \frac{\overline{EN}}{\overline{FN}}$. According to our experience with many examples, the proof for this new conclusion is shorter than the proof for $S_{PQR} = 0$. The following example shows that this rule is also true for the length ratios.

Example 2.37 *A line parallel to the base of trapezoid ABCD meet its two sides and two diagonals at H, G, F, and E. Show that EF = GH.*

We may describe the statements as follows.

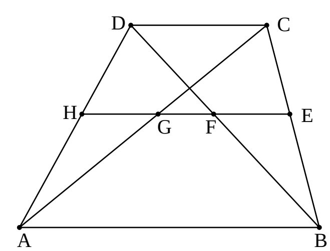


Figure 2-15

> Take arbitrary points A, B, C.

Take a point D such that $DC \parallel AB$.
Take a point E on line BC.
Take the intersection H of line AD
and the line passing through E and parallel AB.
$F = BD \cap EH$.
$G = AC \cap EF$.
Prove that $\frac{\overline{EF}}{\overline{AB}} = -\frac{\overline{HG}}{\overline{AB}}$.

The machine proof

$$\frac{\frac{\overline{EF}}{\overline{AB}}}{-\frac{\overline{HG}}{\overline{AB}}}$$

$$\overset{G}{=} \frac{S_{ABC}}{-S_{ACH}} \cdot \frac{\overline{EF}}{\overline{AB}}$$

$$\overset{F}{=} \frac{S_{BDE}\cdot S_{ABC}}{-S_{ACH}\cdot S_{ABD}}$$

$$\overset{H}{=} \frac{-S_{BDE}\cdot S_{ABC}\cdot(-S_{ABD})}{(-S_{ACD}\cdot S_{ABE})\cdot S_{ABD}}$$

$$\overset{simplify}{=} \frac{-S_{BDE}\cdot S_{ABC}}{S_{ACD}\cdot S_{ABE}}$$

$$\overset{E}{=} \frac{-(-S_{BCD}\cdot\frac{\overline{BE}}{\overline{BC}})\cdot S_{ABC}}{S_{ACD}\cdot S_{ABC}\cdot\frac{\overline{BE}}{\overline{BC}}}$$

$$\overset{simplify}{=} \frac{S_{BCD}}{S_{ACD}}$$

$$\overset{D}{=} \frac{-S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}}{-S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{HG}}{\overline{AB}} \overset{G}{=} \frac{S_{ACH}}{S_{ABC}}$$

$$\frac{\overline{EF}}{\overline{AB}} \overset{F}{=} \frac{S_{BDE}}{S_{ABD}}$$

$$S_{ACH} \overset{H}{=} \frac{S_{ACD}\cdot S_{ABE}}{S_{ABD}}$$

$$S_{ABE} \overset{E}{=} S_{ABC}\cdot\frac{\overline{BE}}{\overline{BC}}$$

$$S_{BDE} \overset{E}{=} -\left(S_{BCD}\cdot\frac{\overline{BE}}{\overline{BC}}\right)$$

$$S_{ACD} \overset{D}{=} -\left(S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}\right)$$

$$S_{BCD} \overset{D}{=} -\left(S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}\right)$$

By changing the conclusion to $\frac{\overline{EF}}{\overline{GH}} = 1$, the proof will become much longer.

2.5 More Elimination Techniques

The eleven lemmas (2.19–2.31) provide a complete method of eliminating points from geometry quantities. But if only these lemmas are used, the proofs for some geometry theorems are still too long to be readable. In order to produce short and readable proofs for geometry theorems, we will provide more elimination techniques.

2.5.1 Refined Elimination Techniques

Lemmas 2.19–2.31 only give the elimination result in the general case. In some special cases, the elimination results could be much simpler, e.g., see Excises 2.38, 2.39, and 2.40 below. By only using these refined elimination results, we can obtain much shorter proofs for many geometry theorems.

Exercise 2.38 *Prove the following version of Lemma 2.20. Let $Y = PQ \cap UV$. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \begin{cases} S_{ABU} & \text{if } AB \parallel UV; \\ S_{ABP} & \text{if } AB \parallel PQ; \\ \frac{S_{UBV} \cdot S_{APQ}}{S_{UPVQ}} & \text{if } U, V, A \text{ are collinear;} \\ \frac{S_{AUV} \cdot S_{BPQ}}{S_{UPVQ}} & \text{if } U, V, B \text{ are collinear;} \\ \frac{S_{APQ} \cdot S_{BUV}}{S_{PUQV}} & \text{if } P, Q, A \text{ are collinear;} \\ \frac{S_{PBQ} \cdot S_{AUV}}{S_{PUQV}} & \text{if } P, Q, B \text{ are collinear;} \\ \frac{1}{S_{UPVQ}}(S_{UPQ} \cdot S_{ABV} - S_{VPQ} \cdot S_{ABU}) & \text{if } U \text{ or } V \text{ is on } AB; \\ \frac{1}{S_{PUQV}}(S_{PUV} \cdot S_{ABQ} + S_{QVU} \cdot S_{ABP}) & \text{otherwise.} \end{cases}$$

Exercise 2.39 *Prove the following version of Lemma 2.22. Let Y be the intersection of the line UV and the line passing through R and parallel to line PQ. To eliminate point Y from S_{ABY}, we have*

$$S_{ABY} = \begin{cases} S_{ABU} & \text{if } AB \parallel UV; \\ S_{ABR} & \text{if } AB \parallel PQ; \\ \frac{S_{UBV} \cdot S_{APRQ}}{S_{UPVQ}} & \text{if } U, V, A \text{ are collinear;} \\ \frac{S_{AUV} \cdot S_{BPRQ}}{S_{UPVQ}} & \text{if } U, V, B \text{ are collinear;} \\ \frac{S_{AUV} \cdot S_{BQRP}}{S_{PUQV}} & \text{if } AY \parallel PQ; \\ \frac{S_{BUV} S_{APRQ}}{S_{PUQV}} & \text{if } BY \parallel PQ; \\ \frac{1}{S_{PUQV}}(S_{PUQR} \cdot S_{ABV} + S_{PRQV} \cdot S_{ABU}) & \text{otherwise.} \end{cases}$$

Exercise 2.40 *Prove the following version of Lemma 2.23. Let Y be the intersection of the line passing through point R and parallel to PQ and the line passing through point W and parallel to line UV. We have*

$$S_{ABY} = \begin{cases} S_{ABW} & \text{if } AB \parallel UV; \\ S_{ABR} & \text{if } AB \parallel PQ; \\ \frac{S_{AUWV} \cdot S_{BQRP}}{S_{PUQV}} & \text{if } AY \parallel PQ; \\ \frac{S_{BUWV} \cdot S_{APRQ}}{S_{PUQV}} & \text{if } BY \parallel PQ; \\ \frac{S_{BVWU} \cdot S_{APRQ}}{S_{UPVQ}} & \text{if } AY \parallel UV; \\ \frac{S_{AUWV} \cdot S_{BPRQ}}{S_{UPCQ}} & \text{if } BY \parallel UV; \\ S_{ABY} = \frac{S_{PWQR}}{S_{PUQV}} \cdot S_{AUBV} + S_{ABW} & \text{otherwise.} \end{cases}$$

To use these elimination methods, we have first to decide whether three points are collinear, or whether two lines are parallel. We can use Algorithm 2.32 to do so. But this process is too time consuming. A faster way is to collect all the obvious collinear and parallel relations from the constructions and use them as criteria. For instance, in Ceva's theorem (Example 2.17 on page 61) we can find the following collinear point sets easily: $\{A, B, F\}$, $\{A, C, E\}$, $\{B, C, D\}$, $\{A, D, P\}$, $\{B, E, P\}$, and $\{C, F, P\}$.

Once we obtain all the lines and parallels for a geometry statement, we can use them to simplify some geometry quantities. For instance,

$$S_{P_1P_2P_3P_4} = \begin{cases} 0 & \text{if } P_1P_3 \parallel P_2P_4; \\ S_{P_1P_3P_4} & \text{if } P_1, P_2, P_3 \text{ are collinear;} \\ S_{P_1P_2P_4} & \text{if } P_2, P_3, P_4 \text{ are collinear;} \\ S_{P_1P_2P_3} & \text{if } P_1, P_3, P_4 \text{ are collinear;} \\ S_{P_2P_3P_4} & \text{if } P_1, P_2, P_4 \text{ are collinear.} \end{cases}$$

Example 2.41 *Let A, B, and P be three noncollinear points, and C be a point on line PA. The line passing through C and parallel to AB intersects PB at D. Q is the intersection of AD and BC. M is the intersection of AB and PQ. Show that M is the midpoint of AB.*

The example can be described in the following constructive way.

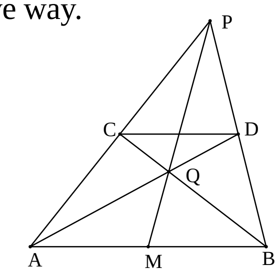


Figure 2-16

Take three arbitrary points A, B, and P.
Take a point C on line AP.
D is the intersection of line BP and the line passing through C and parallel to AB.
Q is the intersection of lines AD and BC.
M is the intersection of lines AB and PQ.
Show that $\frac{\overline{AM}}{\overline{BM}} = -1$.
The ndg conditions: $A \neq P$, $P \notin AB$, $AD \nparallel BC$, and $AB \nparallel PQ$.

The machine proof

$$-\frac{\overline{AM}}{\overline{BM}}$$
$$\stackrel{M}{=} \frac{-S_{APQ}}{S_{BPQ}}$$
$$\stackrel{Q}{=} \frac{-S_{APD}\cdot S_{ABC}\cdot(-S_{ABDC})}{(-S_{BPC}\cdot S_{ABD})\cdot S_{ABDC}}$$
$$\stackrel{simplify}{=} \frac{-S_{APD}\cdot S_{ABC}}{S_{BPC}\cdot S_{ABD}}$$
$$\stackrel{D}{=} \frac{-S_{BPC}\cdot S_{ABC}}{-S_{BPC}\cdot S_{ABC}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AM}}{\overline{BM}} \stackrel{M}{=} \frac{S_{APQ}}{S_{BPQ}}$$
$$S_{BPQ} \stackrel{Q}{=} \frac{S_{BPC}\cdot S_{ABD}}{S_{ABDC}}$$
$$S_{APQ} \stackrel{Q}{=} \frac{S_{APD}\cdot S_{ABC}}{S_{ABDC}}$$
$$S_{ABD} \stackrel{D}{=} S_{ABC}$$
$$S_{APD} \stackrel{D}{=} -(S_{BPC})$$

Our prover first collects the collinear point sets:

$$\{M, P, Q\}; \{Q, A, D\}; \{Q, B, C\}; \{M, A, B\}; \{D, B, P\}; \{C, A, P\};$$

and the parallel lines: $DC \parallel MAB$.

To eliminate D from S_{ABD}, we use the first case of Exercise 2.39: $S_{ABD} = S_{ABC}$. To eliminate D from S_{APD}, since P, B, and D are collinear we can use the fourth case of Exercise 2.39:

$$S_{APD} = \frac{S_{ABP} S_{PACB}}{S_{BAPB}} = -S_{PACB}$$

Since P, A, and C are collinear, $S_{PACB} = S_{PAC} + S_{PCB} = S_{BPC}$.

2.5.2 The Two-line Configuration

Another commonly used technique is the *two-line configuration*. This trick works only when there exist at least five free or semi-free points and these points are on two lines l_1 and l_2. If $l_1 \parallel l_2$, let α be the oriented distance from line l_1 to line l_2. If $l_1 \nparallel l_2$ let $\beta = \sin(\measuredangle(l_1, l_2))$ and O be the intersection of l_1 and l_2.

Let l_1 and l_2 be two lines satisfying the above conditions. Then we have the following elimination procedure.

Case 1. $G = S_{ABC}$.

$$S_{ABC} = \begin{cases} 0 & \text{if } A, B, C \text{ are collinear} \\ \frac{1}{2}\alpha\overline{BC} & \text{if } l_1 \parallel l_2,\ A \in l_1, \text{ and } B, C \in l_2 \\ -\frac{1}{2}\alpha\overline{BC} & \text{if } l_1 \parallel l_2,\ A \in l_2, \text{ and } B, C \in l_1 \\ \frac{1}{2}\beta\overline{OA}\cdot\overline{BC} & \text{if } l_1 \nparallel l_2,\ A \in l_1, \text{ and } B, C \in l_2 \\ -\frac{1}{2}\beta\overline{OA}\cdot\overline{BC} & \text{if } l_1 \nparallel l_2,\ A \in l_2, \text{ and } B, C \in l_1. \end{cases}$$

Case 2. $G = \frac{\overline{AB}}{\overline{CD}}$. We have $G = (\frac{\overline{AB}}{\overline{CD}}) = \overline{AB}/\overline{CD}$, i.e., we break one geometry quantity into the ratio of two quantities.

Case 3. $G = \overline{AB}$

$$\overline{AB} = \begin{cases} \overline{OB} - \overline{OA} & \text{if } l_1 \text{ and } l_2 \text{ intersects at } O \\ \overline{O_1B} - \overline{O_1A} & \text{if } l_1 \parallel l_2,\ A \in l_1 \text{ and } B \in l_1 \\ \overline{O_2B} - \overline{O_2A} & \text{if } l_1 \parallel l_2,\ A \in l_2 \text{ and } B \in l_2 \end{cases}$$

where O_1 and O_2 are fixed points on l_1 and l_2 respectively. ∎

For the constructive description of Menelaus' theorem on page 73, we have the following machine proof using the two-line trick.

$$\frac{\overline{CE}}{\overline{AE}} \cdot \frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{AF}}{\overline{BF}}$$

$$\stackrel{F}{=} \frac{-S_{ADE}}{-S_{BDE}} \cdot \frac{\overline{CE}}{\overline{AE}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{2lines}{=} \frac{\overline{CE} \cdot \overline{BD} \cdot (-\overline{CD} \cdot \overline{AE} \cdot \beta) \cdot (2)}{(-\overline{CE} \cdot \overline{BD} \cdot \beta) \cdot (2) \cdot \overline{CD} \cdot \overline{AE}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}} \stackrel{F}{=} \frac{S_{ADE}}{S_{BDE}}$$

$$S_{BDE} = -\frac{1}{2}(\overline{CE} \cdot \overline{BD} \cdot \beta)$$

$$S_{ADE} = -\frac{1}{2}(\overline{CD} \cdot \overline{AE} \cdot \beta)$$

Example 2.42 (Pappus' Theorem) *Let points A, B and C be on one line, and A_1, B_1 and C_1 be on another line. Let $P = AB_1 \cap A_1B$, $Q = AC_1 \cap A_1C$, and $S = BC_1 \cap B_1C$. Show that P, Q, and S are collinear.*

The input to the program.

Take arbitrary points A, A_1, B, B_1.
Take a point C on line AB.
Take a point C_1 on line A_1B_1.
$P = A_1B \cap AB_1$.
$Q = AC_1 \cap A_1C$.
$S = B_1C \cap BC_1$.
$T = B_1C \cap PQ$.
Prove that $\frac{\overline{B_1S}}{\overline{CS}} = \frac{\overline{B_1T}}{\overline{CT}}$.

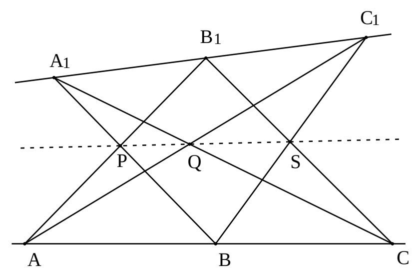


Figure 2-17

The machine proof

$$\left(\frac{\overline{B_1S}}{\overline{CS}}\right) / \left(\frac{\overline{B_1T}}{\overline{CT}}\right)$$

$$\stackrel{T}{=} \frac{S_{CPQ}}{S_{B_1PQ}} \cdot \frac{\overline{B_1S}}{\overline{CS}}$$

$$\stackrel{S}{=} \frac{(-S_{BB_1C_1}) \cdot S_{CPQ}}{S_{B_1PQ} \cdot (-S_{BCC_1})}$$

$$\stackrel{Q}{=} \frac{S_{BB_1C_1} \cdot S_{A_1CP} \cdot S_{ACC_1} \cdot S_{AA_1C_1C}}{(-S_{B_1C_1P} \cdot S_{AA_1C}) \cdot S_{BCC_1} \cdot (-S_{AA_1C_1C})}$$

$$\stackrel{simplify}{=} \frac{S_{BB_1C_1} \cdot S_{A_1CP} \cdot S_{ACC_1}}{S_{B_1C_1P} \cdot S_{AA_1C} \cdot S_{BCC_1}}$$

$$\stackrel{P}{=} \frac{S_{BB_1C_1} \cdot S_{A_1BC} \cdot S_{AA_1B_1} \cdot S_{ACC_1} \cdot S_{AA_1B_1B}}{(-S_{A_1BB_1} \cdot S_{AB_1C_1}) \cdot S_{AA_1C} \cdot S_{BCC_1} \cdot (-S_{AA_1B_1B})}$$

$$\stackrel{simplify}{=} \frac{S_{BB_1C_1} \cdot S_{A_1BC} \cdot S_{AA_1B_1} \cdot S_{ACC_1}}{S_{A_1BB_1} \cdot S_{AB_1C_1} \cdot S_{AA_1C} \cdot S_{BCC_1}}$$

$$\stackrel{2lines}{=} \frac{\overline{B_1C_1} \cdot \overline{OB} \cdot \beta \cdot (-\overline{BC} \cdot \overline{OA_1} \cdot \beta) \cdot \overline{A_1B_1} \cdot \overline{OA} \cdot \beta \cdot (-\overline{AC} \cdot \overline{OC_1} \cdot \beta) \cdot ((2))^4}{(-\overline{A_1B_1} \cdot \overline{OB} \cdot \beta) \cdot \overline{B_1C_1} \cdot \overline{OA} \cdot \beta \cdot \overline{AC} \cdot \overline{OA_1} \cdot \beta \cdot (-\overline{BC} \cdot \overline{OC_1} \cdot \beta) \cdot ((2))^4}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{B_1T}}{\overline{CT}} \stackrel{T}{=} \frac{S_{B_1PQ}}{S_{CPQ}}$$

$$\frac{\overline{B_1S}}{\overline{CS}} \stackrel{S}{=} \frac{S_{BB_1C_1}}{S_{BCC_1}}$$

$$S_{B_1PQ} \stackrel{Q}{=} \frac{-S_{B_1C_1P} \cdot S_{AA_1C}}{S_{AA_1C_1C}}$$

$$S_{CPQ} \stackrel{Q}{=} \frac{S_{A_1CP} \cdot S_{ACC_1}}{-S_{AA_1C_1C}}$$

$$S_{B_1C_1P} \stackrel{P}{=} \frac{-S_{A_1BB_1} \cdot S_{AB_1C_1}}{S_{AA_1B_1B}}$$

$$S_{A_1CP} \stackrel{P}{=} \frac{S_{A_1BC} \cdot S_{AA_1B_1}}{-S_{AA_1B_1B}}$$

$$S_{BCC_1} = -\frac{1}{2}(\overline{BC} \cdot \overline{OC_1} \cdot \beta)$$

$$S_{AA_1C} = \frac{1}{2}(\overline{AC} \cdot \overline{OA_1} \cdot \beta)$$

$$S_{AB_1C_1} = \frac{1}{2}(\overline{B_1C_1} \cdot \overline{OA} \cdot \beta)$$

$$S_{A_1BB_1} = -\frac{1}{2}(\overline{A_1B_1} \cdot \overline{OB} \cdot \beta)$$

$$S_{ACC_1} = -\frac{1}{2}(\overline{AC} \cdot \overline{OC_1} \cdot \beta)$$

$$S_{AA_1B_1} = \frac{1}{2}(\overline{A_1B_1} \cdot \overline{OA} \cdot \beta)$$

$$S_{A_1BC} = -\frac{1}{2}(\overline{BC} \cdot \overline{OA_1} \cdot \beta)$$

$$S_{BB_1C_1} = \frac{1}{2}(\overline{B_1C_1} \cdot \overline{OB} \cdot \beta)$$

2.6 Area Method and Affine Geometry

We shall first discuss briefly the relationship between geometry and algebra, beginning with passages from E. *Artin*'s book "*Geometric Algebra*", [5]:

> We are all familiar with analytic geometry where a point in a plane is described by a pair (x,y) of *real* numbers, a straight line by a linear, a conic by a quadratic equation. Analytic geometry enables us to reduce any elementary geometric problem to a mere algebraic one. The intersection of a straight line and a circle suggests, however, enlarging the system by introducing a new plane whose points are pairs of *complex* numbers. An obvious generalization of this procedure is the following. Let k be a given field; construct a plane whose "points" are the pairs (x,y) of elements of k and define lines by linear equations. ...
>
> A much more fascinating problem is, however, the converse. Given a plane geometry whose objects are the elements of two sets, the set of points and the set of lines; assume that certain axioms of geometric nature are true. Is it possible to find a field k such that the points of our geometry can be described by coordinates from k and the lines by linear equations?

These passages suggest that there are two approaches to defining geometry.

The Algebraic Approach. Starting from a number system $\mathcal{E}$ (usually fields), we can define geometry objects and relations between those objects in the *Cartesian product* $\mathcal{E}^n$ (or $\mathcal{E}^n/E^*$ in projective geometry). In modern geometry, especially in algebraic geometry, this approach indisputably prevails. If we take this approach, then there are only a few differences between algebra and geometry; geometry can be regarded as a part of algebra.

However, the second approach suggested by Artin is more attractive from the point of view of traditional proofs of geometry theorems.

The Geometric Approach. By this approach we mean the one that was used by *Euclid* and *Hilbert*. In the Euclid-Hilbert system, number systems are developed as parts of the geometry. For each model of a *theory of geometry*, we can prove the existence of a number system (usually a field) inherent to that geometry. This field is called the *field associated with* that geometry. That geometry then can be represented as the *Cartesian product* of its *associated field*. Though beautiful and elegant, the Euclid-Hilbert approach is on the other hand a heavy burden to develop.

The axiom systems which we have adopted in this chapter is a mixture of the above approaches. First we take the number systems for granted. On the other hand we use a geometric language instead of an algebraic one. This system is a modification of an axiom system developed by J.Z. Zhang for the purpose of geometry education [40]. It has the advantage of providing simple but also general methods of solving geometric problems, a virtue the algebraic and the geometric approaches do not possess.

2.6.1 Affine Plane Geometry

Affine geometry is the study of *incidence* and *parallelism*. There are two kinds of geometric objects: points and lines. The only basic relation in this geometry is that of *incidence*, i.e., a point A is on a line l, or equivalently, a line l passes through (contains) a point A. Two lines which do not have a point in common are called *parallel* lines. The following is a group of axioms of affine plane geometry [5].

Axiom H.1. *Given two distinct points P and Q, there exists a unique line passing through both P and Q.*

Axiom H.2. *Given a line l and a point P not on l, there exists one and only one line m such that P lies on m and such that m is parallel to l.*

Axiom H.3. *There exist three distinct points A, B, C such that C does not lie on the line passing through A and B.*

Axiom H.4 (Desargues' Axiom). *Let l_1, l_2, l_3 be distinct lines which are either parallel or meet in a point S. Let A, A_1 be points on l_1, B, B_1 points on l_2 and C, C_1 points on l_3 which are distinct from S if our lines meet. We assume line $AB \parallel A_1B_1$ and $BC \parallel B_1C_1$. Show that $AC \parallel A_1C_1$.*

Axiom H.5 (Pascalian Axiom). *Let l and l_1 be two distinct lines, and A, B, C and A_1, B_1, C_1 be distinct points on l and l_1, respectively. If $BC_1 \parallel B_1C$ and $AB_1 \parallel A_1B$, then $AC_1 \parallel A_1C$.*

A geometry in which all the above five axioms hold is called an *affine geometry.*

The above is a geometric approach for defining affine geometry. Now let us start at the other end and give a definition of affine geometry based on the algebraic approach.

Let $\mathcal{E}$ be a field. From $\mathcal{E}$ we can construct a *structure* Ω as follows. Let

$$\tilde{L} = \{(a, b, c) \mid a, b, c \in \mathcal{E},\ a \neq 0 \text{ or } b \neq 0\}.$$

We define a relation $\sim$ in $\tilde{L}$ as: $(a, b, c) \sim (a', b', c')$ if and only if there is a $k \in \mathcal{E}$ such that $k \neq 0$ and $(a, b, c) = (ka', kb', kc')$. It is easy to see that $\sim$ is an equivalence relation. Let $\tilde{L}/\sim$ (the set of all equivalence classes of $\tilde{L}$) be denoted by L. Define $|\Omega|$ to be $\mathcal{E}^2 \cup L$. An element p in $|\Omega|$ is a point if and only if $p \in \mathcal{E}^2$ (i.e., $p = (x, y), x,\ y \in \mathcal{E}$); an element l in $|\Omega|$ is a line if and only if $l \in L$. A point $p = (x, y)$ is on a line $l = (a, b, c)$ if and only if $ax + by + c = 0$. Two lines $l_1 = (a, b, c)$ and $l_2 = (a', b', c')$ are parallel if there exists a $k \in \mathcal{E}$ and $k \neq 0$ such that $a = ka', b = kb'$.

It is easy to check the following theorem.

Theorem 2.43 *Axioms H.1-H.5 are valid in the structure Ω.*

Proof. It can be easily checked that the five axioms are valid in Ω. H4 and H5, particularly, can be proved automatically using Wu's method (Example 121 and Example 346 in [12]).∎

The converse of the above theorem is a much deeper result.

Theorem 2.44 *Every geometry G of the theory H.1-H.5 is isomorphic to a structure Ω with some field $\mathcal{E}$.*

The key step of the proof is to introduce the *segment arithmetic* and hence to introduce the field $\mathcal{E}$ inherent to G. The field $\mathcal{E}$, uniquely determined by geometry G up to isomorphism, is called the *field associated with* geometry G. *Desargues' axiom* makes it possible to introduce a division ring $\mathcal{E}$ and *Pascalian axiom* makes $\mathcal{E}$ a commutative field. Each algebraic rule of operation (e.g., associativity of addition) corresponds to a geometry theorem. The process of introducing number systems in this way is the core of the Euclid-Hilbert approach. For details, see [24], [5], and [36].

2.6.2 Area Method and Affine Geometry

Suppose that the number field $\mathcal{E}$ in the six axioms A.1–A.6 is not the real number field **R** but an arbitrary field. We shall show that these six axioms define an *affine geometry*.

Theorem 2.45 *Show that all the five Axioms H.1-H.5 are consequences of Axioms A.1-A.6.*

Proof. Axiom H.1 follows from Corollary 2.6. Axiom H.3 is a consequence of Axioms A.3 and A.4. For Axiom H.2, see Example 2.13 on page 58. H.4 and H.5 can be proved automatically by our prover. For their proofs, see the following examples. ∎

Example 2.46 (Desargues' Axiom) *SAA_1, SBB_1, and SCC_1 are three distinct lines. If $AB \parallel A_1B_1$ and $AC \parallel A_1C_1$ then $BC \parallel B_1C_1$.*

Take arbitrary points S, A, B, and C.
Take a point A_1 on line SA.
Take the intersection B_1 of line SB and
the line passing through A_1 and parallel to AB.
Take the intersection C_1 of line SC and
the line passing through A_1 and parallel to AC.
Prove that $S_{B_1BC} = S_{C_1BC}$.

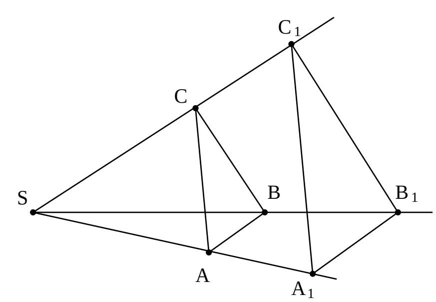


Figure 2-18

The machine proof

$$\frac{S_{BCB_1}}{S_{BCC_1}}$$

$$\overset{C_1}{=} \frac{S_{BCB_1}\cdot S_{SAC}}{S_{ACA_1}\cdot S_{SBC}}$$

$$\overset{B_1}{=} \frac{S_{ABA_1}\cdot S_{SBC}\cdot S_{SAC}}{S_{ACA_1}\cdot S_{SBC}\cdot S_{SAB}}$$

$$\overset{simplify}{=} \frac{S_{ABA_1}\cdot S_{SAC}}{S_{ACA_1}\cdot S_{SAB}}$$

$$\overset{A_1}{=} \frac{(-S_{SAB}\cdot\frac{\overline{SA_1}}{\overline{SA}}+S_{SAB})\cdot S_{SAC}}{(-S_{SAC}\cdot\frac{\overline{SA_1}}{\overline{SA}}+S_{SAC})\cdot S_{SAB}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{BCC_1} \overset{C_1}{=} \frac{S_{ACA_1}\cdot S_{SBC}}{S_{SAC}}$$

$$S_{BCB_1} \overset{B_1}{=} \frac{S_{ABA_1}\cdot S_{SBC}}{S_{SAB}}$$

$$S_{ACA_1} \overset{A_1}{=} -\left(\left(\frac{\overline{SA_1}}{\overline{SA}}-1\right)\cdot S_{SAC}\right)$$

$$S_{ABA_1} \overset{A_1}{=} -\left(\left(\frac{\overline{SA_1}}{\overline{SA}}-1\right)\cdot S_{SAB}\right)$$

The ndg conditions are $S \neq A$, S, A, B and S, A, C are not collinear, which are consequences of the hypotheses of the statement.

Example 2.47 (Pascalian Axiom) *Let A, B and C be three points on one line, and A_1, B_1, C_1 be three points on another line. If $AB_1 \parallel A_1B$ and $AC_1 \parallel A_1C$ then $BC_1 \parallel B_1C$.*

The constructive description.
Take arbitrary points A, B, and A_1.
Take a point C on line AB.
Take a point B_1 such that $B_1A \parallel BA_1$.
Take the intersection C_1 of line A_1B_1 and
the line passing through A and parallel to CA_1.
Prove that $S_{BCB_1} = S_{C_1CB_1}$.

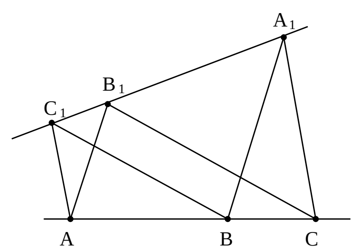


Figure 2-19

The machine proof

$$\frac{S_{BCB_1}}{S_{CB_1C_1}}$$

$$\overset{C_1}{=} \frac{S_{BCB_1}\cdot S_{A_1CB_1}}{-S_{AA_1B_1C}\cdot S_{A_1CB_1}}$$

$$\overset{simplify}{=} \frac{S_{BCB_1}}{-S_{AA_1B_1C}}$$

$$\overset{B_1}{=} \frac{-S_{BA_1C}\cdot\frac{\overline{AB_1}}{\overline{BA_1}}}{-S_{BA_1C}\cdot\frac{\overline{AB_1}}{\overline{BA_1}}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{CB_1C_1} \overset{C_1}{=} -S_{AA_1B_1C}$$

$$S_{AA_1B_1C} \overset{B_1}{=} S_{BA_1C}\cdot\frac{\overline{AB_1}}{\overline{BA_1}}$$

$$S_{BCB_1} \overset{B_1}{=} -\left(S_{BA_1C}\cdot\frac{\overline{AB_1}}{\overline{BA_1}}\right)$$

The ndg conditions are $A \neq B$, $B \neq A_1$, and $A_1B_1 \nparallel CA_1$ which are all in the statements of H.5.

Now we have the converse theorem.

Theorem 2.48 *In the affine geometry associated with any field $\mathcal{E}$, we can define length ratios and areas such that Axioms A.1-A.6 are valid.*

Proof. Let $P_i = (x_i, y_i)$, $i = 1, \cdots, 4$, be four points on a line l such that $P_3 \neq P_4$. Then

$$\frac{\overline{P_1P_2}}{\overline{P_3P_4}} = \begin{cases} \frac{x_1-x_2}{x_3-x_4} & \text{if } y_3 = y_4. \\ \frac{y_1-y_2}{y_3-y_4} & \text{otherwise.} \end{cases}$$

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, be three points. Then define

$$S_{P_1P_2P_3} = k \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where k is any nonzero element in $\mathcal{E}$. Axioms A.1-A.6 can be verified by direct calculation. ▮

Now it is clear that Algorithm 2.32 is for the constructive statements not only in Euclidean geometry but also in affine geometry associated with any field, even finite fields. In other words, the area method works also for *finite geometries*. For examples related to various fields, finite or infinite, see Subsection 2.7.2.

The completeness of Algorithm 2.32 is based on Proposition 2.33. For an arbitrary field $\mathcal{E}$, we have.

Proposition 2.49 *Let $\mathcal{E}$ be an infinite field and P a nonzero polynomial of indeterminates $x_1, \cdots, x_n$ with coefficients in $\mathcal{E}$. Show that we can find elements $e_1, \cdots, e_n$ in $\mathcal{E}$ such that $P(e_1, \cdots, e_n) \neq 0$.*

Proof. We prove the result by induction on n. If $n = 1$, let $P(x_1)$ be of degree d. Then $P(x_1)$ has at most d different roots. Since $\mathcal{E}$ is infinite in any $d + 1$ distinct elements of $\mathcal{E}$ there exists one which is not a root of $P(x_1)$. Suppose that the result is true for $n - 1$. We write P as follows

$$P(x_1, ..., x_n) = a_s(x_1, ..., x_{n-1})x_n^s + ... + a_o(x_1, ..., x_{n-1}).$$

If $s = 0$, we need do nothing. If $s > 0$, by the induction hypotheses there are elements $e_1, .., e_{n-1}$ in $\mathcal{E}$ such that $a_s(e_1, ..., e_{n-1}) \neq 0$. Let $Q(x_n) = P(e_1, ..., e_{n-1}, x_n) \neq 0$. Now the result can be proved similarly to the case $n = 1$. ▮

If $\mathcal{E}$ is a finite field, then the above result is false and we do not know whether there exist *efficient* algorithms to check the existence of such elements. Obviously, a slow algorithm exists: we can check all possible elements in $\mathcal{E}^n$ since $\mathcal{E}$ is finite.

Note that the area is not an invariant in the affine geometry. But due to the following fact the ratios of areas are invariants.

Exercise 2.50 *Let M be a 2×2 matrix, $P_i \in \mathcal{E}^2$, $i = 1, 2, 3$. Let $Q_i = P_i M$, $i = 1, 2, 3$. Then*

$$S_{Q_1Q_2Q_3} = |M| S_{P_1P_2P_3}$$

where $|M|$ is the determinant of M.

So we can use ratios of areas instead of areas as geometry quantities. Also it is worth mentioning that in the proofs of all the examples in this and the preceding chapters, the areas always occur in the form of ratios. This is not a coincidence. Let $C(r_1, \cdots, r_d, a_1, \cdots, a_s) = 0$ be the conclusion of a geometry theorem where the r_i are length ratios and the a_i are areas of triangles. Let $M = \lambda I$ be the multiplication of an indeterminate λ and the unit matrix I. After transforming each point P in the plane to PM, $C = 0$ is still valid. By Exercise 2.50, $C = (r_1, \cdots, r_d, \lambda^2 a_1, \cdots, \lambda^2 a_s) = 0$. Therefore, if $\mathcal{E}$ is an infinite field, each homogeneous component of P in the variables $a_1, \cdots, a_s$ must be zero, i.e., without loss of generality we can assume P is homogeneous in the area variables. That is C can be expressed as a polynomial of the ratio of lengths and the ratio of areas.

2.7 Applications

Besides theorem proving, the area method can be used to solve other geometry problems such as deriving unknown formulas automatically. In this section, we will show the application of the area method in three geometry topics: the formula derivation, the existence of n_3 configurations, and the transversal problems.

2.7.1 Formula Derivation

Algorithm 2.32 can be used to derive unknown formulas. We use a simple example to illustrate how this works.

Example 2.51 *Let L, M, and N be the midpoints of the sides AB, BC, and CA of triangle ABC respectively. Find the area of triangle LMN.*

Solution. Since N is the midpoint of AC, by Proposition 2.9, $S_{LMN}=\frac{1}{2}(S_{CLM} + S_{ALM})$. By the co-side theorem, $S_{ALM}=-\frac{1}{2}(S_{ACL})$, $S_{CLM}=\frac{1}{2}(S_{BCL})$. Then $S_{LMN}=\frac{\frac{1}{2}S_{BCL}-\frac{1}{2}S_{ACL}}{2}$ $= \frac{S_{ABC}}{4}$. ∎

Example 2.52 *Let A_1, B_1, and C_1 be points on the sides BC, CA, and AB of a triangle ABC such that $BA_1/A_1C = r_1$, $CB_1/B_1A = r_2$, and $AC_1/C_1B = r_3$. Show that $\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{r_3r_2r_1+1}{(r_1+1)(r_2+1)(r_3+1)}$.*

Constructive description
Take arbitrary points A, B, and C.
Take a point A_1 such that $\frac{\overline{BA_1}}{\overline{A_1C}} = r_1$.
Take a point B_1 such that $\frac{\overline{CB_1}}{\overline{B_1A}} = r_2$.
Take a point C_1 such that $\frac{\overline{AC_1}}{\overline{C_1B}} = r_3$.
Compute $\frac{S_{A_1B_1C_1}}{S_{ABC}}$

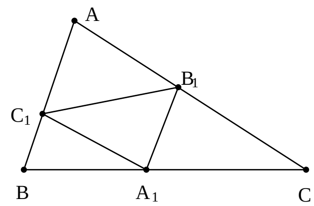


Figure 2-20

The machine derivation.

$$\frac{S_{A_1B_1C_1}}{S_{ABC}}$$

$$\stackrel{C_1}{=} \frac{S_{BA_1B_1}\cdot r_3+S_{AA_1B_1}}{S_{ABC}\cdot(r_3+1)}$$

$$\stackrel{B_1}{=} \frac{-S_{ACA_1}\cdot r_2-S_{ACA_1}+S_{ABA_1}\cdot r_3\cdot r_2^2+S_{ABA_1}\cdot r_3\cdot r_2}{S_{ABC}\cdot(r_3+1)\cdot(r_2+1)^2}$$

$$\stackrel{simplify}{=} \frac{-(S_{ACA_1}-S_{ABA_1}\cdot r_3\cdot r_2)}{S_{ABC}\cdot(r_3+1)\cdot(r_2+1)}$$

$$\stackrel{A_1}{=} \frac{-(-S_{ABC}\cdot r_3\cdot r_2\cdot r_1^2-S_{ABC}\cdot r_3\cdot r_2\cdot r_1-S_{ABC}\cdot r_1-S_{ABC})}{S_{ABC}\cdot(r_3+1)\cdot(r_2+1)\cdot(r_1+1)^2}$$

$$\stackrel{simplify}{=} \frac{r_3\cdot r_2\cdot r_1+1}{(r_3+1)\cdot(r_2+1)\cdot(r_1+1)}$$

The eliminants

$$S_{A_1B_1C_1} \stackrel{C_1}{=} \frac{S_{BA_1B_1}\cdot r_3+S_{AA_1B_1}}{r_3+1}$$

$$S_{AA_1B_1} \stackrel{B_1}{=} \frac{-S_{ACA_1}}{r_2+1}$$

$$S_{BA_1B_1} \stackrel{B_1}{=} \frac{S_{ABA_1}\cdot r_2}{r_2+1}$$

$$S_{ABA_1} \stackrel{A_1}{=} \frac{S_{ABC}\cdot r_1}{r_1+1}$$

$$S_{ACA_1} \stackrel{A_1}{=} \frac{-S_{ABC}}{r_1+1}$$

Remark. As a consequence of Example 2.52, we “discover” Menelaus’ theorem: A_1, B_1, and C_1 are collinear iff $r_1r_2r_3 = -1$.

Example 2.53 *Let A_1, B_1, C_1, D_1 be points on the sides CD, DA, AB, BC of a parallelogram $ABCD$ such that $CA_1/CD = DB_1/DA = AC_1/AB = BD_1/BC = r$. Let $A_2B_2C_2D_2$ be the quadrilateral formed by the lines AA_1 BB_1, CC_1, DD_1. Compute $\frac{S_{ABA_2}}{S_{ABCD}}$ and $\frac{S_{A_2B_2C_2D_2}}{S_{ABCD}}$.*

The constructive description
Take arbitrary points A, B, and C.
Take a point D such that $\frac{\overline{AB}}{\overline{DC}} = 1$.
Take a point A_1 such that $\frac{\overline{CA_1}}{\overline{CD}} = r$.
Take a point B_1 such that $\frac{\overline{DB_1}}{\overline{DA}} = r$.
$A_2 = AA_1 \cap BB_1$.
Compute $\frac{S_{ABA_2}}{S_{ABCD}}$.

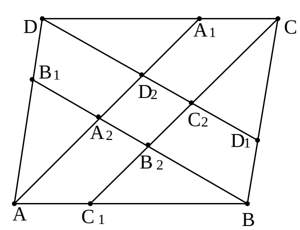


Figure 2-21

The machine derivation.

$\frac{S_{ABA_2}}{S_{ABCD}}$

$\stackrel{A_2}{=} \frac{S_{ABB_1} \cdot S_{ABA_1}}{S_{ABCD} \cdot S_{ABA_1B_1}}$

$\stackrel{B_1}{=} \frac{(-S_{ABD} \cdot r + S_{ABD}) \cdot S_{ABA_1}}{S_{ABCD} \cdot (S_{ADA_1} \cdot r - S_{ADA_1} + S_{ABA_1})}$

$\stackrel{simplify}{=} \frac{-(r-1) \cdot S_{ABD} \cdot S_{ABA_1}}{S_{ABCD} \cdot (S_{ADA_1} \cdot r - S_{ADA_1} + S_{ABA_1})}$

$\stackrel{A_1}{=} \frac{-(r-1) \cdot S_{ABD} \cdot (S_{ABD} \cdot r - S_{ABC} \cdot r + S_{ABC})}{S_{ABCD} \cdot (S_{ACD} \cdot r^2 - 2S_{ACD} \cdot r + S_{ACD} + S_{ABD} \cdot r - S_{ABC} \cdot r + S_{ABC})}$

$\stackrel{D}{=} \frac{-(r-1) \cdot (S_{ABC})^2}{(2S_{ABC}) \cdot (S_{ABC} \cdot r^2 - 2S_{ABC} \cdot r + 2S_{ABC})}$

$\stackrel{simplify}{=} \frac{-(r-1)}{(2) \cdot (r^2 - 2r + 2)}$

The eliminants

$S_{ABA_2} \stackrel{A_2}{=} \frac{S_{ABB_1} \cdot S_{ABA_1}}{S_{ABA_1B_1}}$

$S_{ABA_1B_1} \stackrel{B_1}{=} S_{ADA_1} \cdot r - S_{ADA_1} + S_{ABA_1}$

$S_{ABB_1} \stackrel{B_1}{=} -((r-1) \cdot S_{ABD})$

$S_{ADA_1} \stackrel{A_1}{=} (r-1) \cdot S_{ACD}$

$S_{ABA_1} \stackrel{A_1}{=} S_{ABD} \cdot r - S_{ABC} \cdot r + S_{ABC}$

$S_{ACD} \stackrel{D}{=} S_{ABC}$

$S_{ABCD} \stackrel{D}{=} 2(S_{ABC})$

$S_{ABD} \stackrel{D}{=} S_{ABC}$

Thus $\frac{S_{ABA_2}}{S_{ABCD}} = \frac{1-r}{2(r^2-2r+2)}$. To compute $\frac{S_{A_2B_2C_2D_2}}{S_{ABCD}}$, we have

$$\begin{aligned} S_{A_2B_2C_2D_2} &= S_{ABCD} - S_{ABA_2} - S_{BCB_2} - S_{CDC_2} - S_{DAD_2} \\ &= (1 - 4 \cdot \frac{1-r}{2(r^2-2r+2)}) S_{ABCD} \\ &= \frac{r^2}{r^2 - 2r + 2} S_{ABCD} \end{aligned}$$

Example 2.54 *Let E, F, H, and G be points on sides AB, CD, AD, and BC such that $\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{DF}}{\overline{DC}} = r_1$ and $\frac{\overline{AH}}{\overline{AD}} = \frac{\overline{BG}}{\overline{BC}} = r_2$. Let EF and HG meet in I. Compute $\frac{\overline{EI}}{\overline{EF}}$ and $\frac{\overline{HI}}{\overline{HG}}$.*

Constructive description
Take arbitrary points A, B, C, D.
Take a point E such that $\frac{\overline{AE}}{\overline{AB}} = r_1$.
Take a point F such that $\frac{\overline{DF}}{\overline{DC}} = r_1$.
Take a point H such that $\frac{\overline{AH}}{\overline{AD}} = r_2$.
Take a point G such that $\frac{\overline{BG}}{\overline{BC}} = r_2$.
$I = EF \cap HG$.
Compute $\frac{\overline{HI}}{\overline{GI}}$.

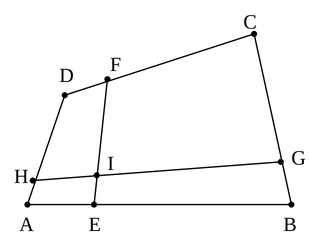


Figure 2-22

The machine proof for this example is a little long. The following proof is the modification of the machine proof.

$\frac{\overline{HI}}{\overline{GI}}$

$= \frac{S_{HEF}}{S_{GEF}}$

$= \frac{r_2 \cdot S_{DEF} + (1-r_2) \cdot S_{AEF}}{r_2 \cdot S_{CEF} + (1-r_2) \cdot S_{BEF}}$

$$= \frac{r_2 \cdot r_1 \cdot S_{DEC} + (1-r_2) \cdot r_1 \cdot S_{ABF}}{r_2 \cdot (1-r_1) \cdot S_{DEC} + (1-r_2) \cdot (1-r_1) \cdot S_{ABF}}$$

$$\stackrel{simplify}{=} \frac{r_1}{1-r_1}$$

Thus $\frac{\overline{HI}}{\overline{EG}} = \frac{\frac{\overline{HI}}{\overline{IG}}}{\frac{\overline{HI}}{\overline{IG}}+1} = r_1$. Similarly $\frac{\overline{EI}}{\overline{EF}} = r_2$.

Example 2.55 *The sides AB and DC of a quadrilateral are cut into* $2n + 1$ *equal segments by points* $P_1, \cdots, P_{2n}$ *and* $Q_1, \cdots, Q_{2n}$ *respectively. Show that*

(1) $S_{P_n P_{n+1} Q_{n+1} Q_n} = \frac{1}{2n+1} S_{ABCD}$.

(2) *If sides BC and AD are cut into* $2m + 1$ *equal segments by points* $R_1, \cdots, R_{2m}$ *and* $S_1, \cdots, S_{2m}$ *respectively, then the area of the quadrilateral formed by the lines* $P_n Q_n$, $P_{n+1} Q_{n+1}$, $R_m S_m$, *and* $R_{m+1} S_{m+1}$ *is* $\frac{1}{(2n+1)(2m+1)} S_{ABCD}$.

Figure 2-23 shows the case $n = m = 2$. Note that in the following machine proof for (1), we use some different names for points $P_n, P_{n+1}, Q_{n+1}, Q_n$. We also use a trick: point Q_{n+1} is introduced two times (Q and V) so that different elimination methods will be used to eliminate Q_{n+1} in different cases.

Constructive description

Take arbitrary points A, B, C, D.

Take a point X such that $\frac{\overline{AX}}{\overline{AB}} = \frac{n}{2n+1}$.

Take a point U such that $\frac{\overline{DU}}{\overline{DC}} = \frac{n}{2n+1}$.

Take a point Q such that $\frac{\overline{DQ}}{\overline{DC}} = \frac{n+1}{2n+1}$.

Take a point V such that $\frac{\overline{UV}}{\overline{DC}} = \frac{1}{2n+1}$.

Take a point Y such that $\frac{\overline{XY}}{\overline{AB}} = \frac{1}{2n+1}$.

Compute $\frac{(S_{AXY}+S_{UXV})}{S_{ABCD}}$.

The eliminants

$S_{XQY} \stackrel{Y}{=} \frac{-S_{ABQ}}{2n+1}$

$S_{XUV} \stackrel{V}{=} \frac{-S_{CDX}}{2n+1}$

$S_{ABQ} \stackrel{Q}{=} \frac{S_{ABD} \cdot n + S_{ABC} \cdot n + S_{ABC}}{2n+1}$

$S_{CDX} \stackrel{X}{=} \frac{S_{BCD} \cdot n + S_{ACD} \cdot n + S_{ACD}}{2n+1}$

$S_{ABCD} = S_{ACD} + S_{ABC}$

$S_{BCD} = S_{ACD} - S_{ABD} + S_{ABC}$

The machine proof

$\frac{-(S_{XQY}+S_{XUV})}{S_{ABCD}}$

$\stackrel{Y}{=} \frac{-(2S_{XUV} \cdot n + S_{XUV} - S_{ABQ})}{S_{ABCD} \cdot (2n+1)}$

$\stackrel{V}{=} \frac{-(-2S_{CDX} \cdot n - S_{CDX} - 2S_{ABQ} \cdot n - S_{ABQ})}{S_{ABCD} \cdot (2n+1)^2}$

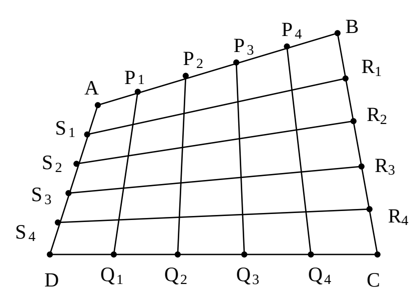


Figure 2-23

$$\overset{simplify}{=} \frac{S_{CDX}+S_{ABQ}}{S_{ABCD}\cdot(2n+1)}$$

$$\overset{Q}{=} \frac{2S_{CDX}\cdot n+S_{CDX}+S_{ABD}\cdot n+S_{ABC}\cdot n+S_{ABC}}{S_{ABCD}\cdot(2n+1)^2}$$

$$\overset{X}{=} \frac{2S_{BCD}\cdot n^2+S_{BCD}\cdot n+2S_{ACD}\cdot n^2+3S_{ACD}\cdot n+S_{ACD}+2S_{ABD}\cdot n^2+S_{ABD}\cdot n+2S_{ABC}\cdot n^2+3S_{ABC}\cdot n+S_{ABC}}{S_{ABCD}\cdot(2n+1)^3}$$

$$\overset{simplify}{=} \frac{S_{BCD}\cdot n+S_{ACD}\cdot n+S_{ACD}+S_{ABD}\cdot n+S_{ABC}\cdot n+S_{ABC}}{S_{ABCD}\cdot(2n+1)^2}$$

$$\overset{area-co}{=} \frac{2S_{ACD}\cdot n+S_{ACD}+2S_{ABC}\cdot n+S_{ABC}}{(S_{ACD}+S_{ABC})\cdot(2n+1)^2}$$

$$\overset{simplify}{=} \frac{1}{2n+1}$$

By Example 2.54, P_nQ_n and $P_{n+1}Q_{n+1}$ are cut into $2m+1$ equal segments by R_iS_i, $i = 1, ..., 2m$ respectively. Now (2) comes from (1) directly. ∎

For more examples of formula derivation, see Examples 1.11, 1.12, 1.13, and the examples in the next subsection.

2.7.2 Existence of n_3 Configurations

We define a plane *configuration* as a system of p points and l lines arranged in a plane in such a way that every point of the system is incident with a fixed number λ of lines of the system and every straight line of the system is incident with a fixed number π of points of the system. We characterize such a configuration by the symbol (p_λ, l_π). For example, the triangle forms the configuration of $(3_2, 3_2)$. The four numbers p, l, λ, and π may not be chosen arbitrarily. For, by the conditions we have stipulated, λp straight lines of the system, in all, pass through the p points; however, every line is counted π times because it passes through π points; thus the number of lines l is equal to $\frac{\lambda p}{\pi}$. Therefore, for every configuration (p_λ, l_π), we have $\lambda p = \pi l$.

We only discuss those configurations in which the number of points is equal to the number of lines, i.e., for which $p = l$. Then it follows from the relation $\lambda p = \pi l$, that $\lambda = \pi$. The symbol for such a configuration is always of the form (p_λ, p_λ). We shall introduce the more concise notation (p_λ) for such a configuration.

We shall further limit the number λ. $\lambda = 1$ yields only the trivial configuration consisting of a point and a line passing through it. The case $\lambda = 2$ is realized by the closed polygons in the plane. On the other hand, the case $\lambda = 3$ includes the most important configurations in projective geometry, the Fano configuration, Desargues' configuration, and Pappus' configuration. In this case the number of points, p, must be at least seven. For through any given point of the configuration there pass three lines, on each of which there must be two further points of the configuration. If in a geometry, there exist p points and lines consisting of a (p_3) configuration, we say that the configuration (p_3) can be *realized* in that geometry.

As an application of the area method, we obtain the sufficient and necessary conditions for the existence of the (7_3), (8_3), and (9_3) configurations. For more complicated configurations, see [152].

Example 2.56 *There exists only one* (7_3) *configuration and this configuration can only be realized in the geometry whose associated field* $\mathcal{E}$ *is of characteristic 2. (Fano plane)*

Proof. Let the seven points be $P_i, i = 1, \cdots, 7$. The only possible (7_3) configuration consists of the lines:

$$\begin{array}{ccccccc} P_1 & P_1 & P_2 & P_2 & P_3 & P_3 & P_1 \\ P_2 & P_4 & P_4 & P_5 & P_4 & P_5 & P_6 \\ P_3 & P_5 & P_6 & P_7 & P_7 & P_6 & P_7 \end{array}$$

Consider the following geometry problem.
Take arbitrary points P_1, P_2, P_4.
Take a point P_3 on line P_1P_2.
Take a point P_5 on line P_1P_4.
$P_6 = P_2P_4 \cap P_3P_5$.
$P_7 = P_2P_5 \cap P_3P_4$.
Compute $S_{P_1P_6P_7}$.

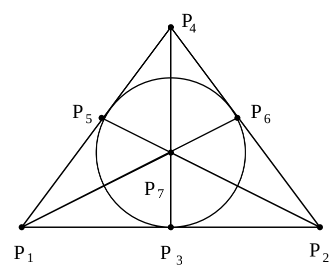


Figure 2-24

Using Algorithm 2.32, we have

$$S_{P_1P_6P_7} = \frac{(2) \cdot (\frac{\overline{P_1P_5}}{\overline{P_1P_4}} - 1) \cdot (\frac{\overline{P_1P_3}}{\overline{P_1P_2}} - 1) \cdot \frac{\overline{P_1P_3}}{\overline{P_1P_2}} \cdot S_{P_1P_2P_4} \cdot \frac{\overline{P_1P_5}}{\overline{P_1P_4}}}{(\frac{\overline{P_1P_5}}{\overline{P_1P_4}} \cdot \frac{\overline{P_1P_3}}{\overline{P_1P_2}} - 1) \cdot (\frac{\overline{P_1P_5}}{\overline{P_1P_4}} - \frac{\overline{P_1P_3}}{\overline{P_1P_2}})}.$$

The (7_3) configuration exists iff $S_{P_1P_6P_7} = 0$, i.e.

$$(2) \cdot (\frac{\overline{P_1P_5}}{\overline{P_1P_4}} - 1) \cdot (\frac{\overline{P_1P_3}}{\overline{P_1P_2}} - 1) \cdot \frac{\overline{P_1P_3}}{\overline{P_1P_2}} \cdot S_{P_1P_2P_4} \cdot \frac{\overline{P_1P_5}}{\overline{P_1P_4}} = 0.$$

If $\frac{\overline{P_1P_5}}{\overline{P_1P_4}} - 1 = 0$, we have $P_4 = P_5$. If $\frac{\overline{P_1P_3}}{\overline{P_1P_2}} - 1 = 0$, we have $P_2 = P_3$. If $\frac{\overline{P_1P_3}}{\overline{P_1P_2}} = 0$, we have $P_1 = P_3$. If $S_{P_1P_2P_4} = 0$, we have P_1, P_2, and P_4 are collinear. If $\frac{\overline{P_1P_5}}{\overline{P_1P_4}} = 0$, we have $P_1 = P_5$. All the above cases will lead to degenerate configurations. Then the (7_3) configuration exists iff $2 = 0$, i.e., the associate field of the geometry is of characteristic 2. ∎

Configuration (8_3) also has only one possible table:

$$\begin{array}{cccccccc} P_1 & P_1 & P_1 & P_2 & P_2 & P_3 & P_3 & P_4 \\ P_2 & P_4 & P_6 & P_3 & P_7 & P_4 & P_5 & P_5 \\ P_5 & P_8 & P_7 & P_6 & P_8 & P_7 & P_8 & P_6 \end{array}$$

Theorem 2.57 *The* (8_3) *configuration only exists in the geometry such that* $\sqrt{-3}$ *belongs to* $\mathcal{E}$*, the field associated with the geometry.*

Proof. Consider the following geometry problem.

Take arbitrary points P_1, P_2, P_4.
Take a point P_5 such that $\frac{\overline{P_1P_5}}{\overline{P_1P_2}} = r_1$.
Take a point P_6 such that $\frac{\overline{P_4P_6}}{\overline{P_4P_5}} = r_2$.
Take a point P_8 such that $\frac{\overline{P_1P_8}}{\overline{P_1P_4}} = r_3$.
$P_7 = P_2P_8 \cap P_1P_6$. $P_3 = P_2P_6 \cap P_5P_8$.
Compute $S_{P_3P_7P_4}$.

Using Algorithm 2.32, $S_{P_3P_7P_4}$ is found to be

$$\frac{(r_3^2(r_2r_2 + r_2r_1 - 2r_2 + r_1r_1 - r_1 + 1) - r_3r_1(r_2 - 2r_1 + 1) + r_1^2)S_{P_1P_2P_3}(r_1 - 1)r_2}{(r_3r_2r_1 - r_2r_1 + r_2 - 1)(r_3(r_2r_1 - r_2 - r_1 + 1) + r_1)}.$$

Then $S_{P_3P_7P_4} = 0$ iff

$$r_3^2(r_2r_2 + r_2r_1 - 2r_2 + r_1r_1 - r_1 + 1) - r_3r_1(r_2 - 2r_1 + 1) + r_1^2 = 0$$

has solutions for r_3. The discriminant of the quadratic equation is $-3(r_2 - 1)^2r_1^2$, and therefore the result. ∎

Contrary to the (7_3) and (8_3) configurations which do not exist in the Euclidean plane, the case $n = 9$ gives rise to three essentially different configurations, all of which can be realized in the Euclidean plane. The first (9_3) configuration is that related to the Pappus theorem, a machine proof of which can be found in Example 2.42 on page 80.

Example 2.58 *Prove the existence of the* (9_3) *configuration as shown in Figure 2-25.*

Proof. Consider the following geometry problem.

Take arbitrary points P_1, P_3, and P_5.
Take a point P_7 such that $\frac{\overline{P_1P_7}}{\overline{P_1P_3}} = r_1$.
Take a point P_8 such that $\frac{\overline{P_1P_8}}{\overline{P_1P_5}} = r_2$.
Take a point P_9 such that $\frac{\overline{P_3P_9}}{\overline{P_3P_5}} = r_3$.
Take a point P_2 such that $\frac{\overline{P_5P_2}}{\overline{P_5P_7}} = r_4$.
$P_4 = P_1P_9 \cap P_2P_8$.
$P_6 = P_3P_8 \cap P_2P_9$.
Compute $S_{P_4P_6P_7}$.

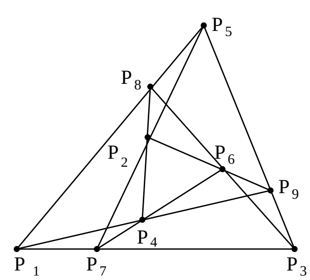


Figure 2-25

Using Algorithm 2.32, $S_{P_4P_6P_7}$ is found to be

$$\frac{(r_4(r_3r_2 + r_3r_1 - 2r_3 - r_2r_1 - r_2 + 2) - 2r_3r_2 + 2r_3 + 2r_2 - 2)r_1r_2(1 - r_1)r_3r_4S_{P_1P_3P_5}}{(r_4r_2r_1 - r_4 + r_3r_2 - r_3 - r_2 + 1)(r_4r_3r_1 - r_4r_3 + r_4 - r_3r_2 + r_3 + r_2 - 1)}.$$

Then $S_{P_4P_6P_7} = 0$ iff $\quad r_4(r_3r_2 + r_3r_1 - 2r_3 - r_2r_1 - r_2 + 2) - 2r_3r_2 + 2r_3 + 2r_2 - 2 = 0,$ or equivalently, iff

$$r_4 = (2r_3r_2 - 2r_3 - 2r_2 + 2)/(r_3r_2 + r_3r_1 - 2r_3 - r_2r_1 - r_2 + 2) \quad ▮$$

Example 2.59 *Show the existence of the* (9_3) *configuration as shown in Figure 2-26.*

Proof. Consider the following geometry problem.

> Take arbitrary points P_1, P_4, and P_7.
> Take a point P_3 such that $\frac{\overline{P_1P_3}}{\overline{P_1P_7}} = r_1$.
> Take a point P_6 such that $\frac{\overline{P_1P_6}}{\overline{P_1P_4}} = r_2$.
> Take a point P_8 such that $\frac{\overline{P_3P_8}}{\overline{P_3P_6}} = r_3$.
> Take a point P_9 such that $\frac{\overline{P_4P_9}}{\overline{P_4P_7}} = r_4$.
> $P_5 = P_1P_8 \cap P_3P_9$.
> $P_2 = P_4P_8 \cap P_6P_9$.
> Compute $S_{P_2P_5P_7}$.

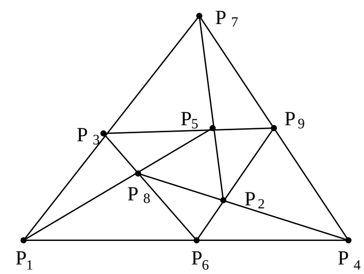


Figure 2-26

Using the program, we have $S_{P_2P_5P_7} = \frac{-f_1 \cdot f_2 \cdot S_{P_1P_4P_7}}{d_1 \cdot d_2}$ where

$$\begin{aligned}
f_1 &= r_4(r_2 - r_1) - r_2r_1 + r_1\\
f_2 &= r_4(r_3^2r_2^2 - r_3^2r_2r_1 + r_3^2r_1^2 - r_3^2r_1 + r_3r_2r_1 - r_3r_2 - 2r_3r_1^2 + 2r_3r_1 + r_1^2 - r_1)\\
&\quad -r_3^2r_1^2 + r_3^2r_1 + 2r_3r_1^2 - 2r_3r_1 - r_1^2 + r_1\\
d_1 &= r_4(r_3(r_2 - r_1) + r_1 - 1) - r_3r_1(r_2 + 1) + r_2r_1 - r_1\\
d_2 &= r_4(r_3(r_2 - r_1) + r_1) - r_3r_1(r_2 + 1) - r_1.
\end{aligned}$$

Using the program again, we may check that $f_1 = 0$ (i.e., $r_4 = (-r_2r_1 + r_1)/(r_1 - r_2)$) implies P_3, P_6, and P_9 are collinear, which is a degenerate case. If $f_2 = 0$, that is,

$$r_4 = \frac{-r_3^2r_1^2 + r_3^2r_1 + 2r_3r_1^2 - 2r_3r_1 - r_1^2 + r_1}{r_3^2r_2^2 - r_3^2r_2r_1 + r_3^2r_1^2 - r_3^2r_1 + r_3r_2r_1 - r_3r_2 - 2r_3r_1^2 + 2r_3r_1 + r_1^2 - r_1}$$

then $S_{P_2P_5P_7} = 0$, and we obtain a realization for the configuration shown in Figure 2-26. ▮

Remark 2.60 *From the above two examples and Example 2.42, the three* 9_3 *configurations can be realized rationally, i.e., they can be realized in the geometry associated with the field of rational numbers.*

2.7.3 Transversals for Polygons

First, Example 1.8 on page 11 can be further generalized to the following form. Notice that in these theorems involving m points, the subscripts are understood to be mod m.

Theorem 2.61 (Ceva's Theorem for an m-polygon) *Let $V_1...V_m$ be an m-polygon , and O a point. Let P_i be the intersection of line OV_i and the side $V_{i+k}V_{i+k+1}$. Then $C(m,k) = \prod_{i=1}^{m} \frac{\overline{V_{i+k}P_i}}{\overline{P_iV_{i+k+1}}} = 1$ iff m is an odd number and $k = \frac{m-1}{2}$.*

Proof. By the co-side theorem,

$$\frac{\overline{V_{i+k}P_i}}{\overline{P_iV_{i+k+1}}} = \frac{S_{OV_iV_{i+k}}}{S_{OV_{i+k+1}V_i}}, \quad i = 1, ..., m.$$

Multiply the above equations together. We have that $C(m,k) = 1$ iff the elements in the numerator are the same as the elements in the denominator. Let us assume that the i-th element in the numerator is the same as the j-th element in the denominator, i.e., $S_{OV_iV_{i+k}} = S_{OV_{j+k+1}V_j}$. Then

$$i = j + k + 1 \ mod\ (m); \quad i + k = j \ mod\ (m).$$

The above two equations have solutions for i and j iff

$$2k + 1 = 0 \ mod\ (m).$$

The only nontrivial solution of the above equation is $2k + 1 = m$ which proves the theorem. ∎

By *polygrams*, we mean the figures formed by the diagonals of polygons. The Menelaus and Ceva type theorems are about the transversals for the sides of polygons. We will discuss some results involving the transversals of polygrams which were discovered by B. Grünbaum and G. C. Shephard using numerical searching ([108]). Using the area method, we can not only prove these results easily but also strengthen some of them.

Example 2.62 *Let $ABCD$ be a quadrilateral and O a point. Let E, F, G, and H be the intersections of lines AO, BO, CO, and DO with the corresponding diagonals of the quadrilateral. Show that $\frac{\overline{AH}}{\overline{HC}}\frac{\overline{CF}}{\overline{FA}}\frac{\overline{BE}}{\overline{ED}}\frac{\overline{DG}}{\overline{GB}} = 1$.*

Constructive description

Take arbitrary points

A, B, C, D, and O.

$E = BD \cap AO$.

$F = AC \cap BO$.

$G = BD \cap CO$.

$H = AC \cap DO$.

Prove that

$\frac{\overline{CF}}{\overline{FA}} \cdot \frac{\overline{BE}}{\overline{ED}} \cdot \frac{\overline{AH}}{\overline{HC}} \cdot \frac{\overline{DG}}{\overline{GB}} = 1.$

The machine proof

$$\frac{\overline{DG}}{\overline{BG}} \cdot \frac{\overline{CF}}{\overline{AF}} \cdot \frac{\overline{BE}}{\overline{DE}} \cdot \frac{\overline{AH}}{\overline{CH}}$$

$$\overset{H}{=} \frac{S_{ADO}}{S_{CDO}} \cdot \frac{\overline{DG}}{\overline{BG}} \cdot \frac{\overline{CF}}{\overline{AF}} \cdot \frac{\overline{BE}}{\overline{DE}}$$

$$\overset{G}{=} \frac{(-S_{CDO}) \cdot S_{ADO}}{S_{CDO} \cdot S_{BCO}} \cdot \frac{\overline{CF}}{\overline{AF}} \cdot \frac{\overline{BE}}{\overline{DE}}$$

$$\overset{simplify}{=} \frac{-S_{ADO}}{S_{BCO}} \cdot \frac{\overline{CF}}{\overline{AF}} \cdot \frac{\overline{BE}}{\overline{DE}}$$

$$\overset{F}{=} \frac{-(-S_{BCO}) \cdot S_{ADO}}{S_{BCO} \cdot S_{ABO}} \cdot \frac{\overline{BE}}{\overline{DE}}$$

$$\overset{simplify}{=} \frac{S_{ADO}}{S_{ABO}} \cdot \frac{\overline{BE}}{\overline{DE}}$$

$$\overset{E}{=} \frac{(-S_{ABO}) \cdot S_{ADO}}{S_{ABO} \cdot (-S_{ADO})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AH}}{\overline{CH}} \overset{H}{=} \frac{S_{ADO}}{S_{CDO}}$$

$$\frac{\overline{DG}}{\overline{BG}} \overset{G}{=} \frac{-S_{CDO}}{S_{BCO}}$$

$$\frac{\overline{CF}}{\overline{AF}} \overset{F}{=} \frac{-S_{BCO}}{S_{ABO}}$$

$$\frac{\overline{BE}}{\overline{DE}} \overset{E}{=} \frac{S_{ABO}}{S_{ADO}}$$

Example 2.62 is a special case of the following result.

Theorem 2.63 *Let an arbitrary polygon $V_1 ... V_m$ and a point O be given, together with a positive integer k such that $1 \le k \le \frac{m}{2}$. Let $P_{i,k}$ be the intersection of line OV_i and line $V_{i-k}V_{i+k}$. Then*

$$\prod_{i=1}^{m} \frac{\overline{V_{i+k}P_{i,k}}}{\overline{P_{i,k}V_{i-k}}} = 1.$$

Proof. By the co-side theorem

$$\frac{\overline{V_{i+k}P_{i,k}}}{\overline{P_{i,k}V_{i-k}}} = \frac{S_{OV_iV_{i+k}}}{S_{OV_{i-k}V_i}}, i = 1, .., m.$$

Multiplying the m equations together and noticing that the elements in the numerator and in the denominator are the same, we prove the result. ∎

The following more intricate extension of Ceva's theorem contains the above example as the special case $j = k$.

Theorem 2.64 *Let an arbitrary polygon $V_1 ... V_m$ and a point O be given, and integers j and k that satisfy $1 \le j \le n-2$, $1 \le k \le n-2$ and $j+k \le n-1$. Let $P_{j,k,i}$ denote the intersection point of the line $V_{i+k}V_{i-j}$ and the line OV_i. We put*

$$C(m, j, k) = \prod_{i=1}^{m} \frac{\overline{V_{i+k}P_{j,k,i}}}{\overline{P_{j,k,i}V_{i-j}}}.$$

Then $C(m, j, k) = (-1)^m C(n, j, m-k) = \frac{1}{C(m,k,j)}$.

Proof. By the co-side theorem

$$C(m, j, k, i) = \frac{\overline{V_{i+k}P_{i,k,j}}}{\overline{P_{i,k,j}V_{i-j}}} = \frac{S_{OV_iV_{i+k}}}{S_{OV_{i-j}V_i}}. \tag{1}$$

Replacing k by $m - k$ in (1), we have

$$C(m, j, m-k, i) = \frac{\overline{V_{i+m-k}P_{i,m-k,j}}}{\overline{P_{i,m-k,j}V_{i-j}}} = \frac{S_{OV_iV_{i+m-k}}}{S_{OV_{i-j}V_i}} = -\frac{S_{OV_{i-k}V_i}}{S_{OV_{i-j}V_i}}. \tag{2}$$

Interchanging k and j in (1), we have

$$C(m, k, j, i) = \frac{S_{OV_iV_{i+j}}}{S_{OV_{i-k}V_i}}. \tag{3}$$

From (1) and (2), it is clear that $C(m, j, k) = (-1)^m C(n, j, m-k)$. From (1) and (3), we have $C(m, j, k) = \frac{1}{C(m,k,j)}$.

Example 2.65 *As shown in Figure 2-28, $ABCDE - PQRST$ is a pentagram. Then* $\frac{\overline{AT}}{\overline{PD}}\frac{\overline{DR}}{\overline{SB}}\frac{\overline{BP}}{\overline{QE}}\frac{\overline{ES}}{\overline{TC}}\frac{\overline{CQ}}{\overline{RA}} = \frac{\overline{AP}}{\overline{TD}}\frac{\overline{DS}}{\overline{RB}}\frac{\overline{BQ}}{\overline{PE}}\frac{\overline{ET}}{\overline{SC}}\frac{\overline{CR}}{\overline{QA}} = 1.$

To prove $\frac{\overline{AT}}{\overline{PD}}\frac{\overline{DR}}{\overline{SB}}\frac{\overline{BP}}{\overline{QE}}\frac{\overline{ES}}{\overline{TC}}\frac{\overline{CQ}}{\overline{RA}} = 1$, we formulate the problem as follows.

Constructive description
Take arbitrary points A, B, C, D, and E.
$P = AD \cap BE$.
$Q = AC \cap BE$.
$R = BD \cap AC$.
$S = BD \cap CE$.
$T = AD \cap CE$.
Show that
$\frac{\overline{AT}}{\overline{AD}}\frac{\overline{DR}}{\overline{DB}}\frac{\overline{BP}}{\overline{BE}}\frac{\overline{ES}}{\overline{EC}}\frac{\overline{CQ}}{\overline{CA}} = \frac{\overline{PD}}{\overline{AD}}\frac{\overline{SB}}{\overline{DB}}\frac{\overline{QE}}{\overline{BE}}\frac{\overline{TC}}{\overline{EC}}\frac{\overline{RA}}{\overline{CA}}$

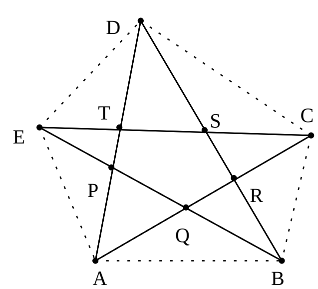


Figure 2-28

The machine proof

$$\frac{-1}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot\frac{\overline{CT}}{\overline{CE}}\cdot\frac{\overline{BS}}{\overline{BD}}\cdot\frac{\overline{AR}}{\overline{AC}}}\cdot\frac{\overline{ES}}{\overline{CE}}\cdot\frac{\overline{DR}}{\overline{BD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}\cdot\frac{\overline{AT}}{\overline{AD}}$$

$$\overset{T}{=}\frac{-S_{ACE}\cdot(-S_{ACDE})}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot(-S_{ACD})\cdot\frac{\overline{BS}}{\overline{BD}}\cdot\frac{\overline{AR}}{\overline{AC}}\cdot S_{ACDE}}\cdot\frac{\overline{ES}}{\overline{CE}}\cdot\frac{\overline{DR}}{\overline{BD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{simplify}{=}\frac{-S_{ACE}}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{ACD}\cdot\frac{\overline{BS}}{\overline{BD}}\cdot\frac{\overline{AR}}{\overline{AC}}}\cdot\frac{\overline{ES}}{\overline{CE}}\cdot\frac{\overline{DR}}{\overline{BD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{S}{=}\frac{-S_{BDE}\cdot S_{ACE}\cdot S_{BCDE}}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{ACD}\cdot S_{BCE}\cdot\frac{\overline{AR}}{\overline{AC}}\cdot(-S_{BCDE})}\cdot\frac{\overline{DR}}{\overline{BD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{simplify}{=}\frac{S_{BDE}\cdot S_{ACE}}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{ACD}\cdot S_{BCE}\cdot\frac{\overline{AR}}{\overline{AC}}}\cdot\frac{\overline{DR}}{\overline{BD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{R}{=}\frac{S_{BDE}\cdot S_{ACD}\cdot S_{ACE}\cdot S_{ABCD}}{\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{ACD}\cdot S_{BCE}\cdot S_{ABD}\cdot(-S_{ABCD})}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{simplify}{=}\frac{S_{BDE}\cdot S_{ACE}}{-\frac{\overline{EQ}}{\overline{BE}}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{BCE}\cdot S_{ABD}}\cdot\frac{\overline{CQ}}{\overline{AC}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{Q}{=}\frac{S_{BDE}\cdot(-S_{BCE})\cdot S_{ACE}\cdot(-S_{ABCE})}{-S_{ACE}\cdot\frac{\overline{DP}}{\overline{AD}}\cdot S_{BCE}\cdot S_{ABD}\cdot S_{ABCE}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{simplify}{=}\frac{-S_{BDE}}{\frac{\overline{DP}}{\overline{AD}}\cdot S_{ABD}}\cdot\frac{\overline{BP}}{\overline{BE}}$$

$$\overset{P}{=}\frac{-S_{BDE}\cdot(-S_{ABD})\cdot S_{ABDE}}{(-S_{BDE})\cdot S_{ABD}\cdot(-S_{ABDE})}$$

$$\overset{simplify}{=}1$$

The eliminants

$$\frac{\overline{CT}}{\overline{CE}}\overset{T}{=}\frac{S_{ACD}}{S_{ACDE}}$$

$$\frac{\overline{AT}}{\overline{AD}}\overset{T}{=}\frac{S_{ACE}}{S_{ACDE}}$$

$$\frac{\overline{BS}}{\overline{BD}}\overset{S}{=}\frac{S_{BCE}}{S_{BCDE}}$$

$$\frac{\overline{ES}}{\overline{CE}}\overset{S}{=}\frac{S_{BDE}}{-S_{BCDE}}$$

$$\frac{\overline{AR}}{\overline{AC}}\overset{R}{=}\frac{S_{ABD}}{S_{ABCD}}$$

$$\frac{\overline{DR}}{\overline{BD}}\overset{R}{=}\frac{S_{ACD}}{-S_{ABCD}}$$

$$\frac{\overline{EQ}}{\overline{BE}}\overset{Q}{=}\frac{S_{ACE}}{-S_{ABCE}}$$

$$\frac{\overline{CQ}}{\overline{AC}}\overset{Q}{=}\frac{-S_{BCE}}{S_{ABCE}}$$

$$\frac{\overline{DP}}{\overline{AD}}\overset{P}{=}\frac{-S_{BDE}}{S_{ABDE}}$$

$$\frac{\overline{BP}}{\overline{BE}}\overset{P}{=}\frac{S_{ABD}}{S_{ABDE}}$$

The second result in Example 2.65 is equivalent to the following statement.

Example 2.66 (Theorem of Pratt-Kasapi) *Let ABCDE be a pentagon.* $A_1B_1 \parallel AC$, $B_1C_1 \parallel BD$, $C_1D_1 \parallel CE$, $D_1E_1 \parallel AD$, $E_1A_1 \parallel EB$. *Show that* $A_1B \cdot B_1C \cdot C_1D \cdot D_1E \cdot E_1A = BB_1 \cdot CC_1 \cdot DD_1 \cdot EE_1 \cdot AA_1$.

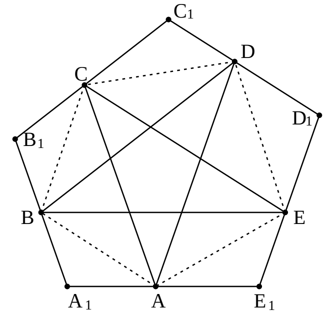


Figure 2-29

The constructive description.

Take arbitrary points A, B, C, D, and E.
Take the intersection A_1 of the line passing through B and parallel to CA and the line passing through A and parallel to EB.
Take the intersection B_1 of the line passing through C and parallel to BD and the line passing through B and parallel to AC.
Take the intersection C_1 of the line passing through D and parallel to CE and the line passing through C and parallel to BD.
Take the intersection D_1 of the line passing through E and parallel to AD and the line passing through D and parallel to CE.
Take the intersection E_1 of the line passing through A and parallel to BE and the line passing through E and parallel to AD.
Show that $\frac{\overline{A_1B}}{\overline{BB_1}}\frac{\overline{B_1C}}{\overline{CC_1}}\frac{\overline{C_1D}}{\overline{DD_1}}\frac{\overline{D_1E}}{\overline{EE_1}}\frac{\overline{E_1A}}{\overline{AA_1}} = 1$.

The machine proof

$$-\frac{\overline{ED_1}}{\overline{EE_1}}\cdot\frac{\overline{DC_1}}{\overline{DD_1}}\cdot\frac{\overline{CB_1}}{\overline{CC_1}}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}\cdot\frac{\overline{AE_1}}{\overline{AA_1}}$$

$$\stackrel{E_1}{=} \frac{-(-S_{BED_1})\cdot(-S_{ADE})}{(-S_{ADA_1})\cdot(-S_{ABE})}\cdot\frac{\overline{DC_1}}{\overline{DD_1}}\cdot\frac{\overline{CB_1}}{\overline{CC_1}}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}$$

$$\stackrel{D_1}{=} \frac{-(-S_{ABDE})\cdot S_{CDE}\cdot(-S_{ADC_1})\cdot S_{ADE}}{S_{ADA_1}\cdot S_{ABE}\cdot(-S_{ADE})\cdot S_{ACDE}}\cdot\frac{\overline{CB_1}}{\overline{CC_1}}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ABDE}\cdot S_{CDE}\cdot S_{ADC_1}}{S_{ADA_1}\cdot S_{ABE}\cdot S_{ACDE}}\cdot\frac{\overline{CB_1}}{\overline{CC_1}}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}$$

$$\stackrel{C_1}{=} \frac{S_{ABDE}\cdot S_{CDE}\cdot S_{ACDE}\cdot S_{BCD}\cdot(-S_{CEB_1})}{S_{ADA_1}\cdot S_{ABE}\cdot S_{ACDE}\cdot S_{CDE}\cdot(-S_{BCDE})}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ABDE}\cdot S_{BCD}\cdot S_{CEB_1}}{S_{ADA_1}\cdot S_{ABE}\cdot S_{BCDE}}\cdot\frac{\overline{BA_1}}{\overline{BB_1}}$$

$$\stackrel{B_1}{=} \frac{S_{ABDE}\cdot S_{BCD}\cdot(-S_{BCDE}\cdot S_{ABC})\cdot(-S_{BDA_1})}{S_{ADA_1}\cdot S_{ABE}\cdot S_{BCDE}\cdot S_{BCD}\cdot(-S_{ABCD})}$$

$$\stackrel{simplify}{=} \frac{-S_{ABDE}\cdot S_{ABC}\cdot S_{BDA_1}}{S_{ADA_1}\cdot S_{ABE}\cdot S_{ABCD}}$$

$$\stackrel{A_1}{=} \frac{-S_{ABDE}\cdot S_{ABC}\cdot S_{ABCD}\cdot S_{ABE}\cdot(-S_{ABCE})}{S_{ABDE}\cdot S_{ABC}\cdot S_{ABE}\cdot S_{ABCD}\cdot S_{ABCE}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AE_1}}{\overline{AA_1}} \stackrel{E_1}{=} \frac{S_{ADE}}{S_{ADA_1}}$$

$$\frac{\overline{ED_1}}{\overline{EE_1}} \stackrel{E_1}{=} \frac{S_{BED_1}}{S_{ABE}}$$

$$\frac{\overline{DC_1}}{\overline{DD_1}} \stackrel{D_1}{=} \frac{S_{ADC_1}}{S_{ADE}}$$

$$S_{BED_1} \stackrel{D_1}{=} \frac{-S_{ABDE}\cdot S_{CDE}}{S_{ACDE}}$$

$$\frac{\overline{CB_1}}{\overline{CC_1}} \stackrel{C_1}{=} \frac{-S_{CEB_1}}{S_{CDE}}$$

$$S_{ADC_1} \stackrel{C_1}{=} \frac{S_{ACDE}\cdot S_{BCD}}{-S_{BCDE}}$$

$$\frac{\overline{BA_1}}{\overline{BB_1}} \stackrel{B_1}{=} \frac{-S_{BDA_1}}{S_{BCD}}$$

$$S_{CEB_1} \stackrel{B_1}{=} \frac{S_{BCDE}\cdot S_{ABC}}{S_{ABCD}}$$

$$S_{ADA_1} \stackrel{A_1}{=} \frac{S_{ABDE}\cdot S_{ABC}}{-S_{ABCE}}$$

$$S_{BDA_1} \stackrel{A_1}{=} \frac{S_{ABCD}\cdot S_{ABE}}{S_{ABCE}}$$

Examples 2.65 and 2.66 are special cases of the following general results. Note that the proofs of these general results are a natural extension of the machine proofs for the two examples.

Let $V_1 \ldots V_m$ be a polygram and $1 \le d \le \frac{m}{2}$, $1 \le j \le \frac{m}{2}$ integers. We denote by $P_{d,j,i}$ the intersections of lines V_iV_{i+d} and lines $V_{i+j}V_{i+j+d}$, $i = 1, ..., m$. Then $P_{d,j,i-j}$ is the intersection of line $V_{i-j}V_{i-j+d}$ and line V_iV_{i+d}. Let

$$T(m, d, j) = \prod_{i=1}^{m} \frac{\overline{V_iP_{d,j,i}}}{\overline{P_{d,j,i-j}V_{i+d}}}; \qquad S(m, d, j) = \prod_{i=1}^{m} \frac{\overline{V_iP_{d,j,i-j}}}{\overline{P_{d,j,i}V_{i+d}}}.$$

The result in Example 2.65 is $T(5, 2, 1) = S(5, 2, 1) = 1$. In general, we have

Theorem 2.67 $T(m, d, j) = 1$ *iff one of the following cases are true*

- $d + 2j = m$;
- $2d + j = m$.

Proof. By the co-side theorem,

$$\frac{\overline{V_iP_{d,j,i}}}{\overline{P_{d,j,i-j}V_{i+d}}} = \frac{\overline{V_iP_{d,j,i}}}{\overline{V_iV_{i+d}}} \cdot \frac{\overline{V_iV_{i+d}}}{\overline{P_{d,j,i-j}V_{i+d}}} = \frac{S_{V_iV_{i+j}V_{i+j+d}}}{S_{V_iV_{i+j}V_{i+d}V_{i+j+d}}} \frac{S_{V_{i-j}V_iV_{i-j+d}V_{i+d}}}{S_{V_{i-j}V_{i-j+d}V_{i+d}}}.$$

Multiplying the m equations together, we see that $T(d, j, m) = 1$ iff the areas of triangles and the areas of quadrilaterals in the numerator are the same as the ones in the denominator respectively. Let us assume that the i-th area in the numerator is the same as the x-th area in the denominator, i.e. $S_{V_iV_{i+j}V_{i+j+d}} = \pm S_{V_{x-j}V_{x-j+d}V_{x+d}}$. Then the point sets $\{V_i, V_{i+j}, V_{i+j+d}\}$ and $\{V_{x-j}, V_{x-j+d}, V_{x+d}\}$ should be the same. Considering the order, there are six possible matches. One of the matches is

$$i + j = x + d; \quad i + j + d = x - j; \quad i = x - j + d$$

where the = is understood to be $mod(m)$. Then it is easy to see that the three equations are true for all i and x iff

$$2d + j = 0 \quad mod(m).$$

Since $1 \le d \le \frac{m}{2}$, $1 \le j \le \frac{m}{2}$, the only possible solution is $2d + j = m$. The other five cases can be treated similarly and $d + 2j = m$ is the only nontrivial solution. ∎

Theorem 2.68 $S(m, d, j) = 1$ *iff one of the following cases are true*

- $d + 2j = m$;
- $2d = j$;

- $2j = d$.

Proof. By the co-side theorem,

$$\frac{\overline{V_iP_{d,j,i-j}}}{\overline{P_{d,j,i}V_{i+d}}} = \frac{\overline{V_iP_{d,j,i-j}}}{\overline{V_iV_{i+d}}} \cdot \frac{\overline{V_iV_{i+d}}}{\overline{P_{d,j,i}V_{i+d}}} = \frac{S_{V_iV_{i-j}V_{i-j+d}}}{S_{V_{i+j}V_{i+j+d}V_{i+d}}}\frac{S_{V_{i+j}V_iV_{i+j+d}V_{i+d}}}{S_{V_iV_{i-j}V_{i+d}V_{i-j+d}}}.$$

Now, we can prove it in a similar way as Theorem 2.67. ∎

At the present time, our prover can not deal with geometry statements about m-polygons for an arbitrary number m. But our prover can prove these results for any concrete m, e.g. quadrilaterals, pentagons, etc.

Summary of Chapter 2

- The following basic propositions are the deductive basis of the area method.

  1. If points C and D are on line AB and P is any point not on line AB, then $\frac{S_{PCD}}{S_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$.
  2. (The co-side theorem) Let M be the intersection of two non parallel lines AB and PQ and $M \neq Q$. Then
  $$\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}};\quad \frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}};\quad \frac{\overline{QM}}{\overline{PQ}} = \frac{S_{QAB}}{S_{PAQB}}.$$
  3. Let R be a point on line PQ. Then
  $$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}}S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}S_{PAB}.$$
  4. $PQ \parallel AB$ iff $S_{PAQB} = S_{PAB} - S_{QAB} = S_{BPQ} - S_{APQ} = 0$.
  5. Let $ABCD$ be a parallelogram, P and Q be two points. Then
  $$S_{APQ} + S_{CPQ} = S_{BPQ} + S_{DPQ} \text{ or } S_{PAQB} = S_{PDQC}.$$
  6. Let $ABCD$ be a parallelogram and P be any point . Then
  $$S_{PAB} = S_{PDC} - S_{ADC} = S_{PDAC}.$$

- The Hilbert intersection point statements are geometry statements whose hypotheses can be described constructively and whose conclusions can be represented by polynomial equations in two geometry quantities: the ratio of collinear or parallel segments and the signed areas of triangles or quadrilaterals.

- The area method can efficiently produce short and readable proofs for the Hilbert intersection point statements. The proving process is to eliminate points from geometry quantities using Lemmas 2.19 - 2.31.
- The area method works for constructive statements in the affine geometry associated with any field.
- The area method is used to solve the following geometry problems: deriving unknown geometry formulas, finding the necessary and sufficient conditions for the existence of n_3 configurations, and proving theorems about the transversals for arbitrary polygons.

Chapter 3

Machine Proof in Plane Geometry

In Chapter 2, we presented an automated theorem proving method for constructive statements involving collinearity and parallelism. In this chapter, we will discuss constructive statements involving perpendicular lines and circles. The key tool for dealing with perpendicularity is the *Pythagoras difference*, which is essentially the same as the inner product. Therefore, the method presented in this chapter is actually for constructive statements in metric geometry.

3.1 The Pythagoras Difference

For three points A, B, and C, the *Pythagoras difference* P_{ABC} is defined as

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$

Note that in the above definition, we use a new geometry object: the *square distance between two points A and B*, i.e., $\overline{AB}^2$. For four points A, B, C, and D, we define

$$P_{ABCD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

For basic properties of the Pythagoras difference, see Section 1.7.

3.1.1 Pythagoras Difference and Perpendicular

Besides collinear and parallel considered in Chapter 2, we now have a new basic geometry relation: line l is *perpendicular* to line l', denoted by $l \perp l'$. Following are some basic properties of the perpendicularity.

1. If $l \perp l'$ then $l' \perp l$.
2. Let P be a point and l a line. Then there exists a unique line l' which passes through point P and is perpendicular to line l.
3. If two distinct lines l' and l'' are both perpendicular to line l then l' is parallel to line l''.
4. (Pythagorean Theorem) $AB \perp BC$ iff $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, i.e., iff $P_{ABC} = 0$.

For the proof of the Pythagorean theorem, see Example 1.42 on page 24 and Proposition 1.55 on page 28. But here, we take Pythagorean theorem as a basic property of the Pythagoras difference. Other propositions in this subsection can be derived from it.

For four points A, B, C, and D, the notation $AB \perp CD$ implies that one of the following conditions is true: $A = B$, or $C = D$, or the line AB is perpendicular to line CD.

Proposition 3.1 *$AB \perp CD$ iff $P_{ACD} = P_{BCD}$ or $P_{ACBD} = 0$.*

Proof. See Proposition 1.62 on page 30. ∎

The above generalized Pythagorean proposition is one of the most useful tools in our mechanical theorem proving method.

Proposition 3.2 *Let D be the foot of the perpendicular drawn from point P to a line AB. Then we have*

$$\frac{\overline{AD}}{\overline{DB}} = \frac{P_{PAB}}{P_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$

Proof. See Proposition 1.65 on page 31. ∎

Proposition 3.3 *Let AB and PQ be two nonperpendicular lines and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then*

$$\frac{\overline{PY}}{\overline{QY}} = \frac{P_{PAB}}{P_{QAB}}, \frac{\overline{PY}}{\overline{PQ}} = \frac{P_{PAB}}{P_{PAQB}}, \frac{\overline{QY}}{\overline{PQ}} = \frac{P_{QAB}}{P_{PAQB}}.$$

Proof. See Proposition 1.66 on page 31. ∎

Proposition 3.4 *Let A, B, and C be three different collinear points. Then for any point P if $P_{PAC} \neq 0$ we have $\frac{P_{PAB}}{P_{PAC}} = \frac{\overline{AB}}{\overline{AC}}$.*

Proof. Let Q be the orthogonal projection from P to line AB. By Proposition 3.1, $P_{PAB} = P_{QAB} = 2\overline{AQ}\cdot\overline{AB}$; $P_{PAC} = P_{QAC} = 2\overline{AQ}\cdot\overline{AC}$. Now the result is clear. ▮

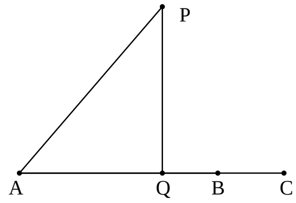


Figure 3-1

Proposition 3.5 *Let R be a point on line PQ with position ratios $r_1 = \frac{\overline{PR}}{\overline{PQ}}, r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ with respect to PQ. Then for any points A and B, we have*

$$\begin{aligned} P_{RAB} &= r_1 P_{QAB} + r_2 P_{PAB} \\ P_{ARB} &= r_1 P_{AQB} + r_2 P_{APB} - r_1 r_2 P_{PQP}. \end{aligned}$$

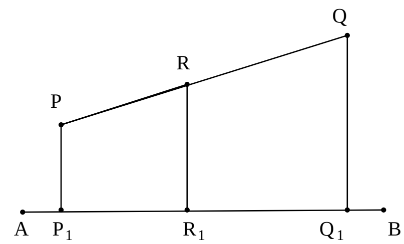


Figure 3-2

Proof. We first assume

$$\overline{RA}^2 = r_1\overline{QA}^2 + r_2\overline{PA}^2 - r_1r_2\overline{PQ}^2 \quad (1)$$
$$\overline{RB}^2 = r_1\overline{QB}^2 + r_2\overline{PB}^2 - r_1r_2\overline{PQ}^2. \quad (2)$$

Then $P_{RAB} = \overline{RA}^2 + \overline{AB}^2 - \overline{RB}^2 = r_1(\overline{QA}^2 + \overline{AB}^2 - \overline{QB}^2) + r_2(\overline{PA}^2 + \overline{AB}^2 - \overline{PB}^2) = r_1P_{QAB} + r_2P_{PAB}$. The second one can be proved similarly. To prove (1), let us first notice that by Proposition 3.4,

$$\frac{P_{APR}}{P_{APQ}} = \frac{\overline{PR}}{\overline{PQ}} = r_1.$$

Then

$$\begin{aligned} r_1\overline{QA}^2 + r_2\overline{PA}^2 - r_1r_2\overline{PQ}^2 &= r_1\overline{QA}^2 + (1-r_1)\overline{PA}^2 - r_1(1-r_1)\overline{PQ}^2 \\ &= \overline{PA}^2 + r_1(\overline{QA}^2 - \overline{PA}^2 - \overline{PQ}) + r_1^2\overline{PQ}^2 \\ &= \overline{PA}^2 + \overline{PR}^2 - r_1P_{APQ} \\ &= \overline{PA}^2 + \overline{PR}^2 - P_{APR} = \overline{AR}^2. \end{aligned}$$ ▮

3.1.2 Pythagoras Difference and Parallel

Proposition 3.6 *For a parallelogram $ABCD$, we have $\overline{AC}^2 + \overline{BD}^2 = 2\overline{AB}^2 + 2\overline{BC}^2$, i.e., $P_{ABC} = -P_{BAD}$.*

Proof. Let O be the intersection of the diagonals AC and BD. By Proposition 3.5, $\overline{AC}^2 = 4\overline{AO}^2 = 4(\frac{1}{2}\overline{AB}^2 + \frac{1}{2}\overline{AD}^2 - \frac{1}{4}\overline{BD}^2) = 2\overline{AB}^2 + 2\overline{AD}^2 - \overline{BD}^2$. ▮

The following proposition shows how the Pythagoras difference changes under a *parallel translation*.

Proposition 3.7 *Let ABCD be a parallelogram. Then for any points P and Q, we have*

$$\begin{array}{rcl} P_{APQ}+P_{CPQ} & = & P_{BPQ}+P_{DPQ} \quad \text{or} \quad P_{APBQ}=P_{DPCQ} \\ P_{PAQ}+P_{PCQ} & = & P_{PBQ}+P_{PDQ}+2P_{BAD} \end{array}$$

Proof. Let O be the intersection of AC and BD. By the first equation of Proposition 3.5, $2P_{OPQ}=P_{APQ}+P_{CPQ}=P_{BPQ}+P_{DPQ}$. By the second equation of Proposition 3.5,

$$2P_{POQ}=P_{PAQ}+P_{PCQ}-\frac{1}{2}P_{ACA}=P_{PBQ}+P_{PDQ}-\frac{1}{2}P_{BDB}.$$

We need only to show that $2P_{BAD}=\frac{1}{2}(P_{ACA}-P_{BDB})$ which is a consequence of Proposition 3.6. ∎

Proposition 3.8 *Let ABCD be a parallelogram and P be any point. Then*

$$\begin{array}{rcl} P_{PAB} & = & P_{PDC}-P_{ADC}=P_{PDAC} \\ P_{APB} & = & P_{APA}-P_{PDAC} \end{array}$$

Proof. By Proposition 3.7, $P_{PAB}=P_{PAC}-P_{PAD}=P_{CADP}=P_{PDAC}=P_{PDC}-P_{ADC}$. For the second equation, $P_{APB}=P_{APA}+P_{APC}-P_{APD}=P_{APA}+P_{CPDA}=P_{APA}-P_{PDAC}$. ∎

Proposition 3.9 *If $PR \parallel AC$ and $QS \parallel BD$ then* $\frac{P_{PQRS}}{P_{ABCD}}=\frac{\overline{PR}}{\overline{AC}}\cdot\frac{\overline{QS}}{\overline{BD}}$.

Proof. Let X and Y be points such that $\overline{PR}=\overline{AX}$, $\overline{QS}=\overline{BY}$. By Propositions 3.7 and 3.4,

$$P_{PQRS}=P_{PBRY}=P_{PBY}-P_{RBY}=\frac{\overline{QS}}{\overline{BD}}(P_{PBD}-P_{RBD})=\frac{\overline{QS}}{\overline{BD}}P_{PBRD}.$$

Similarly we have $P_{PBRD}=\frac{\overline{PR}}{\overline{AC}}P_{ABCD}$. ∎

Proposition 3.10 *Let $AB \parallel CD$. Then* $\frac{\overline{AB}}{\overline{CD}}=\frac{P_{ADBC}}{2\overline{CD}^2}$.

Proof. By Proposition 3.9, $\frac{\overline{AB}}{\overline{CD}}=\frac{\overline{AB}}{\overline{CD}}\cdot\frac{\overline{CD}}{\overline{CD}}=\frac{P_{ACBD}}{P_{CCDD}}=\frac{P_{ADBC}}{2\overline{CD}^2}$. ∎

Exercises 3.11

1. Let $ABCD$ be a parallelogram. Then

   1. $P_{ABC} = P_{ADC} = -P_{BAD} = -P_{BCD}$.
   2. $P_{ABD} = P_{BDC}$; $P_{CBD} = P_{ADB}$.
   3. $P_{ADB} - P_{ADC} = 2\overline{AD}^2$.

2. Let $ABCD$ be a parallelogram and O be the intersection of its diagonals. For any point P, show that

   1. $P_{PAB} + P_{PBC} + P_{PCD} + P_{PDA} = 2(\overline{AB}^2 + \overline{BC}^2)$.
   2. $P_{APD} + P_{BPC} - P_{CPD} - P_{DPD} = 2(\overline{AB}^2 - \overline{BC}^2)$.
   3. $P_{APD} + P_{BPC} + P_{CPD} + P_{DPD} = 8\overline{PO}^2$.

3.1.3 Pythagoras Difference and Area

In this subsection, we will prove the Herron-Qin formula which connects the area and the Pythagoras differences of a triangle.

Definition 3.12 *Let F be the foot of the perpendicular drawn from point R to line PQ. The signed distance from R to PQ, denoted by $h_{R,PQ}$, is a real number which has the same sign as S_{RPQ} and $|h_{R,PQ}| = |RF|$.*

Proposition 3.13 *For any two triangles ABC and RPQ, let $h_A = h_{A,BC}$, $h_R = h_{R,PQ}$. Show that $\frac{S_{ABC}}{|BC|h_A} = \frac{S_{RPQ}}{|PQ|h_R}$.*

Proof. Without loss of generality, we assume that points B, C, P, and Q are on the same line. As in Figure 3-3, let RF be the altitude of triangle RPQ and M be a point on RF such that $AM \parallel BC$. Then $S_{ABC} = S_{MBC}$. By Propositions 2.7 and 2.8,

$$\frac{S_{ABC}}{S_{APQ}} = \frac{\overline{BC}}{\overline{PQ}}; \qquad \frac{S_{MPQ}}{S_{RPQ}} = \frac{\overline{MF}}{\overline{RF}}.$$

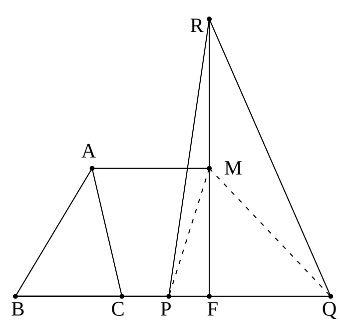


Figure 3-3

Multiplying the two formulas together and noticing that h_A and h_R have the same sign as S_{ABC} and S_{RPQ}, we have $\frac{S_{ABC}}{|BC|h_A} = \frac{S_{RPQ}}{|PQ|h_R}$. ∎

Corollary 3.14 *For a triangle ABC, we have*

$$h_{A,BC}|BC| = h_{B,CA}|AC| = h_{C,AB}|AB|.$$

Proof. Putting $\triangle RPQ$ to be $\triangle BCA$ and $\triangle CAB$ in Proposition 3.13 respectively, we obtain the result. ▮

By Proposition 3.13, we have $S_{ABC} = kh_A|BC| = kh_B|AC| = kh_C|AB|$ where k is a constant which is independent of the triangle ABC. Setting $k = 1/2$, we obtain the usual formula for the areas of triangles.

Proposition 3.15 *For a triangle ABC,*

$$S_{ABC} = \frac{1}{2}h_A|BC| = \frac{1}{2}h_B|AC| = \frac{1}{2}h_C|AB|.$$

Proposition 3.16 (The Herron-Qin Formula) *For a triangle ABC, we have* $16S_{ABC}^2 = 4\overline{AB}^2\overline{AC}^2 - P_{BAC}^2$.

Proof. Let F be the foot of the perpendicular line drawn from point A to lien BC. By Proposition 3.4, $\frac{P_{ABC}}{P_{ABF}} = \frac{\overline{BC}}{\overline{BF}}$. Then

$$P_{ABC} = \frac{\overline{BC}}{\overline{BF}}P_{ABF} = \frac{\overline{BC}}{\overline{BF}}P_{FBF} = 2\overline{BC}\cdot\overline{BF}.$$

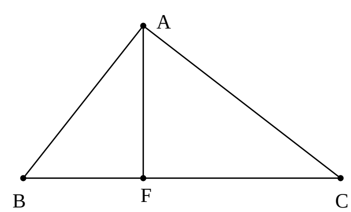


Figure 3-4

Then $16S_{ABC}^2 = 4\overline{AF}^2\cdot\overline{BC}^2 = 4(\overline{AB}^2 - \overline{BF}^2)\overline{BC}^2 = 4\overline{AB}^2\cdot\overline{AC}^2 - P_{BAC}^2$. ▮

Proposition 3.17 (The Herron-Qin Formula for Quadrilaterals) *For any quadrilateral ABCD, we have* $16S_{ABCD}^2 = 4\overline{AC}^2\cdot\overline{BD}^2 - P_{ABCD}^2$.

Proof. Take a point X such that $CXDB$ is a parallelogram. Then $\overline{CX} = \overline{BD}$. By Propositions 2.11, 3.7, and the Herron-Qin formula for triangles,

$$\begin{aligned} S_{ABCD}^2 &= S_{AACX}^2 = S_{XAC}^2 \\ &= \frac{1}{16}(4\overline{AX}^2\cdot\overline{AC}^2 - P_{XAC}^2) \\ &= \frac{1}{16}(4\overline{BD}^2\cdot\overline{AC}^2 - P_{XAAC}^2) \\ &= \frac{1}{16}(4\overline{BD}^2\cdot\overline{AC}^2 - P_{BADC}^2). \end{aligned}$$ ▮

You may compare the proofs for the two the Herron-Qin formulas based on trigonometric functions on page 38. The proof given here is independent of the concept of angles.

Exercises 3.18

1. Prove the following forms of the Herron-Qin formula.

   - $16S_{ABC}^2 = P_{ACB}P_{ABC} + P_{BCB}P_{BAC}$.
   - $16S_{ABC}^2 = P_{BAC}P_{ACB} + P_{ACA}P_{ABC}$.
   - $16S_{ABC}^2 = P_{CAB}P_{CBA} + P_{ABA}P_{ACB}$.

2. The absolute value of the area of a square is equal to the square of its side. Use this result to prove the Pythagorean theorem. (Hint. Use Figure 3-5.)

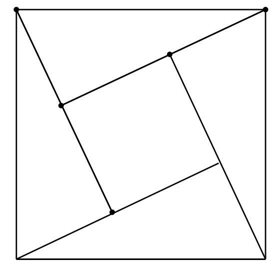


Figure 3-5

3.2 Constructive Geometry Statements

By *constructive geometry statements*, we mean statements which are assertions about configurations that can be drawn using only a ruler and a pair of compasses. More precisely, these configurations can be constructed by first taking arbitrary points, lines and circles and then taking the intersections of two lines and of lines and circles in a prescribed manner. From the constructed points, we can form new lines and circles. By forming the intersections of these new lines and circles, we can obtain new points, etc. Finally, we obtain a configuration consisting of points, lines and circles. The class of constructive statements is denoted by C.

It is clear that the Hilbert intersection point statements belong to class C. In this section, we introduce a new subset of C, i.e., *the linear constructive geometry statement* $\mathbf{C}_{\mathrm{L}}$, which is larger than $\mathbf{C}_{\mathrm{H}}$ and has the advantage that on one hand it contains most of the commonly used geometry theorems and on the other hand its readable proofs can be obtained efficiently by a mechanical method.

3.2.1 Linear Constructive Geometry Statements

Now we have three basic *geometric quantities*:

- the area of a triangle or a quadrilateral,
- the Pythagoras difference of a triangle or a quadrilateral, and
- the ratio of parallel line segments.

By Proposition 3.10, the ratio of parallel line segments can be represented as expressions in Pythagoras differences.

Points are the basic geometry objects. From points, we can introduce two other basic *geometric objects*: lines and circles. A *straight line* can be given in one of the following four forms.

(LINE U V) is the line passing through two points U and V.

(PLINE W U V) is the line passing through point W and parallel to (LINE U V).

(TLINE W U V) is the line passing through point W and perpendicular to (LINE U V).

(BLINE U V) is the perpendicular-bisector of UV.

To make sure that the four kinds of lines are well defined, we need to assume $U \neq V$ which is called the *nondegenerate condition* (ndg) of the corresponding line.

A *circle* with point O as its center and passing through point U is denoted by (CIR O U).

A *construction* is one of the following ways of introducing new points. For each construction, we also give its ndg condition and the degree of freedom for the constructed point.

C1 (POINT[S] $Y_1, \cdots, Y_l$). Take arbitrary points $Y_1, \cdots, Y_l$ in the plane. Each Y_i has two degrees of freedom.

C2 (ON Y ln). Take a point Y on a line ln. The ndg condition of C2 is the ndg condition of the line ln. Point Y has one degree of freedom.

C3 (ON Y (CIR O P)). Take a point Y on a circle (CIR O P). The ndg condition is $O \neq P$. Point Y has one degree of freedom.

C4 (INTER Y $ln1$ $ln2$). Point Y is the intersection of line $ln1$ and line $ln2$. Point Y is a fixed point. The ndg condition is $ln1 \nparallel ln2$. More precisely, we have

1. If $ln1$ is (LINE U V) or (PLINE W U V) and $ln2$ is (LINE P Q) or (PLINE R P Q), then the ndg condition is $UV \nparallel PQ$.
2. If $ln1$ is (LINE U V) or (PLINE W U V) and $ln2$ is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $\neg(UV \perp PQ)$.
3. If $ln1$ is (BLINE U V) or (TLINE W U V) and $ln2$ is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $UV \nparallel PQ$.

C5 (INTER Y ln (CIR O P)). Point Y is the intersection of line ln and circle (CIR O P) other than point P. Line ln could be (LINE P U), (PLINE P U V), and (TLINE P U V). The ndg conditions are $O \neq P$, $Y \neq P$, and line ln is not degenerate. Point Y is a fixed point.

C6 (INTER Y (CIR O_1 P) (CIR O_2 P)). Point Y is the intersection of the circle (CIR O_1 P) and the circle (CIR O_2 P) other than point P. The ndg condition is O_1, O_2, and P are not collinear. Point Y is a fixed point.

C7 (PRATIO Y W U V r). Take a point Y on the line passing through W and parallel to line UV such that $\overline{WY} = r\overline{UV}$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity, then Y is a fixed point; if r is a variable then Y has one degree of freedom. The ndg condition is $U \neq V$. If r is a rational expression in geometry quantities then we will further assume that the denominator of r could not be zero.

C8 (TRATIO Y U V r). Take a point Y on line (TLINE U U V) such that $r = \frac{4S_{UVY}}{P_{UVU}} (= \frac{\overline{UY}}{\overline{UV}})$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity then Y is a fixed point; if r is a variable then Y has one degree of freedom. The ndg condition is the same as that of C7.

The point Y in each of the above constructions is said to be *introduced* by that construction.

Since there are four kinds of lines, constructions C2, C4, and C5 have 4, 10, and 3 possible forms respectively. Thus, in total, we have 22 different forms of constructions.

Now class $\mathbf{C}_{\mathrm{L}}$, the class of the *linear constructive geometry statements*, can be defined similarly as $\mathbf{C}_{\mathrm{H}}$, i.e., a statement in class $\mathbf{C}_{\mathrm{L}}$ is a list

$$S = (C_1, C_2, \ldots, C_k, G)$$

where C_i, $i = 1, \ldots, k$, are constructions such that each C_i introduces a new point from the points introduced before; and $G = (E_1, E_2)$ where E_1 and E_2 are polynomials in geometric quantities of the points introduced by the C_i and $E_1 = E_2$ is the conclusion of the statement.

Let $S = (C_1, C_2, \ldots, C_k, (E_1, E_2))$ be a statement in $\mathbf{C}_{\mathrm{L}}$. The *ndg condition* of S is the set of ndg conditions of the C_i plus the condition that the denominators of the length ratios in E_1 and E_2 are not equal to zero.

We call the statements in $\mathbf{C}_{\mathrm{L}}$ linear, because each of the constructions C1–C8 introduces a unique point. For the constructions involving circles, this fact may not be obvious. See the next subsection for more discussions.

Example 3.19 *The orthocenter theorem on page 32 can be described in the following constructive way.*

((POINTS A B C)
(INTER E (LINE A C) (TLINE B A C))

(INTER F (LINE C B) (TLINE A C B))
(INTER H (LINE A F) (LINE B E))
($P_{ACH} = P_{BCH}$)) $AB \perp CH$.

The ndg condition: $A \neq C$, $C \neq B$, $AF \nparallel BE$.

3.2.2 A Minimal Set of Constructions

There are a total of 22 constructions and three kinds of geometry quantities. So to provide an elimination method for each construction and each geometry quantity, we need to consider 22 * 3 = 66 cases. Instead of considering all these cases, we introduce a minimal set of constructions which are equivalent to all the 22 constructions but much fewer in number.

A *minimal set of constructions* consists of C1, C7, C8 and the following two constructions.

C41 (INTER Y (LINE U V) (LINE P Q)).

C42 (FOOT Y P U V), or equivalently (INTER Y (LINE U V) (TLINE P U V))). The ndg condition is $U \neq V$.

We first show how to represent the four kinds of lines by one kind: (LINE U V).

For ln = (PLINE W U V), we first introduce a new point N by (PRATIO N W U V 1). Then ln = (LINE W N).

For ln = (TLINE W U V), we have two cases: if W, U, V are collinear, ln = (LINE N W) where N is introduced by (TRATIO N W U 1); otherwise ln = (LINE N W) where N is given by (FOOT N W U V).

(BLINE U V) can be written as (LINE N M) where N and M are introduced as follows (MIDPOINT M U V) (i.e., (PRATIO M U U V 1/2)), (TRATIO N M U 1).

Since now there is only one kind of line, to represent all the 22 constructions by the constructions in the minimal set we need only to consider the following cases.

- (ON Y (LINE U V)) is equivalent to (PRATIO Y U U V r) where r is an *indeterminate*.
- (INTER Y (LINE U V) (CIR O U)) is equivalent to two constructions: (FOOT N O U V), (PRATIO Y N N U -1).
- C6 can be reduced to (FOOT N P O_1 O_2) and (PRATIO Y N N P -1).
- For C3, i.e., to take an arbitrary point Y on a circle (CIR O P), we first take an arbitrary point Q. Then Y is introduced by (INTER Y (LINE P Q) (CIR O P)).

Proposition 3.20 *The existence of the point introduced by each of the 22 constructions follows from Axiom A.2 on page 55.*

Proof. We can limit ourselves to the five minimal constructions. Constructions C1, C7, and C41 have been discussed on page 61. Let Y be introduced by (FOOT Y P U V). Then by Proposition 3.2 point Y is a point on UV with the position ratio $\frac{P_{PUV}}{P_{UVU}}$. Hence Y does exist by Axiom A.2. Let Y be introduced by (TRATIO Y U V r). Then Y can also be introduced as follows (check this).

(POINT M); (FOOT N M U V); (PRATIO B U M N 1); (PRATIO Y U U B $\frac{rP_{UVU}}{4S_{UVB}}$).

Thus Y exists. ∎

Exercises 3.21

1. Show that constructions C1, C7, and C8 can also serve as a minimal set of constructions. The reason we use a larger minimal set is that constructions C41 and C42 are used frequently and special treatment of them will lead to short proofs.
2. We introduce a new construction (LRATIO Y U V r) which means taking a point Y on UV such that $\overline{UY} = r\overline{UV}$. Show that C1, C8 and the above construction also form a minimal set of constructions. (See Example 2.13).
3. Show that constructions C1, C7, and C42 could form a minimal set of constructions.

3.2.3 The Predicate Form

The constructive description of geometry statements can be transformed into the commonly used predicate form. In addition to the three predicates POINT, COLL, and PARA introduced on page 63, we introduce two new predicates.

1. Perpendicular (*PERP* P_1, P_2, P_3, P_4): $[(P_1 = P_2) \vee (P_3 = P_4) \vee (P_1P_2$ is perpendicular to $P_3P_4)]$. It is equivalent to $P_{P_1P_3P_2P_4} = 0$.
2. Congruence (*CONG* P_1, P_2, P_3, P_4): Segment P_1P_2 is congruent to P_3P_4. It is equivalent to $P_{P_1P_2P_1} = P_{P_3P_4P_3}$.

To transform constructions into predicate forms, we need only to consider the minimal set of constructions introduced in the preceding subsection. Also constructions C1, C41, and C7 have been discussed in Section 2.3.2. We thus need only to consider C42 and C8.

C42 (FOOT Y P U V) is equivalent to (COLL Y U V), (PERP Y P U V), and $U \neq V$.

C8 (TRATIO Y U V r) is equivalent to (PERP Y U U V), $r = \frac{4S_{UVY}}{P_{UVU}}$, and $U \neq V$.

Now a constructive statement $S = (C_1, \cdots, C_k, (E, F))$ can be transformed into the following predicate form

$$\forall P_i[(P(C_1) \wedge \cdots \wedge P(C_k)) \Rightarrow (E = F)]$$

where $P(C_i)$ is the predicate form for C_i and P_i is the point introduced by C_i.

We will now discuss what geometry properties can be the conclusion of a geometry statement in $\mathbf{C}_{\mathrm{L}}$, i.e., what geometry properties can be represented by polynomial equations of geometry quantities. To illustrate that, let us give an algebraic interpretation for the area and Pythagoras difference. Let A, B, C, and D be four points in the Euclidean plane. Then S_{ABCD} and P_{ABCD} are propositional to the exterior and inner product of the diagonals AC and BD of the quadrilateral ABC: (see Chapter 5 for details)

$$S_{ABCD} = \frac{1}{2}[\overrightarrow{AC}, \overrightarrow{BD}], \quad P_{ABCD} = 2\langle\overrightarrow{AC}, \overrightarrow{BD}\rangle.$$

So any geometry property that can be represented by an equation of the inner and exterior products can be the conclusion of a geometry statement. As examples, we show how to represent several often used geometry properties by the geometry quantities.

(COLLINEAR $A\ B\ C$). Points A, B, and C are collinear iff $S_{ABC} = 0$. For other variants, see the comments after Example 2.36 on page 74.

(PARALLEL $A\ B\ C\ D$). AB is parallel to CD iff $S_{ACD} = S_{BCD}$.

(PERPENDICULAR $A\ B\ C\ D$). AB is perpendicular to CD iff $P_{ACD} = P_{BCD}$.

(MIDPOINT $O\ A\ B$). O is the midpoint of AB iff $\frac{\overline{AO}}{\overline{OB}} = 1$.

(EQDISTANCE $A\ B\ C\ D$). AB has the same length as CD iff $P_{ABA} = P_{CDC}$.

(HARMONIC $A\ B\ C\ D$). A, B and C, D are harmonic points iff $\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$.

(EQ-PRODUCT $A\ B\ C\ D\ P\ Q\ R\ S$). The product of AB and CD is equal to the product of PQ and RS, which is equivalent to $\frac{\overline{AB}}{\overline{PQ}} = \pm\frac{\overline{RS}}{\overline{CD}}$ if $AB \parallel PQ$ and $RS \parallel CD$; $P_{ACBD} = \pm P_{PRQS}$ if $AB \parallel CD$ and $PQ \parallel RS$; $P_{ABA}P_{CDC} = P_{PQP}P_{RSR}$ otherwise.

(TANGENT $O_1\ A\ O_2\ B$). Circle (CIR $O_1\ A$) is tangent to circle (CIR $O_2\ B$) iff $d^2 + r_1^2 + r_2^2 - 2dr_1 - 2dr_2 - 2r_1r_2 = 0$ where $d = \overline{O_1O_2}^2, r_1 = \overline{O_1A}^2, r_2 = \overline{O_2B}^2$.

3.3 Machine Proof for Class $\mathbf{C}_{\mathrm{L}}$

3.3.1 The Algorithm

In Chapter 2, we have seen that the process of proving geometry theorems using the area method actually eliminates points from geometry quantities. To prove geometry theorems in class $\mathbf{C}_L$, we need to eliminate points introduced by constructions: C1, C7, C8, C41, C42 from three geometry quantities: the area, the Pythagoras difference, and the length ratio.

Let $G(Y)$ be one of the following geometry quantities: $S_{ABY}, S_{ABCY}, P_{ABY}$, or P_{ABCY} for distinct points A, B, C, and Y. For three collinear points Y, U, and V, by Propositions 2.9 and 3.5 we have

$$(I) \qquad G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U).$$

We call $G(Y)$ a *linear geometry quantity* for variable Y. Elimination procedures for all linear geometry quantities are similar for constructions C7, C41, and C42.

Lemma 3.22 *Let $G(Y)$ be a linear geometry quantity and point Y be introduced by construction (PRATIO Y W U V r). Then we have*

$$G(Y) = \begin{cases} (\frac{\overline{UW}}{\overline{UV}} + r)G(V) + (\frac{\overline{WV}}{\overline{UV}} - r)G(U) & \text{if } W \text{ is on line } UV. \\ G(W) + r(G(V) - G(U)) & \text{otherwise.} \end{cases}$$

Proof. If W, U, and V are collinear, we have $\frac{\overline{UY}}{\overline{UV}} = \frac{\overline{UW}}{\overline{UV}} + r$; $\frac{\overline{YV}}{\overline{UV}} = \frac{\overline{WV}}{\overline{UV}} - r$. Substituting these into (I), we obtain the first formula. For the second one, take a point S such that $\overline{WS} = \overline{UV}$. By (I)

$$G(Y) = \frac{\overline{WY}}{\overline{WS}} G(S) + \frac{\overline{YS}}{\overline{WS}} G(W) = rG(S) + (1 - r)G(W).$$

By Propositions 2.11 and 3.7, $G(S) = G(W) + G(V) - G(U)$. Substituting this into the above equation, we obtain the result. Notice that in both cases, we need the ndg condition $U \neq V$. ∎

Lemma 3.23 *Let $G(Y)$ be a linear geometry quantity and Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then*

$$G(Y) = \frac{S_{UPQ}G(V) - S_{VPQ}G(U)}{S_{UPVQ}}.$$

Proof. By the co-side theorem, $\frac{\overline{UY}}{\overline{UV}} = \frac{S_{UPQ}}{S_{UPVQ}}$, $\frac{\overline{YV}}{\overline{UV}} = -\frac{S_{VPQ}}{S_{UPVQ}}$. Substituting these into (I), we prove the result. ∎

Lemma 3.24 *Let $G(Y)$ be a linear geometry quantity and Y be introduced by (FOOT Y P U V). Then*

$$G(Y) = \frac{P_{PUV}G(V) + P_{PVU}G(U)}{2\overline{UV}^2}.$$

Proof. By Proposition 3.2, $\frac{\overline{UY}}{\overline{UV}} = \frac{P_{PUV}}{P_{UVU}}$, $\frac{\overline{YV}}{\overline{UV}} = \frac{P_{PVU}}{P_{UVU}}$. Substituting these into (I), we prove the result, ▮

Let $G(Y) = P_{AYB}$. By Proposition 3.5, for three collinear points Y, U, and V

$$(II) \qquad G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U) - \frac{\overline{UY}}{\overline{UV}} \cdot \frac{\overline{YV}}{\overline{UV}}P_{UVU}.$$

We call P_{AYB} a *quadratic geometry quantity* for variable Y. Since in the above three lemmas we have obtained the position ratios $\frac{\overline{UY}}{\overline{UV}}$, $\frac{\overline{YV}}{\overline{UV}}$ for Y when it is introduced by constructions $C7$, $C41$, $C42$, we can substitute them into (II) to eliminate point Y from $G(Y)$. Notice that in the case of construction $C7$, we need to use the second formula of Proposition 3.7. The result is as follows.

Lemma 3.25 *Let Y be introduced by (PRATIO Y W U V r). Then we have*

$$P_{AYB} = P_{AWB} + r(P_{AVB} - P_{AUB} + P_{WUV}) - r(1-r)P_{UVU}.$$

Construction C8 needs special treatment.

Lemma 3.26 *Let Y be introduced by (TRATIO Y P Q r). Then we have* $S_{ABY} = S_{ABP} - \frac{r}{4}P_{PAQB}$.

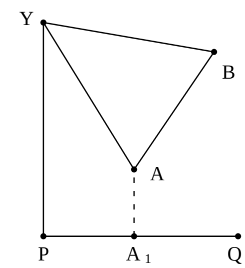


Figure 3-6

Proof. Let A_1 be the orthogonal projection from A to PQ. Then by Propositions 2.10 and 3.2

$$\frac{S_{PAY}}{S_{PQY}} = \frac{S_{PA_1Y}}{S_{PQY}} = \frac{\overline{PA_1}}{\overline{PQ}} = \frac{P_{A_1PQ}}{P_{QPQ}} = \frac{P_{APQ}}{P_{QPQ}}.$$

Thus $S_{PAY} = \frac{P_{APQ}}{P_{QPQ}}S_{PQY} = \frac{r}{4}P_{APQ}$. Similarly, $S_{PBY} = \frac{P_{BPQ}}{P_{QPQ}}S_{PQY} = \frac{r}{4}P_{BPQ}$. Now $S_{ABY} = S_{ABP} + S_{PBY} - S_{PAY} = S_{ABP} - \frac{r}{4}P_{PAQB}$. ▮

Lemma 3.27 *Let Y be introduced by (TRATIO Y P Q r). Then we have* $P_{ABY} = P_{ABP} - 4rS_{PAQB}$.

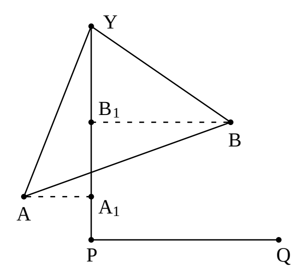


Figure 3-7

Proof. Let the orthogonal projections from A and B to PY be A_1 and B_1. Then

$$\frac{P_{BPAY}}{P_{YPY}} = \frac{P_{B_1PA_1Y}}{P_{YPY}} = \frac{\overline{A_1B_1}}{\overline{PY}} = \frac{S_{PA_1QB_1}}{S_{PQY}} = \frac{S_{PAQB}}{S_{PQY}}.$$

By the Herron-Qin formula, $S^2_{PQY} = \frac{1}{4}\overline{PQ}^2 \cdot \overline{PY}^2$. Then $P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY}$. Therefore $P_{ABY} = P_{ABP} - P_{BPAY} = P_{ABP} - 4rS_{PAQB}$. ∎

Lemma 3.28 *Let Y be introduced by (TRATIO Y P Q r). Then we have*

$$P_{AYB} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ}).$$

Proof. By Lemma 3.27,

$$P_{APY} = 4rS_{APQ}, P_{BPY} = 4rS_{BPQ}.$$

By the Herron-Qin formula,

$$P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY} = r^2 P_{PQP}.$$

Then $P_{AYB} = P_{APB} - P_{APY} - P_{BPY} + P_{YPY} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ})$. ∎

By Proposition 3.10 the ratios of parallel line segments can be represented by Pythagoras differences. Thus, we have given a complete method of eliminating points from geometry quantities. But usually, we consider the length ratios separately in order to obtain short proofs. The methods of eliminating point Y introduced by C41 and C7 from length ratios have been given by Lemmas 2.25 and 2.26. For other constructions, we have

Lemma 3.29 *Let Y be introduced by (FOOT Y P U V). We assume $D \neq U$; otherwise interchange U and V.*

$$G = \frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{P_{PEDF}}{P_{EFE}} & \text{if } D \in UV. \\ \frac{S_{DUV}}{S_{EUFV}} & \text{if } D \notin UV. \end{cases}$$

Proof. If $D \in UV$, let T be a point such that $\overline{DT} = \overline{EF}$. By Propositions 3.2 and 3.8

$$G = \frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DY}}{\overline{DT}} = \frac{P_{PDT}}{P_{DTD}} = \frac{P_{PEDF}}{P_{EFE}}.$$

The second equation is a direct consequence of the co-side theorem. ∎

Lemma 3.30 *Let Y be introduced by (TRATIO Y P Q r).*

$$G = \frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{P_{DPQ}}{P_{EPFQ}} & \text{if } D \notin PY. \\ \frac{S_{DPQ} - \frac{r}{4}P_{PQP}}{S_{EPFQ}} & \text{if } D \in PY. \end{cases}$$

Proof. The first case is a direct consequence of Proposition 3.3. If $D \in PY$, then $\frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DP}}{\overline{EF}} - \frac{\overline{YP}}{\overline{EF}}$. By the co-side theorem,

$$\frac{\overline{DP}}{\overline{EF}} = \frac{S_{DPQ}}{S_{EPFQ}}; \frac{\overline{YP}}{\overline{EF}} = \frac{S_{YPQ}}{S_{EPFQ}} = \frac{rP_{PQP}}{4S_{EPFQ}}.$$

Now the second result follows immediately.

For a geometry statement $S = (C_1, C_2, \ldots, C_k, (E, F))$, after eliminating all the non-free points introduced by C_i from E and F using the above lemmas, we obtain two rational expressions E' and F' in indeterminates, areas and Pythagoras differences of *free points*. These geometric quantities are generally not independent, e.g., for any three points A, B, and C we have the *Herron-Qin formula* (Proposition 3.16):

$$16S_{ABC}^2 = 4\overline{AB}^2\,\overline{AC}^2 - (\overline{AC}^2 + \overline{AB}^2 - \overline{BC}^2)^2.$$

We thus need to reduce E' and F' to expressions in independent variables. To do that, we first introduce three new points O, U, and V such that $UO \perp OV$. We will reduce E' and F' to expressions in the *area coordinates* of the free points with respect to OUV.

Lemma 3.31 *For three points A, B, and C, we have*

1. $S_{ABC} = \frac{1}{S_{OUV}} \begin{vmatrix} S_{OUA} & S_{OVA} & 1 \\ S_{OUB} & S_{OVB} & 1 \\ S_{OUC} & S_{OVC} & 1 \end{vmatrix}.$

2. $P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$

3. $\overline{AB}^2 = \frac{\overline{OU}^2(S_{OVA} - S_{OVB})^2}{S_{OUV}^2} + \frac{\overline{OV}^2(S_{OUA} - S_{OUB})^2}{S_{OUV}^2}.$

4. $S_{OUV}^2 = \frac{1}{4}\overline{OU}^2\,\overline{OV}^2.$

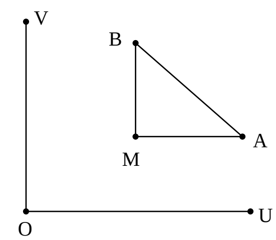


Figure 3-8

Proof. Case 1 is Lemma 2.31. Case 2 is the definition of the Pythagoras difference. For case 3, we introduce a new point M by construction

(INTER M (PLINE A O U) (PLINE B O V)).

Then by the Pythagorean theorem, $\overline{AB}^2 = \overline{AM}^2 + \overline{BM}^2$. By the second case of Lemma 2.26, $\frac{\overline{AM}}{\overline{OU}} = \frac{S_{AOBV}}{S_{OOUV}} = \frac{S_{AOV} - S_{BOV}}{S_{OUV}}$; $\frac{\overline{BM}}{\overline{OV}} = \frac{S_{AOU} - S_{BOU}}{S_{OUV}}$. We have proved case 3. Case 4 is a consequence of Proposition 3.13. ∎

Using Lemma 3.31, E and F can be written as expressions of $\overline{OU}, \overline{OV}$, and the area coordinates of the free points.

Remark 3.32 *In Lemma 3.31, we actually use the Cartesian coordinates of points to represent areas and Pythagoras differences. For a point P, let* $x_P = \frac{2S_{OUP}}{|\overline{OU}|}, y_P = \frac{2S_{OVP}}{|\overline{OV}|}$. *Then the formulas in Lemma 3.31 become*

1'. $S_{ABY} = \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_Y & y_Y & 1 \end{vmatrix}.$

3'. $\overline{AB}^2 = (x_A - x_B)^2 + (y_A - y_B)^2$.

Algorithm 3.33 (PLANE)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a statement in $\mathbf{C}_L$.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \cdots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the ndg conditions of C_i are satisfied. The ndg condition of a construction has three forms: $A \neq B$, $PQ \nparallel UV$, or $PQ \not\perp UV$. For the first case, we check whether $P_{ABA} = 2\overline{AB}^2 = 0$. For the second case, we check whether $S_{PUV} = S_{QUV}$. For the third case, we check whether $P_{PUV} = P_{QUV}$. If a ndg condition of a geometry statement is not satisfied, the statement is *trivially true*. The algorithm terminates.

S3. Let $G_1, \cdots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \cdots, s$ do S4.

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using the lemmas in this section and replace G_j by H_j in E and F to obtain the new E and F.

S5. Now E and F are rational expressions in independent variables. Hence if $E = F$, S is true. Otherwise S is false.

Proof of the correctness. Only the last step needs explanation. If $E = F$, the statement is obviously true. Otherwise, by Proposition 2.33 we can find specific values for the free parameters in E and F such that when substituting them into E and F, we obtain two different numbers, i.e., we have found a counterexample. ∎

For the complexity of the algorithm, let n be the number of the non-free points in a statement which is described using constructions C1–C8. By the analysis in Section 3.2, we will use at most $5n$ constructions in the minimal set to represent the hypotheses (we need five minimal constructions to represent construction (INTER A (BLINE U V) (BLINE P Q))). Then we will use at most $5n$ minimal constructions to describe the statement. Notice that each lemma will replace a geometric quantity by a rational expression with degree less than or equal to three. Then if the conclusion of the geometry statement is of degree d, the output of our algorithm is at most degree $3^{5n}d$. In the last step, we need to represent the area and Pythagoras difference by area coordinates. In the worst case, a geometry quantity (Pythagoras difference) will be replaced by an expression of degree five. Thus the degree of the final polynomial is at most $5d3^{5n}$.

Remark 3.34 *In Section 2.6, we showed that the area method is for affine geometry over any field. Algorithm 3.33 is actually for metric geometry over any field whose characteristic is not* 2*. We have to exclude the case of characteristic* 2*. Otherwise, for three collinear points* A, B*, and* C *we have* $P_{ABC} = 2\overline{AB} \cdot \overline{CB} = 0$*, i.e.,* $AB \perp BC$*. For more details, see Chapter 5.*

3.3.2 Refined Elimination Techniques

We have presented a complete method for proving geometry statements in class $\mathbf{C}_L$ by considering a minimal set of constructions. But if we use only those five constructions, we must introduce many auxiliary points in the description of geometry statements. More points usually mean longer proofs. In this section we will introduce more constructions and more elimination techniques which can be used to obtain shorter proofs.

The elimination lemmas in Section 3.3.1 can be refined in two ways: first we may consider more constructions instead of the minimal set; second for each elimination lemma we may give the elimination results in some special cases of the configuration. As an example of the second way of refinement, prove the following result.

Exercise 3.35 *Let point* Y *be introduced by construction (FOOT Y P U V). Prove the following results.*

$$S_{ABY} = \begin{cases} S_{ABU} & \text{if } AB \parallel UV; \\ S_{ABP} & \text{if } AB \perp UV; \\ \frac{S_{UBV}P_{PUAV}}{P_{UVU}} & \text{if } U, V, \text{ and } A \text{ are collinear;} \\ \frac{S_{AUV}S_{PUBV}}{S_{UVU}} & \text{if } U, V, \text{ and } B \text{ are collinear.} \end{cases}$$

$$P_{ABY} = \begin{cases} P_{ABP} & \text{if } AB \parallel UV; \\ P_{ABU} & \text{if } AB \perp UV; \\ \frac{P_{ABU}P_{PBU}}{P_{UBU}} & \text{if } U, V, \text{ and } B \text{ are collinear.} \end{cases}$$

$$P_{AYB} = \begin{cases} \frac{16S_{PUV}^2}{P_{UVU}} & \text{if } A = B = P; \\ \frac{P_{PUV}^2}{P_{UVU}} & \text{if } A = B = U; \\ \frac{P_{PVU}^2}{P_{UVU}} & \text{if } A = B = V; \\ \frac{-P_{PVU}P_{PUV}}{P_{UVU}} & \text{if } A = U, B = V. \end{cases}$$

To use the above elimination technique, similar to in Section 2.5.1, we need to find the collinear point sets, parallel lines, and perpendicular lines which are obvious from the constructive description of the geometry statement. See the following example.

Example 3.36 *The following machine proof of the orthocenter theorem on page 32 uses the above exercise.*

Constructive description	The machine proof	The eliminants
((POINTS A B C)	$\frac{P_{ACH}}{P_{BCH}}$	$P_{BCH} \overset{H}{=} P_{ACB}$
(FOOT E B A C)	$\overset{H}{=} \frac{P_{ACB}}{P_{ACB}}$	$P_{ACH} \overset{H}{=} P_{ACB}$
(FOOT F A B C)	$\overset{simplify}{=} 1$	
(INTER H (LINE A F) (LINE B E))		
(PERPENDICULAR A B C H))		

From the description of the statement, we can find the sets of collinear points:

$$\{H, A, F\}, \{F, C, B\}, \{H, E, B\}, \{E, A, C\},$$

and the sets of perpendicular lines:

$$HAF \perp BCF, HEB \perp EAC.$$

Then by Exercise 3.35, $P_{BCH} = P_{ACB}$ and $P_{ACH} = P_{ACB}$ since $AB \perp CH$ and $CA \perp BH$.

Exercise 3.37 *Let $G(Y)$ be a linear geometry quantity of Y, and $Q(Y)$ a quadratic geometry quantity of Y. If Y is on line UV then*

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U);$$

$$Q(Y) = \frac{\overline{UY}}{\overline{UV}} Q(V) + \frac{\overline{YV}}{\overline{UV}} Q(U) - 2 \cdot \frac{\overline{UY}}{\overline{UV}} \cdot \frac{\overline{YV}}{\overline{UV}} \cdot \overline{UV}^2.$$

$$\frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{S_{DUV}}{S_{EUFV}} & \text{if } D \notin UV; \\ \frac{\frac{\overline{DU}}{\overline{UV}} + \frac{\overline{UY}}{\overline{UV}}}{\frac{\overline{EF}}{\overline{UV}}} & \text{if } D \in UV. \end{cases}$$

For several constructions, we need to compute the position ratio of Y with respect to UV and substitute them into the above formulas to eliminate Y. Precisely, we have

1. *If Y is introduced by (INTER Y (LINE U V) (PLINE R P Q)) then* $\frac{\overline{UY}}{\overline{UV}} = \frac{S_{UPRQ}}{S_{UPVQ}}, \frac{\overline{YV}}{\overline{UV}} = -\frac{S_{VPRQ}}{S_{UPVQ}}.$
2. *If Y is introduced by (INTER Y (LINE U V) (TLINE P P Q)) then* $\frac{\overline{UY}}{\overline{UV}} = \frac{P_{UPQ}}{P_{UPVQ}}, \frac{\overline{YV}}{\overline{UV}} = -\frac{P_{VPQ}}{P_{UPVQ}}.$
3. *If Y is introduced by (INTER Y (LINE U V) (BLINE P Q)) then* $\frac{\overline{UY}}{\overline{UV}} = \frac{P_{UPQ} - \overline{PQ}^2}{P_{UPVQ}}, \frac{\overline{YV}}{\overline{UV}} = -\frac{P_{VPQ} - \overline{PQ}^2}{P_{UPVQ}}.$
4. *If Y is introduced by (INTER Y (LINE U V) (CIR O U)) then* $\frac{\overline{UY}}{\overline{UV}} = 2 \cdot \frac{P_{OUV}}{P_{UVU}}, \frac{\overline{YV}}{\overline{UV}} = \frac{P_{OVO} - P_{OUO}}{P_{UVU}}.$

Exercise 3.38 *If Y is introduced by (INTER Y (PLINE W U V) (PLINE R P Q)) then*

$$G(Y) = G(W) + r(G(V) - G(U));$$
$$Q(Y) = Q(W) + r(G(V) - G(U)) - 2r(1 - r)\overline{UV}^2$$

where $r = \frac{S_{WPRQ}}{S_{UPVQ}}$.

Exercise 3.39 *Of the 22 constructions, there are still eight which are not discussed (Take points on a BLINE, a PLINE, or a TLINE; Take the intersections of two TLINEs, two BLINEs, a TLINE and a PLINE, a TLINE and a BLINE, and a PLINE and a BLINE; Take the intersection of a circle and a PLINE or a TLINE.) Try to eliminate a point introduced by one the eight constructions from a geometry quantity.*

3.4 The Ratio Constructions

By the *ratio constructions*, we mean the constructions PRATIO and TRATIO and other constructions which can be reduced to them. These constructions are the most subtle constructions; appropriate use of the ratio constructions may lead to elegant proofs for geometry statements.

3.4.1 More Ratio Constructions

We first introduce the following constructions for convenience.

C9 (MIDPOINT Y U V). Y is the midpoint of UV. It is equivalent to (PRATIO Y U U V 1/2).

C10 (SYMMETRY Y U V). Y is the symmetry of point V with respect to point U. It is equivalent to (PRATIO Y U U V -1).

C11 (LRATIO Y U V r). Y is a point on UV such that $\frac{\overline{UY}}{\overline{UV}} = r$. It is equivalent to (PRATIO Y U U V r).

C12 (MRATIO Y U V r). Y is a point on UV such that $\frac{\overline{UY}}{\overline{YV}} = r$. It is equivalent to (PRATIO Y U U V $\frac{r}{1+r}$).

Four collinear points A, B, C, and D are said to form a *harmonic sequence* if

$$\frac{\overline{CA}}{\overline{CB}} = -\frac{\overline{DA}}{\overline{DB}}.$$

Given two points A and B, we have the following ways of introducing points C and D such that A, B, C, and D form a harmonic sequence.

$$(\text{ON } C \text{ (LINE } A\ B)), (\text{LRATIO } D\ A\ B\ \frac{\frac{\overline{CA}}{\overline{CB}}}{\frac{\overline{CA}}{\overline{CB}}+1});$$
$$(\text{MRATIO } C\ A\ B\ r), (\text{MRATIO } D\ A\ B\ -r).$$

For convenience, we can introduce a new construction

C13 (HARMONIC $D\ C\ A\ B$) introduces a point D such that, for three collinear points A, B, and C, A, B, C, and D form a harmonic consequence.

Another important geometry concept related to ratios is *inversion*. Suppose that we have given a circle with a center O and a radius $r \neq 0$. Let P and Q be any two points collinear with O such that

$$\overline{OP} \cdot \overline{OQ} = r^2.$$

Then P is said to be the *inversion* of Q with regard to the circle and O is called the inversion center. We introduce a new construction as follows

C14 (INVERSION $P\ Q\ O\ A$) means that P is the inversion of Q with regard to circle (CIR $O\ A$). This construction is equivalent to

$$(\text{LRATIO } P\ O\ A\ \frac{\overline{OA}}{\overline{OQ}}) \qquad \text{if } Q \in OA,$$
$$(\text{LRATIO } P\ O\ Q\ \frac{P_{OAO}}{P_{OQO}}) \qquad \text{otherwise.}$$

The ratio r in the ratio constructions could be rational numbers, variables, or expressions in geometry quantities. Now we allow r to be any algebraic numbers by adding a new special construction.

C15 (CONSTANT $p(r)$) where $p(r)$ is an irreducible polynomial in the variable r. This construction introduces an algebraic number r which is a root of $p(r) = 0$.

With the help of construction CONSTANT, we can deal with statements involving special angles such as 30°, 45°, and 60°, etc. In the rest of this section, we will use several examples to show how to solve geometry problems using the ratio constructions.

The construction TRATIO can be used to express geometry statements involving *squares* easily.

Example 3.40 *The following is the machine proof of Example 1.69 on page 32.*

Constructive description	The machine proof	The eliminants
((POINTS $A\ B\ C$)	$\frac{P_{BGE}}{P_{BGC}}$	$P_{BGC} \overset{G}{=} P_{BAC}+P_{ACA}-4S_{ABC}$
(TRATIO $E\ A\ B\ 1$)	$\overset{G}{=} \frac{P_{ACA}+4S_{ACE}-4S_{ABC}}{P_{BAC}+P_{ACA}-4S_{ABC}}$	$P_{BGE} \overset{G}{=} P_{ACA}+4S_{ACE}-4S_{ABC}$
(TRATIO $G\ A\ C\ -1$)	$\overset{E}{=} \frac{P_{BAC}+P_{ACA}-4S_{ABC}}{P_{BAC}+P_{ACA}-4S_{ABC}}$	$S_{ACE} \overset{E}{=} \frac{1}{4}(P_{BAC})$
(PERPENDICULAR $E\ C\ G\ B$))	$\overset{simplify}{=} 1$	

Example 3.41 *On the two sides AC and BC of triangle ABC, two squares $ACDE$ and $BCFG$ are drawn. M is the midpoint of AB. Show that CM is perpendicular to DF. (Figure 3-9)*

Constructive description

((POINTS A B C)
 (TRATIO D C A 1)
 (TRATIO F C B −1)
 (MIDPOINT M A B)
 (PERPENDICULAR D F C M))

The machine proof

$$\frac{P_{DCM}}{P_{FCM}} \overset{M}{=} \frac{\frac{1}{2}P_{BCD}}{\frac{1}{2}P_{ACF}} \overset{F}{=} \frac{P_{BCD}}{4S_{ABC}} \overset{D}{=} \frac{4S_{ABC}}{(4)\cdot S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$P_{FCM} \overset{M}{=} \frac{1}{2}(P_{ACF})$

$P_{DCM} \overset{M}{=} \frac{1}{2}(P_{BCD})$

$P_{ACF} \overset{F}{=} 4(S_{ABC})$

$P_{BCD} \overset{D}{=} 4(S_{ABC})$

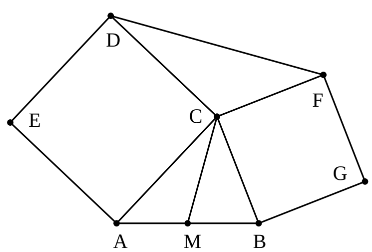


Figure 3-9

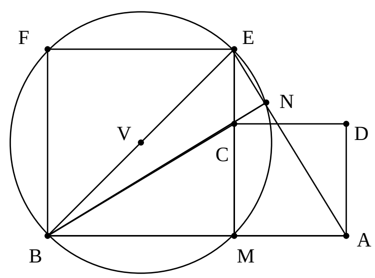


Figure 3-10

Example 3.42 [1] *Let M be a point on line AB. Two squares $AMCD$ and $BMEF$ are drawn on the same side of AB. Let U and V be the center of the squares $AMCD$ and $BMEF$. Line BC and circle VB meet in N. Show that A, E, and N are collinear. (Figure 3-10)*

Constructive description

((POINTS A B) (ON M (LINE A B)) (TRATIO C M A 1)
 (TRATIO E M B −1) (MIDPOINT V E B) (INTER N (LINE B C) (CIR V B))
 (INTER T (LINE B C) (LINE A E)) ($\frac{\overline{BN}}{\overline{CN}} = \frac{\overline{BT}}{\overline{CT}}$))

---

[1]This is a problem from the 1959 International Mathematical Olympiad.

The machine proof

$$\left(\frac{\overline{BN}}{\overline{CN}}\right)/\left(\frac{\overline{BT}}{\overline{CT}}\right)$$

$$\stackrel{T}{=} \frac{-S_{ACE}}{-S_{ABE}} \cdot \frac{\overline{BN}}{\overline{CN}}$$

$$\stackrel{N}{=} \frac{P_{CBV}\cdot S_{ACE}}{S_{ABE}\cdot(P_{CBV}-\frac{1}{2}P_{BCB})}$$

$$\stackrel{V}{=} \frac{(\frac{1}{2}P_{CBE})\cdot S_{ACE}}{S_{ABE}\cdot(\frac{1}{2}P_{CBE}-\frac{1}{2}P_{BCB})}$$

$$\stackrel{E}{=} \frac{(P_{MBC}+4S_{BMC})\cdot(\frac{1}{4}P_{MABC}-S_{AMC})}{(-\frac{1}{4}P_{ABM})\cdot(P_{MBC}-P_{BCB}+4S_{BMC})}$$

$$\stackrel{C}{=} \frac{-(P_{BMB}-P_{AMB})\cdot(P_{BMB}+P_{AMA}-P_{ABM})}{P_{ABM}\cdot(-P_{AMB}-P_{AMA})}$$

$$\stackrel{M}{=} \frac{(-P_{ABA}\cdot\frac{\overline{AM}}{\overline{AB}}+P_{ABA})\cdot(2P_{ABA}\cdot(\frac{\overline{AM}}{\overline{AB}})^2-P_{ABA}\cdot\frac{\overline{AM}}{\overline{AB}})}{(-P_{ABA}\cdot\frac{\overline{AM}}{\overline{AB}}+P_{ABA})\cdot(2P_{ABA}\cdot(\frac{\overline{AM}}{\overline{AB}})^2-P_{ABA}\cdot\frac{\overline{AM}}{\overline{AB}})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BT}}{\overline{CT}} \stackrel{T}{=} \frac{S_{ABE}}{S_{ACE}}$$

$$\frac{\overline{BN}}{\overline{CN}} \stackrel{N}{=} \frac{P_{CBV}}{(\frac{1}{2})\cdot(2P_{CBV}-P_{BCB})}$$

$$P_{CBV} \stackrel{V}{=} \frac{1}{2}\left(P_{CBE}\right)$$

$$S_{ABE} \stackrel{E}{=} -\frac{1}{4}\left(P_{ABM}\right)$$

$$S_{ACE} \stackrel{E}{=} \frac{1}{4}\left(P_{MABC}-4S_{AMC}\right)$$

$$P_{CBE} \stackrel{E}{=} P_{MBC}+4S_{BMC}$$

$$P_{BCB} \stackrel{C}{=} P_{BMB}+P_{AMA}$$

$$S_{AMC} \stackrel{C}{=} -\frac{1}{4}\left(P_{AMA}\right)$$

$$P_{MABC} \stackrel{C}{=} P_{BMB}-P_{ABM}$$

$$S_{BMC} \stackrel{C}{=} -\frac{1}{4}\left(P_{AMB}\right)$$

$$P_{MBC} \stackrel{C}{=} P_{BMB}$$

$$P_{ABM} \stackrel{M}{=} -\left((\frac{\overline{AM}}{\overline{AB}}-1)\cdot P_{ABA}\right)$$

$$P_{AMA} \stackrel{M}{=} P_{ABA}\cdot(\frac{\overline{AM}}{\overline{AB}})^2$$

$$P_{AMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}}-1)\cdot P_{ABA}\cdot\frac{\overline{AM}}{\overline{AB}}$$

$$P_{BMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}}-1)^2\cdot P_{ABA}$$

For more examples involving squares, see Section 9.4.

Example 3.43 *Given four points A, B, C, and D which form a harmonic sequence and a point O outside the line AB, any transversal cuts the four lines OA, OB, OC, and OD in four harmonic points.*

Constructive description

((POINTS $O\ A\ B\ X\ Y$)
(MRATIO $C\ A\ B\ r$)
(MRATIO $D\ A\ B\ -r$)
(INTER P (LINE $O\ A$) (LINE $X\ Y$))
(INTER Q (LINE $O\ B$) (LINE $X\ Y$))
(INTER R (LINE $O\ C$) (LINE $X\ Y$))
(INTER S (LINE $O\ D$) (LINE $X\ Y$))
(HARMONIC $P\ Q\ S\ R$))

The machine proof

$$(-\frac{\overline{PS}}{\overline{QS}})/(\frac{\overline{PR}}{\overline{QR}})$$

$$\overset{S}{=} \frac{-S_{ODP}}{\frac{\overline{PR}}{\overline{QR}}\cdot S_{ODQ}}$$

$$\overset{R}{=} \frac{-S_{ODP}\cdot S_{OCQ}}{S_{OCP}\cdot S_{ODQ}}$$

$$\overset{Q}{=} \frac{-S_{ODP}\cdot(-S_{OXY}\cdot S_{OBC})\cdot S_{OXBY}}{S_{OCP}\cdot(-S_{OXY}\cdot S_{OBD})\cdot S_{OXBY}}$$

$$\overset{simplify}{=} \frac{-S_{ODP}\cdot S_{OBC}}{S_{OCP}\cdot S_{OBD}}$$

$$\overset{P}{=} \frac{-(-S_{OXY}\cdot S_{OAD})\cdot S_{OBC}\cdot S_{OXAY}}{(-S_{OXY}\cdot S_{OAC})\cdot S_{OBD}\cdot S_{OXAY}}$$

$$\overset{simplify}{=} \frac{-S_{OAD}\cdot S_{OBC}}{S_{OAC}\cdot S_{OBD}}$$

$$\overset{D}{=} \frac{-(-S_{OAB}\cdot r)\cdot S_{OBC}\cdot(-r+1)}{S_{OAC}\cdot(-S_{OAB})\cdot(-r+1)}$$

$$\overset{simplify}{=} \frac{-r\cdot S_{OBC}}{S_{OAC}}$$

$$\overset{C}{=} \frac{-r\cdot(-S_{OAB})\cdot(r+1)}{S_{OAB}\cdot r\cdot(r+1)} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{PS}}{\overline{QS}} \overset{S}{=} \frac{S_{ODP}}{S_{ODQ}}$$

$$\frac{\overline{PR}}{\overline{QR}} \overset{R}{=} \frac{S_{OCP}}{S_{OCQ}}$$

$$S_{ODQ} \overset{Q}{=} \frac{-S_{OXY}\cdot S_{OBD}}{S_{OXBY}}$$

$$S_{OCQ} \overset{Q}{=} \frac{-S_{OXY}\cdot S_{OBC}}{S_{OXBY}}$$

$$S_{OCP} \overset{P}{=} \frac{-S_{OXY}\cdot S_{OAC}}{S_{OXAY}}$$

$$S_{ODP} \overset{P}{=} \frac{-S_{OXY}\cdot S_{OAD}}{S_{OXAY}}$$

$$S_{OBD} \overset{D}{=} \frac{S_{OAB}}{r-1}$$

$$S_{OAD} \overset{D}{=} \frac{S_{OAB}\cdot r}{r-1}$$

$$S_{OAC} \overset{C}{=} \frac{S_{OAB}\cdot r}{r+1}$$

$$S_{OBC} \overset{C}{=} \frac{-S_{OAB}}{r+1}$$

For more examples involving harmonic sequences, see Section 6.3.

Example 3.44 *The inverse of a circle passing through the center of inversion is a line.*

Constructive description

((points $O\ A\ X$)
(lratio $P\ O\ A\ r_1$)
(inversion $Q\ P\ O\ A$)
(midpoint $U\ P\ O$)
(inter R (l $O\ X$) (cir $U\ O$))
(inversion $G\ R\ O\ A$)
(perpendicular $G\ Q\ O\ A$))

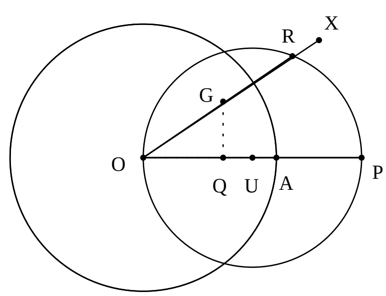


Figure 3-12

The machine proof

$$\frac{P_{AOG}}{P_{AOQ}}$$

$$= \frac{P_{AOR}\cdot P_{OAO}}{P_{AOQ}\cdot P_{ORO}}$$

$$\overset{R}{=} \frac{(2P_{XOU}\cdot P_{AOX})\cdot P_{OAO}\cdot P_{OXO}}{P_{AOQ}\cdot(4P_{XOU}^2)\cdot P_{OXO}}$$

$$\overset{simplify}{=} \frac{P_{AOX}\cdot P_{OAO}}{(2)\cdot P_{AOQ}\cdot P_{XOU}}$$

$$\overset{U}{=} \frac{P_{AOX}\cdot P_{OAO}}{(2)\cdot P_{AOQ}\cdot(\frac{1}{2}P_{XOP})}$$

$$= \frac{P_{AOX}\cdot P_{OAO}}{\frac{\overline{OA}}{\overline{OP}}\cdot P_{OAO}\cdot P_{XOP}}$$

$$\overset{simplify}{=} \frac{P_{AOX}}{\frac{\overline{OA}}{\overline{OP}}\cdot P_{XOP}}$$

$$\overset{P}{=} \frac{P_{AOX}\cdot r_1}{P_{AOX}\cdot r_1} \overset{simplify}{=} 1$$

The eliminants

$$P_{AOG} = \frac{P_{AOR}\cdot P_{OAO}}{P_{ORO}}$$

$$P_{ORO} \overset{R}{=} \frac{(4)\cdot(P_{XOU})^2}{P_{OXO}}$$

$$P_{AOR} \overset{R}{=} \frac{(2)\cdot P_{XOU}\cdot P_{AOX}}{P_{OXO}}$$

$$P_{XOU} \overset{U}{=} \frac{1}{2}\left(P_{XOP}\right)$$

$$P_{AOQ} = \frac{\overline{OA}}{\overline{OP}}\cdot P_{OAO}$$

$$P_{XOP} \overset{P}{=} P_{AOX}\cdot r_1$$

$$\frac{\overline{OA}}{\overline{OP}} \overset{P}{=} \frac{1}{r_1}$$

For more examples involving inversions, see Section 10.3.

The ratio constructions are used extensively in the following example. Also notice that the construction CONSTANT is used to describe equilateral triangles.

Example 3.45 *Three equilateral triangles A_1BC, AB_1C, ABC_1 are erected on the three sides of triangle ABC. Show that $CA_1C_1B_1$ is a parallelogram.*

Constructive description
((points A B C)
(constant r^2-3)
(midpoint E A C)
(tratio B_1 E A r)
(midpoint F B C)
(tratio A_1 F C r)
(midpoint G A B)
(tratio C_1 G B $-r$)
(parallel A_1 C_1 C B_1))

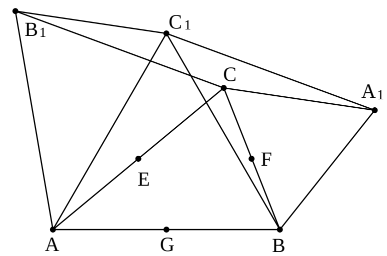


Figure 3-13

The machine proof

$$\frac{S_{CB_1A_1}}{S_{CB_1C_1}}$$

$$\overset{C_1}{=} \frac{S_{CB_1A_1}}{\frac{1}{4}P_{B_1BCG}\cdot r+S_{CB_1G}}$$

$$\overset{G}{=} \frac{(4)\cdot S_{CB_1A_1}}{-\frac{1}{2}P_{BCB_1}\cdot r+\frac{1}{2}P_{ACB_1}\cdot r+2S_{BCB_1}+2S_{ACB_1}}$$

$$\overset{A_1}{=} \frac{(-8)\cdot(-\frac{1}{4}P_{B_1CF}\cdot r+S_{CB_1F})}{P_{BCB_1}\cdot r-P_{ACB_1}\cdot r-4S_{BCB_1}-4S_{ACB_1}}$$

$$\overset{F}{=} \frac{(2)\cdot(\frac{1}{2}P_{BCB_1}\cdot r-2S_{BCB_1})}{P_{BCB_1}\cdot r-P_{ACB_1}\cdot r-4S_{BCB_1}-4S_{ACB_1}}$$

$$\overset{B_1}{=} \frac{P_{CABE}\cdot r+P_{BCE}\cdot r-4S_{BCE}+4S_{ABE}\cdot r^2}{P_{CABE}\cdot r+P_{CAE}\cdot r+P_{BCE}\cdot r-P_{ACE}\cdot r-4S_{BCE}+4S_{ABE}\cdot r^2}$$

$$\overset{E}{=} \frac{\frac{1}{2}P_{BCB}\cdot r+\frac{1}{2}P_{ACB}\cdot r-\frac{1}{2}P_{ABC}\cdot r+2S_{ABC}\cdot r^2-2S_{ABC}}{\frac{1}{2}P_{BCB}\cdot r+\frac{1}{2}P_{ACB}\cdot r-\frac{1}{2}P_{ABC}\cdot r+2S_{ABC}\cdot r^2-2S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{CB_1C_1} \overset{C_1}{=} \frac{1}{4}(P_{B_1BCG}\cdot r+4S_{CB_1G})$$
$$S_{CB_1G} \overset{G}{=} \frac{1}{2}(S_{BCB_1}+S_{ACB_1})$$
$$P_{B_1BCG} \overset{G}{=} -\frac{1}{2}(P_{BCB_1}-P_{ACB_1})$$
$$S_{CB_1A_1} \overset{A_1}{=} -\frac{1}{4}(P_{B_1CF}\cdot r-4S_{CB_1F})$$
$$S_{CB_1F} \overset{F}{=} \frac{1}{2}(S_{BCB_1})$$
$$P_{B_1CF} \overset{F}{=} \frac{1}{2}(P_{BCB_1})$$
$$S_{ACB_1} \overset{B_1}{=} -\frac{1}{4}(P_{CAE}\cdot r)$$
$$P_{ACB_1} \overset{B_1}{=} P_{ACE}$$
$$S_{BCB_1} \overset{B_1}{=} -\frac{1}{4}(P_{CABE}\cdot r-4S_{BCE})$$
$$P_{BCB_1} \overset{B_1}{=} P_{BCE}+4S_{ABE}\cdot r$$
$$P_{ACE} \overset{E}{=} \frac{1}{2}(P_{ACA})$$
$$P_{CAE} \overset{E}{=} \frac{1}{2}(P_{ACA})$$
$$S_{ABE} \overset{E}{=} \frac{1}{2}(S_{ABC})$$
$$S_{BCE} \overset{E}{=} \frac{1}{2}(S_{ABC})$$
$$P_{BCE} \overset{E}{=} \frac{1}{2}(P_{ACB})$$
$$P_{CABE} \overset{E}{=} \frac{1}{2}(P_{BCB}-P_{ABC})$$

Note that the condition $r^2 = 3$ is not needed in the proof, i.e., the result is true if triangles B_1AC, A_1BC, and C_1AB are similar isosceles triangles. The above proof is used to illustrate the use of ratio constructions. For a much shorter proof of this example, see Example 5.61 on page 253.

3.4.2 Mechanization of Full-Angles

As an application of the construction TRATIO, we will present an automated theorem proving method for geometry theorems involving full-angles. The formal definition of full-angles is as follows.

Definition 3.46 *An ordered pair of lines AB and CD determines a full-angle, denoted by* $\angle[AB, CD]$, *which satisfies*

1. $\angle[AB, CD] = \angle[PQ, UV]$ *if and only if*

$$S_{ACBD} P_{PUQV} = S_{PUQV} P_{ACBD}.$$

   Thus the tangent function for the full-angle,

$$\tan(\angle[AB, CD]) = \frac{4S_{ACBD}}{P_{ADBC}}$$

   is a well defined geometry quantity.

2. *For all parallel lines* $AB \parallel PQ$, $\angle[0] = \angle[AB, PQ]$ *is a constant.*

3. *For all perpendicular lines* $AB \perp PQ$, $\angle[1] = \angle[AB, PQ]$ *is a constant.*

4. *There exists an operation "+" for full-angles which is associative and commutative. Furthermore, we have*

   - $\angle[1] + \angle[1] = \angle[0]$.
   - *If* $PQ \parallel UV$ *then* $\angle[AB, PQ] + \angle[UV, CD] = \angle[AB, CD]$.
   - *The tangent function of the sum of two full-angles is defined as follows*

$$\tan(\angle[AB, CD] + \angle[PQ, UV]) = \frac{\tan(\angle[AB, CD]) + \tan(\angle[PQ, UV])}{1 - \tan(\angle[AB, CD]) \tan(\angle[PQ, UV])}.$$

You can find the geometric background for the above definition in Section 1.10. For three points A, B, and C, let $\angle[ABC] = \angle[AB, BC]$.

Remark 3.47 *According to the above definition,* $\angle[AB, CD] = \angle[PQ, UV]$ *if and only if one of the following conditions holds.*

1. (PARA $A\ B\ C\ D$) and (PARA $P\ Q\ U\ V$);
2. (PERP $A\ B\ C\ D$) and (PERP $P\ Q\ U\ V$);
3. $A \neq B, C \neq D, P \neq Q, U \neq V$ and the full-angle $\angle[AB, CD]$ is equal to the full-angle $\angle[PQ, UV]$.

Proposition 3.48 (The Co-angle Theorem) *In triangles ABC and XYZ, if* $\angle[ABC] = \angle[XYZ]$, $\angle[ABC] \neq \angle[1]$, *and* $\angle[ABC] \neq \angle[0]$, *then*

$$\frac{S_{ABC}}{S_{XYZ}} = \frac{P_{ABC}}{P_{XYZ}} = \lambda \quad where \quad \lambda^2 = \frac{\overline{AB}^2 \cdot \overline{BC}^2}{\overline{XY}^2 \cdot \overline{ZY}^2}.$$

Proof. By Definition 3.46 if $\angle[ABC] = \angle[XYZ]$ then $\frac{S_{ABC}}{S_{XYZ}} = \frac{P_{ABC}}{P_{XYZ}} = \lambda$. By the Herron-Qin formula

$$16S_{ABC}^2 + P_{ABC}^2 = 4\overline{AB}^2 \cdot \overline{CB}^2, \quad 16S_{XYZ}^2 + P_{XYZ}^2 = 4\overline{XY}^2 \cdot \overline{ZY}^2.$$

Set $S_{ABC} = \lambda S_{XYZ}, P_{ABC} = \lambda P_{XYZ}$ in the first equation we have

$$\lambda^2 = \frac{4\overline{AB}^2 \cdot \overline{CB}^2}{16S_{AYX}^2 + P_{AYZ}^2} = \frac{4\overline{AB}^2 \cdot \overline{CB}^2}{4\overline{XY}^2 \cdot \overline{ZY}^2}. \quad \blacksquare$$

With the concept of full-angles, the constructive geometry statements can be extended as follows. First we have a new geometry quantity: the tangent function of full-angles. Since the tangent function can be represented by the area and Pythagoras difference, we do not need to introduce new elimination techniques to eliminate points from it. Its main contribution is that we may now prove assertions like $\angle[AB, CD] = \angle[PQ, UV]$ and $\angle[AB, CD] = \angle[PQ, UV] + \angle[XY, WZ]$.

The second extension is more interesting: we can introduce a new representation for lines.

(ALINE $P\ Q\ U\ W\ V$) which is the line l passing through P such that $\angle[PQ, l] = \angle[UW, WV]$.

With this new type of lines, we can introduce seven new constructions:

1. (ON Y (ALINE $P\ Q\ L\ M\ N$)). Take an arbitrary point on an ALINE; the ndg conditions are $P \neq Q$, $L \neq M$, and $N \neq M$.

2. (INTER Y ln (ALINE $P\ Q\ L\ M\ N$)). Take the intersection of ln and (ALINE $P\ Q\ L\ M\ N$).
   - If ln = (LINE $U\ V$) or ln = (PLINE $W\ U\ V$), then the ndg condition is $\angle[PQ, UV] \neq \angle[LM, MN]$.
   - If ln = (BLINE $U\ V$) or ln = (TLINE $W\ U\ V$), then the ndg condition is $\angle[UV, PQ] + \angle[LM, MN] \neq \angle[1]$.
   - If ln = (ALINE $U\ V\ X\ Y\ Z$), then the ndg condition is $\angle[UV, PQ] \neq \angle[NM, ML] + \angle[XY, YZ]$.

3. (INTER Y (CIR O P) (ALINE P Q L M N)). The ndg conditions are $Y \neq P$, $P \neq O$, $P \neq Q$, $L \neq M$, and $N \neq M$.

To provide methods of eliminating points introduced by these new constructions from geometry quantities, we need only to reduce an ALINE to a LINE.

Proposition 3.49 *If UW is not perpendicular to WV, line l =(ALINE P Q U W V) is the same as (LINE P R) where R is introduced by construction (TRATIO R Q P $\frac{4S_{UWV}}{P_{UWV}}$).*

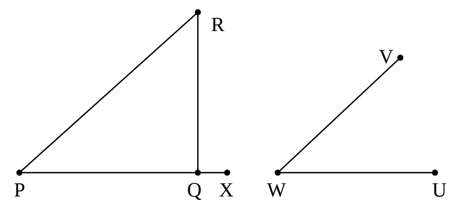


Figure 3-14

If $UW \perp WV$, line l is (TLINE P P Q).

Proof. Let the line passing through point Q and perpendicular to PQ meet line l in R. Then R is introduced by construction (TRATIO R Q P r), where

$$r = \frac{4S_{RQP}}{P_{QPQ}} = \frac{4S_{QPR}}{P_{QPR}} = \tan(\angle[RPQ]) = \tan(\angle[VWU]) = \frac{4S_{UWV}}{P_{UWV}}. \quad \blacksquare$$

Remark 3.50 *From the above proposition we see that the construction (TRATIO Y P Q r) is actually to take a point Y such that $YP \perp PQ$ and* $\tan(\angle[YPQ]) = r$.

Now, we introduce the following predicates.

(EQANGLE A B C D E F). $\angle[ABC] = \angle[DEF]$ iff $S_{ABC}P_{DEF} = S_{DEF}P_{ABC}$.

(COCIRCLE A B C D). Points A, B, C, and D are co-circle iff $\angle[CAD] = \angle[CBD]$, or equivalently, $S_{CAD}P_{CBD} = P_{CAD}P_{CBD}$.

Example 3.51 *If N, M are points on the sides AC, AB of a triangle ABC and the lines BN, CM intersect at a point J which is on the altitude AD, show that AD is the bisector of the angle MDN.*

Constructive description
((POINTS A B C)
(FOOT D A B C)
(ON J (LINE A D))
(INTER M (LINE A B) (LINE C J))
(INTER N (LINE A C) (LINE B J))
(EQANGLE M D A A D N))

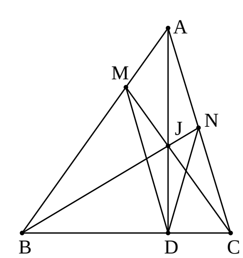


Figure 3-15

The ndg conditions: $B \neq C$, $A \neq D$, $AB \nparallel CJ$, $AC \nparallel BJ$.

The machine proof

$$\frac{(-S_{ADM})\cdot P_{ADN}}{S_{ADN}\cdot P_{ADM}}$$

$$\stackrel{N}{=} \frac{(-S_{ADM})\cdot(-P_{ADJ}\cdot S_{ABC})\cdot S_{ABCJ}}{(-S_{ACD}\cdot S_{ABJ})\cdot P_{ADM}\cdot(-S_{ABCJ})}$$

$$\stackrel{simplify}{=} \frac{S_{ADM}\cdot P_{ADJ}\cdot S_{ABC}}{S_{ACD}\cdot S_{ABJ}\cdot P_{ADM}}$$

$$\stackrel{M}{=} \frac{(-S_{ACJ}\cdot S_{ABD})\cdot P_{ADJ}\cdot S_{ABC}\cdot(-S_{ACBJ})}{S_{ACD}\cdot S_{ABJ}\cdot P_{ADJ}\cdot S_{ABC}\cdot S_{ACBJ}}$$

$$\stackrel{simplify}{=} \frac{S_{ACJ}\cdot S_{ABD}}{S_{ACD}\cdot S_{ABJ}}$$

$$\stackrel{J}{=} \frac{S_{ACD}\cdot\frac{\overline{AJ}}{\overline{AD}}\cdot S_{ABD}}{S_{ACD}\cdot S_{ABD}\cdot\frac{\overline{AJ}}{\overline{AD}}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{ADN} \stackrel{N}{=} \frac{-S_{ACD}\cdot S_{ABJ}}{S_{ABCJ}}$$

$$P_{ADN} \stackrel{N}{=} \frac{P_{ADJ}\cdot S_{ABC}}{S_{ABCJ}}$$

$$P_{ADM} \stackrel{M}{=} \frac{P_{ADJ}\cdot S_{ABC}}{-S_{ACBJ}}$$

$$S_{ADM} \stackrel{M}{=} \frac{-S_{ACJ}\cdot S_{ABD}}{S_{ACBJ}}$$

$$S_{ABJ} \stackrel{J}{=} S_{ABD}\cdot\frac{\overline{AJ}}{\overline{AD}}$$

$$S_{ACJ} \stackrel{J}{=} S_{ACD}\cdot\frac{\overline{AJ}}{\overline{AD}}$$

Example 3.52 (The Inscribed Angle Theorem) *Let A, B, C, and D be four points on a circle with center O. Then $\angle[ACB] = \angle[ADB]$ and $\angle[AOB] = 2\angle[ACB]$.*

Proof. We first use our program to compute $\tan(\angle[ACB])$.

Constructive description

((POINTS A B)
 (ON O (BLINE A B))
 (TRATIO P B A r)
 (INTER C (LINE A P) (CIR O A))
 ((TANGENT A C B)))

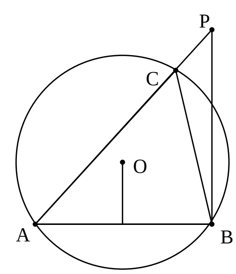


Figure 3-16

The machine proof

$$\frac{(-4)\cdot S_{ABC}}{P_{ACB}}$$

$$\stackrel{C}{=} \frac{(-4)\cdot(2P_{OAP}\cdot S_{ABP})\cdot P_{APA}}{(-2P_{OPO}\cdot P_{OAP}+2P_{OAP}\cdot P_{APB}+2P_{OAP}\cdot P_{AOA})\cdot P_{APA}}$$

$$\stackrel{simplify}{=} \frac{(4)\cdot S_{ABP}}{P_{OPO}-P_{APB}-P_{AOA}}$$

$$\stackrel{P}{=} \frac{(4)\cdot(-\frac{1}{4}P_{ABA}\cdot r)}{P_{BOB}-P_{AOA}+8S_{ABO}\cdot r}$$

$$\stackrel{O}{=} \frac{-P_{ABA}\cdot r}{8S_{ABO}\cdot r}$$

$$\stackrel{simplify}{=} \frac{-P_{ABA}}{(8)\cdot S_{ABO}}$$

The eliminants

$$P_{ACB} \stackrel{C}{=} \frac{(-2)\cdot(P_{OPO}-P_{APB}-P_{AOA})\cdot P_{OAP}}{P_{APA}}$$

$$S_{ABC} \stackrel{C}{=} \frac{(2)\cdot P_{OAP}\cdot S_{ABP}}{P_{APA}}$$

$$P_{APB} \stackrel{P}{=} P_{ABA}\cdot(r)^2$$

$$P_{OPO} \stackrel{P}{=} P_{BOB}+P_{ABA}\cdot r^2+8S_{ABO}\cdot r$$

$$S_{ABP} \stackrel{P}{=} -\frac{1}{4}\left(P_{ABA}\cdot r\right)$$

From the above computation, it is clear that $\tan(\angle[ACB])$ is independent of point P and C, i.e., $\tan(\angle[ACB]) = \tan(\angle[ADB])$.

$$\begin{aligned}\tan(2\angle[ABC]) &= \frac{2\tan(\angle[ABC])}{1-\tan(\angle[ABC])^2} = \frac{8\overline{AB}^2 S_{AOB}}{16S_{AOB}^2 - \overline{AB}^4} \\ &= \frac{8\overline{AB}^2 S_{AOB}}{4\overline{OA}^2\cdot\overline{OB}^2 - P_{AOB}^2 - \overline{AB}^4} \\ &= \frac{4S_{AOB}}{2\overline{AO}^2 - \overline{AB}^2} = \frac{4S_{AOB}}{P_{AOB}} = \tan(\angle[AOB])\end{aligned}$$

i.e., $2\angle[ABC] = \angle[AOB]$. ∎

Example 3.53 (Morley's Theorem) *The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.*

Constructive description
((POINTS *B C L*)
(ON *X* (ALINE *B L L B C*))
(ON *Y* (ALINE *B X L B C*))
(ON *Z* (ALINE *C L L C B*))
(ON *W* (ALINE *C Z L C B*))
(INTER *A* (LINE *B Y*) (LINE *C W*))
(MIDPOINT *O L C*)
(CONSTANT r^2 3)
(TRATIO *T O C r*)
(INTER *Q* (LINE *B L*) (LINE *T C*))
(ON *H* (ALINE *A B T L Q*))
(INTER *M* (LINE *B X*) (LINE *A H*)) (ON *G* (ALINE *A C Q L T*))
(INTER *N* (LINE *C Z*) (LINE *A G*)) ((TANGENT *M L N*) = *r*))

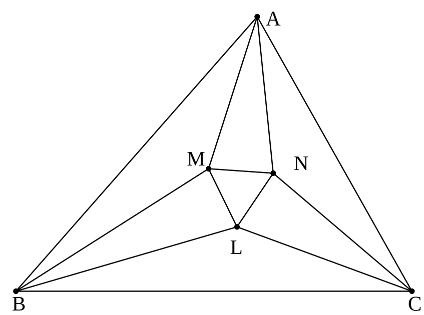


Figure 3-17

Morley's theorem is among the most difficult problems proved by our method. The proof for it is still too long to be considered readable. Our method need further improvement to produce a readable proof for this theorem.

Exercises 3.54

1. Note that the sine and cosine functions for full-angles are meaningless. But we can define their squares. For a full-angle α, let $\sin^2(\alpha) = \frac{\tan^2(\alpha)}{1+\tan^2(\alpha)}$, $\cos^2(\alpha) = \frac{1}{1+\tan^2(\alpha)}$. Then

$$S_{ABCD}^2 = \frac{1}{4}\overline{AC}^2\cdot\overline{BD}^2\sin^2(\angle[AC,BD]), \quad P_{ABCD}^2 = 4\overline{AC}^2\cdot\overline{BD}^2\cos^2(\angle[AC,BD]).$$

(Use the Herron-Qin formula.)

2. Try to eliminate Y from S_{ABY} if Y is introduced by (INTER Y (LINE U V) (ALINE P Q L M N)). Show that the ndg conditions ensure that all the geometric quantities occurring in the elimination have geometric meaning.
3. If Y is introduced by (INTER Y (TLINE U U O) (TLINE V V O)) then

$$S_{ABY} = \begin{cases} S_{ABU} & \text{if } AB \perp OU; \\ S_{ABV} & \text{if } AB \perp OV; \\ \frac{P_{OUV}P_{OVU}}{-16S_{OUV}} & \text{if } A = U, B = V; \\ \frac{P_{OVU}P_{OUO}}{16S_{OUV}} & \text{if } A = O, B = U; \\ \frac{P_{OUV}P_{OVO}}{16S_{OVU}} & \text{if } A = O, B = V; \end{cases}$$

$$P_{ABY} = \begin{cases} P_{ABU} & \text{if } AB \parallel OU; \\ P_{ABV} & \text{if } AB \parallel OV; \\ P_{OUV} & \text{if } A = U, B = V; \end{cases}$$

$$P_{AYB} = \begin{cases} \frac{-P_{UOV}P_{OVU}P_{OUV}}{16S_{OUV}^2} & \text{if } A = U, B = V; \\ \frac{P_{OUO}P_{OVU}^2}{16S_{OUV}^2} & \text{if } A = O, B = U \text{ or } \quad A = B = U; \\ \frac{P_{OVO}P_{OUV}^2}{16S_{OUV}^2} & \text{if } A = O, B = V \text{ or } \quad A = B = V; \\ \frac{P_{OUO}P_{OVO}P_{UVU}}{16S_{OUV}^2} & \text{if } A = B = O. \end{cases}$$

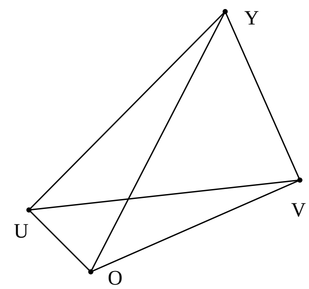


Figure 3-18

In the general case, this construction can be reduced to the following construction

$$\text{(TRATIO } Y\ U\ O\ \tfrac{P_{OVU}}{4S_{OVU}}\text{)}.$$

Notice that this construction is actually to introduce the antipodal point of O with respect to the circumcircle of triangle OUV.

3.5 Area Coordinates

3.5.1 Area Coordinate Systems

In Lemma 3.31, we use an orthogonal coordinate system. In order to do this, we have to introduce three auxiliary points O, U, and V. In this section, we will develop a *skew area coordinate system* in which any three free points can be used as the reference points. As a consequence, we obtain a new version of Lemma 3.31 and consequently a new version of Algorithm 3.33. We will also prove some interesting features of the skew area coordinate system.

Let O, U, and V be three non-collinear points. For any point A, let

$$x_A = \frac{S_{OUA}}{S_{OUV}}, \quad y_A = \frac{S_{VOA}}{S_{OUV}}, \quad z_A = \frac{S_{UVA}}{S_{OUV}}$$

be the area coordinates of A with respect to OUV. It is clear that $x_A + y_A + z_A = 1$. Below are some results proved before.

Proposition 3.55 *The points in the plane are in a one to one correspondence with the triples* (x, y, z) *such that* $x + y + z = 1$.

Proposition 3.56 *For any points* A, B, *and* C, *we have*

$$S_{ABC} = -S_{OUV}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = -S_{OUV}\begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}.$$

As a consequence of Proposition 3.56, we can give the line equation in the area coordinate system. Let P be a point on line AB. Then the area coordinates of P must satisfy

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_P & y_P & 1 \end{vmatrix} = 0$$

which is the equation for line AB. Notice that this is the same as the line equation in the Cartesian coordinate system. Another interesting fact is the position ratio formula.

Proposition 3.57 *Let* R *be a point on line* PQ *and* $r_1 = \frac{\overline{PR}}{\overline{PQ}}$ *and* $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$. *Then*

$$x_R = r_1 x_Q + r_2 x_P; \quad y_R = r_1 y_Q + r_2 y_P; \quad z_R = r_1 z_Q + r_2 z_P.$$

Proof. This is a consequence of Proposition 2.9. ∎

We will now prove the distance formula between two points.

Proposition 3.58 *For two points* A *and* B, *we have*

$$\overline{AB}^2 = \overline{OV}^2(x_B - x_A)^2 + \overline{OU}^2(y_B - y_A)^2 + (x_B - x_A)(y_B - y_A)P_{UOV}.$$

Proof. Let M and N be points such that $MA \parallel OU$, $MB \parallel OV$, $NA \parallel OV$, and $NB \parallel OU$. Then by Proposition 3.6.

(1) $\overline{AB}^2 = \overline{AM}^2 + \overline{BM}^2 - P_{AMB}$
$= \overline{OU}^2(\frac{\overline{AM}}{\overline{OU}})^2 + \overline{OV}^2(\frac{\overline{BM}}{\overline{OV}})^2 + P_{NAM}$.

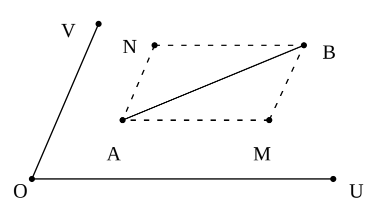


Figure 3-19

By Lemma 2.28, $\frac{\overline{AM}}{\overline{OU}} = \frac{S_{BOAV}}{S_{UOV}} = y_B - y_A$, $\frac{\overline{BM}}{\overline{OV}} = \frac{S_{BOAU}}{S_{OVU}} = x_A - x_B$. By Example 3.9, $P_{NAM} = \frac{\overline{AN}}{\overline{OV}} \cdot \frac{\overline{AM}}{\overline{OU}} \cdot P_{UOV} = (y_B - y_A)(x_B - x_A)P_{UOV}$. Substituting these into (1), we prove the result. ∎

Corollary 3.59 *Show that*

$$2\overline{AB}^2 = P_{OVU}(x_B - x_A)^2 + P_{OUV}(y_B - y_A)^2 + P_{UOV}(z_B - z_A)^2.$$

Proof. We need only to observe that $z_A - z_B = (x_B - x_A) + (y_B - y_A)$. ∎

In Algorithm 3.33, let E be an expression in areas and Pythagoras differences of free points. Instead of using Lemma 3.31, we can use the following procedure to transform E into an expression in independent variables. If there are fewer than three points involved in E then we need do nothing; if there are more than two points, choose three free points O, U, and V from the points occurring in E, and apply Propositions 3.56 and 3.58 to E to transform the areas and Pythagoras differences into area coordinates with respect to OUV. Now the new E is an expression in area coordinates of free points, $\overline{OU}^2, \overline{OV}^2, \overline{UV}^2$, and S_{OUV}. The only algebraic relation among these quantities is the Herron-Qin formula (Proposition 3.16):

$$16S_{OUV}^2 = 4\overline{OU}^2\overline{OV}^2 - (\overline{OV}^2 + \overline{OU}^2 - \overline{UV}^2)^2 = 4\overline{OU}^2\overline{OV}^2 - P_{UOV}^2.$$

Substituting S_{OUV}^2 into E, we obtain an expression in independent variables.

If we assume that $OU \perp OV$ and $|OU| = |OV| = 1$ then the area coordinate system becomes the Cartesian coordinate system.

Exercises 3.60

1. The process of transforming the areas and Pythagoras differences of free points into expressions in independent variables presented in this section becomes particularly simple if there are only three free points in a geometry statement. Let the three free points be O, U, and V. Show that any polynomial g in the areas and Pythagoras differences involving O, U, and V can be transformed into a polynomial of the following form $g' = f + S_{OUV}h$ where f and h are polynomials of $\overline{OU}^2, \overline{OV}^2$, and $\overline{UV}^2$. We also have that $g = 0$ iff $h = f = 0$.
2. If in a geometry statement there are only two free points U and V, and the third point O is introduced by the construction (ON O (BLINE U V)) or (ON O (TLINE U V)), design a simple method of transforming a polynomial in the areas and Pythagoras differences involving points O, U, and V into a polynomial of independent variables. (Chose $\overline{OU}^2$ and $\overline{UV}^2$ as independent variables.)

3.5.2 Area Coordinates and Special Points of Triangles

We introduce a new construction.

C16 (ARATIO A O U V r_O r_U r_V). Take a point A such that

$$r_O = \frac{S_{AUV}}{S_{OUV}}, \quad r_U = \frac{S_{OAV}}{S_{OUV}}, \quad r_V = \frac{S_{OUA}}{S_{OUV}}$$

are the *area coordinates* of A with respect to OUV. The r_O, r_U, and r_V could be rational numbers, rational expressions in geometric quantities, indeterminates, or algebraic numbers. The ndg condition is that O, U, and V are not collinear. The degree of freedom for A is dependent on the number of indeterminates in $\{r_O, r_U, r_V\}$.

Lemma 3.61 *Let $G(Y)$ be a linear geometry quantity and Y be introduced by (ARATIO Y O U V r_O r_U r_V). Then*

$$G(Y) = r_O G(O) + r_U G(U) + r_V G(V).$$

Proof. Without loss of generality, let OY intersect UV at T. If OY is parallel to UV, we may consider the intersection of UY and OV or the intersection of VY and OU since one of them must exist. By Proposition 2.10,

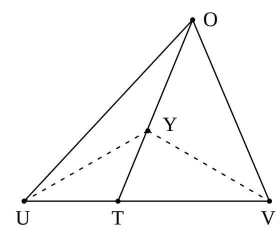


Figure 3-20

$$G(Y) = \frac{\overline{OY}}{\overline{OT}} G(T) + \frac{\overline{YT}}{\overline{OT}} G(O) = \frac{\overline{OY}}{\overline{OT}} \left(\frac{\overline{UT}}{\overline{UV}} G(V) + \frac{\overline{TV}}{\overline{UV}} G(U)\right) + \frac{\overline{YT}}{\overline{OT}} G(O).$$

By the co-side theorem, $\frac{\overline{YT}}{\overline{OT}} = r_O$; $\frac{\overline{OY}}{\overline{OT}} = \frac{S_{OUYV}}{S_{OUV}}$; $\frac{\overline{UT}}{\overline{UV}} = \frac{S_{OUY}}{S_{OUYV}}$; $\frac{\overline{TV}}{\overline{UV}} = \frac{S_{OYV}}{S_{OUYV}}$. Substituting these into the above formula, we obtain the desired result. ∎

Lemma 3.62 *Let $G(Y)$ be a quadratic geometry quantity and Y be introduced by (ARATIO Y O U V r_O r_U r_V). Then*

$$G(Y) = r_O G(O) + r_U G(U) + r_V G(V) - 2(r_O r_U \overline{OU}^2 + r_O r_V \overline{OV}^2 + r_U r_V \overline{UV}^2).$$

Proof. Continue from the proof of Lemma 3.61, By (II) on page 116

$$G(Y) = \frac{\overline{OY}}{\overline{OT}} G(T) + \frac{\overline{YT}}{\overline{OT}} G(O) - \frac{\overline{OY}}{\overline{OT}} \frac{\overline{YT}}{\overline{OT}} P_{OTO}$$

$$G(T) = \frac{\overline{UT}}{\overline{UV}} G(V) + \frac{\overline{TV}}{\overline{UV}} G(U) - \frac{\overline{UT}}{\overline{UV}} \frac{\overline{TV}}{\overline{UV}} P_{UVU}.$$

Substituting $G(T)$ into $G(Y)$, we have

$$G(Y) - r = -\frac{\overline{OY}}{\overline{OT}} \frac{\overline{UT}}{\overline{UV}} \frac{\overline{TV}}{\overline{UV}} P_{UVU} - \frac{\overline{OY}}{\overline{OT}} \frac{\overline{YT}}{\overline{OT}} P_{OTO} = -r_V \frac{\overline{TV}}{\overline{UV}} P_{UVU} - r_A \frac{\overline{OY}}{\overline{OT}} P_{OTO},$$

where $r = r_O G(O) + r_U G(U) + r_V G(V)$. By (II),

$$P_{OTO} = \frac{\overline{UT}}{\overline{UV}} P_{OVO} + \frac{\overline{TV}}{\overline{UV}} P_{OUO} - \frac{\overline{UT}}{\overline{UV}} \frac{\overline{TV}}{\overline{UV}} P_{UVU}.$$

Then

$$\begin{aligned} & G(Y) - r \\ = \ & -r_V \frac{\overline{TV}}{\overline{UV}} P_{UVU} - r_O \frac{\overline{OY}}{\overline{OT}} \frac{\overline{UT}}{\overline{UV}} P_{OVO} - r_O \frac{\overline{OY}}{\overline{OT}} \frac{\overline{TV}}{\overline{UV}} P_{OUO} + r_O \frac{\overline{OY}}{\overline{OT}} \frac{\overline{UT}}{\overline{UV}} \frac{\overline{TV}}{\overline{UV}} P_{UVU} \\ = \ & -r_O r_V P_{OVO} - r_O r_U P_{OUO} - r_U r_V \left(-\frac{S_{YUV}}{S_{OUYV}} + \frac{S_{OUV}}{S_{OUYV}}\right) P_{UVU} \\ = \ & -r_O r_V P_{OVO} - r_O r_U P_{OUO} - r_U r_V P_{UVU}. \quad \blacksquare \end{aligned}$$

If Y is introduced by construction ARATIO, then we need rarely to eliminate Y from $G = \frac{\overline{DY}}{\overline{EF}}$. Because points D, E, and F are introduced before point Y, we generally do not know whether DY is parallel to EF or not. But in some special cases, we still need to eliminate point Y from G. This can be done as follows. One of O, U, and V, say O, satisfies the condition that D, Y, and O are not collinear. Then $G = \frac{S_{ODY}}{S_{OEDF}}$. Now, we can use Lemma 3.61 to eliminate Y.

By using the construction ARATIO, we can treat some special points of triangles easily.

Proposition 3.63 *The area coordinate of the centroid of a triangle ABC is* $(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$.

Proof. Let G be the centroid of $\triangle ABC$ and M be the midpoint of BC. Since M is the midpoint of BC, we have $S_{ABM} = S_{AMC}$ and $S_{GBM} = S_{GMC}$. Then $S_{GAB} = S_{ABM} - P_{GBM} = S_{AMC} - S_{GMC} = S_{GCA}$. Similarly $S_{GBC} = S_{GAB} = S_{GCA} = \frac{1}{3} S_{ABC}$. $\blacksquare$

Proposition 3.64 *The area coordinate of the orthocenter of triangle ABC is*

$$\left(\frac{P_{ABC}P_{ACB}}{16S_{ABC}^2} \quad \frac{P_{BAC}P_{BCA}}{16S_{ABC}^2} \quad \frac{P_{CAB}P_{CBA}}{16S_{ABC}^2}\right).$$

Proof. As in the figure, let H be the intersection of the altitudes CD and AE. Then

$$\begin{aligned} \frac{r_B}{r_A} &= \frac{S_{AHC}}{S_{HBC}} = \frac{\overline{AD}}{\overline{DB}} = \frac{P_{CAB}}{P_{ABC}} \\ \frac{r_B}{r_C} &= \frac{S_{AHC}}{S_{ABH}} = \frac{\overline{CE}}{\overline{EB}} = \frac{P_{BCA}}{P_{ABC}}. \end{aligned}$$

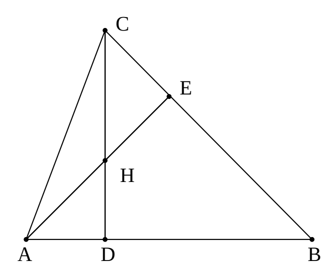


Figure 3-21

Then

$$r_A : r_B : r_C = P_{ABC}P_{BCA} : P_{CAB}P_{BCA} : P_{ABC}P_{CAB}.$$

By the Herron-Qin Theorem (Exercise 3.18), $P_{ABC}P_{BCA}+P_{CAB}P_{BCA}+P_{ABC}P_{CAB} = 2\overline{AB}^2P_{BCA}+P_{ABC}P_{CAB} = 16S_{ABC}^2$. Now the result is clear. ∎

Exercises 3.65

1. The area coordinate of the circumscribed center of triangle ABC is

$$(\frac{P_{BCB}P_{BAC}}{32S_{ABC}^2} \quad \frac{P_{ACA}P_{ABC}}{32S_{ABC}^2} \quad \frac{P_{ABA}P_{ACB}}{32S_{ABC}^2}).$$

   (Use full-angles.)

2. Let I be the incenter or one of the excenters of $\triangle ABC$. The area coordinate of C with respect to IBC is

$$(-\frac{2P_{IAB}P_{IBA}}{P_{AIB}P_{ABA}} \quad \frac{P_{IAB}P_{IBI}}{P_{AIB}P_{ABA}} \quad \frac{P_{IBA}P_{IAI}}{P_{AIB}P_{ABA}}).$$

   (Use full-angles.)

3. Construction C6 on page 111 is equivalent to

$$(\text{ARATIO } Y\ P\ O_1\ O_2\ -1 \quad \frac{2P_{PO_2O_1}}{P_{O_1O_2O_1}} \quad \frac{2P_{PO_1O_2}}{P_{O_1O_2O_1}})$$

   Now we can use the following new constructions:

C17 (CENTROID G A B C). G is the centroid of triangle ABC.

C18 (ORTHOCENTER H A B C). H is the orthocenter of the triangle ABC.

C19 (CIRCUMCENTER O A B C). O is the circumcenter of triangle ABC.

C20 (INCENTER C I A B). I is the center of the inscribed circle of triangle ABC. This construction is to construct point C from points A, B, and I.

Construction C20 needs some explanation. If three vertices of a triangle are given and we need to find the coordinates of the incenter, we generally have an equation of degree four in the coordinates of the incenter. The reason is that we can not distinguish the incenter and the three excenters without using inequalities. What we do here is to reverse the problem: when the incenter (or an excenter) and two vertices of a triangle are given, the third vertex is uniquely determined and can be introduced using the constructions given before.

Remark 3.66 *In this section, we actually use the centroid theorem (Example 1.12 on page 13), the orthocenter theorem (Example 1.67 on page 32), and the incenter theorem (Example 6.144 on page 333) in the proof of more complicated theorems. The four theorems* *themselves can be proved using the basic propositions. In general, we can use different constructions to describe the same geometry statement, and the more basic the constructions used the less prerequisite is needed in the proof of the theorem, and generally, the longer the proof is. On the contrary, the more complicated the constructions used, the more prerequisite is needed, and generally, the shorter the proof is.*

For constructions C17, C18, C19, and C20, the eliminating results for some special cases are very simple. As an example, we have

Exercise 3.67 *For (***CIRCUMCENTER** *O A B C), we have*

1. $P_{OAO} = \frac{P_{ABA}P_{ACA}P_{BCB}}{64S_{ABC}^2}$.
2. $P_{ABO} = \overline{AB}^2$
3. $P_{AOB} = \frac{P_{ABA}(P_{ACA}P_{BCB}-32S_{ABC}^2)}{64S_{ABC}^2}$.
4. $S_{PQO} = \frac{S_{PQA}}{2} + \frac{S_{PQB}}{2}$, *if* $PQ \perp AB$.

Example 3.68 *Let H be the orthocenter of triangle ABC. Then the circumcenters of the four triangles ABC, ABH, ACH, and BCH are such that each is the orthocenter of the triangle formed by the remaining three.*

Constructive description

((points A B C)
(orthocenter H A B C)
(circumcenter O A B C)
(circumcenter A_1 B C H)
(circumcenter B_1 A C H)
(circumcenter C_1 A B H)
(parallel B_1 C_1 B C))

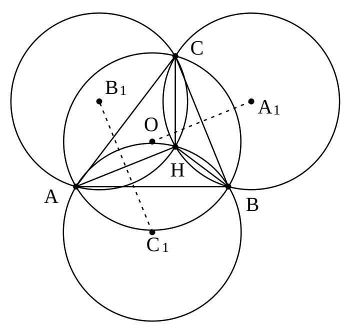


Figure 3-22

The machine proof

$$\frac{S_{BCB_1}}{S_{BCC_1}}$$

$$\stackrel{C_1}{=} \frac{S_{BCB_1}\cdot(2)}{S_{BCH}+S_{ABC}}$$

$$\stackrel{B_1}{=} \frac{(2)\cdot(S_{BCH}+S_{ABC})}{(S_{BCH}+S_{ABC})\cdot(2)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{BCC_1} \stackrel{C_1}{=} \frac{1}{2}(S_{BCH}+S_{ABC})$$

$$S_{BCB_1} \stackrel{B_1}{=} \frac{1}{2}(S_{BCH}+S_{ABC})$$

The above machine proof uses the fourth result of Exercise 3.67.

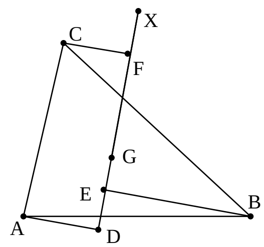


Figure 3-23

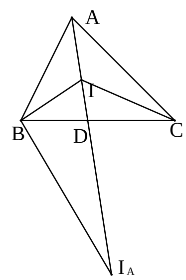


Figure 3-24

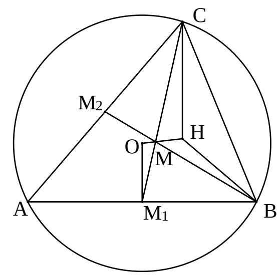


Figure 3-25

Example 3.69 *A line is drawn through the centroid of a triangle. Show that the sum of the distances of the line from the two vertices of the triangle situated on the same side of the line is equal to the distance of the line from the third vertex. (Figure 3-23)*

Constructive description

((points $A\ B\ C\ X$)
 (centroid $G\ A\ B\ C$)
 (foot $D\ A\ G\ X$)
 (foot $E\ B\ G\ X$)
 (foot $F\ C\ G\ X$)
 ($\frac{\overline{EB}}{\overline{DA}}+\frac{\overline{FC}}{\overline{DA}} = -1$))

The machine proof

$$-\left(\frac{\overline{CF}}{\overline{AD}}+\frac{\overline{BE}}{\overline{AD}}\right)$$

$$\stackrel{F}{=} \frac{-\frac{\overline{BE}}{\overline{AD}}\cdot S_{AXG}-S_{CXG}}{-(-S_{AXG})}$$

$$\stackrel{E}{=} \frac{-(-S_{CXG}\cdot S_{AXG}-S_{BXG}\cdot S_{AXG})}{S_{AXG}\cdot(-S_{AXG})}$$

$$\stackrel{simplify}{=} \frac{-(S_{CXG}+S_{BXG})}{S_{AXG}}$$

$$\stackrel{G}{=} \frac{-(3S_{ACX}+3S_{ABX})\cdot(3)}{(-S_{ACX}-S_{ABX})\cdot((3))^2}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{CF}}{\overline{AD}} \stackrel{F}{=} \frac{S_{CXG}}{S_{AXG}}$$

$$\frac{\overline{BE}}{\overline{AD}} \stackrel{E}{=} \frac{S_{BXG}}{S_{AXG}}$$

$$S_{AXG} \stackrel{G}{=} -\frac{1}{3}\left(S_{ACX}+S_{ABX}\right)$$

$$S_{BXG} \stackrel{G}{=} -\frac{1}{3}\left(S_{BCX}-S_{ABX}\right)$$

$$S_{CXG} \stackrel{G}{=} \frac{1}{3}\left(S_{BCX}+S_{ACX}\right)$$

Example 3.70 *Two tritangent centers divide the bisector on which they are located, harmonically (Figure 3-24).*

Constructive description

((POINTS $B\ C\ I$)
 (INCENTER $A\ I\ C\ B$)
 (INTER D (LINE $A\ I$) (LINE $B\ C$))
 (INTER I_A (LINE $A\ I$) (TLINE $B\ B\ I$))
 (HARMONIC $A\ D\ I\ I_A$))

The machine proof

$$\left(-\frac{\overline{IA}}{\overline{ID}}\right)/\left(\frac{\overline{AI_A}}{\overline{DI_A}}\right)$$

$$\stackrel{I_A}{=} \frac{P_{IBD}}{P_{IBA}}\cdot -\frac{\overline{IA}}{\overline{ID}}$$

$$\stackrel{D}{=} \frac{-(-S_{BICA})\cdot P_{CBI}\cdot S_{BIA}}{P_{IBA}\cdot S_{BICA}\cdot S_{BCI}}$$

$$\stackrel{simplify}{=} \frac{P_{CBI}\cdot S_{BIA}}{P_{IBA}\cdot S_{BCI}}$$

$$\stackrel{A}{=} \frac{P_{CBI}\cdot(-P_{BIB}\cdot P_{BCI}\cdot S_{BCI})\cdot P_{BIC}\cdot P_{BCB}}{(-P_{CBI}\cdot P_{BIB}\cdot P_{BCI})\cdot S_{BCI}\cdot P_{BIC}\cdot P_{BCB}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AI_A}}{\overline{DI_A}} \stackrel{I_A}{=} \frac{P_{IBA}}{P_{IBD}}$$

$$P_{IBD} \stackrel{D}{=} \frac{P_{CBI}\cdot S_{BIA}}{S_{BICA}}$$

$$\frac{\overline{IA}}{\overline{ID}} \stackrel{D}{=} \frac{-S_{BICA}}{S_{BCI}}$$

$$P_{IBA} \stackrel{A}{=} \frac{-P_{CBI}\cdot P_{BIB}\cdot P_{BCI}}{P_{BIC}\cdot P_{BCB}}$$

$$S_{BIA} \stackrel{A}{=} \frac{-P_{BIB}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$$

Example 3.71 (Euler's Theorem) *The centroid of a triangle is on the segment determined by the circumcenter O and the orthocenter H of the same triangle, and divides OH in the ratio of 1:2 (Figure 3-25).*

Constructive description	The machine proof	The eliminants
((POINTS *A B C*)	$\frac{P_{ABC}}{P_{CBH}}$	$P_{CBH} \stackrel{H}{=} 3P_{CBM}-2P_{CBO}$
(CIRCUMCENTER *O A B C*)	$\stackrel{H}{=} \frac{P_{ABC}}{3P_{CBM}-2P_{CBO}}$	$P_{CBM} \stackrel{M}{=} \frac{1}{3}(P_{BCB}+P_{ABC})$
(CENTROID *M A B C*)	$\stackrel{M}{=} \frac{P_{ABC}\cdot(3)}{-6P_{CBO}+3P_{BCB}+3P_{ABC}}$	$P_{CBO} \stackrel{O}{=} \frac{1}{2}(P_{BCB})$
(LRATIO *H M O* −2)	$\stackrel{O}{=} \frac{-P_{ABC}\cdot(2)}{-2P_{ABC}}$	
(PERPENDICULAR *A H B C*))	$\stackrel{simplify}{=} 1$	

3.6 Trigonometric Functions and Co-Circle Points

The aim of this section is to provide an efficient method for dealing with co-circle points. We will prove the co-circle theorems given in Section 1.9 without using trigonometric functions; on the contrary, we will develop properties of trigonometric functions by using the area and Pythagoras difference only. The readers who are interested primaryly in machine proof may skip the following subsection and proceed directly to Subsection 3.6.2.

3.6.1 The Co-circle Theorems

Let J, A, and B be three points on a circle with center O. In what follows, we will fix J as a reference point. An *oriented chord*, $\widetilde{AB}$, is a directed line segment such that $\widetilde{JA}$ ($A \neq J$) is always positive and $\widetilde{AB} > 0$ iff $S_{JAB} > 0$.

Proposition 3.72 *Let δ be the diameter of the circumcircle of the triangle ABC. Then $S_{ABC} = \frac{\widetilde{AB}\cdot\widetilde{CB}\cdot\widetilde{CA}}{2\delta}$.*

Proof. Let CE be a diameter of circle O. By Proposition 3.48,

$$\frac{S_{ABC}^2}{S_{ACE}^2} = \frac{\overline{AB}^2 \cdot \overline{CB}^2}{\overline{AE}^2 \cdot \delta^2}.$$

Since $S_{ACE}^2 = \frac{1}{4}\overline{AC}^2 \cdot \overline{AE}^2$, we have

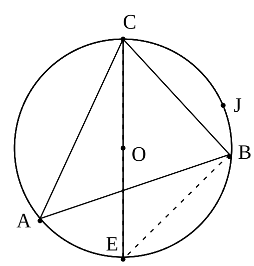


Figure 3-26

$$S_{ABC}^2 = \frac{\overline{AB}^2 \cdot \overline{BC}^2 \cdot \overline{AC}^2}{4\delta^2}, \text{ i.e., } |S_{ABC}| = \frac{|\widetilde{AB}| \cdot |\widetilde{CB}| \cdot |\widetilde{CA}|}{2\delta}.$$

We still need to check whether the signs of both sides of the conclusion equation are the same. At first, it is easy to see that when we interchange two vertices of the triangle ABC, the signs of both sides of the equation will change. Therefore, we need only to check a particular position for A, B, and C, e.g., the case shown in Figure 3-26. In this case, we have $S_{ABC} > 0, \widetilde{AB} > 0, \widetilde{CB} > 0$, and $\widetilde{CA} > 0$. ∎

Let $\widetilde{BC}$ be an oriented chord on circle O and BB' be a diameter. We define the *cochord* of $\widetilde{BC}$ to be $\widehat{BC}$ whose absolute value is equal to $|CB'|$ and has the same sign as P_{BJC}. It is clear that

$$\widetilde{BC}^2 + \widehat{BC}^2 = \delta^2.$$

Proposition 3.73 *For points A, B, and C on circle O, we have $P_{ABC} = \frac{2\widetilde{AB}\cdot\widehat{CB}\cdot\widehat{CA}}{\delta}$.*

Proof. By the Herron-Qin formula and Proposition 3.72, we have

$$P_{ABC}^2 = 4\overline{AB}^2 \cdot \overline{CB}^2 - 16S_{ABC}^2 = \frac{4\overline{AB}^2 \cdot \overline{CB}^2(\delta^2 - \overline{AC}^2)}{\delta^2}.$$

Then $|P_{ABC}| = \frac{2|\widetilde{AB}|\cdot|\widetilde{CB}|\cdot|\widehat{CA}|}{\delta}$. As in Proposition 1.100, we can check that the signs of the two sides of the equation are equal. ∎

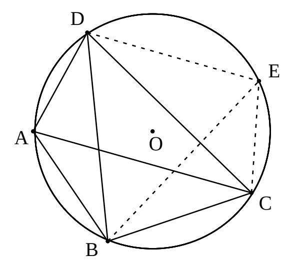


Figure 3-27

Proposition 3.74 *Let A, B, C, and D be four co-circle points, E the intersection of the circle, and the line passing through D and parallel to AC. Then $S_{ABCD} = \frac{\widetilde{AC}\cdot\widetilde{BD}\cdot\widetilde{EB}}{2\delta}$ where δ is the diameter of the circle.*

Proof. If $AC \parallel BD$, we have $E = B$ and $S_{ABCD} = 0$. The result is clearly true. Otherwise, since $DE \parallel AC$, we have

$$S_{ABCD} = \frac{\widetilde{AC}}{\widetilde{ED}} S_{EBD} = \frac{\widetilde{AC} \cdot \widetilde{BD} \cdot \widetilde{EB}}{2\delta}. \quad ∎$$

Proposition 3.75 (Ptolemy's Theorem) *Let A, B, C, and D be four co-circle points. Then $\widetilde{AB} \cdot \widetilde{CD} + \widetilde{BC} \cdot \widetilde{AD} = \widetilde{AC} \cdot \widetilde{BD}$.*

Proof. Let E be the intersection of the circle and the line passing through D and parallel to AC (Figure 3-27). Then by Propositions 3.72 and 3.74,

$$\frac{\widetilde{AC} \cdot \widetilde{BD} \cdot \widetilde{EB}}{2\delta} = S_{ABCD} = S_{BCE} + S_{EAB} = \frac{\widetilde{BC} \cdot \widetilde{EC} \cdot \widetilde{EB} + \widetilde{EA} \cdot \widetilde{BA} \cdot \widetilde{BE}}{2\delta}.$$

Let B be the reference point. Notice that $\widetilde{EB} = -\widetilde{BE}, \widetilde{AE} = -\widetilde{CD}$, and $\widetilde{CE} = -\widetilde{AD}$, we prove the result. ∎

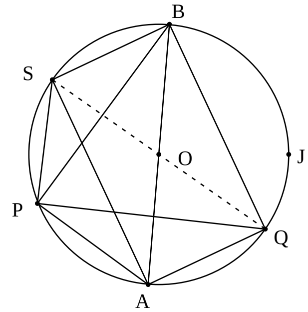


Figure 3-28

Proposition 3.76 *Let $AB = d$ be the diameter of the circle, and P, Q be two points on the circle. Then*

$$d \cdot \widetilde{PQ} = \widetilde{AQ} \cdot \widehat{AP} - \widetilde{AP} \cdot \widehat{AQ}.$$

Proof. Apply Ptolemy's theorem to A, P, B, and Q:

$$\widetilde{AB} \cdot \widetilde{PQ} = \widetilde{AP} \cdot \widetilde{BQ} + \widetilde{AQ} \cdot \widetilde{PB}.$$

Let J be the reference point. If $S_{JAB} < 0$ (Figure 3-28), we have $\widetilde{AB} = -d, \widehat{AQ} = \widetilde{BQ}$, and $\widehat{AP} = \widetilde{BP}$. If $S_{JAB} > 0$ or $J = A$, we have $\widetilde{AB} = d, \widehat{AQ} = \widetilde{QB}$, and $\widehat{AP} = \widetilde{PB}$. The result is always true. ∎

Proposition 3.77 *Let $AB = d$ be the diameter of the circle, and P, Q be two points on the circle. Then*

$$d \cdot \widehat{PQ} = \widetilde{AP} \cdot \widetilde{AQ} + \widehat{AP} \cdot \widehat{AQ}.$$

Proof. Let S be the antipodal of Q (Figure 3-28). By Ptolemy's theorem, we have

$$\widetilde{AB} \cdot \widetilde{PS} = \widetilde{AP} \cdot \widetilde{BS} + \widetilde{AS} \cdot \widetilde{PB}.$$

Let J be the reference point. If $\widetilde{AB}$ and $\widetilde{QS}$ have the same sign, we have

$$\widetilde{AB} \cdot \widetilde{PS} = d\widehat{PQ}, \widetilde{BS} = \widetilde{AQ}, \widetilde{AS} \cdot \widetilde{PB} = \widehat{AQ} \cdot \widehat{AP}.$$

If $\widetilde{AB}$ and $\widetilde{QS}$ have different signs, we have

$$\widetilde{AB} \cdot \widetilde{PS} = -d \cdot \widehat{PQ}, \widetilde{BS} = -\widetilde{AQ}, \widetilde{AS} \cdot \widetilde{PB} = -\widehat{AQ} \cdot \widehat{AP}.$$

The result is true in both cases. ∎

Propositions 3.76 and 3.77 are called *decomposition formulas for chords*. They can be used to reduce any chord or cochord to a polynomial in chords and cochords with a fixed point (A) as an end point.

For points A and B on the circle, let us define

- $\sin(A) = \frac{\widetilde{JA}}{\delta}$; $\cos(A) = \frac{\widehat{JA}}{\delta}$.
- $\sin(AB) = \frac{\widetilde{AB}}{\delta}$; $\cos(AB) = \frac{\widehat{AB}}{\delta}$.

Then we have derived the following properties of trigonometric functions.

$$\begin{aligned}
&\sin(A)^2 + \cos(A)^2 = 1\\
&\sin(AB) = \sin(B)\cos(A) - \sin(A)\cos(B)\\
&\cos(AB) = \cos(B)\cos(A) + \sin(A)\sin(B).
\end{aligned}$$

The $\sin(AB)$ and $\cos(AB)$ are actually the sine and cosine of the inscribed angle in arc AB. But in the above definition of trigonometric functions, we did not mention the the concept of angles.

3.6.2 Eliminating Co-Circle Points

We introduce a new construction.

C21 (CIRCLE $Y_1 \cdots Y_s$), $(s \geq 3)$. Points $Y_1 \cdots Y_s$ are on the same circle. There is no ndg condition for construction C21. The degree of freedom of this construction is $s + 3$.

By Propositions 3.72–3.77, we have

Lemma 3.78 *Let A, B, and C be points on a circle with center O and diameter δ. Then*

$$S_{ABC} = \frac{\widetilde{AB}\cdot\widetilde{CB}\cdot\widetilde{CA}}{2\delta}, \quad P_{ABC} = \frac{2\widetilde{AB}\cdot\widetilde{CB}\cdot\widehat{CA}}{\delta}.$$

$$\widetilde{AC} = \delta\sin(AC), \quad \widehat{AC} = \delta\cos(AC).$$

Using Lemma 3.78, an expression in the areas and Pythagoras differences of points on a circle can be reduced to an expression in the diameter δ of the circle and trigonometric functions of independent angles. Two such expressions have the same value iff when substituting, for each angle α, $(\sin\alpha)^2$ by $1 - (\cos\alpha)^2$ the resulting expression should be the same. We thus have a complete method for this construction. The reader may have noticed that this construction must always come first in the description of a statement. Otherwise, in the next step, we do not know how to eliminate these trigonometric functions.

The proofs of many interesting geometry theorems use this construction.

Example 3.79 (Simson's Theorem) *Let D be a point on the circumscribed circle of triangle ABC. From D three perpendiculars are drawn to the three sides BC, AC, and AB of triangle ABC. Let E, F, and G be the three feet respectively. Show that E, F and G are collinear.*

Here is the input to our program.

((CIRCLE $A\ B\ C\ D$)

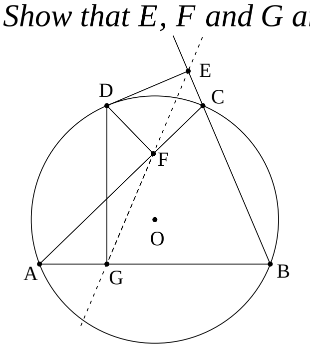


Figure 3-29

(FOOT E D B C)
(FOOT F D A C)
(FOOT G D A B)
(INTER G_1 (LINE E F) (LINE A B))
($\frac{\overline{AG}}{\overline{BG}} = \frac{\overline{AG_1}}{\overline{BG_1}}$))

The ndg conditions: $B \neq C$, $A \neq C$, $A \neq B$, $EF \nparallel AB$, $B \neq G$, $B \neq G_1$.

Here is the machine proof. The last step of the proof uses Lemma 3.78.

The machine proof

$$\left(\frac{\overline{AG}}{\overline{BG}}\right)/\left(\frac{\overline{AG_1}}{\overline{BG_1}}\right)$$

$$\overset{G_1}{=} \frac{S_{BEF}}{S_{AEF}} \cdot \frac{\overline{AG}}{\overline{BG}}$$

$$\overset{G}{=} \frac{P_{BAD} \cdot S_{BEF}}{S_{AEF} \cdot (-P_{ABD})}$$

$$\overset{F}{=} \frac{-P_{BAD} \cdot P_{ACD} \cdot S_{ABE} \cdot P_{ACA}}{(-P_{CAD} \cdot S_{ACE}) \cdot P_{ABD} \cdot P_{ACA}}$$

$$\overset{simplify}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot S_{ABE}}{P_{CAD} \cdot S_{ACE} \cdot P_{ABD}}$$

$$\overset{E}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot P_{CBD} \cdot S_{ABC} \cdot P_{BCB}}{P_{CAD} \cdot (-P_{BCD} \cdot S_{ABC}) \cdot P_{ABD} \cdot P_{BCB}}$$

$$\overset{simplify}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot P_{CBD}}{-P_{CAD} \cdot P_{BCD} \cdot P_{ABD}}$$

$$\overset{co-cir}{=} \frac{(2\widetilde{AD} \cdot \widetilde{AB} \cdot \cos(BD)) \cdot (-2\widetilde{CD} \cdot \widetilde{AC} \cdot \cos(AD)) \cdot (2\widetilde{BD} \cdot \widetilde{BC} \cdot \cos(CD))}{-(2\widetilde{AD} \cdot \widetilde{AC} \cdot \cos(CD)) \cdot (-2\widetilde{CD} \cdot \widetilde{BC} \cdot \cos(BD)) \cdot (-2\widetilde{BD} \cdot \widetilde{AB} \cdot \cos(AD))}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AG_1}}{\overline{BG_1}} \overset{G_1}{=} \frac{S_{AEF}}{S_{BEF}}$$

$$\frac{\overline{AG}}{\overline{BG}} \overset{G}{=} \frac{P_{BAD}}{-P_{ABD}}$$

$$S_{AEF} \overset{F}{=} \frac{-P_{CAD} \cdot S_{ACE}}{P_{ACA}}$$

$$S_{BEF} \overset{F}{=} \frac{P_{ACD} \cdot S_{ABE}}{P_{ACA}}$$

$$S_{ACE} \overset{E}{=} \frac{-P_{BCD} \cdot S_{ABC}}{P_{BCB}}$$

$$S_{ABE} \overset{E}{=} \frac{P_{CBD} \cdot S_{ABC}}{P_{BCB}}$$

$$P_{ABD} = -2(\widetilde{BD} \cdot \widetilde{AB} \cdot \cos(AD))$$

The eliminants

$$P_{BCD} = -2(\widetilde{CD} \cdot \widetilde{BC} \cdot \cos(BD))$$

$$P_{CAD} = 2(\widetilde{AD} \cdot \widetilde{AC} \cdot \cos(CD))$$

$$P_{CBD} = 2(\widetilde{BD} \cdot \widetilde{BC} \cdot \cos(CD))$$

$$P_{ACD} = -2(\widetilde{CD} \cdot \widetilde{AC} \cdot \cos(AD))$$

$$P_{BAD} = 2(\widetilde{AD} \cdot \widetilde{AB} \cdot \cos(BD))$$

Example 3.80 (Pascal's Theorem on a Circle) *Let A, B, C, D, E, and F be six points on a circle. Let $P = AB \cap DF$, $Q = BC \cap EF$, and $S = CD \cap EA$. Show that P, Q, and S are collinear.*

Here is the input to the program.

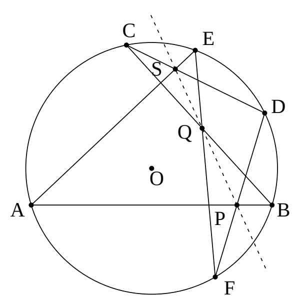


Figure 3-30

((CIRCLE A B C D F E)
(INTER P (LINE D F) (LINE A B))
(INTER Q (LINE F E) (LINE B C))
(INTER S (LINE E A) (LINE C D))
(INTER S_1 (LINE P Q) (LINE C D))
($\frac{\overline{CS}}{\overline{DS}} = \frac{\overline{CS_1}}{\overline{DS_1}}$))

The ndg conditions:
$DF \nparallel AB$, $EF \nparallel BC$, $AE \nparallel CD$, $PQ \nparallel CD$, $D \neq S$, $D \neq S_1$.

The machine proof

$$\left(\frac{\overline{CS}}{\overline{DS}}\right)/\left(\frac{\overline{CS_1}}{\overline{DS_1}}\right)$$

$$\overset{S_1}{=} \frac{S_{DPQ}}{S_{CPQ}} \cdot \frac{\overline{CS}}{\overline{DS}}$$

$$\overset{S}{=} \frac{S_{ACE} \cdot S_{DPQ}}{S_{CPQ} \cdot S_{ADE}}$$

$$\overset{Q}{=} \frac{S_{ACE} \cdot (-S_{DEP} \cdot S_{BCF}) \cdot S_{BFCE}}{(-S_{CFE} \cdot S_{BCP}) \cdot S_{ADE} \cdot (-S_{BFCE})}$$

$$\overset{simplify}{=} \frac{-S_{ACE} \cdot S_{DEP} \cdot S_{BCF}}{S_{CFE} \cdot S_{BCP} \cdot S_{ADE}}$$

$$\overset{P}{=} \frac{-S_{ACE} \cdot (-S_{DFE} \cdot S_{ABD}) \cdot S_{BCF} \cdot S_{ADBF}}{S_{CFE} \cdot (-S_{BDF} \cdot S_{ABC}) \cdot S_{ADE} \cdot (-S_{ADBF})}$$

$$\overset{simplify}{=} \frac{S_{ACE} \cdot S_{DFE} \cdot S_{ABD} \cdot S_{BCF}}{S_{CFE} \cdot S_{BDF} \cdot S_{ABC} \cdot S_{ADE}}$$

The eliminants

$$\frac{\overline{CS_1}}{\overline{DS_1}} \overset{S_1}{=} \frac{S_{CPQ}}{S_{DPQ}}$$

$$\frac{\overline{CS}}{\overline{DS}} \overset{S}{=} \frac{S_{ACE}}{S_{ADE}}$$

$$S_{CPQ} \overset{Q}{=} \frac{-S_{CFE} \cdot S_{BCP}}{S_{BFCE}}$$

$$S_{DPQ} \overset{Q}{=} \frac{S_{DEP} \cdot S_{BCF}}{S_{BFCE}}$$

$$S_{BCP} \overset{P}{=} \frac{-S_{BDF} \cdot S_{ABC}}{S_{ADBF}}$$

$$S_{DEP} \overset{P}{=} \frac{S_{DFE} \cdot S_{ABD}}{S_{ADBF}}$$

$$S_{ADE} = \frac{\widetilde{DE} \cdot \widetilde{AE} \cdot \widetilde{AD}}{(-2) \cdot d}$$

$$S_{ABC} = \frac{\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}}{(-2) \cdot d}$$

$$S_{BDF} = \frac{\widetilde{DF} \cdot \widetilde{BF} \cdot \widetilde{BD}}{(-2) \cdot d}$$

$$S_{CFE} = \frac{\widetilde{FE} \cdot \widetilde{CE} \cdot \widetilde{CF}}{(-2) \cdot d}$$

$$S_{BCF} = \frac{\widetilde{CF} \cdot \widetilde{BF} \cdot \widetilde{BC}}{(-2) \cdot d}$$

The machine proof

$$\overset{co-cir}{=} \frac{(-\widetilde{CE} \cdot \widetilde{AE} \cdot \widetilde{AC}) \cdot (-\widetilde{FE} \cdot \widetilde{DE} \cdot \widetilde{DF}) \cdot (-\widetilde{BD} \cdot \widetilde{AD} \cdot \widetilde{AB}) \cdot (-\widetilde{CF} \cdot \widetilde{BF} \cdot \widetilde{BC}) \cdot ((2d))^4}{(-\widetilde{FE} \cdot \widetilde{CE} \cdot \widetilde{CF}) \cdot (-\widetilde{DF} \cdot \widetilde{BF} \cdot \widetilde{BD}) \cdot (-\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}) \cdot (-\widetilde{DE} \cdot \widetilde{AE} \cdot \widetilde{AD}) \cdot ((2d))^4}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{ABD} = \frac{\widetilde{BD} \cdot \widetilde{AD} \cdot \widetilde{AB}}{(-2) \cdot d}$$

$$S_{DFE} = \frac{\widetilde{FE} \cdot \widetilde{DE} \cdot \widetilde{DF}}{(-2) \cdot d}$$

$$S_{ACE} = \frac{\widetilde{CE} \cdot \widetilde{AE} \cdot \widetilde{AC}}{(-2) \cdot d}$$

Example 3.81 (The General Butterfly Theorem.) *As in the figure, A, B, C, D, E, F are six points on a circle. $M = AB \cap CD$; $N = AB \cap EF$; $G = AB \cap CF$; $H = AB \cap DE$. Show that* $\frac{\overline{MG}}{\overline{AG}} \frac{\overline{BH}}{\overline{NH}} \frac{\overline{AN}}{\overline{MB}} = 1$.

Constructive description

((CIRCLE $A\ B\ C\ D\ E\ F$)

(INTER M (LINE $D\ C$) (LINE $A\ B$))

(INTER N (LINE $E\ F$) (LINE $A\ B$))

(INTER G (LINE $A\ B$) (LINE $C\ F$))

(INTER H (LINE $D\ E$) (LINE $A\ B$))

($\frac{\overline{MG}}{\overline{AG}} \frac{\overline{BH}}{\overline{NH}} = \frac{\overline{BM}}{\overline{AB}} \frac{\overline{BA}}{\overline{AN}}$))

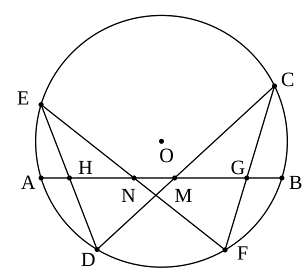


Figure 3-31

The machine proof

$$\frac{\frac{\overline{MG}}{\overline{AG}}\cdot\frac{\overline{BH}}{\overline{NH}}}{-\frac{\overline{BM}}{\overline{AB}}\cdot\frac{\overline{AB}}{\overline{AN}}}$$

$$\overset{H}{=} \frac{S_{BDE}}{-\frac{\overline{BM}}{\overline{AB}}\cdot\frac{\overline{AB}}{\overline{AN}}\cdot S_{DEN}} \cdot \frac{\overline{MG}}{\overline{AG}}$$

$$\overset{G}{=} \frac{-S_{CFM}\cdot S_{BDE}}{\frac{\overline{BM}}{\overline{AB}}\cdot\frac{\overline{AB}}{\overline{AN}}\cdot S_{DEN}\cdot S_{ACF}}$$

$$\overset{N}{=} \frac{-S_{CFM}\cdot S_{BDE}\cdot(-S_{AEBF})\cdot S_{AEF}}{\frac{\overline{BM}}{\overline{AB}}\cdot S_{AEBF}\cdot S_{DEF}\cdot S_{ABE}\cdot S_{ACF}}$$

$$\overset{simplify}{=} \frac{S_{CFM}\cdot S_{BDE}\cdot S_{AEF}}{\frac{\overline{BM}}{\overline{AB}}\cdot S_{DEF}\cdot S_{ABE}\cdot S_{ACF}}$$

$$\overset{M}{=} \frac{(-S_{CDF}\cdot S_{ABC})\cdot S_{BDE}\cdot S_{AEF}\cdot(-S_{ACBD})}{(-S_{BCD})\cdot S_{DEF}\cdot S_{ABE}\cdot S_{ACF}\cdot(-S_{ACBD})}$$

$$\overset{simplify}{=} \frac{S_{CDF}\cdot S_{ABC}\cdot S_{BDE}\cdot S_{AEF}}{S_{BCD}\cdot S_{DEF}\cdot S_{ABE}\cdot S_{ACF}}$$

$$\overset{co-cir}{=} \frac{(-\widetilde{DF}\cdot\widetilde{CF}\cdot\widetilde{CD})\cdot(-\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB})\cdot(-\widetilde{DE}\cdot\widetilde{BE}\cdot\widetilde{BD})\cdot(-\widetilde{EF}\cdot\widetilde{AF}\cdot\widetilde{AE})\cdot((2d))^4}{(-\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC})\cdot(-\widetilde{EF}\cdot\widetilde{DF}\cdot\widetilde{DE})\cdot(-\widetilde{BE}\cdot\widetilde{AE}\cdot\widetilde{AB})\cdot(-\widetilde{CF}\cdot\widetilde{AF}\cdot\widetilde{AC})\cdot((2d))^4}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BH}}{\overline{NH}} \overset{H}{=} \frac{S_{BDE}}{S_{DEN}}$$

$$\frac{\overline{MG}}{\overline{AG}} \overset{G}{=} \frac{S_{CFM}}{S_{ACF}}$$

$$S_{DEN} \overset{N}{=} \frac{S_{DEF}\cdot S_{ABE}}{-S_{AEBF}}$$

$$\frac{\overline{AB}}{\overline{AN}} \overset{N}{=} \frac{S_{AEBF}}{S_{AEF}}$$

$$\frac{\overline{BM}}{\overline{AB}} \overset{M}{=} \frac{S_{BCD}}{S_{ACBD}}$$

$$S_{CFM} \overset{M}{=} \frac{S_{CDF}\cdot S_{ABC}}{S_{ACBD}}$$

$$S_{ACF} = \frac{\widetilde{CF}\cdot\widetilde{AF}\cdot\widetilde{AC}}{(-2)\cdot d}$$

$$S_{ABE} = \frac{\widetilde{BE}\cdot\widetilde{AE}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{DEF} = \frac{\widetilde{EF}\cdot\widetilde{DF}\cdot\widetilde{DE}}{(-2)\cdot d}$$

$$S_{BCD} = \frac{\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC}}{(-2)\cdot d}$$

$$S_{AEF} = \frac{\widetilde{EF}\cdot\widetilde{AF}\cdot\widetilde{AE}}{(-2)\cdot d}$$

$$S_{BDE} = \frac{\widetilde{DE}\cdot\widetilde{BE}\cdot\widetilde{BD}}{(-2)\cdot d}$$

$$S_{ABC} = \frac{\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{CDF} = \frac{\widetilde{DF}\cdot\widetilde{CF}\cdot\widetilde{CD}}{(-2)\cdot d}$$

Example 3.82 (Cantor's Theorem) *The perpendiculars from the midpoints of the sides of a cyclic quadrilateral to the respectively opposite sides are concurrent.*

Constructive description

((CIRCLE A B C D)

(CIRCUMCENTER O A B C)

(MIDPOINT G A D)

(MIDPOINT F A B)

(MIDPOINT E C D)

(PRATIO N E O F 1)

(PERPENDICULAR G N B C))

The machine proof

$$\frac{P_{CBG}}{P_{CBN}}$$

$$\overset{N}{=} \frac{P_{CBG}}{P_{CBE}+P_{CBF}-P_{CBO}}$$

$$\overset{E}{=} \frac{P_{CBG}}{P_{CBF}-P_{CBO}+\frac{1}{2}P_{CBD}+\frac{1}{2}P_{BCB}}$$

$$\overset{F}{=} \frac{(2)\cdot P_{CBG}}{-2P_{CBO}+P_{CBD}+P_{BCB}+P_{ABC}}$$

$$\overset{G}{=} \frac{(-2)\cdot(\frac{1}{2}P_{CBD}+\frac{1}{2}P_{ABC})}{2P_{CBO}-P_{CBD}-P_{BCB}-P_{ABC}}$$

$$\overset{O}{=} \frac{-(P_{CBD}+P_{ABC})\cdot(2)}{-2P_{CBD}-2P_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{CBN} \overset{N}{=} P_{CBE}+P_{CBF}-P_{CBO}$$

$$P_{CBE} \overset{E}{=} \frac{1}{2}\left(P_{CBD}+P_{BCB}\right)$$

$$P_{CBF} \overset{F}{=} \frac{1}{2}\left(P_{ABC}\right);\ P_{CBO} \overset{O}{=} \frac{1}{2}\left(P_{BCB}\right)$$

$$P_{CBG} \overset{G}{=} \frac{1}{2}\left(P_{CBD}+P_{ABC}\right)$$

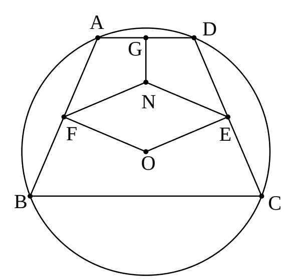


Figure 3-32

The ndg conditions: A, B, C are not collinear; $A \neq D$; $A \neq B$; $C \neq D$; $O \neq F$.

3.7 Machine Proof for Class C

The class C, or the class of constructive geometry statements, are the geometry statements which are assertions about the configurations that can be drawn by rulers and compasses only. To describe the statements in class C, we need two new constructions.

We first introduce a new kind of circle: (CIR O r) is the the circle with center O and radius $\sqrt{r}$. As before, r could be an algebraic number, a rational expression in geometry quantities, or a variable. Thus (CIR O P) is the same as (CIR O $\overline{OP}^2$).

C22 (INTER Y (LINE U V) (CIR O r)). Point Y is one of the intersections of line (LINE U V) and circle (CIR O r). The ndg conditions are $r \neq 0$, $U \neq V$. Point Y is a fixed point and has two possibilities.

C23 (INTER Y (CIR O_1 r_1) (CIR O_2 r_2)). Point Y is one of the intersections of the circle (CIR O_1 r_1) and the circle (CIR O_2 r_2). The ndg conditions are $O_1 \neq O_2$, $r_1 \neq 0$, and $r_2 \neq 0$. Point Y is a fixed point and has two possibilities.

Each of constructions C22 and C23 actually introduces two points, and we generally can not distinguish the two points. This is the main difficulty of dealing with these two constructions.

3.7.1 Eliminating Points from Geometry Quantities

Proposition 3.83 *Let Y be one of the intersection points of line UV and circle (CIR O r). Then we have*

$$(III) \qquad \left(\frac{\overline{UY}}{\overline{UV}}\right)^2 - \frac{P_{OUV}}{\overline{UV}^2}\frac{\overline{UY}}{\overline{UV}} + \frac{\overline{OU}^2 - r}{\overline{UV}^2} = 0.$$

Proof. Let X be another intersection point of the line UV and the circle (CIR O r), and M the midpoint of XY. By Proposition 3.2,

$$\frac{\overline{UY}}{\overline{UV}} + \frac{\overline{UX}}{\overline{UV}} = 2\frac{\overline{UM}}{\overline{UV}} = \frac{P_{MUV}}{\overline{UV}^2} = \frac{P_{OUV}}{\overline{UV}^2}. \qquad (1)$$

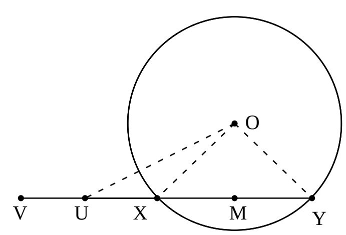


Figure 3-33

By Proposition 3.5,

$$\overline{OU}^2 = \frac{\overline{XU}}{\overline{XY}}\overline{OY}^2 + \frac{\overline{UY}}{\overline{XY}}\overline{OX}^2 - \frac{\overline{XU}}{\overline{XY}} \cdot \frac{\overline{UY}}{\overline{XY}}\overline{XY}^2 = \overline{OX}^2 + \overline{UY} \cdot \overline{UX} = r + \overline{UY} \cdot \overline{UX}. \qquad (2)$$

From (1) and (2), it is easy to obtain the result. ∎

Now we can give the elimination methods for construction C22 easily.

Lemma 3.84 *Let Y be introduced by (INTER Y (LINE U V) (CIR O r)). Then*

$$S_{ABY} = \frac{\overline{UY}}{\overline{UV}} S_{VAUB} + S_{ABU},$$
$$P_{ABY} = \frac{\overline{UY}}{\overline{UV}} P_{VAUB} + P_{ABU},$$
$$P_{AYB} = \frac{\overline{UY}}{\overline{UV}} P_{VAUB} + P_{ABU} - \frac{\overline{UY}}{\overline{UV}}\left(1 - \frac{\overline{UY}}{\overline{UV}}\right) P_{UVU}$$

where $\frac{\overline{UY}}{\overline{UV}}$ satisfies (III).

Proof. By Proposition 2.9, $S_{ABY} = \frac{\overline{UY}}{\overline{UV}} S_{ABV} + (1 - \frac{\overline{UY}}{\overline{UV}}) S_{ABU} = \frac{\overline{UY}}{\overline{UV}} S_{VAUB} + S_{ABU}$. The second and third cases are consequences of Proposition 3.5. ∎

Lemma 3.85 *Let Y be introduced by (INTER Y (LINE U V) (CIR O r)). Then*

$$\frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{\overline{DU}}{\overline{EF}} + \frac{\overline{UY}}{\overline{UV}} \frac{\overline{UV}}{\overline{EF}} & \text{if } D \in UV. \\ \frac{S_{DUV}}{S_{EUFV}} & \text{otherwise.} \end{cases}$$

Proof. The first case is trivial. The second case is a consequence of the co-side theorem. ∎

Proposition 3.86 *Construction C23 is equivalent to the following two constructions*

(LRATIO O O_1 O_2 r) (TRATIO Y O O_1 s)

where $r = \frac{\overline{O_1O_2}^2 + r_1 - r_2}{2\overline{O_1O_2}^2}$, $s^2 = \frac{r_1}{r^2\overline{O_1O_2}^2} - 1$.

Proof. Let O be the foot of the perpendicular dropped from point Y upon the line O_1O_2. By Proposition 3.2,

$$\frac{\overline{O_1O}}{\overline{O_1O_2}} = \frac{P_{YO_1O_2}}{P_{O_1O_2O_1}} = \frac{\overline{O_1O_2}^2 + r_1 - r_2}{2\overline{O_1O_2}^2}.$$

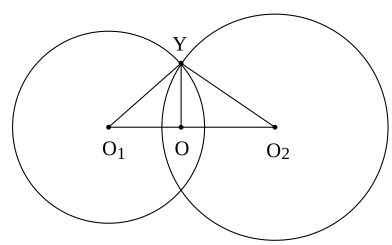


Figure 3-34

For s, we have $s^2 = \frac{\overline{OY}^2}{\overline{OO_1}^2} = \frac{r_1}{\overline{OO_1}^2} - 1 = \frac{r_1}{r^2\overline{O_1O_2}^2} - 1$. ∎

Lemma 3.87 *Let Y be introduced by (INTER Y (CIR O_1 r_1) (CIR O_2 r_2)). Then*

$$S_{ABY} = S_{ABO_1} + rS_{O_2AO_1B} - \frac{rs}{4} P_{O_2AO_1B}$$

where r and s are the same as in Proposition 3.86.

Proof. Let O be the foot of the perpendicular dropped from point Y upon the line O_1O_2. By Lemma 3.26,

$$\begin{aligned} S_{ABY} &= S_{ABO} - \frac{s}{4}P_{OAO_1B} \\ &= rS_{ABO_2} + (1-r)S_{ABO_1} - \frac{s}{4}(rP_{O_2AB} + (1-r)P_{O_1AB} - P_{O_1AB}) \\ &= S_{ABO_1} + rS_{O_2AO_1B} - \frac{rs}{4}P_{O_2AO_1B}. \quad \blacksquare \end{aligned}$$

Lemma 3.88 *Let Y be introduced by (INTER Y (CIR O_1 r_1) (CIR O_2 r_2)). Then*

$$P_{ABY} = P_{ABO_1} + rP_{O_2AO_1B} - 4rsS_{O_2AO_1B}$$

where r and s are the same as in Proposition 3.86.

Proof. Let O be the foot of the perpendicular dropped from point Y upon the line O_1O_2. By Lemma 3.27,

$$\begin{aligned} P_{ABY} &= P_{ABO} - 4sS_{OAO_1B} \\ &= rP_{ABO_2} + (1-r)P_{ABO_1} - 4s(rS_{O_2AB} + (1-r)S_{O_1AB} - S_{O_1AB}) \\ &= P_{ABO_1} + rP_{O_2AO_1B} - 4rsS_{O_2AO_1B}. \quad \blacksquare \end{aligned}$$

The methods of eliminating Y from $\frac{\overline{DY}}{\overline{EF}}$ and P_{AYB} can be found in Lemmas 2.26, 3.30, 3.25, and 3.28.

3.7.2 Pseudo Divisions and Triangular Forms

In this section, we introduce some algebraic tools which will be used in the machine proof for statements in class C. These tools are actually parts of Wu's method of automated geometry theorem proving, more details of which can be found in [36, 12].

Let K be a computable field of characteristic zero (e.g., $\mathbf{Q}$) and $A = K[x_1, ..., x_n]$ be the polynomial ring of the variables $x_1, \cdots, x_n$. For $P \in K[x] - K$, we can write

$$P = c_d x_p^d + ... + c_1 x_p + c_0$$

where $c_i \in B[x_1, ..., x_{p-1}]$, $p > 0$, and $c_d \neq 0$. We call p the *class*, c_d the *initial*, x_p the *leading variable*, and d the *leading degree* of P respectively, or $class(P) = p$, $init(P) = c_d$, $lv(P) = x_p$, $ld(P) = d$. If $P \in K$, we have $class(P) = 0$.

Let $p = class(P) > 0$. A polynomial Q is said to be *reduced* with respect to P if $deg(Q, x_p) < ld(P)$.

Let $f = a_n v^n + \cdots + a_0$ and $h = b_k v^k + \cdots + b_0$ be two polynomials in $A[v]$, where v is a new indeterminate. Suppose k, the leading degree of h in v, is greater than 0. Then the pseudo division proceeds as follows:

Definition 3.89 (Pseudo Division) *First let $r = f$. Then repeat the following process until $m = deg(r, v) < k$: $r = b_k r - c_m v^{m-k} h$, where c_m is the leading coefficient of r. It is easy to see that m strictly decreases after each iteration. Thus the process terminates. At the end, we have the pseudo remainder $prem(f, h, v) = r = r_0$.*

Proposition 3.90 *Continuing from the above definition, we have the following formula,*

$$b_k^s f = qh + r_0, \quad \text{where } s \le n - k + 1 \text{ and } deg(r_0, v) < deg(h, v). \tag{1}$$

Proof. We fix polynomial h and use induction on $n = deg(f, v)$. If $n < k$, then we have $r_0 = f$ and $f = 0 \cdot h + r_0$. Suppose $n \ge k$ and formula (1) is true for those polynomials f, for which $deg(f, v) < n$. After the first iteration, we have $r = b_k f - a_n v^{n-k} h$, where a_n is the leading coefficient of f. Since $deg(r, v) < n$, we have $b_k^t r = q_1 h + r_0$ by the induction hypothesis. Substituting $r = b_k f - a_n v^{n-k} h$ in the last formula, we have (1). ∎

Definition 3.91 *A sequence of polynomials $TS = A_1, ..., A_p$ in $K[X]$ is said to be a triangular form, if either $p = 1$ and $A_1 \ne 0$ or $0 < class(A_i) < class(A_j)$ for $1 \le i < j$.*

For a triangular form $TS = A_1, ..., A_p$, we make a renaming of the variables. If A_i is of class m_i, we rename x_{m_i} as x_i, other variables are renamed as $u_1, ..., u_q$, where $q = n - p$. The variables $u_1, ..., u_q$ are called *the parameter set* of TS. Then TS is like

$$\text{(IV)} \qquad \begin{array}{l} A_1(u_1, \cdots, u_q, x_1) \\ A_2(u_1, \cdots, u_q, x_1, x_2) \\ \quad \cdots \\ A_p(u_1, \cdots, u_q, x_1, \cdots, x_p). \end{array}$$

For another polynomial G, we can define the *successive pseudo division*:

$$R_p = prem(G, A_p, x_p), \ldots, R_1 = prem(R_1, A_1, x_1).$$

$R = R_1$ is called the *final remainder* and is denoted by $prem(G, A_1, \ldots, A_r)$. It is easy to prove the following important proposition:

Proposition 3.92 (The Remainder Formula) *Let TS and R be the same as the above. There are some non-negative integers $s_1, \ldots, s_p$ and polynomials $Q_1, \ldots, Q_p$ such that*

1. $I_1^{s_1} \cdots I_p^{s_p} G = Q_1 A_1 + \cdots + Q_p A_p + R$ *where the I_i are the initials of the A_i.*
2. $deg(R, x_i) < deg(A_i, x_i)$, *for $i = 1, \cdots, p$.*

Proof. We use induction on p. The case $p = 1$ is actually Proposition 3.90. Suppose that $p > 1$ and the proposition is true for $p - 1$. Thus we have:

$$I_1^{s_1} \cdots I_{p-1}^{s_{p-1}} R_{p-1} = Q_1 A_1 + \cdots + Q_{p-1} A_{p-1} + R,$$

with $deg(R, x_i) < deg(A_i, x_i)$, for $i = 1, \ldots, p - 1$. Combining this with $R_{p-1} = I^{s_p} G - Q A_p$, we have 1 and 2. ∎

Theorem 3.93 *Let $TS = A_1, ..., A_p$ be a triangular set, G a polynomial. If $prem(G, TS) = 0$ then*

$$\forall x_i[(A_1 = 0 \wedge \cdots \wedge A_p = 0 \wedge I_1 \neq 0 \wedge \cdots \wedge I_p \neq 0) \Rightarrow G = 0].$$

Proof. Since $prem(G, TS) = 0$, by the remainder formula

$$I_1^{s_1} \cdots I_p^{s_p} G = Q_1 A_1 + \cdots + Q_p A_p.$$

Now it is clear that $A_i = 0$ and $I_i \neq 0$ imply $G = 0$. ∎

Definition 3.94 *A triangular set $A_1, ..., A_p$ of the form (IV) is called irreducible if for each i*

1. *the initial I_i of A_i does not vanish in the polynomial ring $R_i = K(u)[x_1, \cdots, x_i]/(A_1, ..., A_{i-1})$, and*

2. *A_i is irreducible in R_i.*

Thus the sequence

$$F_0 = K(u), F_1 = A_0[x_1]/(A_1), ..., F_p = A_{p-1}[x_p]/(A_p) = A_0[x]/(A_1, ..., A_p)$$

is a tower of field extensions.

Example 3.95 *Let TS be the triangular set $A_1 = x_1^2 - u_1$, $A_2 = x_2^2 - 2x_1x_2 + u_1$. A_1 is irreducible over $A_0 = \mathbf{Q}[u_1]$; but A_2 is reducible over $A_1 = A_0[x_1]/(A_1)$ because $A_2 = (x_2 - x_1)^2$ under $x_1^2 - u_1 = 0$. Thus TS is reducible.*

Proposition 3.96 *Let $TS = A_1, ..., A_p$ of the form (IV) be irreducible, and G be a polynomial in $K[u, x]$. Then the following conditions are equivalent:*

(i) $prem(G, TS) = 0$.

(ii) Let E be an extension field of K. If $\mu = (\eta_1, \cdots, \eta_q, \zeta_1, \cdots, \zeta_p)$ in E^{d+r} is a common zero of $A_1, ..., A_p$ with η_i transcendental over K, then μ is also a zero of G, i.e., $G(\mu) = 0$.

Proof. For any polynomial h, let $\tilde{h}$ be the polynomial obtained from h by substituting u_i, x_i for η_i, ζ_i.

We use induction on k to prove the following assertions (for $0 < k \leq p$):

(U) For any polynomial $P = a_s x_k^s + \cdots + a_0$ ($0 < k \leq p$, $1 \leq s$, $a_i \in K[u, x_1, ..., x_{k-1}]$, $a_s \neq 0$) reduced with respect to $A_1, ..., A_p$, if μ is a zero of P, then $P = 0$.

If $k = 1$, then $P = a_s x_k^s + \cdots + a_0$, with all $a_j \in K[u]$. μ is a zero of P means $\tilde{P} = \tilde{a}_s {\zeta_1}^s + \cdots + \tilde{a}_0 = 0$. Since P is reduced with respect to A_1, $s < deg(A_1, x_1)$. By the uniqueness of representation in *algebraic extension*, we have all $\tilde{a}_j = 0$. Since η_i are transcendental over K, all $a_j = 0$. Hence $P = 0$.

Now we want to prove (U) is true for k assuming it is true for $k-1$. Since μ is a zero of P, $\tilde{P} = \tilde{a}_s \zeta_k{}^s + \cdots + \tilde{a}_0 = 0$. Since $s < deg(A_k, x_k)$, by the uniqueness of representation in *algebraic extension* again, all $\tilde{a}_j = 0$. Thus μ is also a zero of all a_j. Since all a_j are also reduced with respect to $A_1, ..., A_p$, $a_j = 0$ by the induction hypothesis. Hence $P = 0$.

(ii) $\Rightarrow$ (i). Suppose μ is a zero of G. Let $R = prem(G, A_1, ..., A_p)$. We have the remainder formula

$$I_1^{s_1} \cdots I_p^{s_p} G = Q_1 A_1 + \cdots + Q_p A_p + R.$$

Hence μ is a zero of R. Since R is reduced with respect to $A_1, ..., A_p$, $R = 0$.

(i) $\Rightarrow$ (ii). Suppose $prem(G, A_1, ..., A_p) = 0$. Then by the *remainder formula*, we have

$$I_1^{s_1} \cdots I_p^{s_p} G = Q_1 A_1 + \cdots + Q_p A_p.$$

where the I_k the are initials of the A_k. Since $prem(I_k, TS) \neq 0$, μ is not a zero of I_k (by (ii) $\Rightarrow$ (i)). Hence μ is a zero of G. ▮

We call μ in (ii) a *generic point* of that irreducible triangular form in field E.

The theorem is no longer true if $A_1, ..., A_p$ is reducible. We can find such an example by letting A_1, A_2 be the same as in example 3.95 and $G = x_2 - x_1$.

Theorem 3.97 *Let $TS = A_1, ..., A_p$ be an irreducible triangular form, G a polynomial. If*

$$\forall x_i[(A_1 = 0 \wedge \cdots \wedge A_p = 0 \wedge I_1 \neq 0 \wedge \cdots \wedge I_p \neq 0) \Rightarrow G = 0]$$

is true in any extension field of K then $prem(G, TS) = 0$.

Proof. Let $\eta_1, \cdots, \eta_q$ be some elements which are transcendental over K. By the definition of irreducible triangular set, we can find a generic zero $\mu = (\eta_1, \cdots, \eta_q, \zeta_1, \cdots, \zeta_p)$ of TS such that $I_i(\mu) \neq 0, i = 1, \cdots, p$. Since $A_i(\mu) = 0, I_i(\mu) \neq 0$, $i = 1, \cdots, p$, we have $G(\mu) = 0$. By Proposition 3.96, $prem(G, TS) = 0$. ▮

Exercise 3.98 *Let K be the field of the rational numbers, $TS = A_1, ..., A_p$ an irreducible triangular form, and G a polynomial. If*

$$\forall x_i[(A_1 = 0 \wedge \cdots \wedge A_p = 0 \wedge I_1 \neq 0 \wedge \cdots \wedge I_p \neq 0) \Rightarrow G = 0]$$

is true in the field of complex numbers then $prem(G, TS) = 0$.

3.7.3 Machine Proof for Class C

We now return to the theory of machine proof. First, let us restate Algorithm 3.33 using the language of triangular forms and pseudo divisions. Let $S = (C_1, \cdots, C_r, (E, F))$ be a statement in $\mathbf{C}_L$. We denote by $u_1, \cdots, u_q, x_1, \cdots, x_p$ the geometry quantities occurring in the proof of S such that the u_i are the free parameters and the x_i are those quantities from which a point will be eliminated. We arrange the subscripts such that

$$x_i = \frac{U_i(u_1, \cdots, u_q, x_1, \cdots, x_{i-1})}{I_i(u_1, \cdots, u_q, x_1, \cdots, x_{i-1})}, i = 1, \cdots, p.$$

Let $A_i = I_i x_i - U_i$. Then

$$TS = A_1, \cdots, A_p$$

is a triangular form with I_i as the initials of A_i. Furthermore TS is irreducible since $I_i \neq 0$ is true under the ndg conditions of S.

Theorem 3.99 *Use the same notations as above. The statement S is true iff $prem(E - F, TS) = 0$.*

Proof. Notice that the proving process of S is as follows: first replace x_i by $\frac{U_i}{I_i}$, $i = p, \cdots, 1$, in E and F to obtain two polynomials E' and F' in the u_i only. Since the u_i are free parameters, the statement S is true iff $E' = F'$. The above process is equivalent to taking the pseudo remainder of $E - F$ with respect to TS. Therefore S is true iff $prem(E - F, TS) = 0$.
∎

If $S = (C_1, \cdots, C_r, (E, F))$ is a statement in C, then for each i, A_i has two possible forms:

(1) either $A_i = I_i x_i + U_i$, or

(2) $A_i = I_i x_i^2 + U_i x_i + V_i$

where I_i, U_i, and V_i are polynomials in $u_1, \cdots, u_q, x_1, \cdots, x_{i-1}$. $TS = A_1, \cdots, A_p$ is still a triangular form. Let

$$R = prem(E - F, TS).$$

Then we have

Theorem 3.100 *1. If $R = 0$, S is true.*

2. If $R \neq 0$ and TS is irreducible then S is not a theorem in the complex plane.

Proof. For the first case, by Theorem 3.93 we have

$$\forall u_i x_i [(A_1 = 0 \wedge \cdots \wedge A_p = 0 \wedge I_1 \neq 0 \wedge \cdots \wedge I_p \neq 0) \Rightarrow E - F = 0].$$

Under the ndg conditions of S, we have $I_i \neq 0, i = 1, \cdots, p$. Then $E = F$ is true. The second case is a consequence of Theorem 3.97. ∎

Remark 3.101 *In practice, we do not have to take the pseudo remainder of $E - F$ with respect to TS. The better way is to eliminate x_i from E and F separately as usual, so that we can take the advantage of removing the common factors from E and F during the proof. To eliminate x_i from E, if $A_i = I_i x_i - U_i$, we need only to replace x_i in E by $\frac{U_i}{I_i}$; if $A_i = I_i x_i^2 + U_i x_i + V_i$, we need to keep replacing x_i^2 in E by $-\frac{U_i x_i + V_i}{I_i}$ until the degree of x_i in E is less that two.*

If $R \neq 0$ and TS is reducible, we need to factorize TS into irreducible triangular forms. We will not discuss the factorization method in this book; those who are interested in this topic may refer to [36, 12]. Let us assume that for some A_i we have

$$A_i = I_i x_i^2 + U_i x_i + V_i = (I_{i,1} x_i - U_{i,1})(I_{i,2} x_i - U_{i,2})$$

which is true under the condition $A_k = 0, k = 1, \cdots, i-1$. Then TS is factored into two triangular forms TS_1 and TS_2 as follows

$$TS_1 = A_1, ..., A_{i,1}, ..., A_p; TS_2 = A_1, ..., A_{i,2}, ..., A_p$$

where $A_{i,1} = I_{i,1} x_i - U_{i,1}$, $A_{i,2} = I_{i,2} x_i - U_{i,2}$. Geometrically, this means that the two points introduced by a construction of type C22 or C23 can be distinguished and the two triangular forms TS_1 and TS_2 correspond to the two intersections.

If $R \neq 0$ and TS is reducible, let TS be factored into several irreducible triangular forms $TS_1, \cdots, TS_m$. Then we have three possible cases:

- *For each TS_i, $prem(E - F, TS_i) \neq 0$, i.e., $E = F$ is not valid for all triangular forms. In this case, we say that the statement S is generally false.*
- *For some TS_i, $prem(E - F, TS_i) \neq 0$, while for other TS_j, $prem(E - F, TS_j) = 0$. In this case, the statement S is true only for some configurations.*
- *$prem(E - F, TS_i) = 0$ for all i. In this case, S is still true in the Euclidean geometry. This will happen only when we want to introduce the intersection points of two circles or a line and a circle which are tangent to each other.*

With the help of algebraic tools, we have a complete method of machine proof for geometry statements in class C.

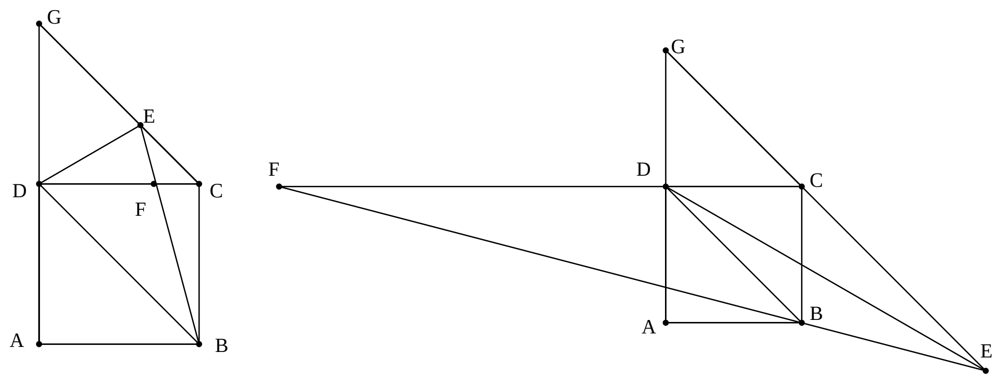


Figure 3-35

Example 3.102 *Let ABCD be a square. CG is parallel to the diagonal BD. Point E is on CG such that BE = BD. F is the intersection of BE and DC. Show that DF = DE.*

Proof. As shown in Figure 3-35, let G be a point on line AD such that $\overline{AD} = \overline{DG}$. Then point E has two possible positions. The following proof shows that $P_{DFD} = P_{DED}$ is true for both positions of E. By Lemmas 3.22 and 3.27, we have

$$P_{BCG} = -P_{DBC}$$
$$P_{DBC} = P_{ADB} = P_{DGD} = P_{BCB} = P_{ADA} = P_{DCD} = P_{ABA}$$
$$P_{CGC} = P_{BDB} = 2P_{ABA}$$

Let $r = \frac{\overline{CE}}{\overline{CG}}$. By the co-side theorem, we have

$$P_{DFD} = \frac{P_{DCD}S^2_{BDE}}{S^2_{BDEC}} = \frac{P_{ABA}S^2_{BDC}}{(S_{BDC} - rS_{DCG})^2} = \frac{P_{ABA}S^2_{BDC}}{(S_{BDC} + rS_{BDC})^2} = \frac{P_{ABA}}{(1+r)^2}.$$

By (II) on page 116,

$$P_{DED} = (1-r)P_{DCD} + rP_{DGD} - (1-r)rP_{CGC} = ((1-r) + r - 2(1-r)r)P_{ABA}.$$

Then $P_{DFD} = P_{DED}$ is true iff

$$(1+r)^2((1-r) + r - 2(1-r)r) = 1$$

and this is the case since by Proposition 3.83,

$$r^2 = \frac{2rP_{BCG} - P_{BCB} + P_{BDB}}{P_{CGC}} = \frac{-2r+1}{2}. \quad \blacksquare$$

Example 3.103 *Let ABC be a triangle such that AC = BC. D is a point on AC; E is a point on BC such that AD = BE. F is the intersection of DE and AB. Show that DF = EF.*

If we describe the statement as follows, it becomes reducible.

((POINTS *A B*)
(ON *C* (BLINE *A B*))
(ON *D* (LINE *A C*))
(INTER *E* (LINE *B C*) (CIR *B* $\overline{AD}^2$))
(INTER *F* (LINE *A B*) (LINE *D E*))
(MIDPOINT *F D E*))

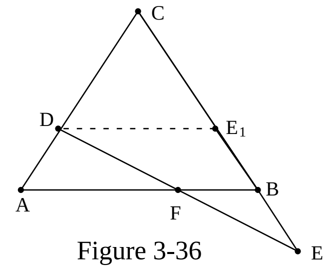


Figure 3-36

By Proposition 3.83,

$$\left(\frac{\overline{BE}}{\overline{BC}}\right)^2 \cdot P_{BCB} - P_{ADA} = 0. \tag{1}$$

Since $P_{ADA}=P_{ACA} \cdot (\frac{\overline{AD}}{\overline{AC}})^2 = P_{BCB} \cdot (\frac{\overline{AD}}{\overline{AC}})^2$, (1) becomes

$$\left(\frac{\overline{BE}}{\overline{BC}}\right)^2 \cdot P_{BCB} - P_{BCB} \cdot \left(\frac{\overline{AD}}{\overline{AC}}\right)^2 = \left(\frac{\overline{BE}}{\overline{BC}} - \frac{\overline{AD}}{\overline{AC}}\right) \cdot \left(\frac{\overline{BE}}{\overline{BC}} + \frac{\overline{AD}}{\overline{AC}}\right) P_{BCB} = 0.$$

Then we have

$$\frac{\overline{BE}}{\overline{BC}} = \frac{\overline{AD}}{\overline{AC}} \quad \text{or} \quad \frac{\overline{BE}}{\overline{BC}} = -\frac{\overline{AD}}{\overline{AC}}$$

which correspond to points E_1 and E in Figure 3-36. In the first case, we have $AB \parallel DE_1$; the nondegenarate condition needed to construct point F is not satisfied. In the second case, the conclusion is true. Here is the proof of the example.

$$\begin{aligned}
&\frac{\overline{DF}}{\overline{FE}} \\
&\overset{F}{=} -\frac{S_{ABD}}{S_{ABE}} \\
&\overset{E}{=} -\frac{S_{ABD}}{\frac{\overline{BE}}{\overline{BC}} S_{ABC}} \overset{E}{=} \frac{S_{ABD}}{\frac{\overline{AD}}{\overline{AC}} S_{ABC}} \\
&\overset{D}{=} \frac{S_{ABC}\frac{\overline{AD}}{\overline{AC}}}{\frac{\overline{AD}}{\overline{AC}} S_{ABC}} \\
&\overset{simplify}{=} 1
\end{aligned}
\qquad
\begin{aligned}
&\frac{\overline{DF}}{\overline{FE}} \overset{F}{=} -\frac{S_{ABD}}{S_{ABE}} \\
&S_{ABE} \overset{E}{=} \frac{\overline{BE}}{\overline{BC}} \cdot S_{ABC} \\
&\frac{\overline{BE}}{\overline{BC}} = -\frac{\overline{AD}}{\overline{AC}} \\
&S_{ABD} \overset{D}{=} S_{ABC} \cdot \frac{\overline{AD}}{\overline{AC}}
\end{aligned}$$

3.8 Geometry Information Bases and Machine Proofs Based on Full-Angles

In many traditional proofs, the relationships among angles are always used directly if possible. This may be one of the main reasons that traditional geometric proofs are very short, skillful and interesting. But the angle is a concept involving the relation of orders, and is thus very difficult to fit into our machine proof system. In Section 1.10, we introduce the concept of full-angles as the basis of machine proof of geometry statements involving angles. In Subsection 3.4.2, we present a machine proof method for geometry statements involving full-angles based on the property of the tangent function of full-angles. But this approach loses some of the unique character of the traditional proofs based on angles.

This section will be devoted to another approach to mechanical generation of proofs based on full-angles. The basic idea for this new approach is that we will build a *geometry information base* (GIB) based on the constructive description of the statement. The GIB for a statement contains some basic geometry relations about the configuration of the statement such as collinear points, parallel lines, perpendicular lines, and cyclic points, etc. In Subsections 2.5.1 (see the paragraph after Exercise 2.40) and 3.3.2, we have touched unon the idea of building some kind of geometry information bases. The purpose of building the GIB in those two cases is that the *refined elimination techniques* need these geometry relations. The GIB actually has much potential in the automated production of traditional proofs for geometry statements. In the following subsections, we will show that elegant proofs for many geometry theorems can be obtained by merely checking a good GIB. This section reports our initial study of this promising approach.

3.8.1 Building the Geometry Information Base

We will use Example 1.118 to illustrate our method.

First, we will check every step in the proof of Example 1.118 to find out how to eliminate points O, D and E from full-angles $\angle[AD, AO]$ and $\angle[AC, CE]$.

We first use rules Q7 and Q9 (on page 46) to eliminate point E from $\angle[AC, CE]$.

(1) $\angle[AC, CE] = \angle[AC, BC] + \angle[BC, CE]$.
Since $BE = CE$ and that E is on line AB, by Q9 we have
(2) $\angle[BC, CE] = \angle[BE, BC] = \angle[BA, BC]$.
To eliminate the points O and D, we first use Q7 to divide $\angle[AD, AO]$ into two parts:
(3) $\angle[AD, AO] = \angle[AD, AC] + \angle[AC, AO]$.
Since $AD \perp BC$, by Q7 we can eliminate D.

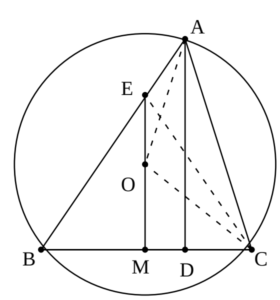


Figure 3-37

(4) $\angle[AD, AC] = \angle[AD, BC] + \angle[BC, AC] = \angle[1] + \angle[BC, AC]$.

The next step is to eliminate O from $\angle[AC, AO]$.

$$
\begin{array}{ll}
(5)\angle[AC,AO] = \angle[CO,AC] & (\text{Q9 and } AO = CO)\\
= \angle[CO,MO] + \angle[MO,AC] & (\text{Q7})\\
= \angle[AC,AB] + \angle[MO,AC] & (\text{Q12, } MB = MC \text{ and } AO = BO = CO)\\
= \angle[AC,AB] + \angle[MO,BC] + \angle[BC,AC] & (\text{Q7})\\
= \angle[BC,AB] + \angle[1]. & (\text{Q7, } MB = MC \text{ and } BO = BC)
\end{array}
$$

Finally, replacing $\angle[AD,AC]$, $\angle[AC,AO]$ in (3) by (4),(5) and $\angle[BC,CE]$ in (1) by (2), we have the conclusion:

$$
\begin{array}{ll}
(6)\angle[AD,AO] + \angle[AC,CE] & \\
= \angle[AC,BC] + \angle[BA,BC] + \angle[1] + \angle[BC,AC] + \angle[BC,AB] + \angle[1] & \\
= \angle[AC,AC] + \angle[BA,AB] + \angle[1] + \angle[1] & (\text{Q7})\\
= \angle[0] & (\text{Q1,Q3 and Q4})
\end{array}
$$

Suppose that a program named QAP could prove the geometry theorem as above. Now we are going to check what kinds of information would be needed for designing the program.

Obviously, the basic properties Q1–Q12 (on page 46) about full-angles are necessary in each step of proof. But we will soon see that it is not enough to prove this geometry theorem by using these rules only. Much more geometric information is needed. Let us check the above proof step by step.

Step (1) seems very easy. But from the point of view of mechanization, it is actually difficult to start this step. The question is why the full-angle $\angle[AC,CE]$ should be divided into $\angle[AC,BC] + \angle[BC,CE]$ but not anything else, for example, $\angle[AC,AD] + \angle[AD,CE]$. Why could our program QAP foresee that the point E will be eliminated from $\angle[BC,CE]$?

To make QAP work like this, we can imagine that a *geometric information base* (GIB) will be generated automatically before QAP proves the theorem. The program will know that $\angle[BC,CE] = \angle[BA,BC]$ by checking GIB. So it chooses step (1) so that point E will be eliminated in the next step.

How do we generate the GIB?

Of course, the hypotheses of the proposition should be put into the GIB first. There are four conditions in Example 1.118:

(G1) The circumcenter of triangle ABC is O. ($OA = OB = OC$)

(G2) AD is an altitude of triangle ABC. ($AD \perp BC$ and $D \in BC$)

(G3) M is the midpoint of BC. ($BM = MC$ and $M \in BC$)

(G4) E is the intersection of AB and MO. ($E \in AB$ and $E \in MO$)

But here (G1)–(G4) are only a small part of the GIB. We must put more information into the GIB. To see what is still needed in the GIB, we check step (2). In this step, the rules on full-angles are not sufficient. We need the hypotheses of the proposition (i.e., conditions (G1)–(G4)). Furthermore, we need some geometric facts which are derived by applying some geometry knowledge to the hypotheses of the statement, (such as deriving $BE = CE$ from $MB = MC, OB = OC$ and E is on the line OM). So we also need a *geometry knowledge base* (GKB) to build the GIB. As an example, in order to obtain $BE = CE$, our GKB should include the following propositions:

(a) If $PB = PC$ and $QB = QC$ then PQ is the perpendicular bisector of BC. (ndg condition: $B \neq C$ and $P \neq Q$).

(b) If P is on the perpendicular bisector of BC, then $BP = CP$.

(c) If $PC = PB$ then $\angle[PC, BC] = \angle[BC, PB]$.

(d) If P is on line AB, then $\angle[PB, XY] = \angle[AB, XY]$.

Applying (a) to G1 and G3, QAP obtains a new information and puts it into the GIB:

(G5) MO is the perpendicular bisector of BC.

Applying (b) to G4 and G5, we have

(G6) $BE = CE$.

Applying (c) to G6, we have

(G7) $\angle[BC, CE] = \angle[BE, BC]$.

Finally, applying (d) to G4 and G7, we have

(G8) $\angle[BE, BC] = \angle[AB, BC]$.

G8 is the deductive basis of step (2).

For step (3), QAP has to foresee that points D and O can be eliminated as in steps (4) and (5) and as a consequence, it decides to split the full-angle $\angle[AD, AO]$ into $\angle[AD, AC]$ and $\angle[AC, AO]$. The following additional geometry knowledge should be included in GKB:

(e) If PQ is perpendicular to UV, then $\angle[PQ, XY] = \angle[1] + \angle[UV, XY]$.

(f) (Another form of Q12) If O is the circumcenter of triangle ABC, then $\angle[AC, AO] = \angle[BC, AB] + \angle[1]$.

Proof of (f). Let D be the intersection of AO and the circle. By the inscribed angle theorem, $\angle[AB, BC] = \angle[AD, CD]$. Since $AC \perp CD$, we have $\angle[AC, AO] = \angle[1] + \angle[DC, AD] = \angle[1] + \angle[AB, BC]$. ∎

Applying (e) to G2 and (f) to G1, QAP will put the following information into GIB:

(G9) $\angle[AD, AC] = \angle[1] + \angle[BC, AC]$.

(G10) $\angle[AC, AO] = \angle[BC, AB] + \angle[1]$.

When QAP finds the information G1-G10, step (6) will be done according to rules of full-angle (Q1-Q12) easily.

Here we mentioned only the information G1–G10 which are useful for proving the statement. In fact, much more information about this statement will be put into the GIB, because QAP does not know what information will be used when it generates the GIB. It will keep applying every rule in the GKB to all the information in GIB to get new information and to put the new information into GIB until nothing new can be obtained.

What geometry knowledge should be included in the GKB? Is it complete? At the present stage, the choices of the rules in the GKB are based on our experience of proving geometry theorems. Its completeness is still not considered. But if the method in this section fails to prove or disprove a statement, we can always use Algorithm 3.33 which is complete for constructive geometry statements. Thus, the GIB-GKB method is actually an expert system of proving geometry statements. Up to now, the following rules have been put into our GKB:

K1 Two points A and B determine one line. (ndg condition: $A \neq B$)

K2 Three points A, B and C determine one circle. (ndg condition: $S_{ABC} \neq 0$)

K3 $\angle[PQ, XY] = \angle[UV, XY]$ if and only if $\angle[PQ, UV] = \angle[0]$. (ndg condition: $X \neq Y$)

K4 Four points A, B, C and D are cyclic if and only if $\angle[AC, BC] = \angle[AD, BD]$. (ndg condition: A, B, C and D are not collinear)

K5 $AB = AC$ if and only if $\angle[AB, BC] = \angle[BC, AC]$. (ndg condition: $S_{ABC} \neq 0$)

K6 $\angle[AB, XY] + \angle[XY, UV] = \angle[AB, UV]$. (ndg condition: $X \neq Y$)

K7 $AB \perp BC$ if and only if AC is the diameter of the circumcircle of triangle ABC. (ndg condition: $S_{ABC} \neq 0$)

K8 AB is the perpendicular bisector of XY if and only if $AX = AY$ and $BX = BY$. (ndg condition: $A \neq B$ and $X \neq Y$)

K9 If point O is the circumcenter of triangle ABC then $\angle[OA, AB] = \angle[1] + \angle[AC, BC]$ (ndg condition: $S_{ABC} \neq 0$)

Suppose that for a geometry statement, the GIB has been generated by using GKB. The next step is how to generate a proof based on the GIB. The key idea is still to eliminate points introduced by constructions of the given statement. But here the eliminating rules are based mainly on the GIB rather than construction. Following are the rules for eliminating point X from the full-angle $\angle[AB, PX]$:

QE1 If X is on line PQ, then $\angle[AB, PX] = \angle[AB, PQ]$.

QE2 If PX is parallel to UV, then $\angle[AB, PX] = \angle[AB, UV]$.

QE3 If PX is perpendicular to UV, then $\angle[AB, PX] = \angle[1] + \angle[AB, UV]$.

QE4 If X is on line UV and U, P, Q and X are cyclic, then $\angle[AB, PX] = \angle[AB, UV] + \angle[UQ, PQ]$.

(because $\angle[AB, PX] = \angle[AB, UV]+\angle[UV, PX]$, $\angle[UV, PX] = \angle[UX, PX] = \angle[UQ, PQ]$.)

QE5 If X is on line UV and $PX = PU$ then $\angle[AB, PX] = \angle[AB, UV]+\angle[PU, UV]$. (because $\angle[PU, UV] = \angle[PU, UX] = \angle[UX, PX] = \angle[UV, PX]$.)

QE6 If X is on line UV and PU is the perpendicular bisector of QX, then $\angle[AB, PX] = \angle[AB, UV] + \angle[PQ, QU]$. (because $\angle[PQ, QU] = \angle[UX, PX] = \angle[UV, PX]$.)

QE7 If X is the circumcenter of triangle PQU, then $\angle[AB, PX] = \angle[AB, PQ]+\angle[UQ, UP]+\angle[1]$. (by K9, $\angle[PQ, PX] = \angle[1] + \angle[UQ, UP]$)

QE8 If $\angle[UV, PX] = \angle[f]$ is known, then $\angle[AB, PX] = \angle[AB, UV] + \angle[f]$.

Here we assume that points A, B, P, Q, U and V are introduced before point X.

Exercise 3.104 *Prove propositions QE1-QE8 based on Q1-Q12.*

3.8.2 Machine Proof Based on the Geometry Information Base

We will use the following new version of Example 1.118 to illustrate how the program works.

Example 3.105 *The circumcenter of triangle ABC is O. AD is the altitude on side BC. Show that* $\angle[AO, DA] = \angle[BA, BC] - \angle[BC, CA]$.

The constructive description is

((POINTS *B C A*)

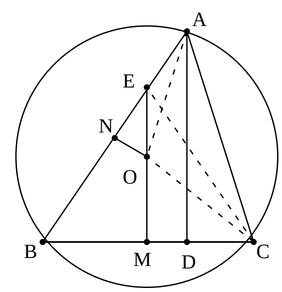


Figure 3-38

(FOOT D A B C)
(MIDPOINT M B C)
(MIDPOINT N A B)
(INTER E (LINE A B) (PLINE M A D))
(INTER O (LINE M E) (TLINE N N B)))

The conclusion is $\angle[AO, DA] + \angle[BC, CA] + \angle[BC, BA] = \angle[0]$ which is equivalent to $\angle[AO, DA] = \angle[BA, BC] - \angle[BC, CA]$.

The GIB for this example contains several groups of geometry relations which will be explained separately below.

(I1) p-list: includes all the points in the statement, listed in the introducing order, i.e., p-list $= (B\ C\ A\ D\ M\ N\ E\ O)$.

(I2) free-points: includes all the free points in the statement, i.e., free-points= $(B\ C\ A)$

(I3) all-lines: GIB can list all[2] the lines in the statement.

$$((O\ N)\ (O\ E\ M)\ (E\ N\ A\ B)\ (D\ A)\ (M\ D\ B\ C))$$

which means that points O, E and M are collinear, etc.

(I4) p-lines: contains all the parallel lines: $((O\ E\ M)\ (D\ A))$ which means that $OEM \parallel DA$.

(I5) t-lines: contains all the perpendicular lines:

$$((O\ N)\ (E\ N\ A\ B))\ (((O\ E\ M)\ (D\ A))\ (M\ D\ B\ C))$$

which means that line ON is perpendicular to line $ENAB$; and lines OEM and DA are both perpendicular to line $MDBC$.

(I6) circles: contains all the circles in the statement;

$((E\ O)\ N\ E\ O)\ ((A\ O)\ N\ A\ O)\ ((B\ O)\ N\ M\ B\ O)\ ((D\ O)\ M\ D\ O)$
$((C\ O)\ M\ C\ O)\ ((D\ E)\ M\ D\ E)\ ((B\ E)\ M\ B\ E)\ \ ((C\ E)\ M\ C\ E)$
$((A\ M)\ D\ A\ M)\ ((B\ A)\ D\ B\ A\ (N))\ ((C\ A)\ D\ C\ A)$
$((B\ C)\ B\ C\ (M))\ (B\ C\ (E))\ (C\ B\ A\ (O))$

which means

EO is a diameter of the circumcircle of triangle NEO;
...;
BA is a diameter and N is the center of circumcircle of triangle DBA;

[2] As we mentioned before, these lines are those which can be obtained directly from the hypotheses of the geometry statement.

BC is a diameter and M is the center of the circle through points B and C;
E is the center of the circle through points B and C; and
O is the circumcenter of triangle CBA.

Using this information, QAP gives the following proof.

$$\begin{aligned}
&\angle[BC,BA]+\angle[BC,CA]+\angle[AO,AD]\\
=\ &\angle[BC,BA]+\angle[BC,CA]+\angle[1]+\angle[BA,BC]+\angle[CA,AD]\\
&\text{(By Q7, } \angle[AO,AD]=\angle[AO,CA]+\angle[CA,AD].\\
&\text{Since } O \text{ is the circumcenter of } ABC,\ \angle[AO,CA]=\angle[1]+\angle[BA,BC].)\\
=\ &\angle[0]+\angle[BC,CA]+\angle[1]+\angle[CA,BC]+\angle[1]\\
&(\angle[BC,BA]+\angle[BA,BC]=\angle[BC,BC]=0.\\
&\text{Since } AD\perp BC,\ \angle[CA,AD]=\angle[CA,BC]+\angle[BC,AD]=\angle[CA,BC]+\angle[1].)\\
=\ &\angle[0].\quad (\angle[1]+\angle[1]=\angle[0] \text{ and } \angle[BC,CA]+\angle[CA,BC]=\angle[0].)
\end{aligned}$$

In what follows, we will use more examples to illustrate the GIB-GKB method. The examples in Section 1.10 were all produced according to the above method by our program.

Example 3.106 (Simson's Theorem) *The same as Example 3.79 on page 144.*

Constructive description
((CIRCLE A B C D)
(FOOT E D B C)
(FOOT F D A C)
(FOOT G D A B)
The conclusion: $\angle[EF,FG]=\angle[0]$.

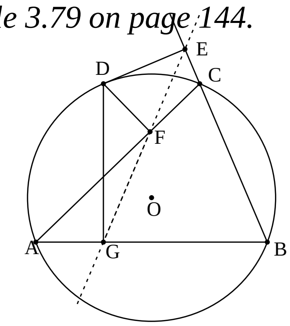


Figure 3-39

Proof. The machine proof

$$\begin{aligned}
\angle[EF,GF] &= \angle[EF,DF]+\angle[DF,GF] && \text{(Q7)}\\
&= \angle[EC,DC]+\angle[DA,GA] && \text{(Q10 } (D,C,E,F;\ A,D,G,F \text{ cyclic.))}\\
&= \angle[BC,DC]+\angle[DA,BA] && \text{(Q8 } (E\in BC;\ G\in AB)).\\
&= \angle[BA,DA]+\angle[DA,BA] && \text{(Q10 } (A,B,C,D \text{ cyclic.))}\\
&= \angle[BA,BA]=\angle[0]. && \blacksquare
\end{aligned}$$

There is a traditional proof which proves the theorem by showing that $\angle EFC=\angle GFA$. This proof is not strict: the fact that points E and G are in different sides of AC is used but not proved.

Before presenting the next example, we introduce a new geometry object: (CIRC A B C) which stands for the circle passing through points A, B, and C.

Example 3.107 (Miquel Point) *Four lines form four triangles. Show that the circumcircles of the four triangles passes through a common point.*

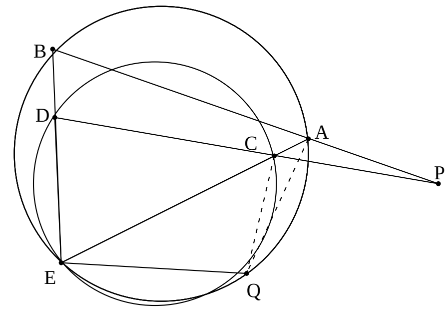


Figure 3-40

Constructive description

((POINT A D E Q)
(INTER B (LINE D E) (CIRC A Q E))
(INTER C (LINE A E) (CIRC D Q E))
(INTER P (LINE A B) (LINE C D))
$\angle[QC,CP]+\angle[AP,AQ]=\angle[0]$

The machine proof

$\angle[QC, CP] + \angle[AP, AQ]$
$= \angle[QC, DC] + \angle[AB, AQ]$
(because $CP \parallel DC$ and $AP \parallel AB$.)
$=\angle[EQ, ED] + \angle[EB, EQ] = \angle[EB, ED] = \angle[0]$.
(because E, Q, D, C and A, B, E, Q are cyclic points respectively.)

Example 3.108 *In a circle, the lines joining the midpoints of two arcs AB and AC meet line AB and AC at D and E. Show that AD = AE.*

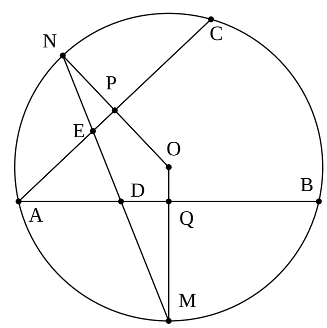


Figure 3-41

Constructive description

((POINTS A M N)
(CIRCUMCENTER O A M N)
(FOOT P A O N)
(FOOT Q A O M)
(INTER D (LINE N M) (LINE A Q))
(INTER E (LINE N M) (LINE A P))
$\angle[AD,DE]+\angle[AE,ED]=\angle[0]$

The machine proof

$\angle[AE, DE] + \angle[AD, DE]$
$= \angle[AP, MN] + \angle[AQ, MN]$
(because $AE \parallel AP$, $DE \parallel MN$, and $AD \parallel AQ$.)
$= \angle[AP, MN] + \angle[1] + \angle[MO, MN]$
(because $AQ \perp MO$.)
$= \angle[1] + \angle[NO, MN] + \angle[1] + \angle[1] + \angle[AM, AN]$
($AP \perp NO$.
$\angle[MO, MN] = \angle[1] + \angle[AM, AN]$, because O is the circumcenter of $\triangle AMN$)
$= \angle[1] + \angle[1] + \angle[AN, AM] + \angle[AM, AN]$
(since O is the circumcenter of $\triangle AMN$, $\angle[NO, MN] = \angle[1] + \angle[AN, AM]$.)
$= 0$

Example 3.109 *From the midpoint C of arc AB of a circle, two secants are drawn meeting line AB at F, G, and the circle at D and E. Show that F, D, E, and G are on the same circle.*

Constructive description
((CIRCLE A C D E)
(CIRCUMCENTER O A C D)
(FOOT M A O C)
(INTER F (LINE A M) (LINE D C))
(INTER G (LINE A M) (LINE C E))
$\angle[CE,FG]+\angle[CD,DE]=\angle[0]$

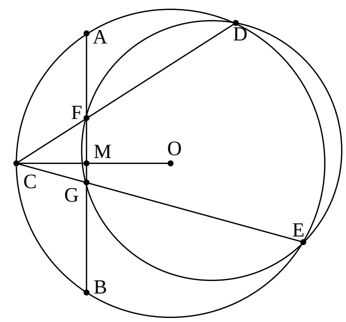


Figure 3-42

The Machine proof

$\angle[CE,FG]+\angle[CD,DE]$
$=\angle[CE,AM]+\angle[AC,AE]$
($FG \parallel AM$; since A,C,D, and E are cyclic, $\angle[CD,DE]=\angle[AC,AE]$.)
$=\angle[1]+\angle[CE,CO]+\angle[AC,AE]$
(since $AM\perp CO$, $\angle[CE,AM]=\angle[1]+\angle[CE,CO]$.)
$=\angle[1]+\angle[1]+\angle[AE,AC]+\angle[AC,AE]$
(since A,C,D, and E are on the circle with center O, $\angle[CE,CO]=\angle[1]+\angle[AE,AC]$.)
$=0$

Example 3.110 *Let Q, S and Y be three collinear points and (O,P) be a circle. Circles SPQ and YPQ meet circle (O,P) again at points R and X, respectively. Show that XY and RS meet on the circle (O,P).*

Constructive description
((CIRCLE R P Q S) (POINT X)
(INTER Y (LINE Q S)(CIRC P Q X))
(INTER I (LINE X Y) (LINE R P))
$\angle[XI,RI]+\angle[RP,XP]=\angle[0]$

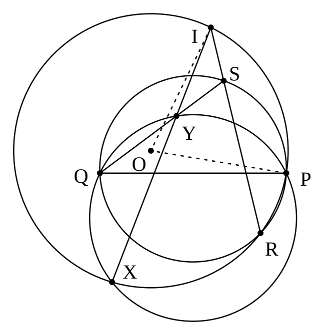


Figure 3-43

The machine proof

$\angle[PR,PX]+\angle[IX,RI]$
$=\angle[PR,PX]+\angle[XY,RS]$
(because $XI \parallel XY$ and $RI \parallel RS$.)
$=\angle[PR,PX]+\angle[XY,QS]+\angle[QS,RS]$
$=\angle[PR,PX]+\angle[XP,QP]+\angle[QS,RS]$
(because X,P,Y, and Q are cyclic, and $QS \parallel QY$.)
(because X,P,Y, and Q are cyclic, and $IX \parallel YI$.)

$= \angle[PR, PQ] + \angle[QS, RS] = \angle[0]$.
(Points R, Q, P, and S are cyclic.)

Example 3.111 *Let ABC be a triangle. Show that the six feet obtained by drawing perpendiculars through the foot of each altitude upon the other two sides are co-circle.*

Constructive description
((POINTS A B C)
(FOOT F C A B)
(FOOT D A B C)
(FOOT E B A C)
(FOOT G F B C)
(FOOT I D A B)
(FOOT H F A C)
(FOOT K E A B)
$\angle[GH,GI]+\angle[AK,HK]=\angle[0]$

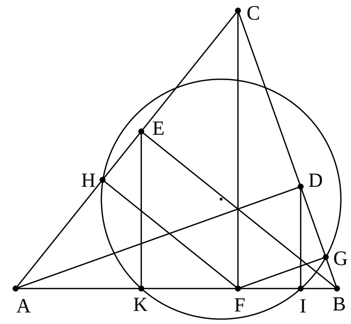


Figure 3-44

The machine proof

$\angle[GH, GI] + \angle[AK, HK]$
$= \angle[GH, GI] + \angle[AB, FI] + \angle[FE, EH]$
(because $AK \parallel AB$, and K, H, F, and E are cyclic.)
$= \angle[FE, EH] + \angle[GH, GI]$
(because $AB \parallel FI$.)
$= \angle[FE, AC] + \angle[CE, GI] + \angle[FG, CF]$
(because $EH \parallel AC$, and H, G, C, and F are cyclic.)
$= \angle[FE, AC] + \angle[CE, BF] + \angle[FD, DG] + \angle[FG, CF]$
(because I, G, F, and D are cyclic.)
$= \angle[FG, CF] + \angle[FE, AC]$
(because A, D, C, and F are cyclic.)
$= \angle[AD, CF] + \angle[FE, AC]$
(because $FG \parallel AD$.)
$= \angle[AD, CF] + \angle[BF, BC]$
(because E, F, C, and B are cyclic.)
$= \angle[1] + \angle[BC, CF] + \angle[BF, BC]$
(because $AD \perp BC$.)
$= \angle[1] + \angle[BF, CF] = \angle[0]$
(because $BF \perp CF$.)

Example 3.112 *The nine-point circle cuts the sides of the triangle at angles* $|B - C|$, $|C - A|$, *and* $|A - B|$.

Constructive description

((POINTS A B C)
(FOOT F C A B)
(MIDPOINT M B C)
(MIDPOINT Q A C)
(MIDPOINT P B A)
(MIDPOINT L F P)
(MIDPOINT S Q P)
(INTER N (TLINE L L P) (TLINE S S P))
$\angle[BC,AB]+\angle[AC,AB]+\angle[FN,LN]=\angle[0]$

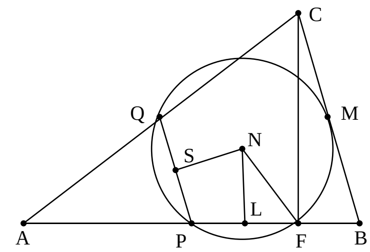


Figure 3-45

The machine proof

$$\angle[FN, LN] + \angle[AC, AB] + \angle[BC, AB]$$
$$= \angle[1] + \angle[FN, PF] + \angle[AC, AB] + \angle[BC, AB]$$
(because $LN \parallel CF$ and $CF \perp PF$.)
$$= \angle[1] + \angle[1] + \angle[FQ, PQ] + \angle[AC, AB] + \angle[BC, AB]$$
(Points F, Q, P are on the circle with center N.)
$$= \angle[FQ, CF] + \angle[CF, PQ] + \angle[AC, AB] + \angle[BC, AB]$$
$$=\angle[1] + \angle[FP, QP] + \angle[FQ, CF] + \angle[AC, AB] + \angle[BC, AB]$$
(because $LN \parallel CF$ and $CF \perp PF$.)
$$=\angle[1] + \angle[MQ, BC] + \angle[FQ, CF] + \angle[AC, AB] + \angle[BC, AB]$$
$$=\angle[1] + \angle[AC, AB] + \angle[FQ, CF]$$
(because $FP \parallel MQ$,$QP \parallel BC$, and $MQ \parallel AB$.)
$$= \angle[1] + \angle[AC, AB] + \angle[1] + \angle[CF, AC] + \angle[AF, CF]$$
(because F, A, C are on the circle with center Q.)
$$= \angle[AC, AB] + \angle[AF, AC]$$
$$= \angle[AC, AB] + \angle[AB, AC] = \angle[0]$$

Summary of Chapter 3

- We have the following formulas for the areas of triangles.

1. $S_{ABC} = \frac{1}{2}|BC|h_A = \frac{1}{2}|AC|h_B = \frac{1}{2}|AB|h_C$.
2. $S_{ABC} = S_{OUV}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$
   where x_A, y_A, x_B, y_B, x_C, and y_C are the area coordinates of points A, B, and C with respect to OUV.
3. $16S_{ABC}^2 = 4\overline{AB}^2\overline{AC}^2 - (\overline{AC}^2 + \overline{AB}^2 - \overline{BC}^2)^2 = 4\overline{AB}^2\overline{AC}^2 - P_{BAC}^2$.

- The following basic propositions and the ones on page 100 are the basis of the area method.

  1. $AB \perp CD$ iff $P_{ACD} = P_{BCD}$ or $P_{ACBD} = 0$.
  2. Let R be a point on line PQ with position ratios $r_1 = \frac{\overline{PR}}{\overline{PQ}}, r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ with respect to PQ. Then
  $$\begin{aligned} P_{RAB} &= r_1 P_{QAB} + r_2 P_{PAB} \\ P_{ARB} &= r_1 P_{AQB} + r_2 P_{APB} - r_1 r_2 P_{PQP}. \end{aligned}$$
  3. Let D be the foot of the perpendicular drawn from point P upon a line AB. Then we have
  $$\frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$
  4. Let AB and PQ be two nonperpendicular lines and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then
  $$\frac{\overline{PY}}{\overline{QY}} = \frac{P_{PAB}}{P_{QAB}}, \frac{\overline{PY}}{\overline{PQ}} = \frac{P_{PAB}}{P_{PAQB}}, \frac{\overline{QY}}{\overline{PQ}} = \frac{P_{QAB}}{P_{PAQB}}.$$
  5. Let $ABCD$ be a parallelogram. Then for any points P and Q, we have
  $$\begin{aligned} P_{APQ} + P_{CPQ} &= P_{BPQ} + P_{DPQ} \quad \text{or} \quad P_{APBQ} = P_{DPCQ} \\ P_{PAQ} + P_{PCQ} &= P_{PBQ} + P_{PDQ} + 2P_{BAD}. \end{aligned}$$
  6. Let $ABCD$ be a parallelogram and P be any point. Then
  $$\begin{aligned} P_{PAB} &= P_{PDC} - P_{ADC} = P_{PDAC} \\ P_{APB} &= P_{APA} - P_{PDAC}. \end{aligned}$$

- A *constructive configuration* is a figure which can be drawn using a ruler and a pair of compasses only. In other words, a constructive configuration can be obtained by first taking some arbitrary points, lines, and circles in the plane, and then taking the intersections of these lines and circles in a prescribed way. A *constructive geometry statement* is an assertion about a constructive configuration, and this assertion can be represented by a polynomial equation of three geometry quantities: the ratios of parallel line segments, the signed areas of triangles or quadrilaterals, and the Pythagoras differences. The set of all constructive statements is denoted by C. We also introduced a subclass of C, i.e., the class of the linear constructive geometry statements, which is denoted by $\mathbf{C}_{\mathrm{L}}$.

- A mechanical theorem proving method for class C was presented. The key idea of the method is to eliminate points from geometry quantities. The method can be used to produce short and readable proofs for geometry statements efficiently.

- We report some initial results on how to use the geometry information base to generate readable proofs for geometry statements using full-angles.

Chapter 4

Machine Proof in Solid Geometry

This chapter deals with the machine proof in solid geometry. Similar to plane geometry, we will consider those geometry statements that can be described constructively using lines, planes, circles, and spheres. In the first three sections, we deal with geometry statements involving collinear and parallel of lines and planes. More precisely, we deal with constructive statements in affine geometry of dimension three. Starting from Section 4, the Pythagoras difference is employed to deal with constructive statements involving perpendicular lines, circles, and spheres.

4.1 The Signed Volume

As before, we denote by $\overline{AB}$ the signed length of the oriented segment from A to B; we denote by S_{ABC} the signed area of the oriented triangle ABC.

In solid geometry, we have a new basic fourfold relation among points, *coplanar*, which will be characterized by Axioms S.1–S.5 about signed volumes.

We assume that Axioms A.1–A.6 (on page 55) are still valid, provided that all the points involved are coplanar. The signed areas of coplanar triangles can be compared, added, or subtracted. For instance, if A, O, U, and V are four coplanar points, by Axiom A.5

$$S_{OUV} = S_{OUA} + S_{OAV} + S_{AUV}. \tag{4.1}$$

A tetrahedron $ABCD$ has two possible orientations. We use the order of its vertices to represent its orientation. If we interchange two neighbor vertices, the orientation of the tetrahedron will be changed.

The *signed volume* V_{ABCD} of an oriented tetrahedron $ABCD$ is a real[1] number which satisfies the following properties.

[1]Here, we can use any number field and the results in this chapter are still valid.

Axiom S.1 *When two neighbor vertices of an oriented tetrahedron are interchanged, the signed volume of the tetrahedron will change signs, e.g.,* $V_{ABCD} = -V_{ABDC}$.

Axiom S.2 *If* A, B, C, *and* D *are four non-coplanar points, we have* $V_{ABCD} \neq 0$.

Axiom S.3 *There exist at least four points* A, B, C, *and* D *such that* $V_{ABCD} \neq 0$.

Axiom S.4 *For five points* A, B, C, D, *and* O *(Figure 4-1), we have*

$$V_{ABCD} = V_{ABCO} + V_{ABOD} + V_{AOCD} + V_{OBCD}.$$

Axioms S.3 and S.4 are called *dimension* axioms. They ensure that we are dealing with a proper three dimensional space: Axiom S.3 says that there are at least four non-coplanar points; Axiom S.4 says that all points must be in the same three dimensional space.

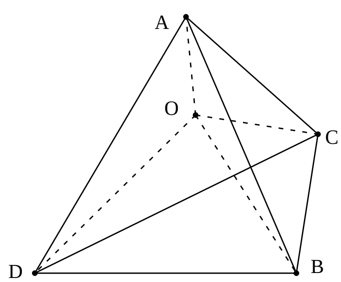


Figure 4-1

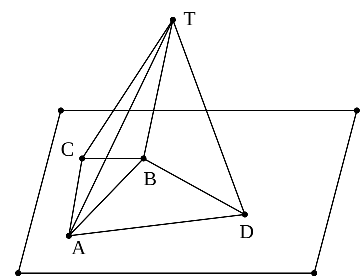


Figure 4-2

Axiom S.5 *If* A, B, C, *and* D *are four coplanar triangles and* $S_{ABC} = \lambda S_{ABD}$ *then for any point* T *we have* $V_{TABC} = \lambda V_{TABD}$. *(Figure 4-2)*

We extend the coplanar to be a geometry relation among any set of points: a set containing fewer than four points is always coplanar, and a set of points is coplanar if any four points in it are coplanar. We thus can introduce a new geometry object, the *plane*, which is a maximal set of coplanar points.

Proposition 4.1 *Four points* A, B, C, *and* D *are coplanar iff* $V_{ABCD} = 0$.

Proof. If $V_{ABCD} = 0$, by Axiom S.2 A, B, C, and D are coplanar. Let us assume A, B, C, and D to be coplanar points. If A, B, C, and D are collinear, we have $S_{ABC} = 0$. Let X be a point not on line AB. Then by Axiom S.5, $V_{ABCD} = \frac{S_{ABC}}{S_{ABX}} V_{ABXD} = 0$. If A, B, and C are not collinear, we have $V_{ABCD} = \frac{S_{BCD}}{S_{ABC}} V_{AABC} = 0$. ∎

Corollary 4.2 *For three non-collinear points* A, B, *and* C, *the set of all the points* D *satisfying* $V_{ABCD} = 0$ *is a plane and is denoted by plane* ABC.

Proof. Let P, Q, R, and S be four points in plane ABC. We need to show that $V_{PQRS} = 0$. We first show that two of the points, say P and Q, are coplanar with A, B, C. By Axiom

S.5,

$$V_{ABPQ} = \frac{S_{ABP}}{S_{ABC}} V_{ABCQ} = 0,$$

i.e., A, B, P, and Q are coplanar. Similarly, we can show that three of P, Q, R, and S are coplanar with A, B, C, and finally P, Q, R, and S are coplanar. ∎

In what follows, when speaking about a plane ABC, we always assume that A, B, and C are not collinear. Similarly, when speaking about a line AB, we assume $A \neq B$.

4.1.1 Co-face Theorem

In this and the next subsections, we will derive some basic properties about volumes which will serve as the basis of the volume method. First, Axiom S4 can be written in the following convenient way.

Proposition 4.3 (The Co-vertex Theorem) *Let ABC and DEF be two proper triangles in the same plane and T be a point not in the plane. Then we have* $\frac{V_{TABC}}{V_{TDEF}} = \frac{S_{ABC}}{S_{DEF}}$.

Proof. By Axiom S.5,

$$\frac{V_{TABC}}{V_{TDEF}} = \frac{V_{TABC}}{V_{TABF}} \frac{V_{TABF}}{V_{TAEF}} \frac{V_{TAEF}}{V_{TDEF}} = \frac{S_{ABC}}{S_{ABF}} \frac{S_{ABF}}{S_{AEF}} \frac{S_{AEF}}{S_{DEF}} = \frac{S_{ABC}}{S_{DEF}}. \quad ∎$$

Before proving the co-face theorem, we need to define the signed volume of a special polyhedron with five vertices.

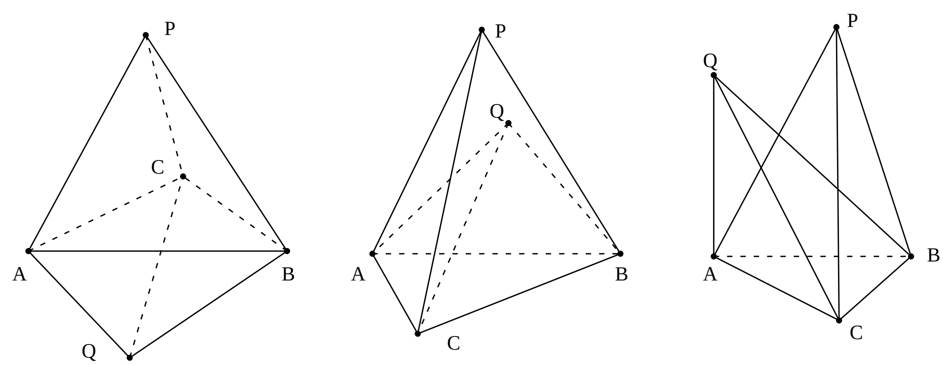


Figure 4-3

The polyhedron formed by five points in space is complicated. By $PABCQ$, we denote the one with faces PAB, PBC, PAC, QAB, QBC, and QAC. Figure 4-3 shows that several possible shapes of $PABCQ$. The volume of $PABCQ$ is defined to be

$$V_{PABCQ} = V_{PABC} - V_{QABC}. \tag{4.2}$$

By Axiom S4, we have

$$V_{PABCQ} = V_{PABQ} + V_{PCAQ} + V_{PBCQ}. \tag{4.3}$$

Proposition 4.4 (The Co-face Theorem) *A line PQ and a plane ABC meet at M. If $Q \neq M$, we have*

$$\frac{\overline{PM}}{\overline{QM}} = \frac{V_{PABC}}{V_{QABC}}; \qquad \frac{\overline{PM}}{\overline{PQ}} = \frac{V_{PABC}}{V_{PABCQ}}; \qquad \frac{\overline{QM}}{\overline{PQ}} = \frac{V_{QABC}}{V_{PABCQ}}.$$

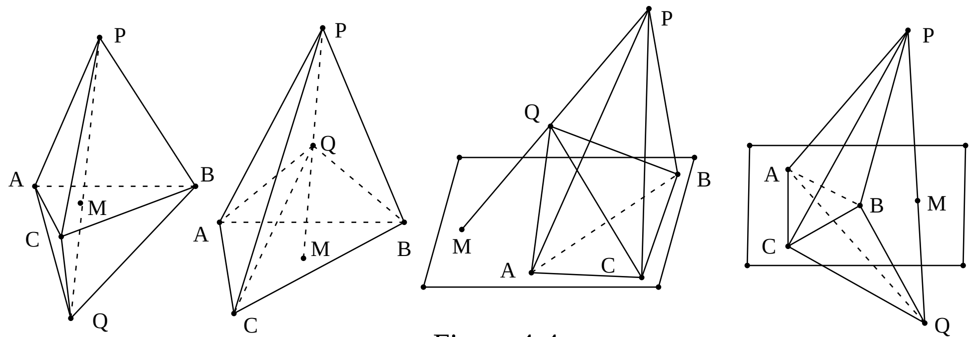


Figure 4-4

Proof. Figure 4-4 shows that several possible configurations of this proposition. Take points A' and B' such that $\overline{MA'} = \overline{CA}, \overline{MB'} = \overline{CB}$. Then $S_{ABC} = S_{A'B'M}$. By Propositions 4.3 and 2.8, we have, $\frac{V_{PABC}}{V_{QABC}} = \frac{V_{PA'B'M}}{V_{QA'B'M}} = \frac{S_{PB'M}}{S_{QB'M}} = \frac{\overline{PM}}{\overline{QM}}$. Other equations are consequences of the first one. ∎

The above proof is a *dimension reduction* process. A quantity in the space ($\frac{V_{PA'B'M}}{V_{QA'B'M}}$) is reduced to a quantity in the plane ($\frac{S_{PB'M}}{S_{QB'M}}$) which is further reduced to a quantity in a line ($\frac{\overline{PM}}{\overline{QM}}$).

Proposition 4.5 *Let R be a point on a line PQ and ABC be a triangle. Then we have*

$$V_{RABC} = \frac{\overline{PR}}{\overline{PQ}} V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}} V_{PABC}.$$

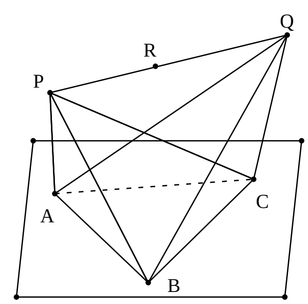


Figure 4-5

Proof. By Proposition 4.4,

$$\frac{V_{PRBC}}{V_{PQBC}} = \frac{\overline{PR}}{\overline{PQ}}, \frac{V_{PARC}}{V_{PAQC}} = \frac{\overline{PR}}{\overline{PQ}}, \frac{V_{PABR}}{V_{PABQ}} = \frac{\overline{PR}}{\overline{PQ}}.$$

By Axiom S4,

$$\begin{aligned} V_{RABC} &= V_{PABC} - V_{PRBC} - V_{PARC} - V_{PABR} \\ &= V_{PABC} - \frac{\overline{PR}}{\overline{PQ}}(V_{PQBC} + V_{PAQC} + V_{PABQ}) \end{aligned}$$

$$
\begin{aligned}
&= V_{PABC} - \frac{\overline{PR}}{\overline{PQ}} V_{PABCQ} \\
&= (1 - \frac{\overline{PR}}{\overline{PQ}}) V_{PABC} + \frac{\overline{PR}}{\overline{PQ}} V_{QABC} \\
&= \frac{\overline{PR}}{\overline{PQ}} V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}} V_{PABC}. \quad \blacksquare
\end{aligned}
$$

Proposition 4.6 *Let R be a point in the plane PQS. Then for three points $A, B,$ and C we have*

$$V_{RABC} = \frac{S_{PQR}}{S_{PQS}} V_{SABC} + \frac{S_{RQS}}{S_{PQS}} V_{PABC} + \frac{S_{PRS}}{S_{PQS}} V_{QABC}.$$

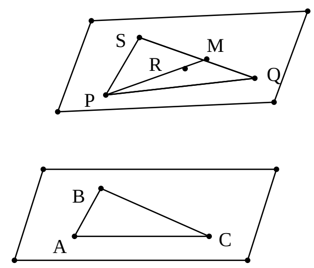


Figure 4-6

Proof. For any point X, let $V_X = V_{XABC}$. Without loss of generality, let M be the intersection of PR and QS. By Proposition 4.5,

$$V_R = \frac{\overline{PR}}{\overline{PM}} V_M + \frac{\overline{RM}}{\overline{PM}} V_P = \frac{\overline{PR}}{\overline{PM}} (\frac{\overline{QM}}{\overline{QS}} V_S + \frac{\overline{MS}}{\overline{QS}} V_Q) + \frac{\overline{RM}}{\overline{PM}} V_P. \quad (1)$$

By the co-side theorem, $\frac{\overline{RM}}{\overline{PM}} = \frac{S_{RQS}}{S_{PQS}}$, $\frac{\overline{QM}}{\overline{QS}} = \frac{S_{PQR}}{S_{PQRS}}$, $\frac{\overline{MS}}{\overline{QS}} = \frac{S_{PRS}}{S_{PQRS}}$, $\frac{\overline{PR}}{\overline{PM}} = \frac{S_{PQRS}}{S_{PQS}}$. Substituting these into (1), we obtain the result. $\blacksquare$

4.1.2 Volumes and Parallels

Two planes or a line and a plane, are said to be *parallel* if they have no point in common. Two lines are said to be parallel if they are in the same plane and do not have a common point.

By the notation $PQ \parallel ABC$, we mean that $A, B, C, P,$ and Q satisfy one of the following conditions: (1) $P = Q$, (2) $A, B,$ and C are collinear, or (3) A, B, C, P, and Q are on the same plane, or (4) line PQ and plane ABC are parallel. According to the above definition, if $PQ \not\parallel ABC$ then line PQ and plane ABC have a normal intersection. For six points $A, B, C, P, Q,$ and R, $ABC \parallel PQR$ iff $AB \parallel PQR$, $BC \parallel PQR$, and $AC \parallel PQR$.

Proposition 4.7 *$PQ \parallel ABC$ iff $V_{PABC} = V_{QABC}$ or equivalently $V_{PABCQ} = 0$.*

Proof. If $V_{PABC} \neq V_{QABC}$, then $P \neq Q$ and $A, B,$ and C are not collinear. Let O be a point on line PQ such that $\frac{\overline{PO}}{\overline{PQ}} = \frac{V_{PABC}}{V_{PABCQ}}$. Thus $\frac{\overline{OQ}}{\overline{PQ}} = -\frac{V_{QABC}}{V_{PABCQ}}$. By Proposition 4.5, $V_{OABC} = \frac{\overline{PO}}{\overline{PQ}} V_{QABC} + \frac{\overline{OQ}}{\overline{PQ}} V_{PABC} = 0$. By Axiom S2, point O is also in plane ABC, i.e., line PQ is not parallel to ABC. Conversely, if $PQ \nparallel ABC$ then $P \neq Q$ and A, B, and C are not collinear. Let O be the intersection of PQ and ABC. By Proposition 4.4, $\frac{\overline{OP}}{\overline{OQ}} = \frac{V_{PABC}}{V_{QABC}} = 1$. Thus $P = Q$, which is a contradiction. ∎

Proposition 4.8 *$PQR \parallel ABC$ iff $V_{PABC} = V_{QABC} = V_{RABC}$.*

Proof. By Proposition 4.7, $V_{PABC} = V_{QABC} = V_{RABC}$ iff lines PQ and PR are parallel to plane ABC. We must show that for any point D in plane PQR, line PD is also parallel to ABC. By Proposition 4.6,

$$\begin{aligned} V_{DABC} &= \frac{S_{PQD}}{S_{PQR}}V_{RABC} + \frac{S_{PDR}}{S_{PQR}}V_{QABC} + \frac{S_{DQR}}{S_{PQR}}V_{PABC} \\ &= V_{PABC}(\frac{S_{PQD}}{S_{PQR}} + \frac{S_{PDR}}{S_{PQR}} + \frac{S_{DQR}}{S_{PQR}}) = V_{PABC} \end{aligned}$$

i.e., $PD \parallel ABC$. ∎

A figure $P_1P_2...P_n$ is said to be a *translation* of $Q_1Q_2...Q_n$ if $\overline{P_iP_{i+1}} = \overline{Q_iQ_{i+1}}$. Let triangle XYZ be a translation of triangle ABC. Then for any points P, Q, and R in plane XYZ, we define

$$\frac{S_{PQR}}{S_{ABC}} = \frac{S_{PQR}}{S_{XYZ}}.$$

For convenience, we use the symbol

$$\frac{S_{PQR}}{S_{ABC}} = \lambda \qquad \text{or} \qquad S_{PQR} = \lambda S_{ABC}$$

to denote the fact that plane PQR is the same as or parallel to plane ABC, and λ is the ratio of the signed areas S_{PQR} and S_{ABC}.

The following propositions about translations of line segments and triangles are often used in the machine proof method, in order to add auxiliary translations of line segments and triangles.

Proposition 4.9 *Let $PQTS$ be a parallelogram. Then for points A, B, and C, we have*

$$V_{PABC} + V_{TABC} = V_{QABC} + V_{SABC} \textit{ or } V_{PABCQ} = V_{SABCT}.$$

Proof. This is a consequence of Proposition 4.5, because both sides of the equation are equal to $2V_{OABC}$ where O is the intersection of PT and SQ. ∎

Proposition 4.10 *Let triangle ABC be a translation of triangle DEF. Then for any point P we have $V_{PABC} = V_{PDEFA}$.*

Proof. By Proposition 4.9 and (4.3), $V_{PABC} = V_{PAEC} - V_{PADC} = V_{PAEF} - V_{PAED} - V_{PADC} = V_{PAEF} - V_{PAED} - V_{PADF} = V_{PDEFA}$. ∎

Corollary 4.11 1. For two parallel planes ABC and PQR and a point T not in ABC we have $\frac{S_{ABC}}{S_{PQR}} = \frac{V_{TABC}}{V_{TPQRA}}$.

2. For two different parallel planes ABC and PQR we have $\frac{S_{ABC}}{S_{PQR}} = -\frac{V_{PABC}}{V_{APQR}}$.

Proof. Let XYZ be a translation of RPQ to plane ABC. By the co-vertex theorem and the preceding proposition,

$$\frac{S_{ABC}}{S_{PQR}} = \frac{S_{ABC}}{S_{XYZ}} = \frac{V_{TABC}}{V_{TXYZ}} = \frac{V_{TABC}}{V_{TPQRA}}.$$

Replacing T by P in the above equation, we prove the second result. ∎

Proposition 4.12 *Let triangle ABC be a translation of triangle DEF. Then for two points P and Q we have*

$$V_{PABC} + V_{QDEF} = V_{QABC} + V_{PDEF} \text{ or } V_{PABCQ} = V_{PDEFQ}.$$

In other words, when ABC moves by a translation the volume of $PABCQ$ remains the same.

Proof. By Proposition 4.10, $V_{PABC} = V_{PDEF} - V_{ADEF}$; $V_{QABC} = V_{QDEF} - V_{ADEF}$ from which we obtain the result immediately. ∎

From Propositions 4.9 and 4.12, we have the following interesting property for the volume V_{PABCQ}.

Corollary 4.13 *Let $\overline{PQ} = r\overline{ST}$ and $S_{ABC} = sS_{EFG}$ then $V_{PABCQ} = rsV_{SEFGT}$.*

4.1.3 Volumes and Affine Geometry of Dimension Three

This subsection has two purposes. First, we will show how to prove geometry theorems using the basic propositions about volumes. Second, we will derive some basic properties of lines and planes in space using the volume method.

Example 4.14 *If points P and Q are in plane ABC then line PQ is also in plane ABC.*

Proof. For any point R on line PQ, by Proposition 4.5, $V_{RABC} = \frac{\overline{PR}}{\overline{PQ}}V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}}V_{PABC} = 0$. By Proposition 4.1, R is in ABC. ∎

Example 4.15 *If two planes have a point in common then they have a line in common.*

Proof. Let planes ABC and RPQ have a point X in common. Without loss of generality, we assume that A, B, C, R, P, and Q are not common to both planes. By Proposition 4.8, two of AB, BC, and AC, say AB and BC could not be parallel to RPQ. Let AB and AC meet RPQ in two distinct points Y and Z respectively. We must show that X, Y, and Z are collinear, i.e., $S_{XYZ} = 0$. By Propositions 4.3 and 4.1, $\frac{S_{XYZ}}{S_{ABC}} = \frac{V_{RXYZ}}{V_{RABC}} = 0$, i.e., $S_{XYZ} = 0$. ∎

Remark 4.16 *The above two examples and Corollary 4.2 are the incidence axioms in the usual axiom system for solid geometry [6]. Therefore, the geometry defined by Axioms A.1-A.6 and S.1-S.5 is an affine geometry of dimension three. The results related to affine plane geometry proved in Section 2.6 are also true for affine geometry of dimension three. So the volume method presented in Section 4.3 below is for constructive statements in affine geometry of dimension three associated with any field.*

We will now prove some basic properties for the parallel.

Example 4.17 *Through any point P not in plane ABC there is one and only one plane parallel to plane ABC.*

Proof. By the Euclidean parallel axiom (Example 2.13 on page 58), there exist points Q and R such that $PABQ$ and $PACR$ are parallelograms. By Propositions 4.7 and 4.8, $PQR \parallel ABC$. To prove the uniqueness, let PTS be another plane parallel to ABC. By Proposition 4.10, $V_{TPQR} = V_{TABC} - V_{PABC} = 0$, i.e., $T \in PAR$. Similarly we have $S \in PAR$. ∎

Example 4.18 *If any two lines are cut by a number of parallel planes, their intercepts are proportional.*

Proof. Let the parallel planes α, β, γ cut two lines in the sets of points A, B, C and P, Q, R respectively. Let X, Y, and Z be three noncollinear points in plane β. By the co-face theorem,

$$\frac{\overline{AB}}{\overline{CB}} = \frac{V_{AXYZ}}{V_{CXYZ}} = \frac{V_{PXYZ}}{V_{RXYZ}} = \frac{\overline{PQ}}{\overline{RQ}}. \quad ∎$$

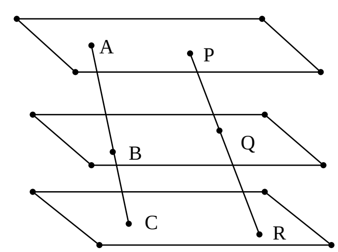


Figure 4-7

Example 4.19 *If a straight line is parallel to a straight line in a plane, it is in or parallel to the plane. Conversely, if a line in one plane is parallel to another plane, it is parallel to their line of intersection.*

Proof. Let AB be parallel to CD and E another point in plane CDE. By Propositions 4.3 and 2.10,

$$\frac{V_{EACD}}{V_{EBCD}} = \frac{S_{ACD}}{S_{BCD}} = 1,$$

i.e., $AB \parallel CDE$. Conversely, let AB be parallel to plane CDE and CD be the intersection of the two planes. By Proposition 4.3, $\frac{S_{ACD}}{S_{BCD}} = \frac{V_{EACD}}{V_{EBCD}} = 1$, i.e., $AB \parallel CD$. ▮

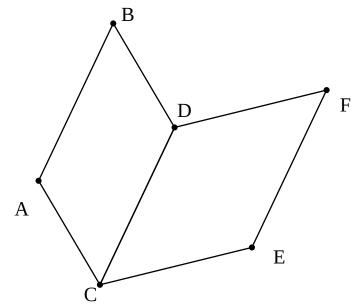


Figure 4-8

Example 4.20 *If two lines are each parallel to a third, they are parallel to one another.*

Proof. Let $ABCD$ and $ABEF$ be two parallelograms. We first prove that C, D, E, and F are coplanar. By Proposition 4.9, $V_{CDEF} = V_{BDEF} - V_{ADEF} = V_{BDAF} - V_{ADBF} = 0$, i.e., C, D, E, and F are coplanar. By Proposition 4.3, $\frac{S_{CEF}}{S_{DEF}} = \frac{V_{ACEF}}{V_{ADEF}} = 1$, i.e., $CD \parallel EF$. ▮

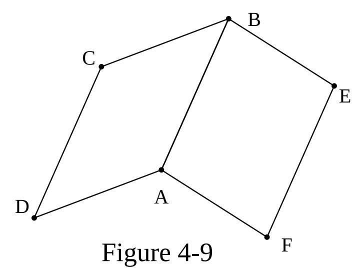


Figure 4-9

Example 4.21 *If a plane cuts two parallel planes, the lines of intersection are parallel.*

Proof. Let plane $ABPQ$ cut planes ABC and RPQ at lines AB and PQ respectively. By Proposition 4.3, $\frac{S_{APQ}}{S_{BPQ}} = \frac{V_{APQR}}{V_{BPQR}} = 1$, i.e., $AB \parallel PQ$. ▮

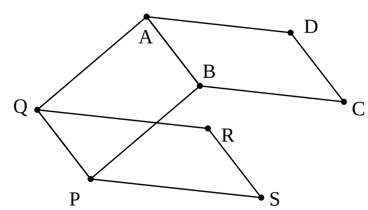


Figure 4-10

Example 4.22 (Co-trihedral Theorem) *If $OW \parallel DA$, $OU \parallel DB$, and $OV \parallel DC$ then* $\frac{V_{OWUV}}{V_{DABC}} = \frac{\overline{OW}}{\overline{DA}} \cdot \frac{\overline{OU}}{\overline{DB}} \cdot \frac{\overline{OV}}{\overline{OC}}$.

Proof. Let R, P, Q be points such that $\overline{DR} = \overline{OW}$, $\overline{DP} = \overline{OU}$, $\overline{DQ} = \overline{OV}$. By the co-face theorem,

$$V_{DABC} = \frac{\overline{DA}}{\overline{DR}} V_{DRBC} = \frac{\overline{DA}}{\overline{OW}} V_{DRBC} = \frac{\overline{DA}}{\overline{OW}} \cdot \frac{\overline{DB}}{\overline{OU}} \cdot \frac{\overline{DC}}{\overline{OV}} V_{DRPQ}.$$

By Propositions 4.9 and 4.10,

$$V_{DRPQ} = V_{ORPQ} - V_{WRPQ} = V_{OWUV} - V_{RWUV} - V_{WRPQ}.$$

Since $RW \parallel PU \parallel QV$, $V_{RWUV} = V_{RWPV} = V_{RWPQ}$ which proves the result. ∎

The above examples are about the basic properties of lines and planes. In what follows, we will prove some relatively non-trivial theorems. The proofs of these theorems are actually modifications of the proofs produced by our program.

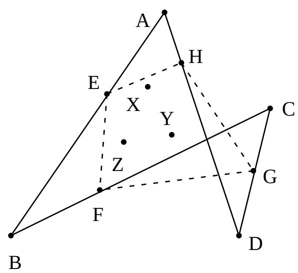


Figure 4-11

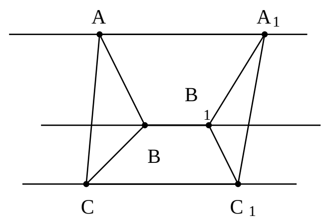


Figure 4-12

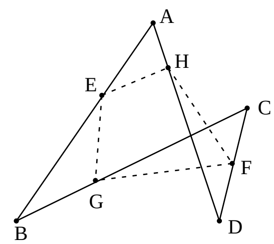


Figure 4-13

Example 4.23 (Menelaus' Theorem for Skew Quadrilaterals) *If the sides AB, BC, CD, and DA of any skew quadrilateral are cut by a plane XYZ in the points E, F, G, and H respectively, then* $\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1$. *(Figure 4-11)*

Proof. By the co-face theorem

$$\frac{\overline{DH}}{\overline{AH}} = \frac{V_{DXYZ}}{V_{AXYZ}}, \frac{\overline{CG}}{\overline{DG}} = \frac{V_{CXYZ}}{V_{DXYZ}}, \frac{\overline{BF}}{\overline{CF}} = \frac{V_{BXYZ}}{V_{CXYZ}}, \frac{\overline{AE}}{\overline{BE}} = \frac{V_{AXYZ}}{V_{BXYZ}}.$$

Then it is clear that $\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1$. For the non-degenerate conditions of this example, see Section 4.2 ∎

Example 4.24 *Let $A_1B_1C_1$ be the parallel projection of any triangle ABC in any plane. Show that the tetrahedra $ABCA_1$ and $A_1B_1C_1A$ are equal in volume. (Figure 4-12)*

Proof. Since CC_1 is parallel to plane AA_1B_1, by Proposition 4.7, $V_{AA_1B_1C_1} = V_{AA_1B_1C}$. Similarly, $V_{AA_1B_1C} = V_{AA_1BC}$. ∎

Example 4.25 *If a plane divides proportionally one pair of opposite sides of a skew quadrilateral, it also divides proportionally the other two sides. (Figure 4-13)*

Proof. This statement is a direct consequence of Example 4.23. The following is a proof produced by our program.[2] Let $r_1 = \frac{\overline{AE}}{\overline{EB}} = \frac{\overline{DF}}{\overline{FC}}$ and $r_2 = \frac{\overline{AH}}{\overline{AD}}$. We need to show that $\frac{\overline{BG}}{\overline{BC}} = r_2$. By the co-face theorem, $\frac{\overline{BG}}{\overline{BC}} = \frac{V_{BEFH}}{V_{BEFH} - V_{CEFH}}$. By Proposition 4.5

[2]This may serve as an example to show the difference between people and the computer in proving theorems. People may use all results available to make the proof short. But the computer does only prescribed steps according to the input.

$$V_{CEFH} = (r_2 - 1)V_{ACEF} = (r_2 - 1)(r_1 - 1)V_{ACDE} = (r_2 - 1)(r_1 - 1)r_1 V_{ABCD};$$
$$V_{BEFH} = r_2 V_{BDEF} = r_2 r_1 V_{BCDE} = r_2 r_1 (r_1 - 1)V_{ABCD}.$$

Then $\frac{\overline{BG}}{\overline{BC}} = \frac{r_2 r_1 (r_1 - 1)}{r_2 r_1 (r_1 - 1) - r_1 (r_1 - 1)(r_2 - 1)} = r_2$.

4.2 Constructive Geometry Statements

In this section, we will introduce a class of constructive geometry statements which is a generalization of the constructive statements in plane geometry.

4.2.1 Constructive Geometry Statements

In this chapter, by a *geometric quantity* we mean one of the following quantities:

1. the ratio of the signed lengths of two oriented segments on one line or on two parallel lines;
2. the ratio of the signed areas of two oriented triangles in the same plane or in two parallel planes; or
3. the signed volume of an oriented tetrahedron.

In Section 4.4, we will introduce more geometry quantities.

We now introduce constructions in space. First, it is clear that most plane constructions can still be used if all related points are in the same plane. We will choose several of them as basic constructions in space.

Definition 4.26 *A construction is one of the following ways of introducing new points in space.*

S1 (*POINTS* $Y_1, \cdots, Y_l$). *Take arbitrary points* $Y_1, \cdots, Y_l$ *in the space. Each* Y_i *has three degrees of freedom.*

S2 (*PRATIO Y W U V r*). *Take a point Y on the line passing through W and parallel to line UV such that* $\overline{WY} = r\overline{UV}$*, where r can be a rational number, a rational expression in geometric quantities, or a variable.*

If r is a fixed quantity, Y is a fixed point; if r is a variable, Y has one degree of freedom. The non-degenerate (ndg) condition is $U \neq V$*. If r is a rational expression of geometry quantities then we will further assume that the denominator of r could not equal to zero.*

S3 *(ARATIO Y L M N r_1 r_2 r_3), where $r_1 = \frac{S_{YMN}}{S_{LMN}}, r_2 = \frac{S_{LYN}}{S_{LMN}}$, and $r_3 = \frac{S_{LMY}}{S_{LMN}}$ are the area coordinates of point Y with respect to LMN. The r_1, r_2 and r_3 could be rational numbers, rational expressions in geometric quantities, or indeterminates satisfying $r_1 + r_2 + r_3 = 1$. The degree of freedom of Y is equal to the number of indeterminates in $\{r_1, r_2, r_3\}$. The ndg conditions are that L, M, and N are not collinear and the denominators of r_1, r_2, and r_3 are not equal to zero.*

S4 *(INTER Y (LINE U V) (LINE P Q)). Point Y is the intersection of line PQ and line UV which are in the same plane. The ndg condition is $PQ \nparallel UV$. Point Y is a fixed point.*

S5 (*INTER Y* (*LINE U V*) (*PLANE L M N*)). *Point Y is the intersection of a line UV and a plane LMN. The ndg condition is that $UV \nparallel LMN$. Point Y is a fixed point.*

S6 *(FOOT2LINE Y P U V) Point Y is the foot from point P to line UV. The ndg condition is $U \neq V$. Point Y is a fixed point.*

Proposition 4.27 *Let Y be introduced by one of the six constructions S1-S6. Show that the existence of Y follows from Axiom A.2.*

Proof. Constructions S1, S2, S4, and S6 have been discussed in Proposition 3.20 on page 113. The existence of the point introduced by S3 follows from Proposition 2.30 on page 68. Let Y be introduced by S5. By the co-face theorem $\frac{\overline{UY}}{\overline{UV}} = \frac{V_{ULMN}}{V_{ULMNV}}$. Since $UV \nparallel LMN$, we have $V_{ULMNV} \neq 0$. By Axiom A.2, Y does exist. ∎

Definition 4.28 *A constructive statement is a list $S = (C_1, C_2, \ldots, C_k, G)$ where*

1. *Each C_i, introduces a new point from the points introduced by the previous constructions; and*

2. *$G = (E_1, E_2)$ where E_1 and E_2 are polynomials in geometric quantities about the points introduced by the C_i, and $E_1 = E_2$ is the conclusion of S.*

The non-degenerate condition of S is the set of non-degenerate conditions of the constructions C_i plus the condition that the geometry quantities in E_1 and E_2 have geometry meanings, i.e., their denominators are not zero.

If the constructions are limited to S1–S5, the corresponding statements are called *Hilbert's intersection point statements* in space. The set of all Hilbert's intersection point statements is denoted by $\mathbf{S}_{\mathrm{H}}$.

The constructive description of geometry statements can be transformed into the commonly used predicate form. Following are several basic predicates.

1. Point ($POINT\ P$): P is a point in the space.
2. Collinear ($COLL\ P_1\ P_2\ P_3$): points P_1, P_2, and P_3 are on the same line.
3. Coplanar ($COPL\ P_1\ P_2\ P_3\ P_4$): P_1, P_2, P_3, and P_4 are in the same plane.
4. Parallel between two lines. ($PRLL\ P_1\ P_2\ P_3\ P_4$): (COPL $P_1\ P_2\ P_3\ P_4$) and $P_1P_2 \parallel P_3P_4$.
5. Parallel between a line and a plane. ($PRLP\ P_1\ P_2\ P_3\ P_4\ P_5$): $P_1P_2 \parallel P_3P_4P_5$.
6. Perpendicular ($PERP\ P_1\ P_2\ P_3\ P_4$): $[(P_1 = P_2) \vee (P_3 = P_4) \vee (P_1P_2$ is perpendicular to $P_3P_4)]$

We will now transform constructions into predicate forms.

S2 (PRATIO $Y\ W\ U\ V\ r$) is equivalent to (PRLL $Y\ W\ U\ V$), $r = \frac{\overline{WY}}{\overline{UV}}$, and $U \neq V$.

S3 (ARATIO $Y\ L\ M\ N\ r_1\ r_2\ r_3$) is equivalent to (COPL $Y\ L\ M\ N$), $r_1 = \frac{S_{YMN}}{S_{LMN}}, r_2 = \frac{S_{LYN}}{S_{LMN}}, r_3 = \frac{S_{LMY}}{S_{LMN}}$, and ¬(COLL $L\ M\ N$).

S4 (INTER Y (LINE $U\ V$) (LINE $P\ Q$)) is equivalent to (COLL $Y\ U\ V$), (COLL $Y\ P\ Q$), and ¬(PRLL $U\ V\ P\ Q$).

S5 (INTER Y (LINE $U\ V$) (PLANE $L\ M\ N$)) is equivalent to (COLL $Y\ U\ V$), (COPL $Y\ L\ M\ N$), and ¬(PRLP $U\ V\ L\ M\ N$).

S6 (FOOT2LINE $Y\ P\ U\ V$) is equivalent to (COLL $Y\ U\ V$), (PERP $Y\ P\ U\ V$), and $U \neq V$.

Now a constructive statement $S = (C_1, \cdots, C_r, (E, F))$ can be transformed into the following predicate form

$$\forall P_1 \cdots \forall P_r((P(C_1) \wedge \cdots \wedge P(C_r)) \Rightarrow E = F)$$

where P_i is the point introduced by C_i and $P(C_i)$ is the predicate form of C_i.

Example 4.23 can be described in the following constructive way.

((POINTS $A\ B\ C\ D\ X\ Y\ Z$)
(INTER E (LINE $A\ B$) (PLANE $X\ Y\ Z$))
(INTER F (LINE $B\ C$) (PLANE $X\ Y\ Z$))
(INTER G (LINE $C\ D$) (PLANE $X\ Y\ Z$))
(INTER H (LINE $A\ D$) (PLANE $X\ Y\ Z$))
($\frac{\overline{AE}}{\overline{BE}} \frac{\overline{BF}}{\overline{CF}} \frac{\overline{CG}}{\overline{DG}} \frac{\overline{DH}}{\overline{AH}} = 1$))
The ndg conditions:

$AB \nparallel XYZ$, $BC \nparallel XYZ$, $CD \nparallel XYZ$, $AD \nparallel XYZ$,
$B \neq E$, $C \neq F$, $D \neq G$, and $A \neq H$.

The predicate form of this example is:

$$\forall A, B, \cdots, H(HYP \Rightarrow CONC)$$

where

$$\begin{aligned}
HYP \quad = \quad & ((COLL\ E\ A\ B) \wedge (COPL\ E\ X\ Y\ Z) \wedge \neg(PRLP\ A\ B\ X\ Y\ Z) \wedge \\
& (COLL\ F\ C\ B) \wedge (COPL\ F\ X\ Y\ Z) \wedge \neg(PRLP\ C\ B\ X\ Y\ Z) \wedge \\
& (COLL\ G\ C\ D) \wedge (COPL\ G\ X\ Y\ Z) \wedge \neg(PRLP\ C\ D\ X\ Y\ Z) \wedge \\
& (COLL\ H\ A\ D) \wedge (COPL\ H\ X\ Y\ Z) \wedge \neg(PRLP\ D\ A\ X\ Y\ Z) \wedge \\
& B \neq E \wedge C \neq F \wedge D \neq G \wedge A \neq H); \\
CONC \quad = \quad & (\frac{\overline{AE}}{\overline{BE}} \frac{\overline{BF}}{\overline{CF}} \frac{\overline{CG}}{\overline{DG}} \frac{\overline{DH}}{\overline{AH}} = 1).
\end{aligned}$$

4.2.2 Constructive Configurations

A geometric figure which can be described by constructions S1–S6 is called a *constructive configuration*. Constructions S1–S6, though simple, can be used to describe most of the commonly used configurations about lines, planes, circles, and spheres. To illustrate, we will introduce more geometry objects.

- We will consider four kinds of lines:

  1. (LINE P Q).
  2. (PLINE R P Q).
  3. (OLINE S P Q R): the line passing through point S and perpendicular to plane PQR. The ndg condition is $\neg$ ($COLL\ P\ Q\ R$).

- We will consider six kinds of planes:

  1. (PLANE L M N).
  2. (PPLANE W L M N): the plane passing through a point W and parallel to plane LMN. The ndg condition is $\neg$($COPL\ L\ M\ N$).
  3. (TPLANE W U V): the plane passing trough a point W and perpendicular to line UV. The ndg condition is $U \neq V$.
  4. (BPLANE U V): the perpendicular-bisector of line UV. The ndg condition is $U \neq V$.

5. (CPLANE $A\ B\ P\ Q\ R$): the plane passing through line AB and perpendicular to plane PQR. The ndg condition is $\neg(AB \perp PQR)$.
6. (DPLANE $A\ B\ P\ Q$): the plane passing through line AB and parallel to line PQ. The ndg condition is $AB \nparallel PQ$.

Now we can consider more constructions:

- (ON $Y\ ln$). Take an arbitrary point Y on a line ln. Line ln could be one of the three kinds of lines.
- (ON $Y\ pl$). Take an arbitrary point Y in a plane pl. Plane pl could be one of the 6 kinds of planes.
- (INTER $Y\ ln1\ ln2$). Take the intersection of two lines $ln1$ and $ln2$ in the same plane.
- (INTER $Y\ ln\ pl$). Take the intersection of line ln and plane pl.
- (INTER $Y\ pl1\ pl2\ pl3$). Take the intersection Y of three planes $pl1$, $pl2$, and $pl3$.

Combining all the possible cases, there are totally $(3 + 6 + 6 + 18 + 56 =)$ 89 constructions. Conveniently, all these constructions can be described by constructions S1–S6. To show that, we need only to reduce all kinds of lines to the form (LINE $P\ Q$) and reduce all kinds of planes to the form (PLANE $R\ P\ Q$).

We first introduce a construction frequently used.

S7 (FOOT2PLANE $Y\ P\ L\ M\ N$) Point Y is the foot of the perpendicular from point P to plane LMN, i.e., Y is the intersection of line (OLINE $P\ L\ M\ N$) and plane (PLANE $L\ M\ N$). The nondegenerate condition is that L, M, and N are not collinear.

Example 4.29 *Construction S7 can be represented by constructions S1-S6.*

Point Y can be introduced by the following sequence of constructions.

(FOOT2LINE $T\ P\ M\ N$)
(FOOT2LINE $F\ L\ M\ N$)
(PRATIO $S\ T\ F\ L$ 1)
(FOOT2LINE $Y\ P\ T\ S$)

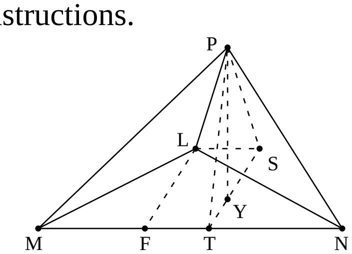


Figure 4-14

Example 4.30 *Find two points P and Q in plane (TPLANE $W\ U\ V$) such that W, P, and Q are not collinear. Then plane (TPLANE $W\ U\ V$) is the same as plane (PLANE $W\ P\ Q$).*

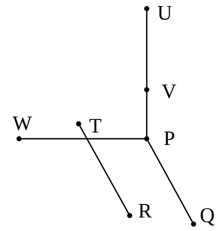


Figure 4-15

Proof. If $W \notin UV$, let P be introduced by (FOOT2LINE P W U V). Take an arbitrary point R in the space. Then Q can be introduced by constructions: (FOOT2PLANE T R W U V); (PRATIO Q P T R 1). It is clear that we need a nondegenerate condition $R \neq T$, or R is not in plane WUV.

If $W \in UV$, let R be an arbitrary point. We introduce point P as follows (FOOT2LINE T R U V); (PRATIO P W T R 1). Now point Q can be introduced as in the first case. ▮

Exercises 4.31

1. For an OLINE *ln*, find two distinct points U and V such that *ln* =(LINE U V).
2. Let *pl* be a plane of the form PPLANE, BPLANE, CPLANE, or DPLANE. Find three noncollinear points W, U, and V such that *pl* =(PLANE W U V).

Example 4.32 *The following results are clear.*

1. Construction (ON Y (LINE U V)) is equivalent to (PRATIO Y U U V r) where r is an indeterminate.
2. Construction (ON Y (PLANE L M N)) is equivalent to (ARATIO Y L M N r_1 r_2 $1 - r_1 - r_2$) where r_1 and r_2 are indeterminates.
3. Construction (INTER Y (PLANE L M N) (PLANE W U V) (PLANE R P Q)) is equivalent to (INTER Y (LINE A B) (PLANE R P Q)) where A and B are introduced as follows

   (INTER A (LINE L M) (PLANE W U V))
   (INTER B (LINE L N) (PLANE W U V))

The ndg conditions are $LM \nparallel WUV, LN \nparallel WUV, AB \nparallel RPQ$.

From Exercise 4.31 and Example 4.32 it is clear that all 89 constructions can be described by constructions S1–S6.

We may also consider circles and spheres. We define (CIR O P Q) to be the circle in the plane OPQ which has O as its center and passes through point P. We define (SPHERE O P) to be the sphere with center O and passing through point P. Then we can introduce the following new constructions.

S8 (ON Y (CIR O U V)). Take an arbitrary point on the circle.

S9 (ON Y (SPHERE O U)). Take an arbitrary point on the sphere.

S10 (INTER Y ln (CIR O W P)). Take the intersection of line ln and circle (CIR O W P) which is different from W. Line ln could be (LINE W V), (PLINE W U V), and (OLINE W L M N). We assume that line ln and the circle are in the same plane.

S11 (INTER Y ln (SPHERE O W)). Take the intersection of line ln and sphere (SPHERE O W) which is different from W. Line ln could be (LINE W V), (PLINE W U V), and (OLINE W R P Q).

S12 (INTER Y (CIR O_1 W U) (CIR O_2 W V)). Take the intersection of circle (CIR O_1 W U) and circle (CIR O_2 W V) which is different from W. We assume that the two circles are in the same plane.

S13 (INTER Y (CIR O_1 U V) (SPHERE O_2 U)). Take the intersection of circle (CIR O_1 U V) and sphere (SPHERE O_2 U) which is different from U.

Here, we introduce another 10 new constructions. In total, we introduced 100 new constructions including 89 constructions about lines and planes, 10 constructions about circles and spheres, and construction S7.

Example 4.33 *All ten constructions involving circles and spheres can be represented by constructions S1-S6.*

Proof. For constructions S8, S10, and S12, see Section 3.2.2. By what we have discussed above, we may assume that a line is always of the form (LINE W U). For construction S11, let Y be introduced by (INTER Y (LINE U V) (SPHERE O U)). Then point Y can also be introduced as follows

$$\text{(FOOT2LINE } N\ O\ U\ V\text{); (PRATIO } Y\ N\ U\ N\ 1\text{).}$$

Let Y be introduced by construction S13. Then Y can be constructed as follows

$$\text{(FOOT2PLANE } M\ O_2\ O_1\ U\ V\text{); (FOOT2LINE } N\ U\ M\ O_1\text{); (PRATIO } Y\ N\ U\ N\ 1\text{).}$$

For S9, we need to take an arbitrary point Y on (SPHERE O U). To do that, we first take an arbitrary point P, and then Y can be introduced by construction S11: (INTER Y (LINE P U) (SPHERE O U)). ∎

In summary, we have

Proposition 4.34 *All 100 constructions introduced in this subsection can be reduced to constructions S1-S6.*

4.3 Machine Proof for Class S_H

In this section, we will present a mechanical proving method for Hilbert's intersection statements in the affine space of dimension three. It is clear that the volume method is a natural generalization of the area method presented in Chapter 2. As with the area method, we must eliminate points from geometry quantities.

4.3.1 Eliminating Points from Volumes

The method of eliminating points from volumes is the basis of the volume method. Two other geometry quantities, the area ratio and the length ratio, will ultimately be reduced to volumes. In this subsection, we will discuss four constructions S2–S5. S1 will be discussed in Subsection 4.3.4.

Lemma 4.35 *Let Y be introduced by (PRATIO Y W U V r). Then we have*

$$V_{ABCY} = \begin{cases} (\frac{\overline{UW}}{\overline{UV}} + r)V_{ABCV} + (\frac{\overline{WV}}{\overline{UV}} - r)V_{ABCU} & \text{if } W \text{ is on line } UV. \\ V_{ABCW} + r(V_{ABCV} - V_{ABCU}) & \text{in all cases.} \end{cases}$$

Proof. If W, U, and V are collinear, by Proposition 4.5 we have

$$V_{ABCY} = \frac{\overline{UY}}{\overline{UV}}V_{ABCV} + \frac{\overline{YV}}{\overline{UV}}V_{ABCU} = (\frac{\overline{UW}}{\overline{UV}} + r)V_{ABCV} + (\frac{\overline{WV}}{\overline{UV}} - r)V_{ABCU}.$$

Otherwise, take a point S such that $\overline{WS} = \overline{UV}$. Then we have

$$V_{ABCY} = \frac{\overline{WY}}{\overline{WS}}V_{ABCS} + \frac{\overline{YS}}{\overline{WS}}V_{ABCW} = rV_{ABCS} + (1 - r)V_{ABCW}.$$

By Proposition 4.9, we have $V_{ABCS} = V_{ABCW} + V_{ABCV} - V_{ABCU}$. Substituting this into the above equation, we obtain the result. Notice that in both cases, we need the ndg condition $U \neq V$. ∎

Lemma 4.36 *Let Y be introduced by (ARATIO Y L M N r_1 r_2 r_3). Then we have*

$$V_{ABCY} = r_1V_{ABCL} + r_2V_{ABCM} + r_3V_{ABCN}.$$

Proof. This lemma is a direct consequence of Proposition 4.6. ∎

Lemma 4.37 *Let Y be introduced by (INTER Y (LINE U V) (LINE I J)). Then we have*

$$V_{ABCY} = \frac{S_{UIJ}}{S_{UIVJ}}V_{ABCV} - \frac{S_{VIJ}}{S_{UIVJ}}V_{ABCU}.$$

Proof. By Propositions 4.5 and the co-side theorem, $V_{ABCY} = \frac{\overline{UY}}{\overline{UV}}V_{ABCV} + \frac{\overline{YV}}{\overline{UV}}V_{ABCU} = \frac{S_{UIJ}V_{ABCV} - S_{VIJ}V_{ABCU}}{S_{UIVJ}}$. Since $UV \nparallel IJ$, we have $S_{UIVJ} \neq 0$. ∎

Lemma 4.38 *Let Y be introduced by (INTER Y (LINE U V) (PLANE L M N)). Then we have*

$$V_{ABCY} = \frac{1}{V_{ULMNV}}(V_{ULMN}V_{ABCV} - V_{VLMN}V_{ABCU})$$

Proof. By Proposition 4.5 and the co-face theorem, $V_{ABCY} = \frac{\overline{UY}}{\overline{UV}}V_{ABCV} + \frac{\overline{YV}}{\overline{UV}}V_{ABCU} = \frac{V_{ULMN}V_{ABCV}-V_{VLMN}V_{ABCU}}{V_{ULMNV}}$. Since $UV \nparallel LMN$, we have $V_{ULMNV} \neq 0$. ∎

Example 4.39 *Let Y be the intersection point of three planes WUV, LMN and RPQ. Then Y can be constructed as follows*

(INTER X (LINE L M) (PLANE R P Q))
(INTER Z (LINE L N) (PLANE R P Q))
(INTER Y (LINE X Z) (PLANE W U V))

By Proposition 4.38, we have

$$\begin{aligned}
V_{ABCY} &= \tfrac{V_{XWUV}}{V_{XWUVZ}}V_{ABCZ} - \tfrac{V_{ZWUV}}{V_{XWUVZ}}V_{ABCX}\\
V_{ABCZ} &= \tfrac{V_{LRPQ}}{V_{LRPQN}}V_{ABCN} - \tfrac{V_{NRPQ}}{V_{LRPQN}}V_{ABCL}\\
V_{ABCX} &= \tfrac{V_{LRPQ}}{V_{LRPQM}}V_{ABCM} - \tfrac{V_{MRPQ}}{V_{LRPQM}}V_{ABCL}\\
V_{XWUV} &= \tfrac{V_{LRPQ}}{V_{LRPQM}}V_{MWUV} - \tfrac{V_{MRPQ}}{V_{LRPQM}}V_{LWUV}\\
V_{ZWUV} &= \tfrac{V_{LRPQ}}{V_{LRPQN}}V_{NWUV} - \tfrac{V_{NRPQ}}{V_{LRPQN}}V_{LWUV}
\end{aligned}$$

From the above formulas, we can express V_{ABCY} as a rational expression of volumes formed by the known points.

Example 4.40 (Steiner's Theorem) *If two opposite edges of a tetrahedron move on two fixed skew lines in any way whatever but remain fixed in length, the volume of the tetrahedron remains constant.*

Constructive description
((POINTS A B C D)
(ON X (LINE A C))
(PRATIO Z X A C 1)
(ON Y (LINE B D))
(PRATIO W Y B D 1)
($V_{XYZW} = V_{ABCD}$))

The ndg conditions: $A \neq C$, $B \neq D$.

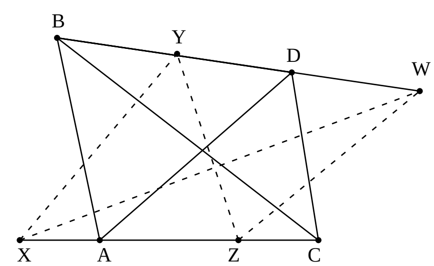


Figure 4-16

Proof. By Lemma 4.35, we can eliminate W

$$V_{XYZW} = V_{DXZY} - V_{BXZY}.$$

By Lemma 4.35 again, $V_{BXZY}=V_{BDXZ}\cdot\frac{\overline{BY}}{\overline{BD}}$; $V_{DXZY}=(\frac{\overline{BY}}{\overline{BD}}-1)\cdot V_{BDXZ}$.
Then $V_{XYZW}=-V_{BDXZ}=V_{XBZD}$. Similarly we can prove $V_{XBZD}=V_{ABCD}$. ∎

Example 4.41 *Show that a plane which bisects two opposite edges of a tetrahedron bisects its volume.*

Constructive description
((POINTS A B C D)
(MIDPOINT P A D)
(MIDPOINT S B C)
(LRATIO Q B D t)
(INTER R (LINE A C) (PLANE P S Q))
($V_{PCSR}-V_{PDCS}-V_{PDSQ}=\frac{1}{2}V_{ABCD}$))

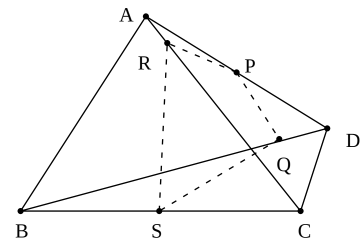


Figure 4-17

The ndg conditions: $A\neq D$; $B\neq C$; $B\neq D$; $AC \nparallel PSQ$.

Proof. Using Lemma 4.38, we can eliminate point R

$$V_{PCSR}=\frac{\overline{RC}}{\overline{AC}}V_{PCSA}=\frac{V_{ACPS}\cdot V_{CPSQ}}{V_{CPSQ}-V_{APSQ}}$$

We can eliminate the remaining points by using Lemma 4.35:

$$\begin{aligned} V_{PCSR} &= \frac{-V_{CPSQ}\cdot V_{ACPS}}{-V_{CPSQ}+V_{APSQ}}=\frac{V_{CDPS}\cdot r\cdot V_{ACPS}}{V_{CDPS}\cdot r+V_{ABPS}\cdot r-V_{ABPS}} \\ &= \frac{(-\frac{1}{2}V_{BCDP})\cdot r\cdot(\frac{1}{2}V_{ABCP})}{-\frac{1}{2}V_{BCDP}\cdot r-\frac{1}{2}V_{ABCP}\cdot r+\frac{1}{2}V_{ABCP}} \\ &= \frac{(-\frac{1}{2}V_{ABCD})\cdot r\cdot(\frac{1}{2}V_{ABCD})}{(2)\cdot(-\frac{1}{2}V_{ABCD})}=\frac{1}{4}(r\cdot V_{ABCD}) \end{aligned}$$

Similarly, we can compute $V_{PDCS}=-\frac{1}{4}V_{ABCD}$, $V_{PDSQ}=\frac{1}{4}(r-1)\cdot V_{ABCD}$.

Then $V_{PCSR}-V_{PDCS}-V_{PDSQ}=(\frac{r}{4}+\frac{1}{4}-\frac{r-1}{4})V_{ABCD}=\frac{1}{2}V_{ABCD}$. ∎

This proof seems to be a little complicated, but the idea behind it is quite simple: one need only to eliminate the points from volumes using Lemmas 4.35–4.38.

4.3.2 Eliminating Points from Area Ratios

The following lemmas provide methods of eliminating Y from $G=\frac{S_{ABY}}{S_{CDE}}$. The proofs of the lemmas are similar: if all the points are in one plane, methods of eliminating Y have been given in Chapter 2. Otherwise, there is a point T which is not in the plane ABY. By

Corollary 4.11,

$$G = \frac{S_{ABY}}{S_{CDE}} = \frac{V_{TABY}}{V_{TCDEA}}. \tag{4.4}$$

Now the lemmas in Section 4.3.1 can be used to eliminate Y from V_{TABY}.

Lemma 4.42 *Let Y be introduced by (PRATIO Y W U V r). Then we have*

$$\frac{\overline{ABY}}{CDE} = \begin{cases} \frac{V_{UABWV}}{V_{UCDEV}} & \text{if } W \text{ is not in plane } ABY \\ \frac{V_{UABW}+rV_{UABV}}{V_{UCDEA}} & \text{if } W \text{ is in plane } ABY \text{ but line } UV \text{ is not.} \\ \frac{S_{ABW}+r(S_{ABV}-S_{ABU})}{S_{CDE}} & \text{if } W, U, V, A, B, \text{ and } Y \text{ are coplanar.} \end{cases}$$

Proof. If $W \notin ABY$, let WS be a parallel translation of UV to line WY. By (4.4),

$$\frac{\overline{ABY}}{CDE} = \frac{V_{WABY}}{V_{WCDEY}} = \frac{1}{r}\frac{V_{WABY}}{V_{UCDEV}}.$$

By Propositions 4.5 and 4.9, $V_{WABY} = \frac{\overline{WY}}{WS} V_{WSAB} = rV_{UABWV}$. We prove the first case. The second case can be proved in a similar manner by replacing W by U. The third case is from Lemma 2.21 on page 65. ∎

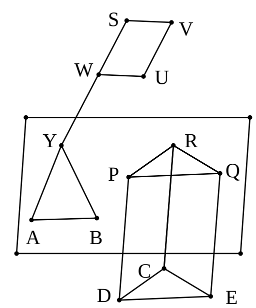


Figure 4-18

Remark 4.43 *The first case of Lemma 4.42 is rarely used in practice, since in this case Y is actually the intersection of (PLINE W U V) and (PPLANE A C D E), i.e., r is a fixed quantity and is not easy to find.*

Lemma 4.44 *Let Y be introduced by (ARATIO Y L M N r_1 r_2 r_3). Then we have*

$$\frac{\overline{ABY}}{CDE} = \begin{cases} \frac{r_2V_{LABM}+r_3V_{LABN}}{V_{LCDEA}} & \text{if one of } L, M, \text{ and } N, \text{ say } L, \text{ is not in } ABY \\ \frac{r_1S_{ABL}+r_2S_{ABM}+r_3S_{ABN}}{S_{CDE}} & \text{if } L, M, \text{ and } N \text{ are in plane } ABY. \end{cases}$$

Proof. If L is not in ABY, $\frac{\overline{ABY}}{CDE} = \frac{V_{LABY}}{V_{LCDEA}}$. Now the result comes from Lemma 4.36. The second case is Lemma 2.31. ∎

Lemma 4.45 *Let Y be introduced by (INTER Y (LINE U V) (LINE I J)). Then we have*

$$\frac{\overline{ABY}}{CDE} = \begin{cases} \frac{S_{UIJ}V_{UABV}}{S_{UIVJ}V_{UCDEA}} & \text{if one of } U, V, I, \text{ and } J, \text{ say } U, \text{ is not in } ABY. \\ \frac{S_{IUV}S_{ABJ}-S_{JUV}S_{ABI}}{S_{CDE}S_{IUJV}} & \text{if } U, V, I, J, A, B, \text{ and } Y \text{ are coplanar.} \end{cases}$$

Proof. If U is not in ABY, $\frac{\overline{ABY}}{\overline{CDE}} = \frac{V_{UABY}}{V_{UCDEA}} = \frac{\overline{UY}}{\overline{UV}}\frac{V_{UABV}}{V_{UCDEA}} = \frac{S_{UIJ}}{S_{UIVJ}}\frac{V_{UABV}}{V_{UCDEA}}$. The second case is Lemma 2.20. ∎

Lemma 4.46 *Let Y be introduced by (INTER Y (LINE U V) (PLANE L M N)). Then we have*

$$\frac{\overline{ABY}}{\overline{CDE}} = \begin{cases} \frac{V_{ULMN}}{V_{ULMNV}}\frac{V_{UABV}}{V_{UCDEA}} & \text{if one of } U \text{ and } V\text{, say } U\text{, is not in } ABY. \\ \frac{V_{ULMN}S_{ABV}-V_{VLMN}S_{ABU}}{S_{CDE}V_{ULMNV}} & \text{if } U \text{ and } V \text{ are in } ABY. \end{cases}$$

Proof. If U is not in ABY, $\frac{\overline{ABY}}{\overline{CDE}} = \frac{V_{UABY}}{V_{UCDEA}} = \frac{\overline{UY}}{\overline{UV}}\frac{V_{UABV}}{V_{UCDEA}} = \frac{V_{ULMN}}{V_{ULMNV}}\frac{V_{UABV}}{V_{UCDEA}}$. The second case is a consequence of Proposition 2.9 and the co-face theorem. ∎

Example 4.47 (Centroid Theorem for Tetrahedra) *The four medians of a tetrahedron meet in a point.*

Constructive description
((POINTS A B C D)
(MIDPOINT S B C)
(LRATIO Y D S 2/3) (Y is the centroid of $\triangle DBC$.)
(LRATIO Z A S 2/3) (Z is the centroid of $\triangle ABC$.)
(INTER G (LINE D Z) (LINE A Y))
(INTER H (LINE C G) (PLANE A B D))
($\frac{S_{ABH}}{S_{ABD}} = 1/3$))

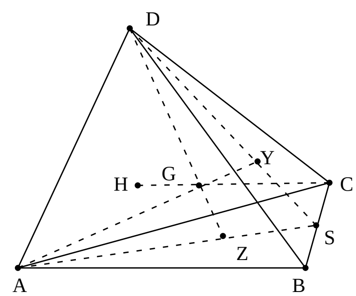


Figure 4-19

The ndg conditions: $B \neq C$, $D \neq S$, $A \neq S$, $DZ \nparallel AY$, $CG \nparallel ABD$, and $S_{ABD} \neq 0$.

Proof. Points will be eliminated in the order H, G, Z, Y, S, D, C, B, and A. By Lemma 4.46,

$$\frac{S_{ABH}}{S_{ABD}} = \frac{V_{CABD}}{V_{CABDG}}\frac{V_{CABG}}{V_{CABDA}} = \frac{V_{ABCG}}{V_{ABDG} + V_{ABCD}}.$$

By Lemma 4.37, $V_{ABDG} = \frac{S_{DAY}}{S_{DAZY}}V_{ABDZ}$, $V_{ABCG} = \frac{S_{ZAY}}{S_{ZADY}}V_{ABCD}$. By Lemma 4.37 again, $V_{ABDZ} = 2/3V_{ABDS} = -1/3V_{ABCD}$. Now

$$\frac{S_{ABH}}{S_{ABD}} = \frac{\frac{S_{ZAY}}{S_{ZADY}}}{1 - 1/3\frac{S_{DAY}}{S_{ZADY}}}.$$

Now all points are in plane ADS. Then by Lemma 4.42, $S_{ZAY} = \frac{1}{3}S_{SAY} = \frac{2}{9}S_{SAD}$; $S_{DAY} = \frac{1}{3}S_{DAS} = -\frac{1}{3}S_{SAD}$; $S_{ZADY} = S_{ZAY} - S_{DAY} = \frac{5}{9}S_{SAD}$. Then $\frac{S_{ABH}}{S_{ABD}} = 1/3$. ∎

Exercise 4.48 *We can also use the following general method to eliminate points from $G = \frac{S_{ABY}}{S_{CDE}}$. If A, B, Y, C, D, and E are coplanar, we need to find a point T that is not in plane ABY. Then*

$$G = \frac{S_{ABY}}{S_{CDE}} = \frac{V_{TABY}}{V_{TCDE}}.$$

Otherwise, by Corollary 4.11

$$G = \frac{V_{CABY}}{V_{ACDE}}.$$

The advantage of this method is that we do not need to use the volume of the polyhedron involving five points in all cases. Prove Lemmas 4.42-4.46 using the above method.

4.3.3 Eliminating Points from Length Ratios

The following lemmas present methods of eliminating point Y introduced by construction C from the length ratio $G = \frac{\overline{DY}}{\overline{EF}}$.

Lemma 4.49 *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, *$C$ =(PRATIO Y W U V r). Then*

$$G = \begin{cases} \frac{\frac{\overline{DW}}{\overline{UV}}+r}{\frac{\overline{EF}}{\overline{UV}}} & \text{if } D \in WY. \\ \frac{V_{DWUV}}{V_{EWUVF}} & \text{if } D \notin WY,\ U \notin DWY. \\ -\frac{V_{UEDWV}}{V_{UEFWV}} & \text{if } D \notin WY,\ E \notin DWY. \\ \frac{S_{DUWV}}{S_{EUFV}} & \text{if all points are coplanar.} \end{cases}$$

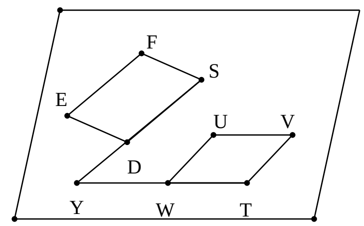


Figure 4-20

Proof. The first and the last cases have been proved in Lemma 2.26. If $U \notin DWY$, take a point S such that $\overline{DS} = \overline{EF}$. By the co-face theorem $G = \frac{\overline{DY}}{\overline{DS}} = \frac{V_{DWUV}}{V_{DWUVS}} = \frac{V_{DWUV}}{V_{EWUVF}}$. If $E \notin DWY$, take a point T such that $\overline{WT} = \overline{UV}$. By the co-side and co-face theorems $G = \frac{\overline{DY}}{\overline{DS}} = \frac{S_{DWT}}{S_{DWST}} = \frac{V_{DWTE}}{V_{DWTES}}$. By Propositions 4.9 and 4.12

$$V_{DWTE} = V_{DWVE} - V_{DWUE} = V_{UDEWV},$$
$$V_{DWTES} = V_{EWTEF} = -V_{FWTE} = -V_{FWVE} + V_{FWUE} = V_{UEFWV}.$$

Then $G = -\frac{V_{UEDWV}}{V_{UEFWV}}$. ∎

Lemma 4.50 *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, *$C$ =(ARATIO Y L M N r_1 r_2 r_3). Then we have*

$$G = \begin{cases} \frac{V_{DLMN}}{V_{ELMNF}} & \text{if } D \notin LMN. \\ \frac{V_{DMNE}-r_1V_{LMNE}}{V_{EMNEF}}. & \text{if } D \in LMN,\ E \notin LMN, \text{and } DY \not\parallel NM. \\ \frac{S_{DMN}-r_1S_{LMN}}{S_{EMFN}} & \text{if all points are coplanar and } DY \not\parallel NM. \end{cases}$$

Proof. If D is not in plane LMN, the result is a direct consequence of Propositions 4.4 and 4.12. For the second case, by Corollary 4.13,

$$G = \frac{V_{DMNEY}}{V_{DMNES}} = \frac{V_{DMNE} - r_1V_{LMNE}}{V_{EMNEF}}.$$

The third case can be proved similarly. ∎

Lemma 4.51 *Let $G = \frac{\overline{DY}}{\overline{EF}}$, C =(INTER Y (LINE U V) (LINE I J)). Then we have*

$$G = \begin{cases} \frac{V_{DUVI}}{V_{EUVIF}} & D \notin UVIJ \text{ and } \neg(\text{COLL } U\ V\ I). \\ \frac{V_{EDUV}}{V_{EFVU}} & D \in UVIJ,\ EF \notin UVIJ, \text{ and } D \notin UV. \\ \frac{S_{DUV}}{S_{EUFV}} & D, E, F \text{ are in } UVIJ, \text{ and } D \notin UV. \end{cases}$$

Proof. The first case is a consequence of the co-face theorem. For the second case, by Corollary 4.13,

$$G = \frac{\overline{DY}}{\overline{EF}} = \frac{V_{DUVEY}}{V_{EUVEF}} = -\frac{V_{DUVE}}{V_{FUVE}}.$$

If all points are coplanar, see Lemma 2.25. ∎

Lemma 4.52 *Let $G = \frac{\overline{DY}}{\overline{EF}}$, C =(INTER Y (LINE U V) (PLANE L M N)). Then we have*

$$G = \begin{cases} \frac{V_{DLMN}}{V_{ELMN} - V_{FLMN}} & \text{If } D \text{ is not in plane } LMN. \\ \frac{V_{DUVL}}{V_{EUVL} - V_{FUVL}} & \text{If } D \in LMN \text{ and one of } L, M, N, \text{ say } L \notin DUV. \end{cases}$$

Proof. If D is not in plane LMN, the result is a direct consequence of the co-face theorem. For the second case, take a point S such that $\overline{DS} = \overline{EF}$. Then we have $G = \frac{\overline{DY}}{\overline{DS}} = \frac{V_{DUVL}}{V_{DUVLS}} = \frac{V_{DUVL}}{V_{EUVLF}}$. ∎

Example 4.53 (Ceva's Theorem for Skew Quadrilaterals) *The planes passing through a point O and the sides AB, BC, CD, and DA of any skew quadrilateral meet the opposite sides of the quadrilateral at G, H, E, and F respectively. Show that $\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1$.*

Constructive description

((POINTS A B C D O)
(INTER E (LINE A B) (PLANE O C D))
(INTER F (LINE B C) (PLANE O A D))
(INTER G (LINE C D) (PLANE O A B))
(INTER H (LINE D A) (PLANE O B C))
($\frac{\overline{AE}}{\overline{EB}} \frac{\overline{BF}}{\overline{FC}} \frac{\overline{CG}}{\overline{GD}} \frac{\overline{DH}}{\overline{HA}} = 1$))

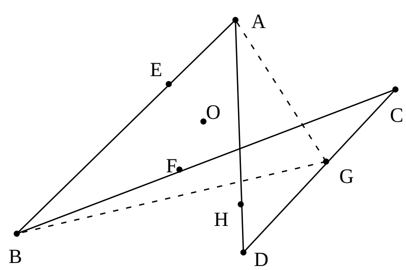


Figure 4-21

The ndg conditions: $AB \nparallel OCD$; $BC \nparallel OAD$; $CD \nparallel OAB$; $AD \nparallel OBC$; $B \neq E$; $C \neq F$; $D \neq G$, $A \neq H$.

Proof. By Lemma 4.52 or just by Proposition 4.4, we have

$$\frac{\overline{AE}}{\overline{BE}} = \frac{V_{ACDO}}{V_{BCDO}}; \quad \frac{\overline{BF}}{\overline{CF}} = \frac{-V_{ABDO}}{-V_{ACDO}}; \quad \frac{\overline{CG}}{\overline{DG}} = \frac{V_{ABCO}}{V_{ABDO}}; \quad \frac{\overline{DH}}{\overline{AH}} = \frac{V_{BCDO}}{V_{ABCO}}.$$

Therefore $\frac{\overline{AE}}{\overline{EB}} \frac{\overline{BF}}{\overline{FC}} \frac{\overline{CG}}{\overline{GD}} \frac{\overline{DH}}{\overline{HA}} = 1$. ∎

Example 4.54 (Centroid of Tetrahedra) *The four medians of a tetrahedron meet in a point which divides each median in the ratio 3:1, the longer segment being on the side of the vertex of the tetrahedron.*

Constructive description
((POINTS A B C D)
(MIDPOINT S B C)
(LRATIO Z A S 2/3)
(LRATIO Y D S 2/3)
(INTER G (LINE D Z) (LINE A Y))
($\left(\frac{\overline{AG}}{\overline{GY}} = 3\right)$))

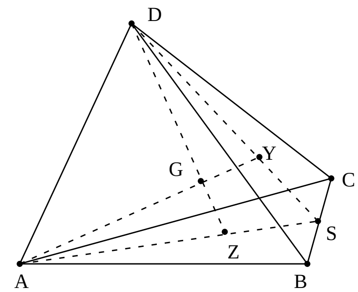


Figure 4-22

The ndg conditions: $B \neq C$, $A \neq S$, $D \neq S$, $DZ \nparallel AY$, $G \neq Y$.

Proof. By Lemma 4.51, $\frac{\overline{AG}}{\overline{YG}} = \frac{S_{ADZ}}{S_{DZY}}$. By Lemma 4.42, $\frac{S_{ADZ}}{S_{DZY}} = \frac{\frac{S_{ADZ}}{S_{DSZ}}}{-2/3}$; $\frac{S_{ADZ}}{S_{DSZ}} = \frac{\overline{AZ}}{\overline{ZS}}\frac{S_{ADS}}{S_{ADS}} = \frac{2}{1}$. Then $\frac{\overline{AG}}{\overline{YG}} = 3$. ∎

Exercise 4.55 *Let point Y be introduced by construction (INTER Y (LINE U V) (PLINE R P Q)). Show that*

- $$V_{ABCY} = \begin{cases} \frac{S_{UPRQ}}{S_{UPVQ}}V_{ABCV} - \frac{S_{VRPQ}}{S_{UPVQ}}V_{ABCU} & \text{if } P, Q, R, U, V \text{ are coplanar.} \\ \frac{V_{PURQ}V_{ABCV} - V_{PVRQ}V_{ABCU}}{V_{URQPV}} & \text{otherwise.} \end{cases}$$

- *If D is on UV*

$$\frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{S_{DPRQ}}{S_{EPFQ}} & \text{If all points are coplanar.} \\ \frac{V_{DPQR}}{V_{EPQRF}} & \text{If } D \notin PAR. \\ \frac{V_{PEDRQ}}{V_{QEFRP}} & \text{If } D \in PQR \text{ and } E, F \notin RUV. \end{cases}$$

- $$\frac{S_{ABY}}{S_{CDE}} = \begin{cases} r'\frac{S_{ABV}}{S_{CDE}} + (1-r')\frac{S_{ABU}}{S_{CDE}} & AB \in RUV \text{ and } PQ \in RUV. \\ r''\frac{S_{ABV}}{S_{CDE}} + (1-r'')\frac{S_{ABU}}{S_{CDE}} & AB \in RUV \text{ and } PQ \notin RUV. \\ r'\frac{V_{XABV}}{V_{XCDEA}} + (1-r')\frac{V_{XABU}}{V_{XCDEA}} & AB \notin RUV \text{ and } PQ \in RUV. \\ r''\frac{V_{XABV}}{V_{XCDEA}} + (1-r'')\frac{V_{XABU}}{V_{XCDEA}} & AB \notin RUV \text{ and } PQ \notin RUV. \end{cases}$$

  where $r' = \frac{V_{UPRQ}}{V_{UPVQ}}$, $r'' = \frac{V_{UPRQ}}{V_{VPRQU}}$ *and X is U, V or R depending on which is not in plane ABY.*

Exercise 4.56 *Try to eliminate Y from the three geometry quantities if Y is introduced by the following constructions.*

(INTER Y (PLINE W U V) (PLINE R P Q))

(INTER Y (PLINE W U V) (PLANE L M N))

4.3.4 Free Points and Volume Coordinates

After applying the above lemmas to any rational expression E in geometric quantities, we can eliminate the non-free points introduced by all constructions from E. Now the new E is a rational expression in indeterminates and volumes of *free points* in space. For more than five free points in the space, the volumes of the tetrahedra formed by them are not independent, e.g. the following equation is always true:

$$V_{ABCD} = V_{ABCO} + V_{ABOD} + V_{AOCD} + V_{OBCD}.$$

To express E and F as expressions in free parameters, we introduce the volume coordinates.

Definition 4.57 *Let X be a point in the space. For four noncoplanar points O, W, U, and V, the volume coordinates of X with respect to $OWUV$ are*

$$r_1 = \frac{V_{OWUX}}{V_{OWUV}}, r_2 = \frac{V_{OWXV}}{V_{OWUV}}, r_3 = \frac{V_{OXUV}}{V_{OWUV}}, r_4 = \frac{V_{XWUV}}{V_{OWUV}}.$$

It is clear that $r_1 + r_2 + r_3 + r_4 = 1$.

Since the sum of the volume coordinates of a point is one, we sometimes omit the last one to obtain an independent set of coordinates.

Exercise 4.58 *Show that the points in the space are in a one to one correspondence with the four-tuples (x, y, z, w) satisfying $x + y + w + z = 1$.*

Lemma 4.59 *Let $G = V_{ABCY}$, and O, W, U, V be four noncoplanar points. Then we have*

$$G = V_{ABCO} + \frac{V_{OABCV}V_{OWUY} + V_{OABCU}V_{OVWY} + V_{OABCW}V_{OUVY}}{V_{OWUV}}$$

Proof. We have

$$V_{ABCY} = V_{ABCO} + V_{ABOY} + V_{AOCY} + V_{OBCY}. \quad (1)$$

Without loss of generality, we assume that YO meets plane WUV at X. (Otherwise, let YW meet plane OUV at X, and so on.) By Proposition 4.4, we have

$$V_{OABY} = \frac{\overline{OY}}{\overline{OX}} V_{OABX} = \frac{V_{OWUVY}V_{OABX}}{V_{OWUV}} \quad (2)$$

By Proposition 4.6,

$$V_{OABX} = \frac{S_{WUX}}{S_{WUV}} V_{OABV} + \frac{S_{WXV}}{S_{WUV}} V_{OABU} + \frac{S_{XUV}}{S_{WUV}} V_{OABW} \tag{3}$$

By Lemma 4.46, we have

$$\frac{S_{WUX}}{S_{WUV}} = \frac{V_{OWUY}}{V_{OWUVY}}; \frac{S_{WXV}}{S_{WUV}} = \frac{V_{OVWY}}{V_{OWUVY}}; \frac{S_{XUV}}{S_{WUV}} = \frac{V_{OUVY}}{V_{OWUVY}}.$$

Substituting them into (3) and (2), we have

$$V_{OABY} = \frac{V_{OWUY}V_{OABV} + V_{OVWY}V_{OABU} + V_{OUVY}V_{OABW}}{V_{OWUV}} \tag{4}$$

Similarly, we have

$$\begin{aligned} V_{OBCY} &= \frac{V_{OWUY}V_{OBCV} + V_{OVWY}V_{OBCU} + V_{OUVY}V_{OBCW}}{V_{OWUV}} \\ V_{OCAY} &= \frac{V_{OWUY}V_{OCAV} + V_{OVWY}V_{OCAU} + V_{OUVY}V_{OCAW}}{V_{OWUV}} \end{aligned}$$

Substituting them into (1) and with (4.3)

$$\begin{aligned} V_{OABV} + V_{OBCV} + V_{OCAV} &= V_{OABCV}, \\ V_{OABU} + V_{OBCU} + V_{OCAU} &= V_{OABCU}, \text{ and} \\ V_{OABW} + V_{OBCW} + V_{OCAW} &= V_{OABCW}, \end{aligned}$$

we can obtain the result. ∎

Corollary 4.60 *Use the same notations as in Lemma 4.59. For any point P let* $x_P = \frac{V_{OWUP}}{V_{OWUV}}, y_P = \frac{V_{OWPV}}{V_{OWUV}}, z_P = \frac{V_{OPUV}}{V_{OWUV}}$. *Then the formula in Lemma 4.59 can be written as*

$$V_{ABCY} = V_{OWUV} \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_Y & y_Y & z_Y & 1 \end{vmatrix}$$

which is quite similar to the formula of the volumes in terms of the Cartesian coordinates of its vertices.

Now we can describe the volume method as follows: for a geometry statement in $\mathbf{S}_H$: $S = (C_1, \cdots, C_r, (E_1, E_2))$, we can use the above lemmas to eliminate all the non-free points and express the volumes of free points as their volume coordinates with respect to four fixed points. Finally, we obtain two rational expressions R_1 and R_2 in independent parameters respectively. S is a correct geometry statement if R_1 is identical to R_2. For the precise description of the algorithm, see Algorithm 4.86 on page 210.

We end this section by a methodological comment. One of the most important ideas in the traditional method of solving problems in solid geometry is reducing a problem of higher dimension to a problem of lower dimension and then solving it using knowledge from plane geometry. This is the so-called "dimension reduction method." But our volume method always does the converse, i.e., it reduces length ratios and area ratios to volumes, because in space the volume is easy to deal with. This can be seen from the three groups of lemmas in Sections 4.3.1, 4.3.2, and 4.3.3. This is also the reason why the method is called *the volume method*.

4.3.5 Working Examples

Example 4.61 *For a tetrahedron ABCD and a point O, let* $P = AO \cap BCD$, $Q = BO \cap ACD$, $R = CO \cap ABD$, *and* $S = DO \cap ABC$. *Show that* $\frac{\overline{OP}}{\overline{AP}} + \frac{\overline{OQ}}{\overline{BQ}} + \frac{\overline{OR}}{\overline{CR}} + \frac{\overline{OS}}{\overline{DS}} = 1$.

Constructive description
((POINTS A B C D O)
(INTER P (LINE A O) (PLANE B C D))
(INTER Q (LINE B O) (PLANE A C D))
(INTER R (LINE C O) (PLANE A B D))
(INTER S (LINE D O) (PLANE A B C))
($\frac{\overline{OP}}{\overline{AP}}+\frac{\overline{OQ}}{\overline{BQ}}+\frac{\overline{OR}}{\overline{CR}}+\frac{\overline{OS}}{\overline{DS}} = 1$))

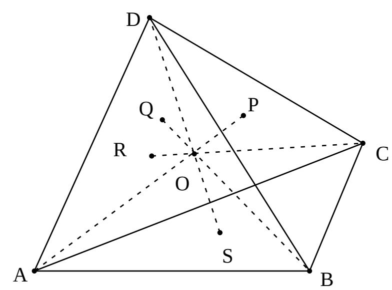


Figure 4-23

The ndg conditions: $AO \nparallel BCD$, $BO \nparallel ACD$, $CO \nparallel ABD$, $DO \nparallel ABC$, $A \neq P$, $B \neq Q$, $C \neq R$, $D \neq S$.

Proof. By the co-face theorem

$$\frac{\overline{OS}}{\overline{DS}} = \frac{V_{ABCO}}{V_{ABCD}}; \quad \frac{\overline{OR}}{\overline{CR}} = \frac{V_{ABOD}}{V_{ABCD}}; \quad \frac{\overline{OQ}}{\overline{BQ}} = \frac{V_{AOCD}}{V_{ABCD}}; \quad \frac{\overline{OP}}{\overline{AP}} = \frac{V_{OBCD}}{V_{ABCD}}.$$

By Lemma 4.59,

$$\frac{\overline{OP}}{\overline{AP}} + \frac{\overline{OQ}}{\overline{BQ}} + \frac{\overline{OR}}{\overline{CR}} + \frac{\overline{OS}}{\overline{DS}} = \frac{V_{OBCD} + V_{AOCD} + V_{ABOD} + V_{ABCO}}{V_{ABCD}} = 1. \quad \blacksquare$$

Example 4.62 *Let ABCD be a tetrahedron and O a point. Let line DO and plane ABC meet in S; line AD and plane OBC meet in P; line BD and plane OAC meet in Q; and line CD and plane OAB meet in R. Show that* $\frac{\overline{DO}}{\overline{OS}} = \frac{\overline{DP}}{\overline{PA}} + \frac{\overline{DQ}}{\overline{QB}} + \frac{\overline{DR}}{\overline{RC}}$.

Constructive description

((POINTS A B C D O)
(INTER S (LINE D O) (PLANE A B C))
(INTER P (LINE D A) (PLANE O B C))
(INTER Q (LINE D B) (PLANE O A C))
(INTER R (LINE D C) (PLANE O A B))
($\frac{\overline{DO}}{\overline{OS}} = \frac{\overline{DP}}{\overline{PA}}+\frac{\overline{DQ}}{\overline{QB}}+\frac{\overline{DR}}{\overline{RC}}$))

The eliminants

$\frac{\overline{DR}}{\overline{CR}} \overset{R}{=} \frac{V_{ABDO}}{V_{ABCO}}$

$\frac{\overline{DQ}}{\overline{BQ}} \overset{Q}{=} \frac{V_{ACDO}}{-V_{ABCO}}$

$\frac{\overline{DP}}{\overline{AP}} \overset{P}{=} \frac{V_{BCDO}}{V_{ABCO}}$

$\frac{\overline{DO}}{\overline{OS}} \overset{S}{=} \frac{V_{ABCO}-V_{ABCD}}{-V_{ABCO}}$

$V_{BCDO}=V_{ACDO}-V_{ABDO}+V_{ABCO}-V_{ABCD}$

The machine proof

$$\frac{\frac{\overline{DO}}{\overline{OS}}}{-(\frac{\overline{DR}}{\overline{CR}}+\frac{\overline{DQ}}{\overline{BQ}}+\frac{\overline{DP}}{\overline{AP}})}$$

$$\overset{R}{=} \frac{V_{ABCO}}{-(V_{ABDO}+V_{ABCO}\cdot\frac{\overline{DQ}}{\overline{BQ}}+V_{ABCO}\cdot\frac{\overline{DP}}{\overline{AP}})} \cdot \frac{\overline{DO}}{\overline{OS}}$$

$$\overset{Q}{=} \frac{V_{ABCO}\cdot(-V_{ABCO})}{-(V_{ACDO}\cdot V_{ABCO}-V_{ABDO}\cdot V_{ABCO}-V_{ABCO}^2\cdot\frac{\overline{DP}}{\overline{AP}})} \cdot \frac{\overline{DO}}{\overline{OS}}$$

$$\overset{simplify}{=} \frac{V_{ABCO}}{V_{ACDO}-V_{ABDO}-V_{ABCO}\cdot\frac{\overline{DP}}{\overline{AP}}} \cdot \frac{\overline{DO}}{\overline{OS}}$$

$$\overset{P}{=} \frac{(V_{ABCO})^2}{-V_{BCDO}\cdot V_{ABCO}+V_{ACDO}\cdot V_{ABCO}-V_{ABDO}\cdot V_{ABCO}} \cdot \frac{\overline{DO}}{\overline{OS}}$$

$$\overset{simplify}{=} \frac{V_{ABCO}}{-(V_{BCDO}-V_{ACDO}+V_{ABDO})} \cdot \frac{\overline{DO}}{\overline{OS}}$$

$$\overset{S}{=} \frac{(V_{ABCO}-V_{ABCD})\cdot V_{ABCO}}{-(V_{BCDO}-V_{ACDO}+V_{ABDO})\cdot(-V_{ABCO})}$$

$$\overset{simplify}{=} \frac{V_{ABCO}-V_{ABCD}}{V_{BCDO}-V_{ACDO}+V_{ABDO}} \overset{volume-co}{=} \frac{V_{ABCO}-V_{ABCD}}{V_{ABCO}-V_{ABCD}} \overset{simplify}{=} 1$$

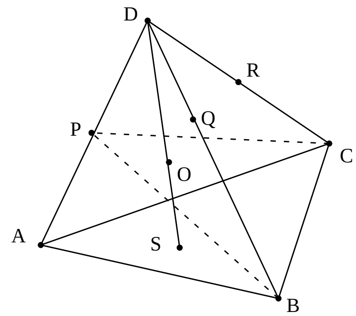


Figure 4-24

In the above proof, $a \overset{volume-co}{=} b$ means that b is the result obtained by replacing each volume using the volume coordinates with respect to four fixed points.

Example 4.63 *Let a line l meet four coaxial planes in $A, B, C,$ and D respectively. The cross-ratio of l for the four planes is defined to be $(ABCD) = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}$. Show that for any line l the cross ratio is fixed.*

Constructive description

((POINTS X Y A B C_1 D_1)
(INTER C (LINE A B) (PLANE C_1 X Y))
(INTER D (LINE A B) (PLANE D_1 X Y))
(INTER A_1 (LINE C_1 D_1) (PLANE A X Y))
(INTER B_1 (LINE C_1 D_1) (PLANE B X Y))
($\frac{\overline{AC}}{\overline{AD}}\frac{\overline{BD}}{\overline{BC}} = \frac{\overline{A_1C_1}}{\overline{A_1D_1}}\frac{\overline{B_1D_1}}{\overline{B_1C_1}}$))

The eliminants

$$\frac{\overline{D_1B_1}}{\overline{C_1B_1}} \overset{B_1}{=} \frac{V_{XYBD_1}}{V_{XYBC_1}}$$
$$\frac{\overline{C_1A_1}}{\overline{D_1A_1}} \overset{A_1}{=} \frac{V_{XYAC_1}}{V_{XYAD_1}}$$
$$\frac{\overline{AC}}{\overline{AD}} \overset{D}{=} \frac{V_{XYD_1C}+V_{XYAD_1}}{V_{XYAD_1}}$$
$$\frac{\overline{BD}}{\overline{BC}} \overset{D}{=} \frac{V_{XYBD_1}}{V_{XYD_1C}+V_{XYBD_1}}$$
$$V_{XYD_1C} \overset{C}{=} \frac{V_{XYBD_1}\cdot V_{XYAC_1}-V_{XYBC_1}\cdot V_{XYAD_1}}{V_{XYBC_1}-V_{XYAC_1}}$$

The machine proof

$$\frac{\frac{\overline{BD}}{\overline{BC}}\cdot\frac{\overline{AC}}{\overline{AD}}}{\frac{\overline{D_1B_1}}{\overline{C_1B_1}}\cdot\frac{\overline{C_1A_1}}{\overline{D_1A_1}}}$$

$$\overset{B_1}{=} \frac{-V_{XYBC_1}}{(-V_{XYBD_1})\cdot\frac{\overline{C_1A_1}}{\overline{D_1A_1}}} \cdot \frac{\overline{BD}}{\overline{BC}}\cdot\frac{\overline{AC}}{\overline{AD}}$$

$$\overset{A_1}{=} \frac{V_{XYBC_1}\cdot(-V_{XYAD_1})}{V_{XYBD_1}\cdot(-V_{XYAC_1})} \cdot \frac{\overline{BD}}{\overline{BC}}\cdot\frac{\overline{AC}}{\overline{AD}}$$

$$\overset{D}{=} \frac{V_{XYBD_1}\cdot(V_{XYD_1C}+V_{XYAD_1})\cdot V_{XYBC_1}\cdot V_{XYAD_1}}{V_{XYBD_1}\cdot V_{XYAC_1}\cdot V_{XYAD_1}\cdot(V_{XYD_1C}+V_{XYBD_1})}$$

$$\overset{simplify}{=} \frac{(V_{XYD_1C}+V_{XYAD_1})\cdot V_{XYBC_1}}{V_{XYAC_1}\cdot(V_{XYD_1C}+V_{XYBD_1})}$$

$$\overset{C}{=} \frac{(-V_{XYBD_1}\cdot V_{XYAC_1}+V_{XYAD_1}\cdot V_{XYAC_1})\cdot V_{XYBC_1}\cdot(-V_{XYBC_1}+V_{XYAC_1})}{V_{XYAC_1}\cdot(-V_{XYBD_1}\cdot V_{XYBC_1}+V_{XYBC_1}\cdot V_{XYAD_1})\cdot(-V_{XYBC_1}+V_{XYAC_1})}$$

$$\overset{simplify}{=} 1$$

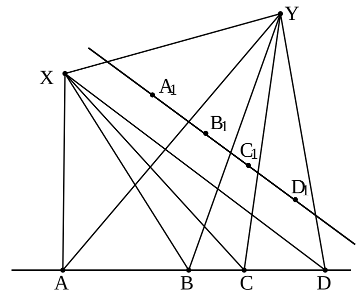


Figure 4-25

Example 4.64 [3] *Let ABCD be a tetrahedron and G the centroid of triangle ABC. The lines passing through points A, B, and C and parallel to line DG meet their opposite face in P, Q, and R respectively. Show that* $V_{GPQR} = 3V_{ABCD}$.

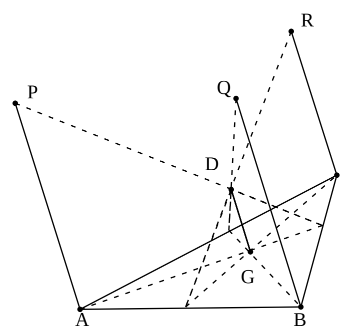


Figure 4-26

[3]This is a problem from the 1964 International Mathematical Olympiad.

Constructive description

((POINTS $A\ B\ C\ D$)
(CENTROID $G\ A\ B\ C$)
(INTER P (PLINE $A\ D\ G$) (PLANE $B\ C\ D$))
(INTER Q (PLINE $B\ D\ G$) (PLANE $A\ C\ D$))
(INTER R (PLINE $C\ D\ G$) (PLANE $A\ B\ D$))
($3V_{ABCD} = V_{GPQR}$))

The eliminants

$V_{CDGP} = -V_{ACDG}$

$V_{BDGP} = -V_{ABDG}$

$V_{BCGP} = V_{ABCD}$

$V_{DGPQ} = -V_{BDGP}$

$V_{CGPQ} = \frac{V_{CDGP} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG}}{V_{ACDG}}$

$V_{GPQR} = \frac{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}{V_{ABDG}}$

The machine proof

$$\frac{(3) \cdot V_{ABCD}}{V_{GPQR}}$$

$$\stackrel{R}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG}}{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}$$

$$\stackrel{Q}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot (V_{ACDG})^2}{-V_{CDGP} \cdot V_{ACDG} \cdot V_{ABDG} \cdot V_{ABCD} - V_{BDGP} \cdot V_{ACDG}^2 \cdot V_{ABCD} + V_{BCGP} \cdot V_{ACDG}^2 \cdot V_{ABDG}}$$

$$\stackrel{simplify}{=} \frac{(-3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot V_{ACDG}}{V_{CDGP} \cdot V_{ABDG} \cdot V_{ABCD} + V_{BDGP} \cdot V_{ACDG} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG} \cdot V_{ABDG}}$$

$$\stackrel{P}{=} \frac{(-3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot V_{ACDG} \cdot (V_{BCDG})^3}{-3V_{BCDG}^3 \cdot V_{ACDG} \cdot V_{ABDG} \cdot V_{ABCD}}$$

$$\stackrel{simplify}{=} 1$$

In the above proof the fact that G is the centroid of triangle ABC is not used. We thus have the following extension of Example 4.64.

Example 4.65 *The result of Example 4.64 is still true if point G is any point in plane ABC.*

We further ask whether the result of Example 4.64 is true or not if point G is an arbitrary point.

Constructive description

((POINTS $A\ B\ C\ D\ G$)
(INTER P (PLINE $A\ D\ G$) (PLANE $B\ C\ D$))
(INTER Q (PLINE $B\ D\ G$) (PLANE $A\ C\ D$))
(INTER R (PLINE $C\ D\ G$) (PLANE $A\ B\ D$))
($3V_{ABCD} = V_{GPQR}$))

The eliminants

$V_{CDGP} = -V_{ACDG}$

$V_{BDGP} = -V_{ABDG}$

$V_{BCGP} = -(V_{ABCG} - V_{ABCD})$

$V_{DGPQ} = -V_{BDGP}$

$V_{CGPQ} = \frac{V_{CDGP} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG}}{V_{ACDG}}$

$V_{GPQR} = \frac{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}{V_{ABDG}}$

The machine proof

$$\frac{(3) \cdot V_{ABCD}}{V_{GPQR}}$$

$$\stackrel{R}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG}}{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}$$

$$\stackrel{Q}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot (V_{ACDG})^2}{-V_{CDGP} \cdot V_{ACDG} \cdot V_{ABDG} \cdot V_{ABCD} - V_{BDGP} \cdot V_{ACDG}^2 \cdot V_{ABCD} + V_{BCGP} \cdot V_{ACDG}^2 \cdot V_{ABDG}}$$

$$\stackrel{simplify}{=} \frac{(-3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot V_{ACDG}}{V_{CDGP} \cdot V_{ABDG} \cdot V_{ABCD} + V_{BDGP} \cdot V_{ACDG} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG} \cdot V_{ABDG}}$$

$$\stackrel{P}{=} \frac{(-3)\cdot V_{ABCD}\cdot V_{ABDG}\cdot V_{ACDG}\cdot (V_{BCDG})^3}{V_{BCDG}^3\cdot V_{ACDG}\cdot V_{ABDG}\cdot V_{ABCG}-3V_{BCDG}^3\cdot V_{ACDG}\cdot V_{ABDG}\cdot V_{ABCD}}$$
$$\stackrel{simplify}{=} \frac{(-3)\cdot V_{ABCD}}{V_{ABCG}-3V_{ABCD}}$$

We thus obtain the following extension of Example 4.65: $V_{GPQR} = 3V_{ABC}$ iff G is in plane ABC.

4.4 Pythagoras Differences in Space

4.4.1 Pythagoras Difference and Perpendicularity

From now on, the concept of the *square-distance* between two points will be used. The definition for the Pythagoras difference is the same as in the plane geometry, i.e., for $\triangle ABC$

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$

For a skew quadrilateral $ABCD$, we define

$$P_{ABCD} = P_{ABD} - P_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

Properties of the Pythagoras difference in space are quite similar to those in a plane. Propositions 3.4, 3.2, and 3.3 are true in space, since all the points involved are in the same plane. Surprisingly, Propositions 3.1, 3.5, and 3.7 are also true in space even if the involving points are not coplanar.

Proposition 4.66 *Let R be a point on line PQ with position ratio $r_1 = \frac{\overline{PR}}{\overline{PQ}}, r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ with respect to PQ. Then for any points A and B in space, we have*

$$\begin{aligned} P_{RAB} &= r_1P_{QAB} + r_2P_{PAB} \\ P_{ARB} &= r_1P_{AQB} + r_2P_{APB} - r_1r_2P_{PQP}. \end{aligned}$$

Proof. Since points P, Q, R, A and P, Q, R, B are two groups of coplanar points, we may use Proposition 3.5 on them separately:

$$\begin{aligned} \overline{RA}^2 &= r_1\overline{QA}^2 + r_2\overline{PA}^2 - r_1r_2\overline{PQ}^2 \\ \overline{RB}^2 &= r_1\overline{QB}^2 + r_2\overline{PB}^2 - r_1r_2\overline{PQ}^2. \end{aligned}$$

Then $P_{RAB} = \overline{RA}^2 + \overline{AB}^2 - \overline{RB}^2 = r_1(\overline{QA}^2 + \overline{AB}^2 - \overline{QB}^2) + r_2(\overline{PA}^2 + \overline{AB}^2 - \overline{PB}^2) = r_1P_{QAB} + r_2P_{PAB}$. The second equation can be proved similarly. ∎

Proposition 4.67 *Let R be a point in the plane PQS, and $r_1 = \frac{S_{PQR}}{S_{PQS}}$, $r_2 = \frac{S_{RQS}}{S_{PQS}}$, and $r_3 = \frac{S_{PRS}}{S_{PQS}}$. Then for points A and B we have*

$$\begin{aligned} P_{RAB} &= r_1 P_{SAB} + r_2 P_{PAB} + r_3 P_{QAB} \\ P_{ARB} &= r_1 P_{ASB} + r_2 P_{APB} + r_3 P_{AQB} - 2(r_1 r_2 \overline{PS}^2 + r_1 r_3 \overline{QS}^2 + r_2 r_3 \overline{PQ}^2) \end{aligned}$$

Proof. The proof for proposition is similar to that for Propositions 3.61 and 3.62. ∎

Proposition 4.68 *Let $ABCD$ be a parallelogram. Then for any points P and Q, we have*

$$\begin{aligned} P_{APQ} + P_{CPQ} &= P_{BPQ} + P_{DPQ} \quad \text{or} \quad P_{APBQ} = P_{DPCQ} \\ P_{PAQ} + P_{PCQ} &= P_{PBQ} + P_{PDQ} + 2P_{BAD} \end{aligned}$$

Proof. The proof is the same as for Proposition 3.7. ∎

As in plane geometry, we use the notation $AB \perp CD$ to denote the fact that four points A, B, C, and D satisfy one of the following conditions: $A = B$, or $C = D$, or line AB is perpendicular to line CD.

Proposition 4.69 *$AB \perp CD$ iff $P_{ACD} = P_{BCD}$ or $P_{ACBD} = 0$.*

Proof. Let E be a point such that $\overline{AE} = \overline{CD}$. By Proposition 4.68, $P_{ACBD} = P_{AABE} = P_{BAE}$. By the Pythagorean theorem, $AB \perp CD$ iff $P_{ACBD} = P_{BAE} = 0$. ∎

As consequences, Proposition 3.10 and Example 3.9 are still true in space.

Just as the volume makes the machine proof for geometry theorems involving collinear and parallel possible, the Pythagoras difference will make the machine proof for geometry theorems involving perpendicular possible. Before presenting the method of machine proof, let us first get acquainted with the Pythagoras differences by proving some basic properties of the perpendicular.

Example 4.70 *If a given line is perpendicular to a pair of non-parallel lines in a plane, it is perpendicular to every lines in the given plane, i.e., the line is perpendicular to the plane.*

Proof. As in Figure 4-27, line PQ is perpendicular to line OU and OV. Let W be a point in plane OUV. We need to show that $PQ \perp OW$. By Propositions 4.67 and 4.69

$$\begin{aligned} P_{PQW} &= \frac{S_{OUW}}{S_{OUV}} P_{PQV} + \frac{S_{OWV}}{S_{OUV}} P_{PQU} + \frac{S_{WUV}}{S_{OUV}} P_{PQO} \\ &= \left(\frac{S_{OUW}}{S_{OUV}} + \frac{S_{OWV}}{S_{OUV}} + \frac{S_{WUV}}{S_{OUV}}\right) P_{PQO} = P_{PQO}. \end{aligned}$$

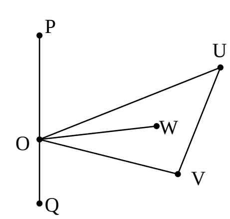


Figure 4-27

By Proposition 4.69 again, $PQ \perp OW$. ∎

Example 4.71 (The Theorem of Three Perpendiculars) *A line PQ is perpendicular to a line AB iff PQ is perpendicular to the orthogonal projection of AB to a plane containing PQ.*

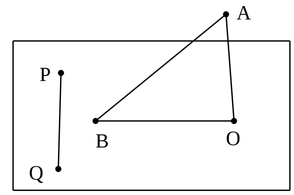


Figure 4-28

Proof. Let O be the orthogonal projection of point A to plane PQB. Then $AO \perp PQ$. If $PQ \perp BO$, we have $P_{PQA} = P_{PQO} = P_{PQB}$, i.e., $PQ \perp AB$. Conversely if $PQ \perp AB$, we have $P_{PQB} = P_{PQA} = P_{PQO}$, i.e., $PQ \perp BO$. ∎

Example 4.72 (The Orthocenter Theorem for Tetrahedra) *If two pairs of opposite edges of a tetrahedron are at right angles to one another, the third pair are at right angles; and the altitudes are concurrent, and pass through the orthocenters of the opposite faces.*

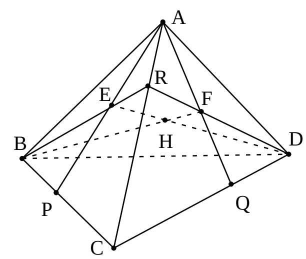


Figure 4-29

Proof. Let $AB \perp CD$ and $AC \perp BD$. Then $P_{ABC} = P_{ABD} = P_{CBD} = P_{DBC}$, i.e., $AD \perp BC$.
Let the two altitudes AQ and DR of triangle ACD meet in F. Then $P_{BAQ} - P_{FAQ} = P_{BAD} - P_{FAC} = P_{CAD} - P_{DAC} = 0$, i.e., $BF \perp AQ$. Similarly, $BF \perp DR$. Therefore BF is the altitude from B to plane ACD.

To prove that the four altitudes are concurrent, we first show that $BR \perp AC$ which follows from $P_{ACR} = P_{ACD} = P_{ACB}$. Let altitudes AP and BR of triangle ABC meet in E and DE and BF meet in H. We need to show that $AH \perp BCD$. By Proposition 4.69, $P_{HDC} = P_{BDC} = P_{ADC}$, i.e., $AH \perp DC$. Similarly $AH \perp BC$. Thus $AH \perp BCD$. ∎

Example 4.73 *Show that the construction (FOOT2PLANE A P L M N) is equivalent to the construction (ARATIO A L M N r_1 r_2 r_3) where*

$$r_1 = \frac{S_{YMN}}{S_{LMN}} = \frac{-P_{PMN}P_{LMN}+2\overline{MN}^2P_{PML}}{4\overline{LN}^2\cdot\overline{LM}^2-P_{MLN}^2}$$
$$r_2 = \frac{S_{LYN}}{S_{LMN}} = \frac{-P_{PNL}P_{LNM}+2\overline{NL}^2P_{PNM}}{4\overline{LN}^2\cdot\overline{LM}^2-P_{MLN}^2}$$
$$r_3 = \frac{S_{LMY}}{S_{LMN}} = \frac{-P_{PLM}P_{MLN}+2\overline{ML}^2P_{PLN}}{4\overline{LN}^2\cdot\overline{LM}^2-P_{MLN}^2}$$

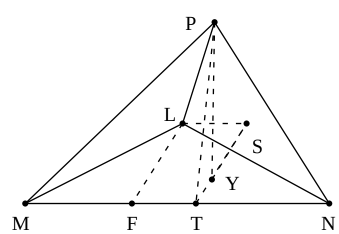


Figure 4-30

Proof. We construct point A as in Example 4.29. Then

$$\frac{S_{YMN}}{S_{LMN}} = \frac{\overline{YT}}{\overline{ST}} = \frac{P_{PTS}}{P_{TST}} = \frac{P_{PTS}}{P_{LFL}}.$$

By Propositions 4.66 and 4.69,

$$\begin{aligned} P_{PTS} &= P_{PTL} = \frac{P_{PMN}P_{PNL} + P_{PNM}P_{PML} - P_{PMN}P_{PNM}}{P_{MNM}} \\ &= \frac{P_{PMN}P_{PNL} + (2\overline{MN}^2 - P_{PMN})P_{PML} - P_{PMN}P_{PNM}}{P_{MNM}} \\ &= \frac{-P_{PMN}P_{NML} + 2\overline{MN}^2 P_{PML}}{P_{MNM}} \\ P_{LFL} &= \frac{16S^2_{LMN}}{P_{MNM}} = \frac{4\overline{LN}^2 \cdot \overline{LM}^2 - P^2_{MLN}}{P_{MNM}}. \end{aligned}$$

We have proved the first case. Other cases are similar. ∎

4.4.2 Pythagoras Difference and Volume

With the concept of perpendicularity, we can obtain the exact measurement for the volume of a tetrahedron.

Definition 4.74 *Let F be the foot of the perpendicular dropped from the point R upon the plane LMN. The distance from R to LMN, denoted by $h_{R,LMN}$, is a real number which has the same sign as V_{RLMN} and $|h_{R,LMN}| = |RF|$.*

Proposition 4.75 *For any two tetrahedra $ABCD$ and $RLMN$, let $h_A = h_{A,BCD}$, $h_R = h_{R,LMN}$. Show that $\frac{V_{ABCD}}{|S_{BCD}|h_A} = \frac{V_{RLMN}}{|S_{LMN}|h_R}$.*

Proof. Without loss of generality, we assume that points B, C, D, L, M, and N are in the same plane. As in Figure 4-31, let RF be the altitude of the tetrahedron $RLMN$ and S be a point on RF such that $AS \parallel BCD$. Then $V_{ABCD} = V_{SBCD}$. By Propositions 4.3 and 4.4,

$$\frac{V_{ABCD}}{V_{ALMN}} = \frac{S_{BCD}}{S_{LMN}}; \quad \frac{V_{SLMN}}{V_{RLMN}} = \frac{\overline{SF}}{\overline{RF}}.$$

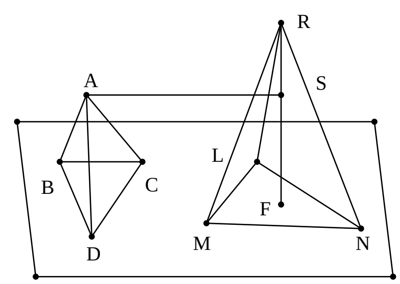


Figure 4-31

Multiplying the two formulas together and noting that h_A and h_R have the same sign as V_{ABCD} and V_{RLMN}, we prove the result. ∎

Corollary 4.76 *For a tetrahedron $ABCD$, we have*

$$h_{A,BCD}|S_{BCD}| = h_{B,CDA}|S_{CDA}| = h_{C,DAB}|S_{DAB}| = h_{D,ABC}|S_{ABC}|.$$

Proof. Replacing the tetrahedron $RLMN$ in the Proposition 4.75 by $BCDA$, we obtain the first equation, etc. ∎

By Proposition 4.75, we have

$$V_{ABCD} = kh_{A,BCD}|S_{BCD}| = kh_{B,CDA}|S_{CDA}| = kh_{C,DAB}|S_{DAB}| = kh_{D,ABC}|S_{ABC}|$$

where k is a constant which is independent of the tetrahedron $ABCD$. Setting $k = 1/3$, we obtain the usual formula for the volumes of tetrahedra.

Proposition 4.77 *We have*

$$V_{ABCD} = \frac{1}{3}h_{A,BCD}|S_{BCD}| = \frac{1}{3}h_{B,CDA}|S_{CDA}| = \frac{1}{3}h_{C,DAB}|S_{DAB}| = \frac{1}{3}h_{D,ABC}|S_{ABC}|.$$

Proposition 4.78 (The Herron-Qin Formula for Tetrahedra) *Prove the following formula*

$$144V_{ABCD}^2 = 4\overline{AB}^2 \cdot \overline{AC}^2 \cdot \overline{AD}^2 - \overline{AB}^2 P_{DAC}^2 - \overline{AC}^2 P_{BAD}^2 - \overline{AD}^2 P_{BAC}^2 + P_{BAC}P_{BAD}P_{CAD}.$$

Proof. Similar to Example 4.29, we construct the altitude BO as follows.

(FOOT2LINE F D A C)
(FOOT2LINE H B A C)
(PRATIO G H F D 1)
(FOOT2LINE O B H G)

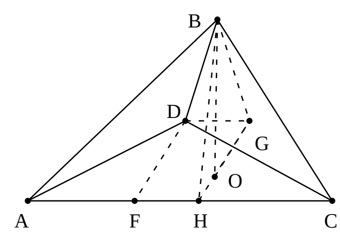


Figure 4-32

Then

$$\overline{BO}^2 = \overline{BH}^2 - \overline{HO}^2 = \frac{4S_{ABC}^2}{\overline{AC}^2} - \overline{HO}^2. \tag{1}$$

By Propositions 3.1, 3.2, and 4.66

$$\begin{aligned}
\overline{OH}^2 &= \left(\frac{\overline{OH}}{\overline{HG}}\right)^2 \overline{HG}^2 = \left(\frac{P_{BHG}}{P_{HGH}}\right)^2 \overline{HG}^2 = \frac{P_{BHG}^2}{4\overline{HG}^2} = \frac{P_{BHD}^2}{4\overline{DF}^2} \\
&= (P_{BAC}P_{BCD} + P_{BCA}P_{BAD} - P_{BAC}P_{BCA})^2/(4\overline{AC}^4 \cdot 4\overline{DF}^2) \\
&= (P_{BAC}P_{BCD} + (2\overline{AC}^2 - P_{BAC})P_{BAD} - P_{BAC}P_{BCA})^2/(64\overline{AC}^2 S_{DAC}^2) \\
&= (-P_{BAC}P_{CAD} + 2\overline{AC}^2 P_{BAD})^2/(64\overline{AC}^2 S_{DAC}^2)
\end{aligned}$$

Substitute this into (1). By the Herron-Qin formula for triangles (on page 108), we have

$$\begin{aligned}
&144V_{ABCD}^2 \\
= \ & 144(1/3)^2\overline{BO}^2 S_{ACD}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(16)^2 S_{BAC}^2 S_{DAC}^2 - (-P_{BAC}P_{CAD} + 2\overline{AC}^2 P_{BAD})^2}{4\overline{AC}^2} \\
&= \frac{(4\overline{AB}^2 \cdot \overline{AC}^2 - P_{BAC}^2)(4\overline{AD}^2 \cdot \overline{AC}^2 - P_{DAC}^2) - (-P_{BAC}P_{CAD} + 2\overline{AC}^2 P_{BAD})^2}{4\overline{AC}^2} \\
&= 4\overline{AB}^2 \cdot \overline{AC}^2 \cdot \overline{AD}^2 - \overline{AB}^2 P_{DAC}^2 - \overline{AC}^2 P_{BAD}^2 - \overline{AD}^2 P_{BAC}^2 + P_{BAC}P_{BAD}P_{CAD}. \quad \blacksquare
\end{aligned}
$$

Corollary 4.79 (The Cayley-Menger Formula) *We have the following commonly used version of the Herron-Qin formula.*

$$
288V_{P_1P_2P_3P_4} = \begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & 1 \\ r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & 1 \\ r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}
$$

where $r_{ij} = |\overline{P_iP_j}|$.

Proof. In the above determinant, subtracting the first row from the second, the third, and the fourth rows and subtracting the first column from the second, the third, and the fourth columns, the determinant becomes

$$
\begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_{21}^2 & -2r_{12}^2 & -P_{213} & -P_{214} & 0 \\ r_{31}^2 & -P_{312} & -2r_{13}^2 & -P_{314} & 0 \\ r_{41}^2 & -P_{412} & -P_{413} & -2r_{14}^2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2r_{12}^2 & P_{213} & P_{214} \\ P_{312} & 2r_{13}^2 & P_{314} \\ P_{412} & P_{413} & 2r_{14}^2 \end{vmatrix}.
$$

Expanding the last determinant and comparing the formula in Proposition 4.78, we prove the result. $\blacksquare$

4.5 The Volume Method

Since we have a new geometry quantity, the constructive statements can be enlarged in the following way: the conclusion of a geometry statement could be the equation of two polynomials of length ratios, area ratios, volumes and Pythagoras differences.

4.5.1 The Algorithm

Now we have six constructions S1–S6 and four geometry quantities. We need to give a method to eliminate the point introduced by each of the constructions S1–S6 from each of the four quantities. This section deals with the cases which are not discussed in Section 4.3.

Lemma 4.80 *Let $G = P_{ABY}$. Then*

$$G = \begin{cases} P_{ABW} + r(P_{ABV} - P_{ABU}) \\ \quad \text{if } Y \text{ is introduced by (PRATIO } Y\ W\ U\ V\ r) \\ r_1 P_{ABL} + r_2 P_{ABM} + r_3 P_{ABN} \\ \quad \text{if } Y \text{ is introduced by (ARATIO } Y\ L\ M\ N\ r_1\ r_2\ r_3) \\ \frac{S_{UIJ}}{S_{UIVJ}} P_{ABV} - \frac{S_{VIJ}}{S_{UIVJ}} P_{ABU} \\ \quad \text{if } Y \text{ is introduced by (INTER } Y \text{ (LINE } U\ V) \text{ (LINE } I\ J)) \\ \frac{1}{V_{ULMNV}}(V_{ULMN} P_{ABV} - V_{VLMN} P_{ABU}) \\ \quad \text{if } Y \text{ is introduced by (INTER } Y \text{ (LINE } U\ V) \text{ (PLANE } L\ M\ N)) \\ \frac{P_{PUV} P_{ABV} + P_{PVU} P_{ABU}}{2\overline{UV}^2} \\ \quad \text{if } Y \text{ is introduced by (FOOT2LINE } Y\ P\ U\ V) \end{cases}$$

Proof. We only need to find the position ratio of Y with respect to UV and substitute it into the first equation of Proposition 4.66. For the second case, see Lemma 4.36. For other cases, see Subsection 4.3.1 for details. ∎

From the above lemma and Proposition 4.66, it is easy to eliminate Y from P_{AYB}. We leave this as an exercise.

Exercise 4.81 *Try to eliminate Y from P_{AYB} if Y is introduced by each of the constructions S2-S6.*

Lemma 4.82 *If Y is introduced by (FOOT2LINE Y P U V) then*

$$V_{ABCY} = \frac{P_{PUV}}{P_{UVU}} V_{ABCV} + \frac{P_{PVU}}{P_{UVU}} V_{ABCU}.$$

Proof. This is a consequence of Propositions 4.5 and 3.2. ∎

Lemma 4.83 *Let Y be introduced by (FOOT2LINE Y P U V). Then*

$$\frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{P_{PUDV}}{P_{EUFV}} & \text{if } D \in UV. \\ \frac{V_{DPUV}}{V_{EPUVF}} & D \notin PUV \\ \frac{V_{DUVE}}{V_{EUVF}} & \text{if } D \in PUV \text{ and } E \notin PUV \\ \frac{S_{DUV}}{S_{EUFV}} & \text{if all points are coplanar} \end{cases}$$

In all cases, we assume P is not on line UV; otherwise $P = Y$ and $\frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DP}}{\overline{EF}}$.

Proof. The first and last cases are from Lemma 3.29. The second case is a consequence of the co-face theorem. For the third case, let T be a point such that $\overline{DT} = \overline{EF}$. Then $\frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DY}}{\overline{DT}} = \frac{S_{DUV}}{S_{DUTV}} = \frac{V_{DUVE}}{V_{DUVET}} = \frac{V_{DUVE}}{V_{EUVEF}} = -\frac{V_{DUVE}}{V_{FUVE}}$. ▮

Lemma 4.84 *Let Y be introduced by (FOOT2LINE Y P U V). Then*

$$\frac{S_{ABY}}{S_{CDE}} = \begin{cases} \frac{P_{PUV}V_{PABV}+P_{PVU}V_{PABU}}{2\overline{UV}^2 V_{PCDEA}} & \text{if } P \text{ is not in } ABY. \\ \frac{V_{UABV}}{V_{UCDEV}} & \text{if } UV \not\parallel ABY. \\ \frac{P_{PUV}S_{ABV}+P_{PVU}S_{ABU}}{2\overline{UV}^2 S_{CDE}} & \text{if } P, U, \text{ and } V \text{ are in } ABY. \end{cases}$$

Proof. If P is not in ABY, by Proposition 4.3, $\frac{S_{ABY}}{S_{CDE}} = \frac{V_{PABY}}{V_{PCDEA}}$. Now the result comes from Lemma 4.82. For the second case

$$\frac{S_{ABY}}{S_{CDE}} = \frac{V_{UABYV}}{V_{UCDEV}} = \frac{V_{UABV}}{V_{UCDEV}}.$$

The last case is Lemma 3.24. ▮

By now, we have given methods of eliminating points introduced by constructions S2-S6. We still need to deal with free points. By Lemma 4.59, volumes of tetrahedra can be reduced to volume coordinates with respect to four non-coplanar points. The following lemma will reduce the Pythagoras difference of free points to volume coordinates.

Lemma 4.85 *Let O, W, U, and V be four points such that $OW \perp OUV$, $OU \perp OWV$, and $OV \perp OWU$. Then*

(1) $\overline{AB}^2 = \overline{OW}^2 (\frac{V_{AOUVB}}{V_{OWUV}})^2 + \overline{OU}^2 (\frac{V_{AOWVB}}{V_{OWUV}})^2 + \overline{OV}^2 (\frac{V_{AOWUB}}{V_{OWUV}})^2$.

(2) $V_{OWUV}^2 = \frac{1}{36}\overline{OW}^2\overline{OU}^2\overline{OV}^2$.

Proof. (2) is from Propositions 4.77 and 3.15. For (1), let D, E, and F be points such that $AD \parallel OW$, $BE \parallel OV$, $BF \parallel OU$, $DE \parallel OU$, and $DF \parallel OV$. Then

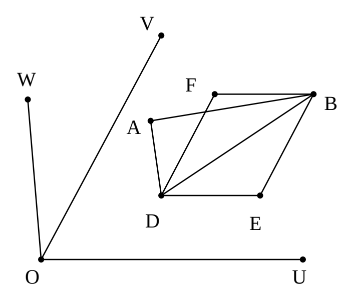


Figure 4-33

$$\begin{aligned} \overline{AB}^2 &= \overline{AD}^2 + \overline{BD}^2 \\ &= \overline{AD}^2 + \overline{BE}^2 + \overline{BF}^2 \\ &= \overline{OW}^2 (\frac{\overline{AD}}{\overline{OW}})^2 + \overline{OU}^2 (\frac{\overline{BF}}{\overline{OU}})^2 + \overline{OV}^2 (\frac{\overline{BE}}{\overline{OV}})^2. \end{aligned}$$

By the co-face theorem, $\frac{\overline{AD}}{\overline{OW}} = -\frac{V_{AOUVD}}{V_{WOUV}} = \frac{V_{AOUVB}}{V_{OWUV}}$, $\frac{\overline{BE}}{\overline{OV}} = \frac{V_{BOWUA}}{V_{OWUV}}$, and $\frac{\overline{BF}}{\overline{OU}} = \frac{V_{BOWVA}}{V_{OWUV}}$. ▮

Now we have the main algorithm.

Algorithm 4.86 (Solid)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a constructive geometric statement.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \cdots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the nondegenerate conditions of C_i are satisfied. The nondegenerate condition of a construction has three forms: $A \neq B$, $PQ \nparallel UV$, $PQ \nparallel WUV$. For the first case, we check whether $P_{ABA} = 2\overline{AB}^2 = 0$. For the second case, we check whether $V_{PQUV} = 0$ and $S_{PUV} = S_{QUV}$. For the third case, we check whether $V_{PWUV} = V_{QWUV}$. If a nondegenerate condition of a geometry statement is not satisfied, the statement is *trivially true*. The algorithm terminates.

S3. Let $G_1, \cdots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \cdots, s$ do S4

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using the lemmas in this chapter and replace G_j by H_j in E and F to obtain the new E and F.

S5. Now E and F are rational expressions of independent variables. Hence if $E = F$, S is true under the nondegenerate conditions. Otherwise S is false in Euclidean solid geometry.

Proof. The algorithm is correct, because the volume coordinates of free points are independent parameters. ∎

For the complexity of the algorithm, let n be the number of the non-free points in a statement. If the conclusion of the geometry statement is of degree d, the output of our algorithm is at most of degree $5d3^n$.

4.5.2 Working Examples

Example 4.87 *If a line divides two opposite sides of a skew quadrilateral in direct proportion, and a second line divides the other two opposite sides in direct proportion, the two lines are coplanar.*

Constructive description
((POINTS A B C D)

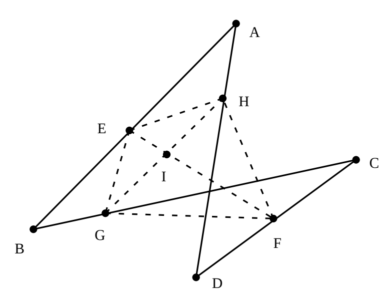


Figure 4-34

(LRATIO E A B r_1)
(LRATIO F D C r_1)
(LRATIO H A D r_2)
(LRATIO G B C r_2)
($V_{EFHG} = 0$))

The machine proof

V_{EFHG}

$\overset{n}{=} -V_{CEFH}\cdot r_2+V_{BEFH}\cdot r_2-V_{BEFH}$

$\overset{n}{=} -(-V_{BDEF}\cdot r_2^2+V_{BDEF}\cdot r_2+V_{ACEF}\cdot r_2^2-V_{ACEF}\cdot r_2)$

$\overset{simplify}{=} (r_2-1)\cdot(V_{BDEF}-V_{ACEF})\cdot r_2$

$\overset{n}{=} (r_2-1)\cdot(V_{BCDE}\cdot r_1-V_{ACDE}\cdot r_1+V_{ACDE})\cdot r_2$

$\overset{n}{=} (r_2-1)\cdot(0)\cdot r_2$

$\overset{simplify}{=} 0$

The eliminants

$V_{EFHG}\overset{G}{=}-\left(V_{CEFH}\cdot r_2-V_{BEFH}\cdot r_2+V_{BEFH}\right)$

$V_{BEFH}\overset{H}{=}V_{BDEF}\cdot r_2$

$V_{CEFH}\overset{H}{=}(r_2-1)\cdot V_{ACEF}$

$V_{ACEF}\overset{F}{=}(r_1-1)\cdot V_{ACDE}$

$V_{BDEF}\overset{F}{=}V_{BCDE}\cdot r_1$

$V_{ACDE}\overset{E}{=}V_{ABCD}\cdot r_1$

$V_{BCDE}\overset{E}{=}(r_1-1)\cdot V_{ABCD}$

Example 4.88 *Continue from Example 4.87. Let the two lines EF and GH intersect at I. Then* $\frac{\overline{EI}}{\overline{EF}} = \frac{\overline{AH}}{\overline{AD}}$.

Constructive description
((POINTS A B C D) (LRATIO E A B r_1) (LRATIO F D C r_1) (LRATIO H A D r_2)
(LRATIO G B C r_2) (INTER I (LINE E F) (LINE H G)) ($\frac{\overline{EI}}{\overline{EF}} = r_2$))

The machine proof

$\frac{\frac{\overline{EI}}{\overline{EF}}}{r_2}$

$\overset{I}{=} \frac{1}{r_2\cdot(-\frac{S_{FHG}}{S_{EHG}}+1)}$

$\overset{G}{=} \frac{-(-V_{BCEH})}{r_2\cdot(-V_{BCFH}+V_{BCEH})}$

$\overset{H}{=} \frac{-(-V_{BCDE}\cdot r_2)}{r_2\cdot(V_{BCDE}\cdot r_2+V_{ABCF}\cdot r_2-V_{ABCF})}$

$\overset{simplify}{=} \frac{V_{BCDE}}{V_{BCDE}\cdot r_2+V_{ABCF}\cdot r_2-V_{ABCF}}$

$\overset{F}{=} \frac{V_{BCDE}}{V_{BCDE}\cdot r_2-V_{ABCD}\cdot r_2\cdot r_1+V_{ABCD}\cdot r_2+V_{ABCD}\cdot r_1-V_{ABCD}}$

$\overset{E}{=} \frac{V_{ABCD}\cdot r_1-V_{ABCD}}{V_{ABCD}\cdot r_1-V_{ABCD}} \overset{simplify}{=} 1$

The eliminants

$\frac{\overline{EI}}{\overline{EF}}\overset{I}{=}\frac{1}{-(\frac{S_{FHG}}{S_{EHG}}-1)}$

$\frac{S_{FHG}}{S_{EHG}}\overset{G}{=}\frac{V_{BCFH}}{V_{BCEH}}$

$V_{BCFH}\overset{H}{=}(r_2-1)\cdot V_{ABCF}$

$V_{BCEH}\overset{H}{=}-\left(V_{BCDE}\cdot r_2\right)$

$V_{ABCF}\overset{F}{=}-\left((r_1-1)\cdot V_{ABCD}\right)$

$V_{BCDE}\overset{E}{=}(r_1-1)\cdot V_{ABCD}$

Example 4.89 *The sides AB and DC of a skew quadrilateral are cut into $2n+1$ equal segments by points $P_1,\cdots,P_{2n}$ and $Q_1,\cdots,Q_{2n}$ respectively. Show that*

(1) $V_{P_nP_{n+1}Q_{n+1}Q_n} = \frac{1}{(2n+1)^2}V_{ABCD}$.

(2) *If sides BC and AD are cut into* $2m+1$ *equal segments by points* $R_1, \cdots, R_{2m}$ *and* $S_1, \cdots, S_{2m}$ *respectively, then the area of the quadrilateral formed by the lines* P_nQ_n, $P_{n+1}Q_{n+1}$, R_mS_m, *and* $R_{m+1}S_{m+1}$ *is* $\frac{1}{(2n+1)^2(2m+1)^2}V_{ABCD}$.

Figure 4-35 shows the case $n = m = 2$. Note that in the following machine proof for (1), we use some different names for points $P_n, P_{n+1}, Q_{n+1}, Q_n$.

Constructive description

((POINTS A B C D)
(LRATIO X A B $\frac{n}{2n+1}$)
(LRATIO Y A B $\frac{n+1}{2n+1}$)
(LRATIO U D C $\frac{n}{2n+1}$)
(LRATIO V D C $\frac{n+1}{2n+1}$)
($V_{XYVU} = V_{ABCD}$))

The eliminants

$$V_{XYUV} \stackrel{V}{=} \frac{-(V_{DXYU}\cdot n+V_{CXYU}\cdot n+V_{CXYU})}{2n+1}$$
$$V_{CXYU} \stackrel{U}{=} \frac{(n+1)\cdot V_{CDXY}}{2n+1}$$
$$V_{DXYU} \stackrel{U}{=} \frac{-V_{CDXY}\cdot n}{2n+1}$$
$$V_{CDXY} \stackrel{Y}{=} \frac{-(V_{BCDX}\cdot n+V_{BCDX}+V_{ACDX}\cdot n)}{2n+1}$$
$$V_{ACDX} \stackrel{X}{=} \frac{V_{ABCD}\cdot n}{2n+1}$$
$$V_{BCDX} \stackrel{X}{=} \frac{-(n+1)\cdot V_{ABCD}}{2n+1}$$

The machine proof

$$\frac{-V_{XYUV}}{V_{ABCD}}$$
$$\stackrel{V}{=} \frac{-(-V_{DXYU}\cdot n-V_{CXYU}\cdot n-V_{CXYU})}{V_{ABCD}\cdot(2n+1)}$$
$$\stackrel{U}{=} \frac{4V_{CDXY}\cdot n^2+4V_{CDXY}\cdot n+V_{CDXY}}{V_{ABCD}\cdot(2n+1)^3}$$
$$\stackrel{simplify}{=} \frac{V_{CDXY}}{V_{ABCD}\cdot(2n+1)}$$
$$\stackrel{Y}{=} \frac{-V_{BCDX}\cdot n-V_{BCDX}-V_{ACDX}\cdot n}{V_{ABCD}\cdot(2n+1)^2}$$
$$\stackrel{X}{=} \frac{-(-4V_{ABCD}\cdot n^2-4V_{ABCD}\cdot n-V_{ABCD})}{V_{ABCD}\cdot(2n+1)^4}$$
$$\stackrel{simplify}{=} \frac{1}{(2n+1)^2}$$

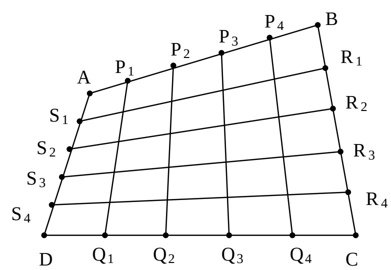


Figure 4-35

By Example 4.88, P_nQ_n and $P_{n+1}Q_{n+1}$ are cut into $2m+1$ equal segments by R_iS_i, $i = 1, ..., 2m$ respectively. Now (2) comes from (1) directly. ▮

A line joining the mid-points of two opposite edges of a tetrahedron will be called a *bimedian* of the tetrahedron relative to the pair of edges considered. The common perpendicular to the two opposite edges of a tetrahedron is called the *bialtitude* of the tetrahedron relative to these edges.

Example 4.90 *The bialtitude relative to one pair of opposite edges of a tetrahedron is perpendicular to the two bimedians relative to the two other pairs of opposite edges.*

Constructive description
((POINTS $X\ Y\ A\ C$)
(FOOT2LINE $S\ A\ X\ Y$)
(ON B (LINE $S\ A$))
(FOOT2LINE $T\ C\ X\ Y$)
(ON D (LINE $T\ C$))
(MIDPOINT $N\ B\ C$)
(MIDPOINT $Q\ A\ D$)
(PERPENDICULAR $N\ Q\ X\ Y$))

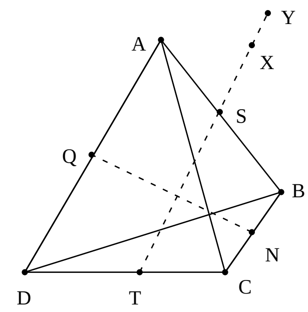


Figure 4-36

The machine proof

$$\frac{P_{YXN}}{P_{YXQ}}$$
$$\overset{Q}{=} \frac{P_{YXN}}{\frac{1}{2}P_{YXD}+\frac{1}{2}P_{YXA}}$$
$$\overset{P}{=} \frac{(2)\cdot P_{YXN}}{P_{YXD}+P_{YXA}}$$
$$\overset{N}{=} \frac{(2)\cdot(\frac{1}{2}P_{YXB}+\frac{1}{2}P_{YXC})}{P_{YXD}+P_{YXA}}$$
$$\overset{M}{=} \frac{P_{YXB}+P_{YXC}}{P_{YXD}+P_{YXA}}$$
$$\overset{D}{=} \frac{P_{YXB}+P_{YXC}}{-P_{YXT}\cdot\frac{\overline{TD}}{\overline{TC}}+P_{YXT}+P_{YXC}\cdot\frac{\overline{TD}}{\overline{TC}}+P_{YXA}}$$
$$= \frac{-(P_{YXB}+P_{YXC})}{-P_{YXC}-P_{YXA}}$$
$$\overset{B}{=} \frac{-P_{YXS}\cdot\frac{\overline{SB}}{\overline{SA}}+P_{YXS}+P_{YXC}+P_{YXA}\cdot\frac{\overline{SB}}{\overline{SA}}}{P_{YXC}+P_{YXA}}$$
$$= \frac{-(-P_{YXC}-P_{YXA})}{P_{YXC}+P_{YXA}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$P_{YXS}=P_{YXA}$$
$$P_{YXB}=-\left(P_{YXS}\cdot\frac{\overline{SB}}{\overline{SA}}-P_{YXS}-P_{YXA}\cdot\frac{\overline{SB}}{\overline{SA}}\right)$$
$$P_{YXT}=P_{YXC}$$
$$P_{YXD}=-\left(P_{YXT}\cdot\frac{\overline{TD}}{\overline{TC}}-P_{YXT}-P_{YXC}\cdot\frac{\overline{TD}}{\overline{TC}}\right)$$
$$P_{YXN}=\frac{1}{2}\left(P_{YXB}+P_{YXC}\right)$$
$$P_{YXQ}=\frac{1}{2}\left(P_{YXD}+P_{YXA}\right)$$

Example 4.91 [4] *A plane parallel to AB and CD meets the edges AD, AC, BD, and BC in P, Q, S, and R respectively. The plane divides the tetrahedron into two parts. Let r be the ratio of the distances between AB, CD and the plane PQS. Find the ratio of the volumes of the two parts.*

First by the co-face theorem,

$$r = \frac{V_{APQS}}{V_{DPQS}} = \frac{\overline{AP}}{\overline{PD}}.$$

Since the volume V_1 of $AB - PQRS$ is equal to $V_{ABSR} + V_{APRQ} + V_{APSR}$, we will compute $\frac{V_{ABSR}}{V_{ABCD}}$, $\frac{V_{APRQ}}{V_{ABCD}}$, and $\frac{V_{APSR}}{V_{ABCD}}$ separately.

[4]This is a problem from the 1965 International Mathematical Olympiad.

Constructive description

((POINTS $A\ B\ C\ D$)
(LRATIO $P\ A\ D\ \frac{r}{1+r}$)
(INTER Q (LINE $A\ C$) (PLINE $P\ C\ D$))
(INTER S (LINE $B\ D$) (PLINE $P\ A\ B$))
(INTER R (LINE $B\ C$) (PLANE $P\ Q\ S$))
$(\frac{V_{ABSR}}{V_{ABDC}})$)

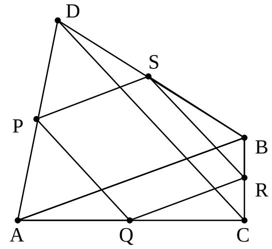


Figure 4-37

The eliminants

$$\frac{S_{ACD}}{S_{CDP}}=r+1$$
$$V_{BCDP}=\frac{-V_{ABCD}}{r+1}$$
$$\frac{S_{ABD}}{S_{ABP}}=\frac{r+1}{r}$$
$$V_{ABCP}=\frac{V_{ABCD}\cdot r}{r+1}$$
$$V_{BDPQ}=\frac{(\frac{S_{ACD}}{S_{CDP}}-1)\cdot V_{BCDP}}{\frac{S_{ACD}}{S_{CDP}}}$$
$$V_{BCPQ}=\frac{-V_{ABCP}}{\frac{S_{ACD}}{S_{CDP}}}$$
$$V_{BPQS}=\frac{V_{BDPQ}}{\frac{S_{ABD}}{S_{ABP}}}$$
$$V_{ABCS}=\frac{V_{ABCD}}{\frac{S_{ABD}}{S_{ABP}}}$$
$$V_{CPQS}=\frac{(\frac{S_{ABD}}{S_{ABP}}-1)\cdot V_{BCPQ}}{-\frac{S_{ABD}}{S_{ABP}}}$$
$$V_{ABSR}=\frac{V_{BPQS}\cdot V_{ABCS}}{V_{CPQS}-V_{BPQS}}$$

The machine proof

$$\frac{V_{ABSR}}{-V_{ABCD}}$$

$$\stackrel{R}{=}\frac{-V_{BPQS}\cdot V_{ABCS}}{-V_{ABCD}\cdot(-V_{CPQS}+V_{BPQS})}$$

$$\stackrel{S}{=}\frac{-(-V_{BDPQ})\cdot(-V_{ABCD})\cdot((-\frac{S_{ABD}}{S_{ABP}}))^2}{V_{ABCD}\cdot(-V_{BDPQ}\cdot\frac{S_{ABD}}{S_{ABP}}-V_{BCPQ}\cdot\frac{S_{ABD}}{S_{ABP}}^2+V_{BCPQ}\cdot\frac{S_{ABD}}{S_{ABP}})\cdot((-\frac{S_{ABD}}{S_{ABP}}))^2}$$

$$\stackrel{simplify}{=}\frac{V_{BDPQ}}{(V_{BDPQ}+V_{BCPQ}\cdot\frac{S_{ABD}}{S_{ABP}}-V_{BCPQ})\cdot\frac{S_{ABD}}{S_{ABP}}}$$

$$\stackrel{Q}{=}\frac{(V_{BCDP}\cdot\frac{S_{ACD}}{S_{CDP}}-V_{BCDP})\cdot(\frac{S_{ACD}}{S_{CDP}})^2}{(V_{BCDP}\cdot\frac{S_{ACD}}{S_{CDP}}^2-V_{BCDP}\cdot\frac{S_{ACD}}{S_{CDP}}-V_{ABCP}\cdot\frac{S_{ACD}}{S_{CDP}}\cdot\frac{S_{ABD}}{S_{ABP}}+V_{ABCP}\cdot\frac{S_{ACD}}{S_{CDP}})\cdot\frac{S_{ABD}}{S_{ABP}}\cdot\frac{S_{ACD}}{S_{CDP}}}$$

$$\stackrel{simplify}{=}\frac{(\frac{S_{ACD}}{S_{CDP}}-1)\cdot V_{BCDP}}{(V_{BCDP}\cdot\frac{S_{ACD}}{S_{CDP}}-V_{BCDP}-V_{ABCP}\cdot\frac{S_{ABD}}{S_{ABP}}+V_{ABCP})\cdot\frac{S_{ABD}}{S_{ABP}}}$$

$$\stackrel{P}{=}\frac{(r)^3\cdot(-V_{ABCD})\cdot(r+1)^2}{(-V_{ABCD}\cdot r^3-2V_{ABCD}\cdot r^2-V_{ABCD}\cdot r)\cdot(r+1)^2}$$

$$\stackrel{simplify}{=}\frac{(r)^2}{(r+1)^2}$$

Similarly, we can compute

$$\frac{V_{APRQ}}{V_{ABCD}}=\frac{(r)^2}{(r+1)^3},\frac{V_{APSR}}{V_{ABCD}}=\frac{(r)^2}{(r+1)^3}.$$

Thus

$$V_1=(\frac{(r)^2}{(r+1)^2}+\frac{(r)^2}{(r+1)^3}+\frac{(r)^2}{(r+1)^3})V_{ABCD}=\frac{(r)^2(r+3)}{(r+1)^3}V_{ABCD}.$$

Let $V_2=V_{ABCD}-V_1=\frac{3r+1}{(r+1)^3}V_{ABCD}$. Finally we have $\frac{V_1}{V_2}=\frac{r^2(r+3)}{3r+1}$. ∎

Example 4.92 (Monge's Theorem) *The six planes through the midpoints of the edges of a tetrahedron and perpendicular to the edges respectively opposite have a point in common. This point is called the Monge point of the tetrahedron.*

Constructive description
((POINTS A B C D)
(MIDPOINT L A B)
(FOOT2LINE L_1 L C D)
(FOOT2LINE A_1 A C D)
(PRATIO L_2 L_1 A_1 A 1)
(MIDPOINT R A C)
(FOOT2LINE R_1 R B D)
(FOOT2LINE A_2 A B D)
(PRATIO R_2 R_1 A_2 A 1)
(INTER P (LINE L L_1) (PLANE R R_1 R_2)) (INTER Q (LINE L L_2) (PLANE R R_1 R_2))
(MIDPOINT S B C) (FOOT2LINE S_1 S A D) (FOOT2LINE B_1 B A D) (PRATIO S_2 S_1 B_1 B 1)
(INTER M (LINE P Q) (PLANE S S_1 S_2)) (MIDPOINT N C D) (PERPENDICULAR N M A B))

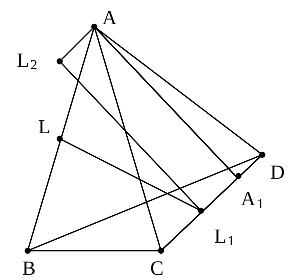


Figure 4-38

The proof for this theorem is too long to print here. We need more elimination techniques to produce short and readable proofs for problems like this one.

4.6 Volume Coordinate System

In Lemma 4.85, we use an orthogonal coordinate system, which is essentially the same as the usual Cartesian coordinate system. In order to do this, we have to introduce four auxiliary points O, W, U, and V. In this section, we will develop some properties for a *skew volume coordinate system* in which any four free points can be selected as the reference points. As a consequence, we obtain a new proof of Lemma 4.85 and hence a new version of Algorithm 4.86.

Let O, W, U, and V be four non-coplanar points. Then for any point A, we will denote its volume coordinates with respect to $OWUV$ as

$$x_A = \frac{V_{OWUA}}{V_{OWUV}}, \quad y_A = \frac{V_{OWAV}}{V_{OWUV}}, \quad z_A = \frac{V_{OAUV}}{V_{OWUV}}, \quad w_A = \frac{V_{AWUV}}{V_{OWUV}}.$$

It is clear that $x_A + y_A + z_A + w_A = 1$. Following are some known results.

Proposition 4.93 *The points in the space are in a one to one correspondence with the four-tuples (x, y, z, w) such that $x + y + z + w = 1$.*

Proposition 4.94 *For any points A, B, C, and D, we have*

$$V_{ABCD} = V_{OWUV}\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}.$$

As a consequence of Proposition 4.94, we can give the equation for planes in the volume coordinate system. Let P be a point in plane ABC. Then the volume coordinates of P must satisfy

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_P & y_P & z_P & 1 \end{vmatrix} = 0$$

which is the equation for the plane ABC.

The position ratio formula is still true

Proposition 4.95 *Let R be a point on line PQ and $r_1 = \frac{\overline{PR}}{\overline{PQ}}$ and $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ be the position ration of R with respect to PQ. Then*

$$x_R = r_1 x_Q + r_2 x_P; \quad y_R = r_1 y_Q + r_2 y_P; \quad z_R = r_1 z_Q + r_2 z_P; \quad w_R = r_1 w_Q + r_2 w_P.$$

Proof. This is a consequence of Proposition 4.5. ∎

We now develop the formula for the square distance between two points.

Proposition 4.96 *Let $OXZ_1Y - ZY_1O_1X_1$ be a parallelepiped. We have*

$$\overline{OO_1}^2 = \overline{OX}^2 + \overline{OY}^2 + \overline{OZ}^2 + P_{XOY} + P_{XOZ} + P_{YOZ}.$$

Proof. By Proposition 3.6,

$$\begin{aligned} &\overline{OX_1}^2 = 2\overline{OZ}^2 + 2\overline{OY}^2 - \overline{ZY}^2. \\ &\overline{OY_1}^2 = 2\overline{OX}^2 + 2\overline{OZ}^2 - \overline{XZ}^2. \\ &\overline{OZ_1}^2 = 2\overline{OX}^2 + 2\overline{OY}^2 - \overline{XY}^2. \qquad (1) \\ &\overline{XX_1}^2 + \overline{YY_1}^2 = 2\overline{OZ}^2 + 2\overline{XY}^2. \\ &\overline{OO_1}^2 + \overline{ZZ_1}^2 = 2\overline{OZ}^2 + 2\overline{OZ_1}^2 \\ &\quad = 4\overline{OX}^2 + 4\overline{OY}^2 + 2\overline{OZ}^2 - 2\overline{XY}^2. \end{aligned}$$

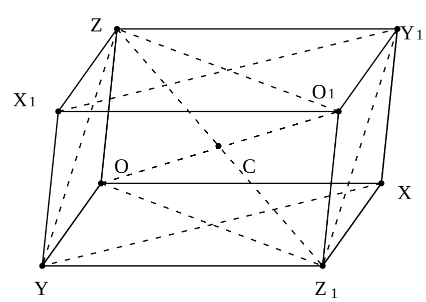


Figure 4-39

Let C be the center of the parallelepiped. By Proposition 4.66, $\overline{OO_1}^2 = 4\overline{OC}^2 = 4(\frac{1}{2}\overline{OY}^2 + \frac{1}{2}\overline{OY_1}^2 - \frac{1}{4}\overline{YY1}^2)$. Then

$$\overline{OO_1}^2 + \overline{YY_1}^2 = 4\overline{OX}^2 + 2\overline{OY}^2 + 4\overline{OZ}^2 - 2\overline{XY}^2. \tag{2}$$

Similarly, we have

$$\overline{OO_1}^2 + \overline{XX_1}^2 = 2\overline{OX}^2 + 4\overline{OY}^2 + 4\overline{OZ}^2 - 2\overline{ZY}^2. \tag{3}$$

Solving the linear system (1), (2), (3), we have

$$\begin{aligned}\overline{OO_1}^2 &= 3\overline{OX}^2 + 3\overline{OY}^2 + 3\overline{OZ}^2 - \overline{ZY}^2 - \overline{ZX}^2 - \overline{XY}^2\\ &= \overline{OX}^2 + \overline{OY}^2 + \overline{OZ}^2 + P_{XOY} + P_{XOZ} + P_{YOZ}. \quad ∎\end{aligned}$$

Proposition 4.97 *Let O, W, U, and V be four free points. Then*

$$\begin{aligned}\overline{AB}^2 &= \overline{OU}^2(x_B - x_A)^2 + \overline{OV}^2(y_B - y_A)^2 + \overline{OW}^2(z_B - z_A)^2 +\\ &\quad (y_B - y_A)(x_B - x_A)P_{UOV} + (z_B - z_A)(y_B - y_A)P_{WOV} +\\ &\quad (z_B - z_A)(x_B - x_A)P_{WOU}.\end{aligned}$$

Proof. We form a parallelepiped $AMLN - RPBQ$ such that $AR \parallel OW$, $AM \parallel OU$, and $AN \parallel OV$. By Proposition 4.96,

$$\overline{AB}^2 = \overline{AM}^2 + \overline{AN}^2 + \overline{AR}^2 + P_{RAN} + P_{RAM} + P_{NAM}.$$

By Lemma 4.49,

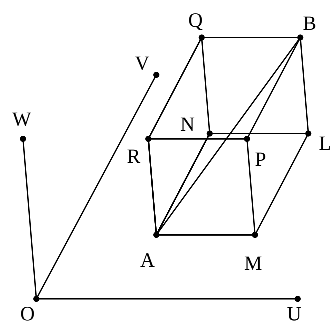


Figure 4-40

$$\overline{AR}^2 = \overline{OW}^2(\frac{\overline{AR}}{\overline{OW}})^2 = \overline{OW}^2(\frac{V_{BOUVA}}{V_{WOUV}})^2 = \overline{OW}^2(z_B - z_A)^2.$$

$\overline{AN}^2$ and $\overline{AM}^2$ can be computed similarly. By Proposition 3.9,

$$P_{NAM} = \frac{\overline{AM}}{\overline{OU}}\frac{\overline{AN}}{\overline{OV}}P_{UOV} = (y_B - y_A)(x_B - x_A)P_{UOV}.$$

We can compute P_{RAN} and P_{RAM} similarly. ∎

Once again with Algorithm 4.86, let E be an expression in volumes and Pythagoras differences of free points. Instead of using Lemma 4.85, we can use the following procedure to transform E into an expression in independent variables: if there are fewer than four points occurring in E then we need do nothing. Otherwise choose four free points O, W, U, and V from the points occurring in E and apply Propositions 4.94 and 4.97 to E to transform the volumes and Pythagoras differences into volume coordinates with respect to $OWUV$. Now the new E is an expression in volume coordinates of free points, $\overline{OW}^2$, $\overline{OU}^2, \overline{OV}^2, \overline{UV}^2$, and V_{OWUV}. The only algebraic relation among these quantities is the Herron-Qin formula (Proposition 4.78). Substituting V^2_{OWUV} into E, we obtain an expression in independent variables.

Example 4.98 *The above process of transforming an expression in volumes and Pythagoras differences of free points into an expression in free parameters becomes very simple when there exist exactly four free points O, W, U, and V in a geometry statement. In this case, we first transform the square of the volume V_{OWUV} into Pythagoras differences using the Herron-Qin formula (Proposition 4.78), and then express the Pythagoras differences in terms of the six square-distances, $\overline{OW}^2, \overline{OU}^2, \overline{OV}^2, \overline{UV}^2, \overline{UW}^2$, and $\overline{WV}^2$, which form a set of free parameters for this geometry statement.*

Exercises 4.99

1. If $OV \perp OU$, $OV \perp OW$, $OW \perp OU$, and $\overline{OU}^2 = \overline{OV}^2 = \overline{OW}^2 = 1$, the volume coordinate system developed in this section becomes the standard Cartesian coordinate system. Prove the following formulas in the Cartesian coordinate system.
    1. $\overline{AB}^2 = (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2$.
    2. $P_{ABC} = 2((x_B - x_A)(x_B - x_C) + (y_B - y_A)(y_B - y_C) + (z_B - z_A)(z_B - z_C))$.
    3. $$V_{ABCD} = \frac{1}{6}\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}.$$

2. Prove the Herron-Qin formula for tetrahedra (see Propositions 4.78 and 4.79) using the three formulas in the preceding exercise. (First notice that
$$V_{ABCD} = -\frac{1}{6}\begin{vmatrix} x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \\ x_D - x_A & y_D - y_A & z_D - z_A \end{vmatrix}.$$
Let M be the matrix in the above formula. Then $V^2_{ABCD} = \frac{1}{36}|M \times M^*|$ where M^* is the transpose of M.)

Summary of Chapter 4

- Signed volumes and Pythagoras differences are used to describe some basic geometry relations in solid geometry: collinear, coplanar, parallel, perpendicular, and congruence of line segments.

1. Four points A, B, C, and D are coplanar iff $V_{ABCD} = 0$.
2. $PQ \parallel ABC$ iff $V_{PABC} = V_{QABC}$ or equivalently iff $V_{PABCQ} = 0$.
3. $PQR \parallel ABC$ iff $V_{PABC} = V_{QABC} = V_{RABC}$.
4. $PQ \perp AB$ iff $P_{PAQB} = P_{PAB} - P_{QAB} = 0$.

- We have the following formulas for the volumes of tetrahedra.
  1. $V_{ABCD} = \frac{1}{3} h_{A,BCD} |S_{BCD}|$ where $h_{A,BCD}$ is the signed altitude from point A to plane BCD.
  2. $$V_{ABCD} = V_{OWUV} \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}$$
     where x_A, y_A, z_A are the volume coordinates of point A with respect to points O, W, U, and V.
  3. (The Herron-Qin Formula)
     $$144V^2_{P_1P_2P_3P_4} = 4r^2_{12}r^2_{13}r^2_{14} - r^2_{12}P_{314} - r^2_{13}P_{214} - r^2_{14}P_{312} + P_{314}P_{214}P_{312}$$
     where $r_{ij} = |\overline{P_iP_j}|, P_{ijk} = P_{P_iP_jP_k}$.
  4. (The Cayley-Menger Formula)
     $$288V^2_{P_1P_2P_3P_4} = \begin{vmatrix} 0 & r^2_{12} & r^2_{13} & r^2_{14} & 1 \\ r^2_{21} & 0 & r^2_{23} & r^2_{24} & 1 \\ r^2_{31} & r^2_{32} & 0 & r^2_{34} & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

- The following basic propositions are the basis of the volume method.

1. (The Co-vertex Theorem) Let ABC and DEF be two proper triangles in the same plane and T be a point not in the plane. Then we have $\frac{V_{TABC}}{V_{TDEF}} = \frac{S_{ABC}}{S_{DEF}}$.
2. (The Co-face Theorem) A line PQ and a plane ABC meet at M. If $Q \neq M$, we have
   $$\frac{\overline{PM}}{\overline{QM}} = \frac{V_{PABC}}{V_{QABC}}; \frac{\overline{PM}}{\overline{PQ}} = \frac{V_{PABC}}{V_{PABCQ}}; \frac{\overline{QM}}{\overline{PQ}} = \frac{V_{QABC}}{V_{PABCQ}}.$$
3. Let R be a point on line PQ. Then for any points A, B, and C, we have
   $$V_{RABC} = \frac{\overline{PR}}{\overline{PQ}} V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}} V_{PABC}.$$

4. Let R be a point in the plane PQS. Then for any points A, B, and C we have

$$V_{RABC} = \frac{S_{PQR}}{S_{PQS}} V_{SABC} + \frac{S_{RQS}}{S_{PQS}} V_{PABC} + \frac{S_{PRS}}{S_{PQS}} V_{QABC}.$$

5. Let $PQTS$ be a parallelogram. Then for points A, B, and C, we have $V_{PABC} + V_{TABC} = V_{QABC} + V_{SABC}$, or $V_{PABCQ} = V_{SABCT}$.
6. Let triangle ABC be a parallel translation of triangle DEF. Then for points P and Q we have $V_{PABC} = V_{PDEFA}$ and $V_{PABCQ} = V_{PDEFQ}$.

Also notice that all the propositions on page 168 about Pythagoras differences are also valid if the points involved are in the space.

- We present a mechanical proving method which can produce short and readable proofs for many constructive geometry statements in the space.

Chapter 5

Vectors and Machine Proofs

In Section 2.6, we mentioned that there are two approaches to defining geometries: the geometric approach and the algebraic approach. In this chapter, we will show how to prove geometry theorems automatically in a geometry that is defined using the algebraic approach. The modern source for affine and metric geometries based on the algebraic approach is *linear algebra* or the *theory of vector spaces* (see [5, 16, 33]). In this approach, the metric is introduced by the inner product of vectors, while the areas and volumes are represented by the exterior product of vectors. It is interesting to note that two of the most important concepts in this linear algebra approach to geometry, the *inner* and *exterior products* of vectors, are essentially the same as the two basic geometry quantities used by us: the *Pythagoras difference* and the *area (or volume)*. This strongly suggests that the method based on areas, volumes, and Pythagoras differences can be stated in the language of vectors. Moreover, the vector approach based on inner and exterior products has the advantage that it is easy to develop; it uses more geometry quantities such as the vector itself; and it covers more geometries such as the Minkowskian geometry. But on the other hand, the vector approach needs more algebraic prerequisites. Also, the proofs produced by the vector approach generally do not have such clear geometric meaning as do those produced by the volume-Pythagoras difference approach.

5.1 Metric Vector Spaces of Dimension Three

Let $\mathcal{E}$ be a *field* with characteristic different from two. A *vector space* over $\mathcal{E}$ is a set $\boldsymbol{V}$ with two structures:

- $\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}$, denoted by $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow \boldsymbol{x} + \boldsymbol{y}$ and
- $\mathcal{E} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}$, denoted by $(\alpha, \boldsymbol{x}) \longrightarrow \alpha \boldsymbol{x}$

which satisfy the following properties:

V1 $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ (commutative law)

V2 $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ (associate law)

V3 There exists a zero-element $\boldsymbol{o}$ such that $\boldsymbol{x}+\boldsymbol{o}=\boldsymbol{x}$ for every $\boldsymbol{x}\in E$. $\boldsymbol{o}$ is called the *origin* of $\boldsymbol{V}$.

V4 To every element $\boldsymbol{x}$ there exists an *inverse element:* $-\boldsymbol{x}$ such that $\boldsymbol{x}+(-\boldsymbol{x})=\boldsymbol{o}$.

V5 $(\alpha\beta)\boldsymbol{x}=\alpha(\beta\boldsymbol{x})$

V6 $(\alpha+\beta)\boldsymbol{x}=\alpha\boldsymbol{x}+\beta\boldsymbol{x}$; $\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ (distributive laws)

V7 $1\cdot\boldsymbol{x}=\boldsymbol{x}$.

The elements of $\boldsymbol{V}$ are called *vectors* and are denoted by $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\cdots$; the elements of $\mathcal{E}$ are called *scalars* and are denoted by $\alpha,\beta,a,b,\ldots$. Scalars are always written on the left of the vectors.

The vector space $\boldsymbol{V}$ is called n-dimensional if there exist n elements $\boldsymbol{e_1},\cdots,\boldsymbol{e_n}$ in $\boldsymbol{V}$ such that

- for any $\boldsymbol{x}\in\boldsymbol{V}$ there exist scalars $\alpha_1,\cdots,\alpha_n$ such that $\boldsymbol{x}=\alpha_1\boldsymbol{e_1}+\cdots+\alpha_n\boldsymbol{e_n}$ and
- if $\alpha_1\boldsymbol{e_1}+\cdots+\alpha_n\boldsymbol{e_n}=0$ then $\alpha_i=0, i=1,\cdots,n$.

The n elements $\boldsymbol{e_1},\cdots,\boldsymbol{e_n}$ satisfying the above property form a *basis* for the vector space $\boldsymbol{V}$. If $\boldsymbol{x}=\alpha_1\boldsymbol{e_1}+\cdots+\alpha_n\boldsymbol{e_n}$, we say that $\boldsymbol{x}$ is a linear combination of the vectors $\boldsymbol{e_1},\ldots,\boldsymbol{e_n}$ and $(\alpha_1,\ldots,\alpha_n)$ are the *coordinates* of $\boldsymbol{x}$ with respect to the basis $\boldsymbol{e_1},\cdots,\boldsymbol{e_n}$.

Given an ordered basis for $\boldsymbol{V}$, we can associate with each vector $\boldsymbol{x}$ the unique n-tuple of coordinates $(\alpha_1,\ldots,\alpha_n)$ such that $\boldsymbol{x}=\alpha_1\boldsymbol{e_1}+\cdots+\alpha_n\boldsymbol{e_n}$. This establishes a one-to-one correspondence between the vectors in $\boldsymbol{V}$ and the elements in $\mathcal{E}^n$ which is the Cartesian product of $\mathcal{E}$ with itself n times. It is easy to show that this correspondence preserves the structure of the vector spaces, i.e., it is an isomorphism between vector spaces. So we may safely assume that $\boldsymbol{V}$ is actually $\mathcal{E}^n$.

A nonempty subset $\boldsymbol{W}$ of a vector space $\boldsymbol{V}$ is called a *subspace* of $\boldsymbol{V}$ if the following conditions hold:

1. If $\boldsymbol{x}\in\boldsymbol{W}$ and $\boldsymbol{y}\in\boldsymbol{W}$, then $\boldsymbol{x}+\boldsymbol{y}\in\boldsymbol{W}$.
2. If $\boldsymbol{x}\in\boldsymbol{W}$, then $\alpha\boldsymbol{x}\in\boldsymbol{W}$ for $\alpha\in\mathcal{E}$.

Let $\boldsymbol{f_1},\cdots,\boldsymbol{f_m}$ be vectors in $\boldsymbol{V}$. Then the set of all the vectors like

$$\sum_{i=1}^{m}\alpha_i\boldsymbol{f_i},\quad \alpha_i\in\mathcal{E}$$

is a subspace of $\boldsymbol{V}$ and is called the subspace generated by vectors $\boldsymbol{f_1}, \cdots, \boldsymbol{f_m}$.

This chapter uses some basic knowledge from linear algebra, e.g., the first five chapters of [22].

5.1.1 Inner Products and Metric Vector Space

In what follows, we assume that $n = 3$ and an ordered basis for $\boldsymbol{V}$ has been given. For $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$, and $\alpha \in \mathcal{E}$, we thus have

$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, \alpha x_3) \quad \text{and} \quad \boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

Definition 5.1 *An inner product on* $\boldsymbol{V}$ *is a map*

$\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \mathcal{E}$, *denoted by* $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle$

which satisfies the following properties

I1 $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$.

I2 $\langle \alpha\boldsymbol{x} + \beta\boldsymbol{y}, \boldsymbol{z} \rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{z} \rangle + \beta\langle \boldsymbol{y}, \boldsymbol{z} \rangle$ *where* α *and* β *are scalars.*

Proposition 5.2 *Let* $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ *be a basis of* $\boldsymbol{V}$, $\boldsymbol{x} = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3}$, *and* $\boldsymbol{y} = y_1\boldsymbol{y_1} + y_2\boldsymbol{y_2} + y_3\boldsymbol{y_3}$. *Show that*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = (x_1, x_2, x_3)\mathcal{M}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $\mathcal{M} = (\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle)$ *is a symmetric matrix.*

Proof.

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i,j=1}^{3} x_i y_j \langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = (x_1, x_2, x_3)\mathcal{M}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $\mathcal{M} = (\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle)$ is a symmetric matrix. ∎

Definition 5.3 *A vector space with an inner product is called a metric vector space.*

Definition 5.4 *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are perpendicular if* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

A metric vector space is called *nonsingular* if its origin is the only vector which is orthogonal to all vectors.

Proposition 5.5 *Metric vector space* $\boldsymbol{V}$ *is nonsingular if and only if* $|\mathcal{M}| = det(\mathcal{M}) \neq 0$.

Proof. If $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = (x_1, x_2, x_3)\mathcal{M}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Then $\boldsymbol{x}$ is perpendicular to all vectors in $\boldsymbol{V}$ iff

$$(x_1, x_2, x_3)\mathcal{M}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

for all possible y_i, i.e., iff

$$(x_1, x_2, x_3)\mathcal{M} = (0, 0, 0).$$

The above linear system has nonzero solution iff $|\mathcal{M}| = 0$. ▮

A vector $\boldsymbol{x}$ is called *isotropic* if $\boldsymbol{x}$ is perpendicular to itself, or equivalently if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$.

The origin $\boldsymbol{o}$ is always isotropic. Even if $\boldsymbol{V}$ is nonsingular, there may be many nonzero isotropic vectors. This is seen by observing that a vector is isotropic iff its coordinates satisfy the equation

$$\sum_{i,j=1}^{n} m_{i,j} x_i x_j = 0$$

where $\mathcal{M} = (m_{i,j})$. The solutions for the above equation (if exist) consist of a cone which is referred as the *light cone*. The word light cone is originated from physics. More physics background can be found in [38].

For $\boldsymbol{x} = (x_1, x_2, x_3)$, let the *square* of $\boldsymbol{x}$ be

$$\boldsymbol{x}^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle.$$

Suppose that $\boldsymbol{V}$ is a metric vector space, but that we only know the value of $\boldsymbol{x}^2$ for each vector $\boldsymbol{x} \in \boldsymbol{V}$. Can we compute the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$? The answer is affirmative.

Proposition 5.6 *In a metric vector space, the square function* $\boldsymbol{x}^2$ *determines the inner product completely.*

Proof. For $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, by I1 and I2,

$$(\boldsymbol{x} + \boldsymbol{y})^2 = \boldsymbol{x}^2 + 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle + \boldsymbol{y}^2.$$

Since $\mathcal{E}$ is not of characteristic 2, we have

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \frac{1}{2}(\boldsymbol{x}^2 + \boldsymbol{y}^2 - (\boldsymbol{x} + \boldsymbol{y})^2). \quad ▮$$

Corollary 5.7 (Pythagorean Theorem) $\boldsymbol{x} \perp \boldsymbol{y}$ *iff* $\boldsymbol{x}^2 + \boldsymbol{y}^2 - (\boldsymbol{x} + \boldsymbol{y})^2 = 0$.

Definition 5.8 *A coordinate system* $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ *of* $\boldsymbol{V}$ *is called a rectangular coordinate system if* $\boldsymbol{e_i} \perp \boldsymbol{e_j}$ *for* $j \neq i$.

For a rectangular basis $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$, the matrix defining the inner product is diagonal, i.e.,

$$\mathcal{M} = \begin{pmatrix} \langle \boldsymbol{e_1}, \boldsymbol{e_1} \rangle & & \\ & \langle \boldsymbol{e_2}, \boldsymbol{e_2} \rangle & \\ & & \langle \boldsymbol{e_3}, \boldsymbol{e_3} \rangle \end{pmatrix}.$$

For $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, we have

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{e_1}, \boldsymbol{e_1} \rangle x_1 y_1 + \langle \boldsymbol{e_2}, \boldsymbol{e_2} \rangle x_2 y_2 + \langle \boldsymbol{e_3}, \boldsymbol{e_3} \rangle x_3 y_3.$$

Proposition 5.9 *A metric vector space* $\boldsymbol{V}$ *always has a rectangular coordinate system.*

Proof. We prove the proposition using induction on the dimension of $\boldsymbol{V}$. If $\boldsymbol{V}$ is of dimension one, any basis of it is a rectangular basis. Suppose that the result is true for all vector spaces with dimension less than n. Let $\boldsymbol{V}$ be a vector space of dimension n. If all the vectors in $\boldsymbol{V}$ are isotropic, by the Pythagorean theorem any basis is a rectangular basis. Otherwise, let $\boldsymbol{e_1}, \cdots, \boldsymbol{e_n}$ be a basis of $\boldsymbol{V}$ such that $\boldsymbol{e_1}$ is a non-isotropic vector. Let

$$\boldsymbol{e}'_{\boldsymbol{i}} = \boldsymbol{e_i} - \frac{\langle \boldsymbol{e_i}, \boldsymbol{e_1} \rangle}{\boldsymbol{e_1}^2} \boldsymbol{e_1}, i = 2, \cdots, n.$$

Then it is clear that $\boldsymbol{e_1}, \boldsymbol{e}'_{\boldsymbol{2}}, \cdots, \boldsymbol{e}'_{\boldsymbol{n}}$ are also a basis of $\boldsymbol{V}$ and $\boldsymbol{e_1} \perp \boldsymbol{e}'_{\boldsymbol{i}}, i = 2, \cdots, n$. By the induction hypothesis, the vector space generated by $\boldsymbol{e}'_{\boldsymbol{i}}, i = 2, \cdots, n$, has a rectangular basis $\boldsymbol{f_i}, i = 2, \cdots, n$. Then it is easy to check that $\boldsymbol{e_1}, \boldsymbol{f_2}, \cdots, \boldsymbol{f_n}$ form a rectangular basis for $\boldsymbol{V}$. ▮

Exercises 5.10

1. If $\mathcal{E}$ is the field of real numbers and the inner product of $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ is defined to be

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

   the resulting space is the *Euclidean space of dimension three*. Show that the Euclidean space as defined above is a nonsingular metric vector space satisfying

   I3 $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ iff $\boldsymbol{x} = (0, 0, 0)$.

2. The n-dimensional ($n \leq 3$) *Minkowskian space* is a metric vector space whose inner product for $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \ldots + x_{n-1} y_{n-1} - x_n y_n.$$

   Show that the Minkowskian space is a nonsingular metric vector space in which there exist nonzero isotropic vectors.

3. Show that if $\mathcal{E}$ is the field of real numbers and $n = 3$, every nonsingular metric vector space has a coordinate system such that its matrix is one of the following form.

$$\begin{aligned}\mathcal{M}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \mathcal{M}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathcal{M}_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.\end{aligned}$$

   The matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ determine, respectively, the Euclidean and the Minkowskian spaces. We call the geometries determined by $\mathcal{M}_4$ and $\mathcal{M}_3$ the *negative Euclidean space* and the *negative Minkowskian space* respectively.

4. Let $(\boldsymbol{e_1}, ..., \boldsymbol{e_n})$ and $(\boldsymbol{f_1}, ..., \boldsymbol{f_n})$ be two different bases for $\boldsymbol{V}$. Then there is a nonsingular matrix $\mathcal{P}$ such that

$$(\boldsymbol{f_1}, ..., \boldsymbol{f_n}) = (\boldsymbol{e_1}, ..., \boldsymbol{e_n})\mathcal{P}.$$

   Let $\mathcal{M}$ and $\mathcal{M}'$ be the matrices of the inner products corresponding to the bases $(\boldsymbol{e_1}, ..., \boldsymbol{e_n})$ and $(\boldsymbol{f_1}, ..., \boldsymbol{f_n})$. Show that

$$\mathcal{M}' = \mathcal{P}^*\mathcal{M}\mathcal{P}$$

   where $\mathcal{P}^*$ is the transpose of $\mathcal{P}$. The two matrices in the above proposition are called *congruent*.

   Thus two matrices are congruent iff they represent the same metric of $\boldsymbol{V}$ relative to different coordinates systems. Therefore, the study of the metric vector space is equivalent to the study of the symmetric matrices under the equivalent relation of congruence.

5. In the algebraic language, Proposition 5.9 is equivalent to the following fact. Let $\mathcal{G}$ be an $n \times n$ symmetric matrix. Then there exists an $n \times n$ nonsingular matrix $\mathcal{P}$ such that $\mathcal{P}^*G\mathcal{P}$ is a diagonal matrix. Prove the above fact directly.

5.1.2 Exterior Products in Metric Vector Space

In what follows, we always assume that $\boldsymbol{V}$ is a nonsingular metric vector space with a rectangular basis $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$. Thus the matrix that defines the inner product is

$$\mathcal{M} = \begin{pmatrix} \langle \boldsymbol{e_1}, \boldsymbol{e_1}\rangle & & \\ & \langle \boldsymbol{e_2}, \boldsymbol{e_2}\rangle & \\ & & \langle \boldsymbol{e_3}, \boldsymbol{e_3}\rangle \end{pmatrix}.$$

Definition 5.11 *An exterior product on* $\boldsymbol{V}$ *is a map*

$\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}$, *denoted by* $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow [\boldsymbol{x}, \boldsymbol{y}]$

which satisfies the following properties

E1 $[\boldsymbol{x}, \boldsymbol{y}] = -[\boldsymbol{y}, \boldsymbol{x}]$.

E2 $[\alpha\boldsymbol{x} + \beta\boldsymbol{y}, \boldsymbol{z}] = \alpha[\boldsymbol{x}, \boldsymbol{z}] + \beta[\boldsymbol{y}, \boldsymbol{z}]$ *where* α *and* β *are scalars.*

E3 $\boldsymbol{x} \perp [\boldsymbol{x}, \boldsymbol{y}]$.

Note that property E3 is not in the definition for the exterior product in the general case. We add it to make the relation between the inner and exterior products simple.

From E1 and the fact that $\mathcal{E}$ is not of characteristic two, we have

$$[\boldsymbol{x}, \boldsymbol{x}] = 0.$$

Proposition 5.12 *Let* $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ *be a rectangular basis of a nonsingular metric vector space* $\boldsymbol{V}$. *Then*

$$[\boldsymbol{e_1}, \boldsymbol{e_2}] = \frac{\alpha}{\langle \boldsymbol{e_3}, \boldsymbol{e_3}\rangle}\boldsymbol{e_3}, [\boldsymbol{e_2}, \boldsymbol{e_3}] = \frac{\alpha}{\langle \boldsymbol{e_1}, \boldsymbol{e_1}\rangle}\boldsymbol{e_1}, [\boldsymbol{e_3}, \boldsymbol{e_1}] = \frac{\alpha}{\langle \boldsymbol{e_2}, \boldsymbol{e_2}\rangle}\boldsymbol{e_2}$$

where $\alpha = \langle \boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle = \langle \boldsymbol{e_2}, [\boldsymbol{e_3}, \boldsymbol{e_1}]\rangle = \langle \boldsymbol{e_3}, [\boldsymbol{e_1}, \boldsymbol{e_2}]\rangle$.

Proof. Since $[\boldsymbol{e_1}, \boldsymbol{e_2}] \perp e_1$ and $[\boldsymbol{e_1}, \boldsymbol{e_2}] \perp e_2$, it is clear that $[\boldsymbol{e_1}, \boldsymbol{e_2}] = s_1\boldsymbol{e_3}$. Thus $s_1 = \frac{\langle \boldsymbol{e_3}, [\boldsymbol{e_1}, \boldsymbol{e_2}]\rangle}{\langle \boldsymbol{e_3}, \boldsymbol{e_3}\rangle}$. Similarly

$$[\boldsymbol{e_2}, \boldsymbol{e_3}] = s_2\boldsymbol{e_1}, [\boldsymbol{e_3}, \boldsymbol{e_1}] = s_3\boldsymbol{e_2}$$

where $s_2 = \frac{\langle \boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle}{\langle \boldsymbol{e_1}, \boldsymbol{e_1}\rangle}$, $s_3 = \frac{\langle \boldsymbol{e_2}, [\boldsymbol{e_3}, \boldsymbol{e_1}]\rangle}{\langle \boldsymbol{e_2}, \boldsymbol{e_2}\rangle}$. Adding the above two equations, we have

$$[\boldsymbol{e_2} - \boldsymbol{e_1}, \boldsymbol{e_3}] = s_2\boldsymbol{e_1} + s_3\boldsymbol{e_2}.$$

Taking the exterior products of $\boldsymbol{e_2} - \boldsymbol{e_1}$ and the vectors on both sides of the above equation, we have $s_3\langle \boldsymbol{e_2}, \boldsymbol{e_2}\rangle = s_2\langle \boldsymbol{e_1}, \boldsymbol{e_1}\rangle$, i.e., $\langle \boldsymbol{e_2}, [\boldsymbol{e_3}, \boldsymbol{e_1}]\rangle = \langle \boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle$. Similarly, we can prove that $\langle \boldsymbol{e_3}, [\boldsymbol{e_1}, \boldsymbol{e_2}]\rangle = \langle \boldsymbol{e_2}, [\boldsymbol{e_3}, \boldsymbol{e_1}]\rangle$. ∎

Remark 5.13 *The constant* $\alpha = \langle \boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle$ *is a basic quantity related to the exterior product. We always assume that* $\alpha \neq 0$.

Proposition 5.14 *Let* $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ *be a rectangular basis of a nonsingular metric vector space* $\boldsymbol{V}$, $\boldsymbol{x} = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3}$, *and* $\boldsymbol{y} = y_1\boldsymbol{e_1} + y_2\boldsymbol{e_2} + y_3\boldsymbol{e_3}$. *Then*

$$[\boldsymbol{x}, \boldsymbol{y}] = \alpha\left(\frac{1}{m_1}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \frac{1}{m_2}\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \frac{1}{m_3}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right)$$

where $\alpha = \langle \boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle$, $m_1 = \langle \boldsymbol{e_1}, \boldsymbol{e_1}\rangle$, $m_2 = \langle \boldsymbol{e_2}, \boldsymbol{e_2}\rangle$, *and* $m_3 = \langle \boldsymbol{e_3}, \boldsymbol{e_3}\rangle$.

Proof. By E1 and E2,

$$[\boldsymbol{x}, \boldsymbol{y}] = \sum_{i,j=1}^{3} x_iy_j[\boldsymbol{e_i}, \boldsymbol{e_j}] = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}[\boldsymbol{e_2}, \boldsymbol{e_3}] + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}[\boldsymbol{e_3}, \boldsymbol{e_1}] + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}[\boldsymbol{e_1}, \boldsymbol{e_2}].$$

Now the result follows immediately from Proposition 5.12. ∎

Proposition 5.15 *If the metric vector space is not singular, then* $\boldsymbol{x} = \alpha\boldsymbol{y}$ *iff* $[\boldsymbol{x}, \boldsymbol{y}] = 0$.

Proof. By Proposition 5.9, we can chose a rectangular basis for $\boldsymbol{V}$. If $\boldsymbol{x} = \alpha\boldsymbol{y}$ then $[\boldsymbol{x}, \boldsymbol{y}] = \alpha[\boldsymbol{y}, \boldsymbol{y}] = 0$. Conversely, let us assume $[\boldsymbol{x}, \boldsymbol{y}] = 0$. By Proposition 5.14,we have

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \lambda$$

for a scalar λ. Thus $\boldsymbol{x} = \lambda\boldsymbol{y}$. ∎

Definition 5.16 *The triple scalar product for three vectors* $\boldsymbol{x}, \boldsymbol{y}$, *and* $\boldsymbol{z}$ *in* $\boldsymbol{V}$ *is defined as follows.*

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \langle[\boldsymbol{x}, \boldsymbol{y}], \boldsymbol{z}\rangle.$$

Proposition 5.17 *Let* $\boldsymbol{x} = (x_1, x_2, x_3)$,$\boldsymbol{y} = (y_1, y_2, y_3)$, *and* $\boldsymbol{z} = (z_1, z_2, z_3)$. *We have*

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \langle\boldsymbol{e_1}, [\boldsymbol{e_2}, \boldsymbol{e_3}]\rangle \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Proof. Since $\boldsymbol{V}$ has a rectangular basis, this is a direct consequence of Proposition 5.14. ∎

We thus have

T1 $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}) = (\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y}) = -(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}) = -(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) = -(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{z})$.

T2 In a nonsingular metric space, $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0$ iff vectors $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ are coplanar, i.e., iff there exist scalars $\alpha_1, \alpha_2, \alpha_3$ not all zero such that $\alpha_1\boldsymbol{x} + \alpha_2\boldsymbol{y} + \alpha_3\boldsymbol{z} = 0$.

Proposition 5.18

- *(The Lagrange Identity)* $\langle[\boldsymbol{x}, \boldsymbol{y}], [\boldsymbol{u}, \boldsymbol{v}]\rangle = \alpha(\langle\boldsymbol{x}, \boldsymbol{u}\rangle\langle\boldsymbol{y}, \boldsymbol{v}\rangle - \langle\boldsymbol{x}, \boldsymbol{v}\rangle\langle\boldsymbol{y}, \boldsymbol{u}\rangle)$.
- $[[\boldsymbol{x}, \boldsymbol{y}], \boldsymbol{z}] = \alpha(\langle\boldsymbol{x}, \boldsymbol{z}\rangle\boldsymbol{y} - \langle\boldsymbol{y}, \boldsymbol{z}\rangle\boldsymbol{x})$

where $\alpha = \frac{\langle\boldsymbol{e_1},[\boldsymbol{e_2},\boldsymbol{e_3}]\rangle^2}{\langle\boldsymbol{e_1},\boldsymbol{e_1}\rangle\langle\boldsymbol{e_2},\boldsymbol{e_2}\rangle\langle\boldsymbol{e_3},\boldsymbol{e_3}\rangle}$.

Proof. The two formulas can be obtained by direct computation. We leave them as exercises. ∎

Exercise 5.19 *Show that* $[[\boldsymbol{r_1}, \boldsymbol{r_2}], [\boldsymbol{r_3}, \boldsymbol{r_4}]] = \alpha(\langle\boldsymbol{r_4}, [\boldsymbol{r_1}, \boldsymbol{r_2}]\rangle\boldsymbol{r_3} - \langle\boldsymbol{r_3}, [\boldsymbol{r_1}, \boldsymbol{r_2}]\rangle\boldsymbol{r_4})$ *where* α *is the same as in Proposition 5.18.*

5.2 The Solid Metric Geometry

Let $\mathcal{E}$ be a field with characteristic different from two. The vector space $\mathcal{E}^3$ is also called the *affine space* associated with field $\mathcal{E}$.

Definition 5.20 *A non-singular metric vector space $\mathcal{E}^3$ is called a solid metric geometry.*

As usual, elements in $\mathcal{E}^3$ are called points. Let A and B be two points. Then the line passing through A and B is the set

$$\{\alpha A + \beta B | \quad \alpha + \beta = 1\}.$$

The plane passing through three points A, B, and C is the set

$$\{\alpha A + \beta B + \gamma C | \quad \alpha + \beta + \gamma = 1\}.$$

Two points A and B in $\mathcal{E}^3$ determine a new *vector*,

$$\overrightarrow{AB} = B - A$$

A being the *origin* and B being the *endpoint*. Thus two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equal if and only if $A + Q = P + B$. Let O be the origin of $\boldsymbol{V}$. For any point A, let $\overrightarrow{A} = \overrightarrow{OA}$. Thus we also have

$$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}.$$

It is easy to show that line AB can also be written as follows

$$\{A + \beta\overrightarrow{AB} | \quad \beta \in \mathcal{E}\}.$$

Line AB is called *isotropic* if $\overrightarrow{AB}^2 = 0$. Similarly, the plane ABC can be written as

$$\{A + \beta\overrightarrow{AB} + \gamma\overrightarrow{AC} | \quad \beta, \gamma \in \mathcal{E}\}.$$

Line AB is called *parallel* to line PQ if there is a scalar λ such that $\overrightarrow{AB} = \lambda\overrightarrow{PQ}$. If $\overrightarrow{AB} = \lambda\overrightarrow{PQ}$, we say that the ratio of the parallel line segments AB and PQ is λ, i.e.,

$$\frac{\overline{AB}}{\overline{PQ}} = \lambda.$$

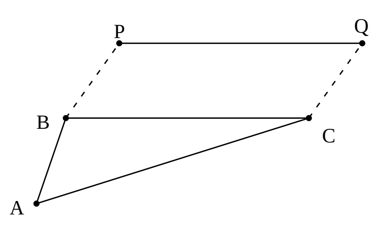


Figure 5-1

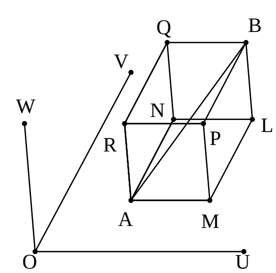


Figure 5-2

To see the geometric meaning of the addition of two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$. Let C be a point such that $\overrightarrow{BC} = \overrightarrow{PQ}$. Then (Figure 5-1)

$$\overrightarrow{AB} + \overrightarrow{PQ} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

We will now give a geometric interpretation of the coordinates of vectors with respect to a basis. Let O, W, U, and V be four points not in the same plane. For any vector $\overrightarrow{AB}$, we form a parallepiped $AMLN - RPBQ$ (Figure 5-2) such that $AR \parallel OW$, $AM \parallel OU$, and $AN \parallel OV$. Then

$$\overrightarrow{AB} = \overrightarrow{AR} + \overrightarrow{AM} + \overrightarrow{AN} = \frac{\overline{AR}}{\overline{OW}}\overrightarrow{OW} + \frac{\overline{AM}}{\overline{OU}}\overrightarrow{OU} + \frac{\overline{AN}}{\overline{OV}}\overrightarrow{OV},$$

i.e., $\overrightarrow{OW}, \overrightarrow{OU}$, and $\overrightarrow{OV}$ form a basis for $\mathcal{E}^3$ and the coordinates of $\overrightarrow{AB}$ with respect to this basis are $(\frac{\overline{AR}}{\overline{OW}}, \frac{\overline{AM}}{\overline{OU}}, \frac{\overline{AN}}{\overline{OV}})$.

Exercise 5.21 *Show that point Y is on line AB iff there is a scalar α such that $\overrightarrow{AY} = \alpha\overrightarrow{AB}$; Point Y is on plane LMN iff there are two scalars α and β such that $\overrightarrow{LY} = \alpha\overrightarrow{LM} + \beta\overrightarrow{LN}$.*

5.2.1 Inner Products and Exterior Products

The *inner product* of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ satisfies

1. $\langle\overrightarrow{AB}, \overrightarrow{CD}\rangle = 0$ if and only if $AB \perp CD$.
2. $\langle\overrightarrow{AB}, \overrightarrow{CD}\rangle = \langle\overrightarrow{CD}, \overrightarrow{AB}\rangle$,
3. $\langle\alpha\overrightarrow{A} + \beta\overrightarrow{B}, \overrightarrow{CD}\rangle = \alpha\langle\overrightarrow{A}, \overrightarrow{CD}\rangle + \beta\langle\overrightarrow{B}, \overrightarrow{CD}\rangle$ where α and β are scalars.

The *square distance* between two points A and B, or the square length of the vector $\overrightarrow{AB}$, is defined to be

$$AB^2 = \overrightarrow{AB}^2 = \langle\overrightarrow{AB}, \overrightarrow{AB}\rangle.$$

By Proposition 5.6,

$$2\langle\overrightarrow{AB}, \overrightarrow{BC}\rangle = AC^2 - AB^2 - BC^2 = -P_{ABC}.$$

Then it is easy to check that

$$P_{ABCD} = -2\langle\overrightarrow{AC}, \overrightarrow{BD}\rangle.$$

Proposition 5.22 (Pythagorean Theorem) *For any points A, B, C, and D*

- $AB \perp BC$ *iff* $AB^2 + BC^2 - AC^2 = 0$.
- $AB \perp CD$ *iff* $P_{ACBD} = AC^2 - CB^2 + BD^2 - AD^2 = 0$.

If four points A, B, C, and D are collinear or $AB \parallel CD$, then the *product of the oriented segments* is

$$\overline{AB} \cdot \overline{CD} = \langle \overrightarrow{AB}, \overrightarrow{CD} \rangle$$

and the *ratio of the oriented segments* is

$$\frac{\overline{AB}}{\overline{CD}} = \frac{\langle \overrightarrow{AB}, \overrightarrow{CD} \rangle}{\langle \overrightarrow{CD}, \overrightarrow{CD} \rangle}.$$

The *exterior product* $[\overrightarrow{AB}, \overrightarrow{CD}]$ of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ satisfies the following properties

1. $[\overrightarrow{AB}, \overrightarrow{CD}] = 0$ iff $AB \parallel CD$.
2. $[\overrightarrow{AB}, \overrightarrow{CD}] = -[\overrightarrow{CD}, \overrightarrow{AB}]$.
3. $[\alpha\overrightarrow{A} + \beta\overrightarrow{B}, \overrightarrow{CD}] = \alpha[\overrightarrow{A}, \overrightarrow{CD}] + \beta[\overrightarrow{B}, \overrightarrow{CD}]$ where α and β are scalars.

By Lagrange's identity,

$$[\overrightarrow{AB}, \overrightarrow{AC}]^2 = \alpha(AB^2 \cdot AC^2 - \langle \overrightarrow{AB}, \overrightarrow{AC} \rangle^2) = \frac{\alpha}{4}(4AB^2 \cdot AC^2 - P_{BAC}^2)$$

where α is the constant defined in Proposition 5.18. Comparing with the Herron-Qin formula on page 108, we see that the length of $[\overrightarrow{AB}, \overrightarrow{AC}]$ is proportional to the area of triangle ABC.

Remark 5.23 *We can determine the exact relation between the area and the exterior product as follows. In Euclidean geometry,* $\alpha = 1$. *By the Herron-Qin formula,*

$$[\overrightarrow{AB}, \overrightarrow{AC}]^2 = 4S_{ABC}^2.$$

Thus $[\overrightarrow{AB}, \overrightarrow{AC}]$ *is a vector* $\overrightarrow{AD}$ *such that* $\overrightarrow{AD}$ *is perpendicular to the plane* ABC*, and pointed in such direction as to make* $(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$ *a right handed triple and* $|AD| = 2|S_{ABC}|$.

We thus define the *signed area of triangle* ABC to be a quantity with the same sign of $[\overrightarrow{AB}, \overrightarrow{AC}]$ and $S_{ABC}^2 = \frac{1}{4}[\overrightarrow{AB}, \overrightarrow{AC}]^2$. Then Heron-Qin's formula in any metric geometry is

$$S_{ABC}^2 = \frac{\alpha}{16}(4AB^2 \cdot AC^2 - P_{BAC}^2).$$

Two planes ABC and PQR are *parallel* if $[\overrightarrow{AB}, \overrightarrow{AC}] \parallel [\overrightarrow{PQ}, \overrightarrow{PR}]$. Let λ be the scalar ratio of these two parallel vectors, i.e., $[\overrightarrow{AB}, \overrightarrow{AC}] = \lambda[\overrightarrow{PQ}, \overrightarrow{PR}]$. Then $\lambda = \frac{S_{ABC}}{S_{PQR}}$. We thus have

$$\lambda = \frac{S_{ABC}}{S_{PQR}} = \frac{\langle[\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle}{\langle[\overrightarrow{PQ}, \overrightarrow{PR}], [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle}.$$

The *volume* of the tetrahedron $ABCD$ is defined to be one sixth of the *triple scalar product*:

$$V_{ABCD} = \frac{1}{6}\langle\overrightarrow{AD}, [\overrightarrow{AB}, \overrightarrow{AC}]\rangle.$$

As a consequence

$$V_{ABCD} = \frac{1}{6}(\langle\vec{D}, [\vec{A}, \vec{B}]\rangle + \langle\vec{D}, [\vec{B}, \vec{C}]\rangle + \langle\vec{D}, [\vec{C}, \vec{A}]\rangle - \langle\vec{A}, [\vec{B}, \vec{C}]\rangle).$$

Starting from several points in the space, we can form vectors and inner and exterior products of these vectors. Since exterior products of vectors are still vectors, we can further form the inner and exterior products of these new vectors. Expressions thus obtained are called *recursive expressions in inner and exterior products* of vectors. It is clear that a recursive expression in inner and exterior products of vectors can be a scalar or a vector. A vector like $\overrightarrow{AB}$ for points A and B is called a *simple vector*.

Proposition 5.24 *Any recursive expression in inner and exterior products of vectors can be represented as a polynomial in inner products of simple vectors, exterior products of simple vectors, and triple scalar products of simple vectors.*

Proof. By Proposition 5.18, for any vectors $\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}$ and $\boldsymbol{r_4}$

1. $[[\boldsymbol{r_1}, \boldsymbol{r_2}], \boldsymbol{r_3}] = \alpha(\langle\boldsymbol{r_1}, \boldsymbol{r_3}\rangle\boldsymbol{r_2} - \langle\boldsymbol{r_2}, \boldsymbol{r_3}\rangle\boldsymbol{r_1})$.
2. (The Lagrange Identity) $\langle[\boldsymbol{r_1}, \boldsymbol{r_2}], [\boldsymbol{r_3}, \boldsymbol{r_4}]\rangle = \alpha(\langle\boldsymbol{r_1}, \boldsymbol{r_3}\rangle\langle\boldsymbol{r_2}, \boldsymbol{r_4}\rangle - \langle\boldsymbol{r_1}, \boldsymbol{r_4}\rangle\langle\boldsymbol{r_2}, \boldsymbol{r_3}\rangle)$.

By repeated use of the above two identities, any recursive expression in inner and exterior products of vectors can be represented as a polynomial of inner products of simple vectors, exterior products of simple vectors, and triple scalar products of simple vectors. ∎

Exercises 5.25

1. Show that with the above definition for the ratio of lengths, signed areas, signed volumes, and the Pythagoras differences, Axioms A.1-A.6, S.1-S.5, and the properties of the Pythagoras difference are true.

2. Prove the following formula of the distance from a point A to a line PQ

$$d(A,PQ)^2 = AP^2 - \frac{\langle \overrightarrow{PA}, \overrightarrow{PQ}\rangle^2}{PQ^2}.$$

3. Prove the following formula of the distance from a point A to a plane LMN.

$$d^2_{A,LMN} = \frac{\langle \overrightarrow{LA}, [\overrightarrow{LM}, \overrightarrow{LN}]\rangle^2}{[\overrightarrow{LM}, \overrightarrow{LN}]^2}.$$

4. Prove the following formula of the distance between two skew lines UV and PQ

$$d^2_{UV,PQ} = \frac{9(\langle \overrightarrow{UV}, [\overrightarrow{UP}, \overrightarrow{UQ}]\rangle)^2}{[\overrightarrow{PQ}, \overrightarrow{UV}]^2}.$$

5.2.2 Constructive Geometry Statements

The constructive statement defined in Section 4.2 can be generalized by considering more constructions and more geometry quantities.

Definition 5.26 *By geometric quantities we mean vectors, the inner or exterior products of vectors, or the quantities which can be represented by the inner and exterior products of vectors.*

With the geometry concepts introduced in the preceding subsection, constructions S1–S7 on page 181 are still meaningful in our metric geometry, except for constructions S6 and S7 whose ndg conditions need modification. We will also introduce a new construction S8.

S6 (FOOT2LINE Y P U V) Point Y is the foot from point P to line UV. Point Y is a fixed point. The ndg condition is $\overrightarrow{UV}^2 \neq 0$. Notice that in the general metric geometry $\overrightarrow{UV}^2 \neq 0$ is not equivalent to $U \neq V$.

S7 (FOOT2PLANE Y P L M N) Point Y is the foot of the perpendicular from point P to plane LMN. The nondegenerate condition is $[\overrightarrow{LM}, \overrightarrow{LN}]^2 \neq 0$.

S8 (SRATIO A L M N r) Take a point A such that $\overrightarrow{LA} = r[\overrightarrow{LM}, \overrightarrow{LN}]$, where r can be a rational number, a rational expression in geometric quantities, or variables.

If r is a fixed quantity, A is a fixed point; otherwise, A has one degree of freedom. The ndg condition is $[\overrightarrow{LM}, \overrightarrow{LN}]^2 \neq 0$.

Two basic geometric relations, parallel and perpendicular, can be easily described by the exterior and inner products. For instance, to represent $AB \parallel CD$ we need two equations $V_{ABCD} = 0$ and $S_{ACD} = S_{BCD}$. But using exterior product, we need only one equation $[\overrightarrow{AB}, \overrightarrow{CD}] = 0$.

Proposition 5.27 *We have*

1. $AB \perp CD \iff \langle \overrightarrow{AB}, \overrightarrow{CD} \rangle = 0$.
2. $AB \parallel CD \iff [\overrightarrow{AB}, \overrightarrow{CD}] = 0$.
3. $A, B,$ *and* C *are collinear* $\iff [\overrightarrow{AB}, \overrightarrow{AC}] = 0$.
4. $AB \perp PQR \iff [\overrightarrow{AB}, [\overrightarrow{PQ}, \overrightarrow{PR}]] = 0$.
5. $AB \parallel PQR \iff \langle \overrightarrow{AB}, [\overrightarrow{PQ}, \overrightarrow{PR}] \rangle = 0$.
6. $ABC \perp PQR \iff \langle [\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}] \rangle = 0$.
7. $ABC \parallel PQR \iff [[\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}]] = 0$.
8. *A, B, C, and D are coplanar* $\iff \langle \overrightarrow{DA}, [\overrightarrow{AB}, \overrightarrow{AC}] \rangle = 0$.

Proof. The first two cases are from the definition. The other cases are consequences of the first two cases. ∎

Example 5.28 (The Centroid Theorem for Tetrahedra) *Let G be the centroid of tetrahedra ABCD. Show that* $\overrightarrow{G} = \frac{\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}+\overrightarrow{D}}{4}$.

This example can be described constructively as follows.

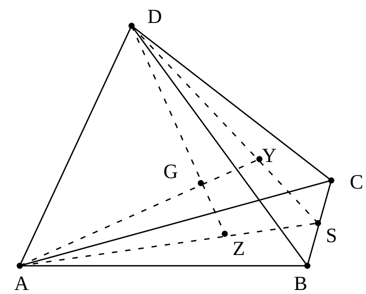


Figure 5-3

Constructive description
((POINTS $A\ B\ C\ D$)
(MIDPOINT $S\ B\ C$)
(LRATIO $Z\ A\ S$ 2/3)
(LRATIO $Y\ D\ S$ 2/3)
(INTER G (LINE $D\ Z$) (LINE $A\ Y$))
($\overrightarrow{G} = \frac{\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}+\overrightarrow{D}}{4}$))

The ndg conditions: $B \neq C$, $A \neq S$, $D \neq S$, and $DZ \nparallel AY$.

5.3 Machine Proof by Vector Calculation

As before, we will give methods of eliminating points from the geometry quantities.

5.3.1 Eliminating Points From Vectors

We first consider how to eliminate points from vectors.

Proposition 5.29 *Let R be a point on line PQ ($P \neq Q$). Then*

$$\overrightarrow{R} = \frac{\overline{PR}}{\overline{PQ}}\overrightarrow{Q} + \frac{\overline{RQ}}{\overline{PQ}}\overrightarrow{P}.$$

Proof. We have $\frac{\overline{PR}}{\overline{PQ}}\overrightarrow{Q} + \frac{\overline{RQ}}{\overline{PQ}}\overrightarrow{P} = \frac{\overline{PR}}{\overline{PQ}}(\overrightarrow{Q} - \overrightarrow{P}) + \overrightarrow{P} = \frac{\overline{PR}}{\overline{PQ}}\overrightarrow{PQ} + \overrightarrow{P} = \overrightarrow{PR} + \overrightarrow{P} = \overrightarrow{R}$. ∎

Lemma 5.30 *Let Y be introduced by (PRATIO $Y\ W\ U\ V\ r$). Then*

$$\overrightarrow{Y} = \overrightarrow{W} + r(\overrightarrow{V} - \overrightarrow{U})$$

Proof. Since $\overrightarrow{WY} = \overrightarrow{W} - \overrightarrow{Y} = r\overrightarrow{UV}$, we have $\overrightarrow{Y} = \overrightarrow{W} - r\overrightarrow{UV}$. ∎

Lemma 5.31 *Let Y be introduced by (ARATIO $Y\ L\ M\ N\ r_1\ r_2\ r_3$). Then*

$$\overrightarrow{Y} = r_1\overrightarrow{L} + r_2\overrightarrow{M} + r_3\overrightarrow{N}.$$

Proof. Since Y is in the plane LMN, we have

$$\overrightarrow{LY} = c_1\overrightarrow{LM} + c_2\overrightarrow{LN}$$

where c_1 and c_2 are some scalars. Then

$$[\overrightarrow{LN}, \overrightarrow{LY}] = c_1[\overrightarrow{LN}, \overrightarrow{LM}] + c_2[\overrightarrow{LN}, \overrightarrow{LN}] = c_1[\overrightarrow{LN}, \overrightarrow{LM}].$$

Thus $c_1 = \frac{S_{LNY}}{S_{LNM}} = r_2$. Similarly $c_2 = r_3$. Thus $\overrightarrow{Y} = \overrightarrow{L} + r_2(\overrightarrow{M} - \overrightarrow{L}) + r_3(\overrightarrow{N} - \overrightarrow{L}) = r_1\overrightarrow{L} + r_2\overrightarrow{M} + r_3\overrightarrow{N}$. ∎

Lemma 5.32 *Let Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then*

$$\overrightarrow{Y} = \frac{S_{UPQ}}{S_{UPVQ}}\overrightarrow{V} - \frac{S_{VPQ}}{S_{UPVQ}}\overrightarrow{U}.$$

Proof. By Proposition 5.29,

$$\overrightarrow{Y} = \frac{\overline{UY}}{\overline{UV}}\overrightarrow{V} - \frac{\overline{VY}}{\overline{UV}}\overrightarrow{U}.$$

Let $r = \frac{\overline{UY}}{\overline{UV}}$. Then $\overrightarrow{UY} = r\overrightarrow{UV}$ and

$$[\overrightarrow{UY}, \overrightarrow{PQ}] = r[\overrightarrow{UV}, \overrightarrow{PQ}].$$

Since $[\overrightarrow{UY}, \overrightarrow{PQ}] = [\overrightarrow{UQ}, \overrightarrow{PQ}] + [\overrightarrow{QY}, \overrightarrow{PQ}] = [\overrightarrow{UQ}, \overrightarrow{PQ}]$, we have $r = \frac{S_{UPQ}}{S_{UPVQ}}$. Similarly $\frac{\overline{VY}}{\overline{UV}} = \frac{S_{VPQ}}{S_{UPVQ}}$. ∎

Lemma 5.33 *Let Y be introduced by (INTER Y (LINE U V) (PLANE L M N)). Then*

$$\overrightarrow{Y} = \frac{V_{ULMN}}{V_{ULMNV}}\overrightarrow{V} - \frac{V_{VLMN}}{V_{ULMNV}}\overrightarrow{U}.$$

Proof. By Proposition 5.29,

$$\overrightarrow{Y} = \frac{\overline{UY}}{\overline{UV}}\overrightarrow{V} + \frac{\overline{YV}}{\overline{UV}}\overrightarrow{U}. \quad (1)$$

Let $r = \frac{\overline{UY}}{\overline{UV}}$. Then $\overrightarrow{UY} = r\overrightarrow{UV}$ and

$$[\overrightarrow{UY}, [\overrightarrow{LM}, \overrightarrow{LN}]] = r[\overrightarrow{UV}, [\overrightarrow{LM}, \overrightarrow{LN}]].$$

Since

$$[\overrightarrow{UY}, [\overrightarrow{LM}, \overrightarrow{LN}]] = [\overrightarrow{UL}, [\overrightarrow{LM}, \overrightarrow{LN}]] + [\overrightarrow{LY}, [\overrightarrow{LM}, \overrightarrow{LN}]] = V_{ULMN},$$

we have $r = \frac{V_{ULMN}}{V_{ULMNV}}$. Similarly $\frac{\overline{VY}}{\overline{UV}} = \frac{V_{VLMN}}{V_{ULMNV}}$. ∎

Lemma 5.34 *Let Y be introduced by (FOOT2LINE Y P U V). Then*

$$\overrightarrow{Y} = \frac{P_{PUV}}{P_{UVU}}\overrightarrow{V} + \frac{P_{PVU}}{P_{UVU}}\overrightarrow{U}.$$

Proof. By Proposition 5.29, $\overrightarrow{Y} = \frac{\overline{UY}}{\overline{UV}}\overrightarrow{V} - \frac{\overline{VY}}{\overline{UV}}\overrightarrow{U}$. Let $r = \frac{\overline{UY}}{\overline{UV}}$, or $\overrightarrow{UY} = r\overrightarrow{UV}$. We have

$$r\langle\overrightarrow{UV},\overrightarrow{UV}\rangle = \langle\overrightarrow{UY},\overrightarrow{UV}\rangle = \langle\overrightarrow{UP},\overrightarrow{UV}\rangle + \langle\overrightarrow{PY},\overrightarrow{UV}\rangle = \langle\overrightarrow{UP},\overrightarrow{UV}\rangle.$$

Then $r = \frac{P_{PUV}}{P_{UVU}}$. Similarly $\frac{\overline{YV}}{\overline{UV}} = \frac{P_{PVU}}{P_{UVU}}$. ∎

Lemma 5.35 *Let Y be introduced by (SRATIO Y L M N r). Then*

$$\overrightarrow{Y} = \overrightarrow{L} + r[\overrightarrow{LM},\overrightarrow{LN}].$$

Proof. This is the definition of the construction SRATIO. ∎

Lemma 5.36 *Let Y be introduced by (FOOT2PLANE Y P L M N). Then*

$$\overrightarrow{Y} = \overrightarrow{P} + \frac{6V_{PLMN}}{[\overrightarrow{LM},\overrightarrow{LN}]^2}[\overrightarrow{LM},\overrightarrow{LN}].$$

Proof. Let $\overrightarrow{PY} = r[\overrightarrow{LM},\overrightarrow{LN}]$. Then $\langle\overrightarrow{LP},\overrightarrow{PY}\rangle = r\langle\overrightarrow{LP},[\overrightarrow{LM},\overrightarrow{LN}]\rangle = 6rV_{LMNP}$. $\langle\overrightarrow{LP},\overrightarrow{PY}\rangle = -PY^2 = -\frac{36V_{PLMN}^2}{[\overrightarrow{LM},\overrightarrow{LN}]^2}$. Thus $r = \frac{6V_{PLMN}}{[\overrightarrow{LM},\overrightarrow{LN}]^2}$. ∎

Example 5.37 *Let Y be introduced by (INTER Y (PLINE W U V) (PLINE R P Q)). Show that* $\overrightarrow{Y} = \overrightarrow{W} + \frac{S_{WPRQ}}{S_{UPVQ}}(\overrightarrow{V} - \overrightarrow{U})$.

Proof. Take points X and S such that $\frac{\overline{WX}}{\overline{UV}} = 1$ and $\frac{\overline{RS}}{\overline{PQ}} = 1$. By Proposition 5.29,

$$\overrightarrow{Y} = r\overrightarrow{X} + (1-r)\overrightarrow{W} = r(\overrightarrow{X} - \overrightarrow{W}) + \overrightarrow{W} = r(\overrightarrow{V} - \overrightarrow{U}) + \overrightarrow{W}$$

where $r = \frac{\overline{WY}}{\overline{WX}} = \frac{S_{WPRQ}}{S_{UPVQ}}$. ∎

Example 5.38 *Continue from Example 5.28. We can actually derive the result of this example without knowing it previously.*

Constructive description

((POINTS A B C D)
(MIDPOINT S B C)
(LRATIO Z A S 2/3)
(LRATIO Y D S 2/3)
(INTER G (LINE D Z) (LINE A Y))
($\overrightarrow{G}$))

The machine proof.

$\overrightarrow{G}$

$$\stackrel{G}{=} \frac{-\overrightarrow{Z}\cdot S_{ADY} + \overrightarrow{D}\cdot S_{AZY}}{-S_{ADYZ}}$$

$$\stackrel{Y}{=} \frac{\frac{2}{3}\overrightarrow{Z}\cdot S_{ADS} + \frac{1}{3}\overrightarrow{D}\cdot S_{ADZ}}{\frac{2}{3}S_{DSZ} + S_{ADZ}}$$

$$\stackrel{Z}{=} \frac{\frac{4}{3}\overrightarrow{S}\cdot S_{ADS} + \frac{2}{3}\overrightarrow{D}\cdot S_{ADS} + \frac{2}{3}\overrightarrow{A}\cdot S_{ADS}}{\frac{8}{3}S_{ADS}}$$

$$\stackrel{simplify}{=} \frac{1}{4}(2\overrightarrow{S} + \overrightarrow{D} + \overrightarrow{A})$$

$$\stackrel{S}{=} \frac{\overrightarrow{D} + \overrightarrow{C} + \overrightarrow{B} + \overrightarrow{A}}{4}$$

The eliminants

$$\overrightarrow{G} \stackrel{G}{=} \frac{\overrightarrow{Z}\cdot S_{ADY} - \overrightarrow{D}\cdot S_{AZY}}{S_{ADYZ}}$$

$$S_{ADYZ} \stackrel{Y}{=} \frac{1}{3}(2S_{DSZ} + 3S_{ADZ})$$

$$S_{AZY} \stackrel{Y}{=} -\frac{1}{3}(S_{ADZ})$$

$$S_{ADY} \stackrel{Y}{=} \frac{2}{3}(S_{ADS})$$

$$S_{DSZ} \stackrel{Z}{=} \frac{1}{3}(S_{ADS})$$

$$S_{ADZ} \stackrel{Z}{=} \frac{2}{3}(S_{ADS})$$

$$\overrightarrow{Z} \stackrel{Z}{=} \frac{1}{3}(2\overrightarrow{S} + \overrightarrow{A})$$

$$\overrightarrow{S} \stackrel{S}{=} \frac{1}{2}(\overrightarrow{C} + \overrightarrow{B})$$

5.3.2 Eliminating Points from Inner and Exterior Products

Let Y be introduced by one of the constructions S1–S8. By Lemmas 5.30–5.36,

$$(I) \qquad \overrightarrow{Y} = \alpha \boldsymbol{r_1} + \beta \boldsymbol{r_2}$$

for vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ and scalars α and β.

To eliminate Y from the inner product, let us note that

$$\langle \overrightarrow{AB}, \overrightarrow{CY} \rangle = \langle \overrightarrow{B}, \overrightarrow{Y} \rangle + \langle \overrightarrow{A}, \overrightarrow{C} \rangle - \langle \overrightarrow{B}, \overrightarrow{C} \rangle - \langle \overrightarrow{A}, \overrightarrow{Y} \rangle.$$

Then we need only to consider how to eliminate Y from $\langle \overrightarrow{A}, \overrightarrow{Y} \rangle$ and $\langle \overrightarrow{Y}, \overrightarrow{Y} \rangle$. If $A \neq Y$,

$$\langle \overrightarrow{A}, \overrightarrow{Y} \rangle = \alpha \langle \overrightarrow{A}, \boldsymbol{r_1} \rangle + \beta \langle \overrightarrow{A}, \boldsymbol{r_2} \rangle.$$

For $\langle \overrightarrow{Y}, \overrightarrow{Y} \rangle$, we have

$$\langle \overrightarrow{Y}, \overrightarrow{Y} \rangle \quad = \quad < \alpha \boldsymbol{r_1} + \beta \boldsymbol{r_2}, \alpha \boldsymbol{r_1} + \beta \boldsymbol{r_2} >= \alpha^2 \langle \boldsymbol{r_1}, \boldsymbol{r_1} \rangle + \beta^2 \langle \boldsymbol{r_2}, \boldsymbol{r_2} \rangle + 2\alpha\beta \langle \boldsymbol{r_1}, \boldsymbol{r_2} \rangle.$$

To eliminate point Y from the exterior product $[\overrightarrow{AB}, \overrightarrow{CY}]$, let us note that

$$[\overrightarrow{AB}, \overrightarrow{CY}] = [\overrightarrow{A}, \overrightarrow{C}] + [\overrightarrow{B}, \overrightarrow{Y}] - [\overrightarrow{A}, \overrightarrow{Y}] - [\overrightarrow{B}, \overrightarrow{C}].$$

If $A = Y$ we have $[\overrightarrow{A}, \overrightarrow{Y}] = 0$; otherwise we have

$$[\overrightarrow{A}, \overrightarrow{Y}] = \alpha [\overrightarrow{A}, \boldsymbol{r_1}] + \beta [\overrightarrow{A}, \boldsymbol{r_2}].$$

Since other geometry quantities can always be represented as a rational expression in inner and exterior products, we can eliminate points introduced by constructions S1–S8 from them. Here are some examples.

Example 5.39 *Let Y be introduced by (SRATIO Y L M N r). By Lemma 5.35, we have*

$$\begin{aligned}
V_{YBCD} &= \langle \overrightarrow{BY}, [\overrightarrow{BC}, \overrightarrow{BD}] \rangle = V_{LBCD} - r \langle [\overrightarrow{LM}, \overrightarrow{LN}], [\overrightarrow{BC}, \overrightarrow{BD}] \rangle \\
\langle \overrightarrow{YB}, \overrightarrow{CD} \rangle &= \langle \overrightarrow{B}, \overrightarrow{CD} \rangle - \langle \overrightarrow{Y}, \overrightarrow{CD} \rangle \\
&= \langle \overrightarrow{LB}, \overrightarrow{CD} \rangle - r \langle \overrightarrow{CD}, [\overrightarrow{LM}, \overrightarrow{LN}] \rangle. \\
\langle \overrightarrow{YB}, \overrightarrow{YC} \rangle &= \langle \overrightarrow{B}, \overrightarrow{C} \rangle + \langle \overrightarrow{Y}, \overrightarrow{Y} \rangle - \langle \overrightarrow{Y}, \overrightarrow{B} \rangle - \langle \overrightarrow{Y}, \overrightarrow{C} \rangle \\
&= \langle \overrightarrow{LB}, \overrightarrow{LC} \rangle + r \langle \overrightarrow{BL} + \overrightarrow{CL}, [\overrightarrow{LM}, \overrightarrow{LN}] \rangle + r^2 [\overrightarrow{LM}, \overrightarrow{LN}]^2 \\
[\overrightarrow{YB}, \overrightarrow{CD}] &= [\overrightarrow{B}, \overrightarrow{CD}] - [\overrightarrow{Y}, \overrightarrow{CD}] \\
&= [\overrightarrow{LB}, \overrightarrow{CD}] + r [\overrightarrow{CD}, [\overrightarrow{LM}, \overrightarrow{LN}]].
\end{aligned}$$

Example 5.40 *Let Y be introduced by (FOOT2PLANE Y P L M N). By Lemma 5.36,*

$$\begin{aligned}
V_{ABCD} &= V_{PBCD} + r\langle[\overrightarrow{PM}, \overrightarrow{PN}], [\overrightarrow{BC}, \overrightarrow{BD}]\rangle. \\
\langle\overrightarrow{AB}, \overrightarrow{CD}\rangle &= \langle\overrightarrow{PB}, \overrightarrow{CD}\rangle - r\langle\overrightarrow{CD}, [\overrightarrow{PM}, \overrightarrow{PN}]\rangle. \\
\langle\overrightarrow{AB}, \overrightarrow{AC}\rangle &= \langle\overrightarrow{PB}, \overrightarrow{PC}\rangle + r\langle\overrightarrow{BP} + \overrightarrow{CP}, [\overrightarrow{PM}, \overrightarrow{PN}]\rangle + r^2[\overrightarrow{PM}, \overrightarrow{PN}]^2
\end{aligned}$$

where $r = \frac{6V_{PLMN}}{[\overrightarrow{LM},\overrightarrow{LN}]^2}$.

To eliminate points introduced by constructions S4, S5, and S6, we do not need to break the inner and exterior products into the sum of several components. In these three constructions, Y is always on a line UV. Remember that a geometry quantity $G(Y)$ is called a *linear quantity* of point Y if

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U).$$

A geometry quantity $G(Y)$ is called a *quadratic geometry quantity* of point Y if

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U) - \frac{\overline{UY}}{\overline{UV}} \frac{\overline{YV}}{\overline{UV}} \overline{UV}^2.$$

Example 5.41 *Show that $\overrightarrow{YB}$, $[\overrightarrow{YB}, \overrightarrow{CD}]$, and $\langle\overrightarrow{YB}, \overrightarrow{CD}\rangle$ are all linear in Y; and $\langle\overrightarrow{YB}, \overrightarrow{YC}\rangle$ is quadratic in Y.*

Proof. By Proposition 5.29, $\overrightarrow{Y} = \frac{\overline{UY}}{\overline{UV}}\overrightarrow{V} + \frac{\overline{YV}}{\overline{UV}}\overrightarrow{U}$.

$$\begin{aligned}
\overrightarrow{YB} &= \overrightarrow{B} - \overrightarrow{Y} = \overrightarrow{B} - \frac{\overline{UY}}{\overline{UV}}\overrightarrow{V} - \frac{\overline{YV}}{\overline{UV}}\overrightarrow{U} \\
&= \frac{\overline{UY}}{\overline{UV}}(\overrightarrow{B} - \overrightarrow{V}) + \frac{\overline{YV}}{\overline{UV}}(\overrightarrow{B} - \overrightarrow{U}) \\
&= \frac{\overline{UY}}{\overline{UV}}\overrightarrow{VB} + \frac{\overline{YV}}{\overline{UV}}\overrightarrow{UB}
\end{aligned}$$

Now it is clear that $[\overrightarrow{YB}, \overrightarrow{CD}]$ and $\langle\overrightarrow{YB}, \overrightarrow{CD}\rangle$ are also linear in Y. For $G(Y) = \langle\overrightarrow{YB}, \overrightarrow{YC}\rangle$, let $r_1 = \frac{\overline{UY}}{\overline{UV}}$ and $r_2 = \frac{\overline{YV}}{\overline{UV}}$. Then

$$\begin{aligned}
G(Y) &= \langle r_1\overrightarrow{VB} + r_2\overrightarrow{UB}, r_1\overrightarrow{VC} + r_2\overrightarrow{UC}\rangle \\
&= r_1^2 G(V) + r_2^2 G(U) + r_1 r_2(\langle\overrightarrow{VB}, \overrightarrow{UC}\rangle + \langle\overrightarrow{UB}, \overrightarrow{VC}\rangle) \\
&= r_1(r_1 + r_2)G(V) + r_2(r_1 + r_2)G(U) - r_1 r_2\langle\overrightarrow{UV}, \overrightarrow{UV}\rangle \\
&= r_1 G(V) + r_2 G(U) - r_1 r_2\langle\overrightarrow{UV}, \overrightarrow{UV}\rangle. \quad \blacksquare
\end{aligned}$$

Therefore to eliminate point Y from the geometry quantities, we need only to find the position ratios of Y with UV, which has been done in Lemmas 5.32, 5.33, and 5.34.

5.3.3 The Algorithm

Algorithm 5.42 (VECTOR)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a constructive geometry statement.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \cdots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the nondegenerate conditions of C_i are satisfied. The nondegenerate conditions of a statement have five forms: $A \neq B$, $AB^2 \neq 0$, $PQ \nparallel UV$, $PQ \nparallel WUV$, and $[\overrightarrow{LM}, \overrightarrow{LN}]^2 \neq 0$. For the first case, we check whether $\overrightarrow{A} = \overrightarrow{B}$. For the second case, we check whether $\langle\overrightarrow{AB}, \overrightarrow{AB}\rangle = 0$. For the third case, we check whether $[\overrightarrow{PQ}, \overrightarrow{UV}] = 0$. For the fourth case, we check whether $V_{PWUV} = V_{QWUV}$. For the fifth case, we check whether $[\overrightarrow{LM}, \overrightarrow{LN}]^2 = 0$. If one of the nondegenerate conditions of a geometry statement is not satisfied, the statement is *trivially true*. The algorithm terminates.

S3. Let $G_1, \cdots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \cdots, s$ do S4

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using the lemmas in this section, and replace G_j by H_j in E and F to obtain the new E and F.

S5. Now there are only free points left. E and F are rational expressions in indeterminates, inner and exterior products of free points. Replacing the inner and exterior products by their coordinate expressions, we obtain E' and F'. Then if $E' = F'$, S is true under the nondegenerate conditions. Otherwise S is false.

In the above algorithm, we represent the ratio of lengths and the ratio of areas as expressions of inner and exterior products, and then eliminate points from the inner and exterior products. In order to obtain short proofs, we may also use the lemmas in Section 4.3 to eliminate points directly from the ratios of lengths or areas. We actually use a hybrid method: inner products, exterior products, length ratios, area ratios, volumes, and the Pythagoras differences are all used in the proof.

Remark 5.43 *Since the inner product, the exterior product, and the triple scalar product are proportional with the Pythagoras difference, the area, and the volume, the volume (area) and the Pythagoras difference method developed in Chapters 3 and 4 are valid for constructive geometry statements in metric geometries associated with any field with characteristic different from* 2. *Thus our method works not only for Euclidean geometry but also for non-Euclidean geometries such as the Minkowskian geometry.*

The vector approach has the advantages that it is easy to develop, and more geometry quantities such as the vector itself can be used. But the proofs produced by the vector approach generally do not have the clear geometric meaning of those produced by the volume-Pythagoras difference approach.

Notice that the elimination results (Lemmas 5.30-5.34) for points introduced by S1–S6 are exactly the same for all metric geometries. From this, one might wonder is there any connection among these metric geometries? But at the last step of the algorithm, we need to replace the inner and exterior products of free points by their coordinate expressions. This step depends on the specific geometries. Thus in order to obtain some meta theorems for different geometries, we need to limit the class of geometry statements.

Definition 5.44 *A constructive geometry statement is called pure constructive if it can be described by constructions S1-S7 and its conclusion can be one of the geometry relations in Proposition 5.27. We further assume that the ratio r in the ratio constructions PRATIO and ARATIO can only be scalars or variables.*

In the predicate form for a pure constructive geometry statement, there are only affine invariants like the ratio of parallel line segments and geometry predicates like COLL, PRLL, PERP, etc.

Proposition 5.45 *Let G_1 and G_2 be two geometries over the same base field $\mathcal{E}$. Then a pure constructive geometry statement is true in G_1 iff it is true in geometry G_2.*

Proof. By Proposition 5.9, we can assume that the matrices for the inner products of G_1 and G_2 are

$$\mathcal{M}_1 = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix}.$$

Let $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{x}', \boldsymbol{y}'$, $\boldsymbol{z}'$ be vectors in G_1 and G_2 with the same coordinates respectively. Any geometry predicate can be represented by the following three quantities.

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3 \qquad \text{in } G_1.$$

$$[\boldsymbol{x}, \boldsymbol{y}] = \alpha_1(a_2a_3(x_2y_2 - x_3y_2), a_1a_3(x_3y_1 - x_1x_3), a_1a_2(x_1y_2 - x_2x_1)) \qquad \text{in } G_1.$$

where α_1 is the constant in Proposition 5.18 for G1.

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \alpha_1 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \qquad \text{in } G_1.$$

$$\langle \boldsymbol{x}', \boldsymbol{y}' \rangle = b_1x_1y_1 + b_2x_2y_2 + b_3x_3y_3 \qquad \text{in } G_2.$$

$$[\boldsymbol{x}', \boldsymbol{y}'] = \alpha_2(b_2b_3(x_2y_2 - x_3y_2), b_1b_3(x_3y_1 - x_1x_3), b_1b_2(x_1y_2 - x_2x_1)) \qquad \text{in } G_2.$$

where α_2 is the constant in Proposition 5.18 for G2.

$$(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') = \alpha_2 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \qquad \text{in } G_2.$$

Let $\mathcal{F}$ be the algebraic closure of field $\mathcal{E}$. Consider the following automorphism of the polynomial ring $\mathcal{F}[x_1, x_2, x_3, ..., z_1, z_2, z_3]$

$$TR : (x_1, x_2, x_3, ..., z_3) \longrightarrow (\sqrt{\frac{a_1}{b_1}}x_1, \sqrt{\frac{a_2}{b_2}}x_2, \sqrt{\frac{a_3}{b_3}}x_3, ..., \sqrt{\frac{a_3}{b_3}}z_3).$$

We have

$$\begin{aligned} &TR(\langle \boldsymbol{x}', \boldsymbol{y}' \rangle) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle, \\ &TR([\boldsymbol{x}', \boldsymbol{y}']) = 0 \iff [\boldsymbol{x}, \boldsymbol{y}] = 0, \\ &TR(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') = 0 \iff (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0. \end{aligned}$$

Also note that an affine invariant is not changed under TR. Since each pure constructive statement can be described by affine invariants and the above geometry quantities, it is clear that a pure constructive geometric statement is true in G_1 iff it is true in G_2. ▮

5.4 Machine Proof in Metric Plane Geometries

Since a metric plane can be seen as a subset of a metric space, the method presented in the preceding section is also valid for plane metric geometries. For an independent study of this topic, see [73]. But for metric plane geometry, the method can be greatly simplified. Without loss of generality, we assume that the metric plane consists of all the points $\boldsymbol{x} = (x_1, x_2, x_3)$ such that $x_3 = 0$. Let

$$\mathcal{M} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

be the matrix defining the inner product. Then for vectors $\boldsymbol{x} = (x_1, x_2, 0), \boldsymbol{y} = (y_1, y_2, 0)$, and $\boldsymbol{z} = (z_1, z_2, 0)$ in the plane, we have

$$\begin{aligned} &\langle \boldsymbol{x}, \boldsymbol{y} \rangle = a_1x_1y_1 + a_2x_2y_2. \\ &[\boldsymbol{x}, \boldsymbol{y}] = (0, 0, \alpha a_1a_2(x_1y_2 - x_2y_1)). \qquad \alpha \text{ is a constant.} \\ &(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0. \end{aligned}$$

Since all the exterior products of vectors in the same plane are parallel, we can just define

$$[\boldsymbol{x}, \boldsymbol{y}] = \alpha a_1a_2(x_1y_2 - x_2y_1).$$

It is easy to check that some of the basic properties of the exterior products such as E1 and E2 are still true. For points A, B, and C, the signed area of triangle ABC is defined to be $\frac{1}{2}[\overrightarrow{AB}, \overrightarrow{AC}]$.

5.4.1 Vector Approach for Euclidean Plane Geometry

For Euclidean geometry, the matrix $\mathcal{M}$ defining the inner product is the unit matrix. Thus if the conclusion of a geometry statement is a polynomial of inner and exterior products, the proofs produced according to the vector approach and the area-Pythagoras difference approach are actually the same. But in the vector approach, we can use a new geometry quantity: the vector itself.

Example 5.46 (The Centroid Theorem) *Show that the three medians of a triangle are concurrent and the medians are divided by their common point in the ratio 2:1.*

Constructive description

((POINTS A B C)
 (MIDPOINT F A C)
 (MIDPOINT E B C)
 (INTER G (LINE A E) (LINE B F))
 ($\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AE}$))

The eliminants

$\overrightarrow{AG} \overset{G}{=} \frac{\overrightarrow{AE}\cdot S_{ABF}}{S_{ABEF}}$

$S_{ABEF} \overset{E}{=} \frac{1}{2}(S_{BCF}+2S_{ABF})$

$S_{BCF} \overset{F}{=} \frac{1}{2}(S_{ABC})$

$S_{ABF} \overset{F}{=} \frac{1}{2}(S_{ABC})$

The machine proof.

$$\frac{\overrightarrow{AG}}{(\frac{2}{3})\cdot\overrightarrow{AE}} \overset{G}{=} \frac{\overrightarrow{AE}\cdot S_{ABF}}{(\frac{2}{3})\cdot\overrightarrow{AE}\cdot S_{ABEF}} \overset{simplify}{=} \frac{(3)\cdot S_{ABF}}{(2)\cdot S_{ABEF}} \overset{E}{=} \frac{(3)\cdot S_{ABF}}{(2)\cdot(\frac{1}{2}S_{BCF}+S_{ABF})} \overset{F}{=} \frac{(3)\cdot(\frac{1}{2}S_{ABC})}{\frac{3}{2}S_{ABC}} \overset{simplify}{=} 1$$

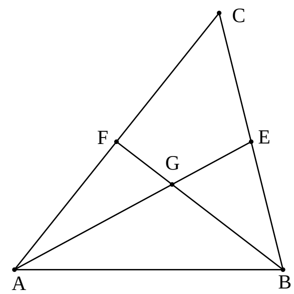


Figure 5-4

Example 5.47 *Let G be the centroid of a triangle ABC. Show that* $\overrightarrow{G} = \frac{1}{3}(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C})$.

Proof. Using our method, we can actually compute this result without knowing it previously. The centroid G of the triangle ABC can be introduced as follows.

(POINTS A B C)
 (MIDPOINT F A C)
 (LRATIO G B F 2/3)

By Lemma 5.30, $\overrightarrow{G} = 2/3\overrightarrow{F} + 1/3\overrightarrow{B} = \frac{1}{3}(\overrightarrow{C} + \overrightarrow{A} + \overrightarrow{B})$. ▮

You might think that the above introduction of G is too tricky. The usual way of introducing G is as follows.

Constructive description

((POINTS $A\ B\ C$)
(MIDPOINT $F\ A\ C$)
(MIDPOINT $E\ B\ C$)
(INTER G (LINE $A\ E$) (LINE $B\ F$))
($\overrightarrow{G}$))

The machine proof.

$$\overrightarrow{G}$$
$$\overset{G}{=} \frac{-\overrightarrow{F}\cdot S_{ABE}+\overrightarrow{B}\cdot S_{AFE}}{-S_{ABEF}}$$
$$\overset{E}{=} \frac{\frac{1}{2}\overrightarrow{F}\cdot S_{ABC}+\frac{1}{2}\overrightarrow{B}\cdot S_{ABF}}{\frac{1}{2}S_{BCF}+S_{ABF}}$$
$$\overset{F}{=} \frac{\frac{1}{2}\overrightarrow{C}\cdot S_{ABC}+\frac{1}{2}\overrightarrow{B}\cdot S_{ABC}+\frac{1}{2}\overrightarrow{A}\cdot S_{ABC}}{\frac{3}{2}S_{ABC}}$$
$$\overset{simplify}{=} \frac{1}{3}(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})$$

The eliminants

$$\overrightarrow{G}\overset{G}{=}\frac{\overrightarrow{F}\cdot S_{ABE}-\overrightarrow{B}\cdot S_{AFE}}{S_{ABEF}}$$
$$S_{ABEF}\overset{E}{=}\frac{1}{2}(S_{BCF}+2S_{ABF})$$
$$S_{AFE}\overset{E}{=}-\frac{1}{2}(S_{ABF})$$
$$S_{ABE}\overset{E}{=}\frac{1}{2}(S_{ABC})$$
$$S_{BCF}\overset{F}{=}\frac{1}{2}(S_{ABC})$$
$$S_{ABF}\overset{F}{=}\frac{1}{2}(S_{ABC})$$
$$\overrightarrow{F}\overset{F}{=}\frac{1}{2}(\overrightarrow{C}+\overrightarrow{A})$$

Example 5.48 *The triangle having for its vertices the midpoints of the sides of a given triangle has the same centroid as the given triangle.*

Constructive description

((POINTS $A\ B\ C\ X\ Y$)
(MIDPOINT $D\ B\ C$)
(MIDPOINT $E\ A\ C$)
(MIDPOINT $F\ A\ B$)
(CENTROID $G\ A\ B\ C$)
(CENTROID $K\ D\ E\ F$)
($\overrightarrow{G} = \overrightarrow{K}$))

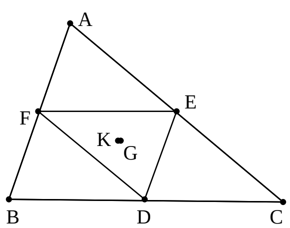


Figure 5-5

The machine proof.

$$\frac{\overrightarrow{G}}{\overrightarrow{K}}$$
$$\overset{K}{=} \frac{\overrightarrow{G}\cdot(3)}{\overrightarrow{F}+\overrightarrow{E}+\overrightarrow{D}}$$
$$\overset{G}{=} \frac{(3)\cdot(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})}{(\overrightarrow{F}+\overrightarrow{E}+\overrightarrow{D})\cdot(3)}$$
$$\overset{F}{=} \frac{\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A}}{\overrightarrow{E}+\overrightarrow{D}+\frac{1}{2}\overrightarrow{B}+\frac{1}{2}\overrightarrow{A}}$$
$$\overset{E}{=} \frac{(2)\cdot(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})}{2\overrightarrow{D}+\overrightarrow{C}+\overrightarrow{B}+2\overrightarrow{A}}$$
$$\overset{D}{=} \frac{(2)\cdot(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})}{2\overrightarrow{C}+2\overrightarrow{B}+2\overrightarrow{A}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\overrightarrow{K}\overset{K}{=}\frac{1}{3}(\overrightarrow{F}+\overrightarrow{E}+\overrightarrow{D})$$
$$\overrightarrow{G}\overset{G}{=}\frac{1}{3}(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})$$
$$\overrightarrow{F}\overset{F}{=}\frac{1}{2}(\overrightarrow{B}+\overrightarrow{A})$$
$$\overrightarrow{E}\overset{E}{=}\frac{1}{2}(\overrightarrow{C}+\overrightarrow{A})$$
$$\overrightarrow{D}\overset{D}{=}\frac{1}{2}(\overrightarrow{C}+\overrightarrow{B})$$

Example 5.49 *The triangle formed by the three lines passing through the three vertices and parallel to the opposite sides of a triangle is called the anticomplementary triangle of the given triangle. Show that a triangle and its anticomplementary triangle have the same centroid.*

Constructive description	The machine proof.	The eliminants
((POINTS A B C)	$\frac{\overrightarrow{G}}{\overrightarrow{K}}$	$\overrightarrow{K} \stackrel{K}{=} \frac{1}{3}(\overrightarrow{R}+\overrightarrow{Q}+\overrightarrow{P})$
(PRATIO P A C B 1)	$\stackrel{K}{=} \frac{\overrightarrow{G}\cdot(3)}{\overrightarrow{R}+\overrightarrow{Q}+\overrightarrow{P}}$	$\overrightarrow{G} \stackrel{G}{=} \frac{1}{3}(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})$
(PRATIO Q A B C 1)	$\stackrel{G}{=} \frac{(3)\cdot(\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A})}{(\overrightarrow{R}+\overrightarrow{Q}+\overrightarrow{P})\cdot(3)}$	$\overrightarrow{R} \stackrel{R}{=} \overrightarrow{C}+\overrightarrow{B}-\overrightarrow{A}$
(PRATIO R B A C 1)	$\stackrel{R}{=} \frac{\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A}}{\overrightarrow{Q}+\overrightarrow{P}+\overrightarrow{C}+\overrightarrow{B}-\overrightarrow{A}}$	$\overrightarrow{Q} \stackrel{Q}{=} \overrightarrow{C}-\overrightarrow{B}+\overrightarrow{A}$
(CENTROID G A B C)	$\stackrel{Q}{=} \frac{\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A}}{\overrightarrow{P}+2\overrightarrow{C}}$	$\overrightarrow{P} \stackrel{P}{=} -(\overrightarrow{C}-\overrightarrow{B}-\overrightarrow{A})$
(CENTROID K P Q R)	$\stackrel{P}{=} \frac{\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A}}{\overrightarrow{C}+\overrightarrow{B}+\overrightarrow{A}}$	
($\overrightarrow{G} = \overrightarrow{K}$))	$\stackrel{simplify}{=} 1$	

With the help of the vector method, we can prove the following theorem about n-polygons. The *centroid* of n points $P_1, \cdots, P_n$ is defined to be $\frac{1}{n}(\overrightarrow{P_1} + \cdots + \overrightarrow{P_n})$.

Example 5.50 (Cantor's First Theorem) *For n points on the circle O, perpendiculars from the centroids of any $n-1$ points taken from the n points to the tangent lines of circle O at the remaining n-th points are concurrent.*

Proof. Let the n points be $P_1, \cdots, P_n$ and $\overrightarrow{P_i} = \overrightarrow{OP_i}$. Let G_1 be the centroid of $P_2, \cdots, P_n$, G_2 the centroid of $P_1, P_3, \cdots, P_n$, and Y the intersection of (PLINE G_1 O P_1) and (PLINE G_2 O P_2). By Example 5.37,

$$\overrightarrow{Y} = \overrightarrow{G_1} + r(\overrightarrow{P_1} - \overrightarrow{O}) = \overrightarrow{G_1} + r\overrightarrow{P_1}.$$

where $r = \frac{S_{G_1OP_2} - S_{G_2OP_2}}{S_{OOP_1P_2}} = \frac{S_{P_1OP_2}}{(n-1)S_{P_1OP_2}} = \frac{1}{n-1}$. Therefore

$$\overrightarrow{Y} = \frac{1}{n-1}(\overrightarrow{P_1} + \cdots + \overrightarrow{P_n})$$

which is a fixed point. ∎

Example 5.51 (Cantor's Second Theorem) *For n points on the circle O, perpendiculars from the centroids of any $n-2$ points taken from the n points to the lines joining the remaining two points are occurrent.*

Proof. Let the n points be $P_1, \cdots, P_n$ and $\overrightarrow{P_i} = \overrightarrow{OP_i}$. Let G_1 be the centroid of $P_3, \cdots, P_n$, G_2 the centroid of $P_1, P_2, P_5 \cdots, P_n$, M the midpoint of P_1P_2, N the midpoint of P_3P_4, and Y the intersection of (PLINE G_1 O M) and (PLINE G_2 N P_2). By Example 5.37,

$$\overrightarrow{Y} = \overrightarrow{G_1} + r(\overrightarrow{M} - \overrightarrow{O}) = \overrightarrow{G_1} + \frac{r}{2}(\overrightarrow{P_1 + P_2}).$$

where $r = \frac{S_{G_1ON}-S_{G_2ON}}{S_{OOMN}} = \frac{2S_{MON}}{(n-2)S_{MON}} = \frac{2}{n-2}$. Therefore

$$\overrightarrow{Y} = \frac{1}{n-2}(\overrightarrow{P_1} + \cdots + \overrightarrow{P_n})$$

which is a fixed point. ∎

5.4.2 Machine Proof in Minkowskian Plane Geometry

In *Minkowskian plane geometry*, $\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the inner product of $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 - x_2y_2.$$

Thus $\boldsymbol{x} \perp \boldsymbol{y}$ iff

$$x_1y_1 - x_2y_2 = 0.$$

We define the exterior product of $\boldsymbol{x}$ and $\boldsymbol{y}$ to be

$$[\boldsymbol{x}, \boldsymbol{y}] = -x_1y_2 + x_2y_1.$$

For points A, B, C, and D in the Minkowskian plane $[\overrightarrow{AC}, \overrightarrow{BD}]$ is also interpreted as twice the area for the quadrilateral $ABCD$, i.e.,

$$S_{ABCD} = \frac{1}{2}[\overrightarrow{AC}, \overrightarrow{BD}].$$

We thus have the *Herron-Qin formula in the Minkowskian geometry*

$$16S_{ABCD}^2 = P_{ABCD}^2 - 4AC^2 \cdot BD^2.$$

In the Minkowskian plane, there exist isotropic lines (vectors). Vector $\boldsymbol{x} = (x_1, x_2)$ is isotropic iff

$$x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0,$$

i.e., the isotropic lines are those which are parallel to one of the following lines

$$x_1 - x_2 = 0 \quad \text{or} \quad x_1 + x_2 = 0.$$

As a consequence of Proposition 5.45, we have

Proposition 5.52 *A pure constructive geometry statement is true in Euclidean geometry if and only if it is true in Minkowskian geometry.*

As a consequence, most of the geometry theorems proved in this book are also valid in Minkowskian geometry. But there are do exist geometry statements in Euclidean geometry which are not true or do not have geometric meaning in Minkowskian geometry. For instance, in Minkowskian geometry there exist no equilateral triangles.

If a geometry statement is in the affine geometry, i.e., the only geometry relations in the statements are incidence and parallel, then it is obvious that this statement is true in Euclidean geometry iff it is true in Minkowskian geometry. The reason is that both geometries are developed from the same affine geometry by adding different metric structures. In other words, the affine part of the two geometries are the same. But for a geometry statement involving geometry relations like perpendicular or measurement, it is not obvious that its validities for both geometries are the same.

Example 5.53 (Orthocenter Theorem in Minkowskian Geometry) *The same as Example 3.19 on page 111.*

The following proof is essentially the same as the proof of Example 3.19. But it is for a different geometry theorem. Compare Figure 1-42 (on page 32) and Figure 5-7.

Constructive description

((POINTS A B C)
(FOOT E B A C)
(FOOT D C A B)
(INTER H (LINE C D) (LINE B E))
(PERPENDICULAR B C A H))

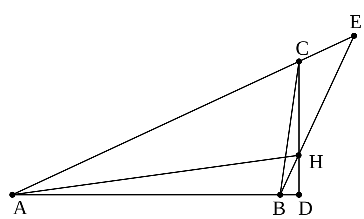


Figure 5-7

The machine proof.

$$\frac{\langle\overrightarrow{AB},\overrightarrow{AH}\rangle}{\langle\overrightarrow{AC},\overrightarrow{AH}\rangle}$$

$$\overset{H}{=} \frac{\langle\overrightarrow{AB},\overrightarrow{AC}\rangle}{\langle\overrightarrow{AB},\overrightarrow{AC}\rangle} \overset{simplify}{=} 1$$

The eliminants

$$\langle\overrightarrow{AC},\overrightarrow{AH}\rangle \overset{H}{=} \langle\overrightarrow{AB},\overrightarrow{AC}\rangle$$

$$\langle\overrightarrow{AB},\overrightarrow{AH}\rangle \overset{H}{=} \langle\overrightarrow{AB},\overrightarrow{AC}\rangle$$

Before giving examples involving circles, let us note that the method presented in Subsection 3.6.2 to eliminate co-circle points is still valid in Minkowskian geometry except that we need to use the *hyperbolic trigonometric* functions instead of the trigonometric functions. A "circle" in Minkowskian geometry is actually a hyperbola:

$$x_1^2 - x_2^2 = r^2.$$

The diameter of the above circle is $\delta = 2r$.

Lemma 5.54 *Let A, B, and C be points on a circle with diameter δ in the Minkowskian plane. Then*

$$S_{ABC} = \frac{\overline{AB}\cdot\overline{CB}\cdot\overline{CA}}{2\delta}, \quad P_{ABC} = \frac{2\overline{AB}\cdot\overline{CB}\cdot\cosh(CA)}{\delta}, \quad \overline{AC} = \delta\sinh(AC).$$

The proof for the above lemma can be carried out as in Section 3.6.

Example 5.55 (Simson's Theorem in Minkowskian Geometry) *The geometry statement of Simson's theorem is exactly the same as in Example 3.79 on page 144.*

Constructive description
((CIRCLE $A\ B\ C\ D$)
(FOOT $E\ D\ B\ C$)
(FOOT $F\ D\ A\ C$)
(FOOT $G\ D\ A\ B$)
($\frac{\overline{AG}}{\overline{GB}}\frac{\overline{BE}}{\overline{EC}}\frac{\overline{CF}}{\overline{FA}} = -1$))

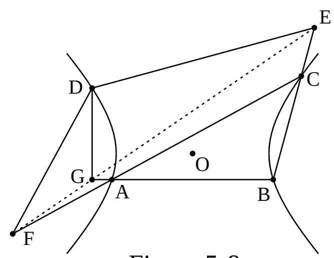


Figure 5-8

The machine proof.

$$\frac{\overline{CF}}{\overline{AF}}\cdot\frac{\overline{BE}}{\overline{CE}}\cdot\frac{\overline{AG}}{\overline{BG}}$$

$$\overset{G}{=} \frac{\langle\overrightarrow{AB},\overrightarrow{AD}\rangle}{-\langle\overrightarrow{BA},\overrightarrow{BD}\rangle}\cdot\frac{\overline{CF}}{\overline{AF}}\cdot\frac{\overline{BE}}{\overline{CE}}$$

$$\overset{F}{=} \frac{-\langle\overrightarrow{CA},\overrightarrow{CD}\rangle\cdot\langle\overrightarrow{AB},\overrightarrow{AD}\rangle}{\langle\overrightarrow{BA},\overrightarrow{BD}\rangle\cdot(-\langle\overrightarrow{AC},\overrightarrow{AD}\rangle)}\cdot\frac{\overline{BE}}{\overline{CE}}$$

$$\overset{E}{=} \frac{\langle\overrightarrow{CA},\overrightarrow{CD}\rangle\cdot\langle\overrightarrow{BC},\overrightarrow{BD}\rangle\cdot\langle\overrightarrow{AB},\overrightarrow{AD}\rangle}{\langle\overrightarrow{BA},\overrightarrow{BD}\rangle\cdot\langle\overrightarrow{AC},\overrightarrow{AD}\rangle\cdot(-\langle\overrightarrow{CB},\overrightarrow{CD}\rangle)}$$

$$\overset{co-cir}{=} \frac{-(-2\overline{CD}\cdot\overline{AC}\cdot\cosh(AD))\cdot(2\overline{BD}\cdot\overline{BC}\cdot\cosh(CD))\cdot(2\overline{AD}\cdot\overline{AB}\cdot\cosh(BD))}{(-2\overline{BD}\cdot\overline{AB}\cdot\cosh(AD))\cdot(2\overline{AD}\cdot\overline{AC}\cdot\cosh(CD))\cdot(-2\overline{CD}\cdot\overline{BC}\cdot\cosh(BD))}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AG}}{\overline{BG}} \overset{G}{=} \frac{\langle\overrightarrow{AB},\overrightarrow{AD}\rangle}{-\langle\overrightarrow{BA},\overrightarrow{BD}\rangle}$$

$$\frac{\overline{CF}}{\overline{AF}} \overset{F}{=} \frac{\langle\overrightarrow{CA},\overrightarrow{CD}\rangle}{-\langle\overrightarrow{AC},\overrightarrow{AD}\rangle}$$

$$\frac{\overline{BE}}{\overline{CE}} \overset{E}{=} \frac{\langle\overrightarrow{BC},\overrightarrow{BD}\rangle}{-\langle\overrightarrow{CB},\overrightarrow{CD}\rangle}$$

$\langle\overrightarrow{CB},\overrightarrow{CD}\rangle = -2\overline{CD}\cdot\overline{BC}\cdot\cosh(BD)$
$\langle\overrightarrow{AC},\overrightarrow{AD}\rangle = 2\overline{AD}\cdot\overline{AC}\cdot\cosh(CD)$
$\langle\overrightarrow{BA},\overrightarrow{BD}\rangle = -2\overline{BD}\cdot\overline{AB}\cdot\cosh(AD)$
$\langle\overrightarrow{AB},\overrightarrow{AD}\rangle = 2\overline{AD}\cdot\overline{AB}\cdot\cosh(BD)$
$\langle\overrightarrow{BC},\overrightarrow{BD}\rangle = 2\overline{BD}\cdot\overline{BC}\cdot\cosh(CD)$
$\langle\overrightarrow{CA},\overrightarrow{CD}\rangle = -2\overline{CD}\cdot\overline{AC}\cdot\cosh(AD)$

By Menelaus' theorem, E, F, and G are collinear.

Example 5.56 (Cantor's Theorem) *The same as Example 3.82 on page 147.*

Constructive description
((CIRCLE $A\ B\ C\ D$)
(CIRCUMCENTER $O\ A\ B\ C$)
(MIDPOINT $G\ A\ D$)
(MIDPOINT $F\ A\ B$)
(MIDPOINT $E\ C\ D$)
(PRATIO $N\ E\ O\ F\ 1$)
(PERPENDICULAR $G\ N\ B\ C$))

The eliminants

$\langle\overrightarrow{BC},\overrightarrow{BN}\rangle \overset{N}{=} \langle\overrightarrow{BC},\overrightarrow{BE}\rangle + \langle\overrightarrow{BC},\overrightarrow{BF}\rangle - \langle\overrightarrow{BC},\overrightarrow{BO}\rangle$
$\langle\overrightarrow{BC},\overrightarrow{BE}\rangle \overset{E}{=} \frac{1}{2}(\langle\overrightarrow{BC},\overrightarrow{BD}\rangle + \langle\overrightarrow{CB},\overrightarrow{CB}\rangle)$
$\langle\overrightarrow{BC},\overrightarrow{BF}\rangle \overset{F}{=} \frac{1}{2}(\langle\overrightarrow{BA},\overrightarrow{BC}\rangle)$
$\langle\overrightarrow{BC},\overrightarrow{BG}\rangle \overset{G}{=} \frac{1}{2}(\langle\overrightarrow{BC},\overrightarrow{BD}\rangle + \langle\overrightarrow{BA},\overrightarrow{BC}\rangle)$
$\langle\overrightarrow{BC},\overrightarrow{BO}\rangle \overset{O}{=} \frac{1}{2}(\langle\overrightarrow{CB},\overrightarrow{CB}\rangle)$

The machine proof.

$$\frac{\langle\overrightarrow{BC},\overrightarrow{BG}\rangle}{\langle\overrightarrow{BC},\overrightarrow{BN}\rangle}$$

$$\stackrel{N}{=} \frac{\langle\overrightarrow{BC},\overrightarrow{BG}\rangle}{\langle\overrightarrow{BC},\overrightarrow{BE}\rangle+\langle\overrightarrow{BC},\overrightarrow{BF}\rangle-\langle\overrightarrow{BC},\overrightarrow{BO}\rangle}$$

$$\stackrel{E}{=} \frac{\langle\overrightarrow{BC},\overrightarrow{BG}\rangle}{\langle\overrightarrow{BC},\overrightarrow{BF}\rangle-\langle\overrightarrow{BC},\overrightarrow{BO}\rangle+\frac{1}{2}\langle\overrightarrow{BC},\overrightarrow{BD}\rangle+\frac{1}{2}\langle\overrightarrow{CB},\overrightarrow{CB}\rangle}$$

$$\stackrel{F}{=} \frac{(2)\cdot\langle\overrightarrow{BC},\overrightarrow{BG}\rangle}{-2\langle\overrightarrow{BC},\overrightarrow{BO}\rangle+\langle\overrightarrow{BC},\overrightarrow{BD}\rangle+\langle\overrightarrow{CB},\overrightarrow{CB}\rangle+\langle\overrightarrow{BA},\overrightarrow{BC}\rangle}$$

$$\stackrel{G}{=} \frac{(-2)\cdot(\frac{1}{2}\langle\overrightarrow{BC},\overrightarrow{BD}\rangle+\frac{1}{2}\langle\overrightarrow{BA},\overrightarrow{BC}\rangle)}{2\langle\overrightarrow{BC},\overrightarrow{BO}\rangle-\langle\overrightarrow{BC},\overrightarrow{BD}\rangle-\langle\overrightarrow{CB},\overrightarrow{CB}\rangle-\langle\overrightarrow{BA},\overrightarrow{BC}\rangle}$$

$$\stackrel{O}{=} \frac{-(\langle\overrightarrow{BC},\overrightarrow{BD}\rangle+\langle\overrightarrow{BA},\overrightarrow{BC}\rangle)\cdot(2)}{-2\langle\overrightarrow{BC},\overrightarrow{BD}\rangle-2\langle\overrightarrow{BA},\overrightarrow{BC}\rangle}$$

$$\stackrel{simplify}{=} 1$$

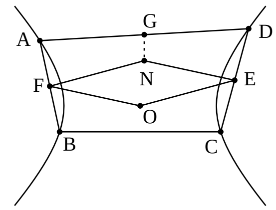


Figure 5-9

In the last step, let T be the midpoint of BC. Then we have $BC \perp TO$, and hence

$$\langle\overrightarrow{BC},\overrightarrow{BO}\rangle = \langle\overrightarrow{BC},\overrightarrow{BT}\rangle = \frac{1}{2}(\langle\overrightarrow{CB},\overrightarrow{CB}\rangle).$$

5.5 Machine Proof Using Complex Numbers

In addition to vectors, complex numbers may be used to generate readable proofs for geometry theorems. Complex numbers are often used to solve difficult geometry theorems and to interpret various different geometries. See [11, 38]. They have also been used to mechanical geometry theorem proving in [151] based on the Gröbner basis computation. In this section, we will show that it is possible to obtain readable proofs for many geometry theorems using complex numbers.

Complex numbers may be looked as vectors. But we can also multiply two complex numbers. We will start our discussion with this special property of complex numbers. Let

$$\boldsymbol{x} = x_1 + x_2 i \text{ and } \boldsymbol{y} = y_1 + y_2 i$$

be two complex numbers, where $i = \sqrt{-1}$. The *conjugate* of $\boldsymbol{y} = y_1 + y_2 i$ is $\tilde{\boldsymbol{y}} = y_1 - y_2 i$. Then

$$\boldsymbol{x} \cdot \tilde{\boldsymbol{y}} = x_1 y_1 + x_2 y_2 - (x_1 y_2 - x_2 y_1) i.$$

If the complex number $\boldsymbol{x} = x_1 + x_2 i$ is seen as the same as the vector $\boldsymbol{x} = (x_1, x_2)$, then it is clear that

$$\boldsymbol{x} \cdot \tilde{\boldsymbol{y}} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle - [\boldsymbol{x}, \boldsymbol{y}] i.$$

Thus
$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \frac{1}{2}(\boldsymbol{x}\cdot\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{x}}\cdot\boldsymbol{y}); \quad [\boldsymbol{x}, \boldsymbol{y}] = \frac{1}{2i}(\tilde{\boldsymbol{x}}\cdot\boldsymbol{y} - \boldsymbol{x}\cdot\tilde{\boldsymbol{y}}).$$
That is, the inner product and the exterior product can be expressed by the multiplications of complex numbers.

In geometric language, for each point P, let $\overrightarrow{P}$ be the corresponding complex number, and $\tilde{P}$ the conjugate of $\overrightarrow{P}$. Then the above two equations become

$$P_{ABC} = (\overrightarrow{B} - \overrightarrow{A})(\tilde{C} - \tilde{A}) + (\tilde{B} - \tilde{A})(\overrightarrow{C} - \overrightarrow{A})$$
$$S_{ABC} = \tfrac{1}{4i}((\tilde{B} - \tilde{A})(\overrightarrow{C} - \overrightarrow{A}) - (\overrightarrow{B} - \overrightarrow{A})(\tilde{C} - \tilde{A})).$$

Therefore, the vector approach for Euclidean geometry can be translated into the language of complex numbers. But the proofs thus produced are generally longer than those produced by the vector approach. The reason is that in the vector approach the area and Pythagoras difference are treated like one-term variables, while in the complex number approach they are expressions of several terms. This complex number approach is essentially the same as the Wu's method in the case of constructive geometry statements [56, 63].

But in many cases, the complex number approach does give short proofs.

Example 5.57 *As shown in Figure 5-10, on two sides AC and BC of triangle ABC, two similar triangles PAC and QCB are drawn. $RPCQ$ is a parallelogram. Show that triangle RAB is similar with triangle PAC.*

For two points A and B, let $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$. Two triangles PAB and QCB are similar and have the same orientation iff

$$\frac{\overrightarrow{PA}}{\overrightarrow{AC}} = \frac{\overrightarrow{QC}}{\overrightarrow{CB}} \quad \text{or} \quad \overrightarrow{PA}\cdot\overrightarrow{CB} = \overrightarrow{AC}\cdot\overrightarrow{QC}. \tag{5.1}$$

We thus can use a new construction

(SIM-TRIANGLE $Q\ C\ B\ P\ A\ C$)

which introduces a point Q such that (5.1) is true.

Now Example 5.57 can be described constructively as follows.

Constructive description

((POINTS $A\ B\ C\ P$)

(SIM-TRIANGLE $Q\ B\ C\ P\ C\ A$)

(PRATIO $R\ Q\ C\ P$ 1)

($\overrightarrow{PA}\cdot\overrightarrow{AB} = \overrightarrow{RA}\cdot\overrightarrow{AC}$))

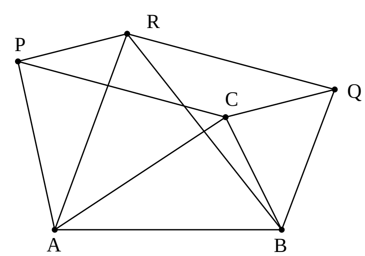


Figure 5-10

The machine proof.

$$\frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{\overrightarrow{AR}\cdot\overrightarrow{AC}}$$

$$\overset{R}{=} \frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{(\overrightarrow{AQ}+\overrightarrow{AP}-\overrightarrow{AC})\cdot\overrightarrow{AC}}$$

$$\overset{Q}{=} \frac{\overrightarrow{AP}\cdot\overrightarrow{AB}\cdot(-\overrightarrow{AC})}{(\overrightarrow{CP}\cdot\overrightarrow{BC}-\overrightarrow{B}\cdot\overrightarrow{AC}-\overrightarrow{AP}\cdot\overrightarrow{AC}+\overrightarrow{AC}^2+\overrightarrow{AC}\cdot\overrightarrow{A})\cdot\overrightarrow{AC}}$$

$$\overset{simplify}{=} \frac{-\overrightarrow{AP}\cdot\overrightarrow{AB}}{\overrightarrow{CP}\cdot\overrightarrow{BC}-\overrightarrow{B}\cdot\overrightarrow{AC}-\overrightarrow{AP}\cdot\overrightarrow{AC}+\overrightarrow{AC}^2+\overrightarrow{AC}\cdot\overrightarrow{A}}$$

$$= \frac{-(\overrightarrow{P}-\overrightarrow{A})\cdot(\overrightarrow{B}-\overrightarrow{A})}{-\overrightarrow{P}\cdot\overrightarrow{B}+\overrightarrow{P}\cdot\overrightarrow{A}+\overrightarrow{B}\cdot\overrightarrow{A}-\overrightarrow{A}^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\overrightarrow{AR} \overset{R}{=} \overrightarrow{AQ}+\overrightarrow{AP}-\overrightarrow{AC}$$

$$\overrightarrow{AQ} \overset{Q}{=} \frac{\overrightarrow{CP}\cdot\overrightarrow{BC}-\overrightarrow{B}\cdot\overrightarrow{AC}+\overrightarrow{AC}\cdot\overrightarrow{A}}{-\overrightarrow{AC}}$$

$$\overrightarrow{AC} = \overrightarrow{C}-\overrightarrow{A}$$

$$\overrightarrow{BC} = \overrightarrow{C}-\overrightarrow{B}$$

$$\overrightarrow{CP} = \overrightarrow{P}-\overrightarrow{C}$$

$$\overrightarrow{AB} = \overrightarrow{B}-\overrightarrow{A}$$

$$\overrightarrow{AP} = \overrightarrow{P}-\overrightarrow{A}$$

Let $\omega = e^{\frac{2i\pi}{3}}$. Then the three points corresponding to the three complex numbers $1, \omega$, and ω^2 form an equilateral triangle with positive orientation (Figure 5-11).

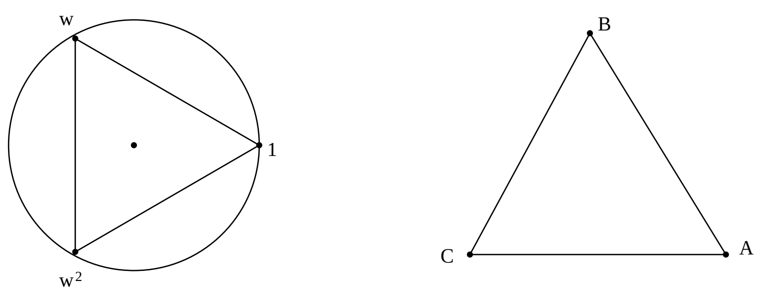


Figure 5-11

Thus ABC is an *equilateral triangle* with positive orientation (Figure 5-11) iff

$$\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \frac{\omega - 1}{\omega^2 - 1}.$$

From the above equation, we have

$$(\omega^3 - \omega^2)\overrightarrow{B} + (\omega - 1)\overrightarrow{C} + (\omega^2 - \omega)\overrightarrow{A} = 0.$$

Dividing $\omega - 1$ from both sides of the above equation, we have that ABC is an equilateral triangle with positive orientation iff

$$\overrightarrow{C} + \omega^2\overrightarrow{B} + \omega\overrightarrow{A} = 0.$$

Similarly ABC is an equilateral triangle with negative orientation iff

$$\overrightarrow{C} + \omega\overrightarrow{B} + \omega^2\overrightarrow{A} = 0.$$

We thus introduce two new constructions.

(PE-TRIANGLE C B A) introduces a point C such that ABC is an equilateral triangle with positive orientation, i.e., $\overrightarrow{C} + \omega^2\overrightarrow{B} + \omega\overrightarrow{A} = 0$.

(NE-TRIANGLE C B A) introduces a point C such that ABC is an equilateral triangle with negative orientation, i.e, $\overrightarrow{C} + \omega\overrightarrow{B} + \omega^2\overrightarrow{A} = 0$.

Example 5.58 (Echols' First Theorem[1]) *If ABC and PQR are equilateral triangles with the same orientation, then the triangle formed by the midpoints of AP, BQ, and CR is also an equilateral triangle.*

Constructive description

((POINTS A B P Q)
(PE-TRIANGLE C B A)
(PE-TRIANGLE R Q P)
(MIDPOINT L A P)
(MIDPOINT M B Q)
(MIDPOINT N C R)
(PE-TRIANGLE N M L))

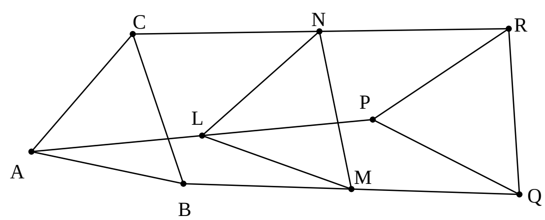


Figure 5-12

The machine proof.

$$\overrightarrow{N} + \overrightarrow{M}\cdot w + \overrightarrow{L}\cdot w^2$$
$$\overset{n}{=} \overrightarrow{M}\cdot w + \overrightarrow{L}\cdot w^2 + \tfrac{1}{2}\overrightarrow{R} + \tfrac{1}{2}\overrightarrow{C}$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(2\overrightarrow{L}\cdot w^2 + \overrightarrow{R} + \overrightarrow{C} + \overrightarrow{Q}\cdot w + \overrightarrow{B}\cdot w)$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(\overrightarrow{R} + \overrightarrow{C} + \overrightarrow{Q}\cdot w + \overrightarrow{P}\cdot w^2 + \overrightarrow{B}\cdot w + \overrightarrow{A}\cdot w^2)$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(\overrightarrow{C} + \overrightarrow{B}\cdot w + \overrightarrow{A}\cdot w^2)$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(0)$$
$$\overset{simplify}{=} 0$$

The eliminants

$$\overrightarrow{N} \overset{N}{=} \frac{1}{2}(\overrightarrow{R} + \overrightarrow{C})$$
$$\overrightarrow{M} \overset{M}{=} \frac{1}{2}(\overrightarrow{Q} + \overrightarrow{B})$$
$$\overrightarrow{L} \overset{L}{=} \frac{1}{2}(\overrightarrow{P} + \overrightarrow{A})$$
$$\overrightarrow{R} \overset{R}{=} -((\overrightarrow{Q} + \overrightarrow{P}\cdot w)\cdot w)$$
$$\overrightarrow{C} \overset{C}{=} -((\overrightarrow{B} + \overrightarrow{A}\cdot w)\cdot w)$$

Example 5.59 (Echols' Second Theorem) *If ABC, PQR, and XYZ are equilateral triangles with the same orientation, then the triangle formed by the centroids of triangle APX, BQY, and CRZ is an equilateral triangle.*

[1]American Mathematical Monthly, 39, 1932, p.46

Constructive description

((POINTS $A\ B\ P\ Q\ X\ Y$)
(PE-TRIANGLE $C\ B\ A$)
(PE-TRIANGLE $R\ Q\ P$)
(PE-TRIANGLE $Z\ Y\ X$)
(CENTROID $L\ A\ P\ X$)
(CENTROID $M\ B\ Q\ Y$)
(CENTROID $N\ C\ R\ Z$)
(PE-TRIANGLE $N\ M\ L$))

The eliminants

$\overrightarrow{N} \stackrel{N}{=} \frac{1}{3}(\overrightarrow{Z}+\overrightarrow{R}+\overrightarrow{C})$

$\overrightarrow{M} \stackrel{M}{=} \frac{1}{3}(\overrightarrow{Y}+\overrightarrow{Q}+\overrightarrow{B})$

$\overrightarrow{L} \stackrel{L}{=} \frac{1}{3}(\overrightarrow{X}+\overrightarrow{P}+\overrightarrow{A})$

$\overrightarrow{Z} \stackrel{Z}{=} -((\overrightarrow{Y}+\overrightarrow{X}\cdot w)\cdot w)$

$\overrightarrow{R} \stackrel{R}{=} -((\overrightarrow{Q}+\overrightarrow{P}\cdot w)\cdot w)$

$\overrightarrow{C} \stackrel{C}{=} -((\overrightarrow{B}+\overrightarrow{A}\cdot w)\cdot w)$

The machine proof.

$\overrightarrow{N}+\overrightarrow{M}\cdot w+\overrightarrow{L}\cdot w^2$

$\stackrel{n}{=} 3\overrightarrow{M}\cdot w+3\overrightarrow{L}\cdot w^2+\overrightarrow{Z}+\overrightarrow{R}+\overrightarrow{C}$

$\stackrel{n}{=} 9\overrightarrow{L}\cdot w^2+3\overrightarrow{Z}+3\overrightarrow{R}+3\overrightarrow{C}+3\overrightarrow{Y}\cdot w+3\overrightarrow{Q}\cdot w+3\overrightarrow{B}\cdot w$

$\stackrel{n}{=} (3)\cdot(3\overrightarrow{Z}+3\overrightarrow{R}+3\overrightarrow{C}+3\overrightarrow{Y}\cdot w+3\overrightarrow{X}\cdot w^2+3\overrightarrow{Q}\cdot w+3\overrightarrow{P}\cdot w^2+3\overrightarrow{B}\cdot w+3\overrightarrow{A}\cdot w^2)$

$\stackrel{n}{=} (9)\cdot(\overrightarrow{R}+\overrightarrow{C}+\overrightarrow{Q}\cdot w+\overrightarrow{P}\cdot w^2+\overrightarrow{B}\cdot w+\overrightarrow{A}\cdot w^2)$

$\stackrel{n}{=} (9)\cdot(\overrightarrow{C}+\overrightarrow{B}\cdot w+\overrightarrow{A}\cdot w^2)$

$\stackrel{simplify}{=} 0$

Example 5.60 (Echols' Theorem in General Form) *Let $P_{i,1}...P_{i,n}$, $i = 1,...,m$ be m regular n-polygons with the same orientation. Then the centroids of the m-polygons $P_{1,j}...P_{m,j}$, $j = 1,...,n$, form a regular n-polygon.*

Proof. From the proofs of Examples 5.58 and 5.59, we need only to show that $A_1...A_n$ is a regular n-polygon iff points $A_1,...,A_n$ satisfy some linear relations $R_k(A_1,...,A_n) = 0, k = 1,...s$. We leave the details to the reader. ∎

Example 5.61 *The same as Example 3.45 on page 127. The following proof is much shorter. (Figure 5-13)*

Constructive description

((POINTS A B C)
(CONSTANT w^2+w+1)
(PE-TRIANGLE B_1 A C)
(PE-TRIANGLE A_1 C B)
(NE-TRIANGLE C_1 B A)
($\overrightarrow{A_1C_1}-\overrightarrow{CB_1}=0$))

The machine proof.

$\overrightarrow{A_1C_1}-\overrightarrow{CB_1}$

$\stackrel{n}{=} -\overrightarrow{A_1}-\overrightarrow{CB_1}-\overrightarrow{B}\cdot w^2-\overrightarrow{A}\cdot w$

$\stackrel{n}{=} -(\overrightarrow{CB_1}-\overrightarrow{C}\cdot w+\overrightarrow{A}\cdot w)$

$\stackrel{n}{=} -(-\overrightarrow{C}\cdot w^2-\overrightarrow{C}\cdot w-\overrightarrow{C})$

$\stackrel{simplify}{=} (w^2+w+1)\cdot\overrightarrow{C}$

$\stackrel{n}{=} 0$

$\stackrel{simplify}{=} 0$

The eliminants

$\overrightarrow{A_1C_1} \stackrel{C_1}{=} -(\overrightarrow{A_1}+\overrightarrow{B}\cdot w^2+\overrightarrow{A}\cdot w)$

$\overrightarrow{A_1} \stackrel{A_1}{=} -((\overrightarrow{C}+\overrightarrow{B}\cdot w)\cdot w)$

$\overrightarrow{CB_1} \stackrel{B_1}{=} -(\overrightarrow{C}\cdot w^2+\overrightarrow{C}+\overrightarrow{A}\cdot w)$

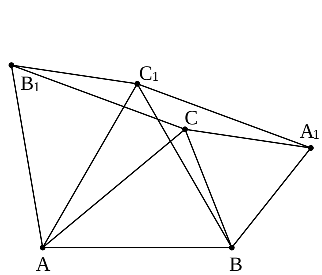


Figure 5-13

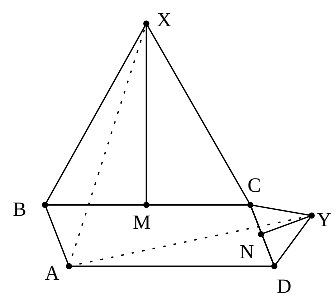


Figure 5-14

Example 5.62 *The converse of Example 5.61. (Figure 5-14)*

Constructive description

((POINTS A B C)
(CONSTANT w^2+w+1)
(PRATIO D A B C 1)
(PE-TRIANGLE X B C)
(PE-TRIANGLE Y C D)
(PE-TRIANGLE A Y X))

The machine proof.

$\overrightarrow{Y}\cdot w+\overrightarrow{X}\cdot w^2+\overrightarrow{A}$

$\stackrel{n}{=} \overrightarrow{X}\cdot w^2-\overrightarrow{D}\cdot w^3-\overrightarrow{C}\cdot w^2+\overrightarrow{A}$

$\stackrel{n}{=} -\overrightarrow{D}\cdot w^3-\overrightarrow{C}\cdot w^4-\overrightarrow{C}\cdot w^2-\overrightarrow{B}\cdot w^3+\overrightarrow{A}$

$\stackrel{n}{=} -(\overrightarrow{C}\cdot w^4+\overrightarrow{C}\cdot w^3+\overrightarrow{C}\cdot w^2+\overrightarrow{A}\cdot w^3-\overrightarrow{A})$

$\stackrel{simplify}{=} -(w^2+w+1)\cdot(\overrightarrow{C}\cdot w^2+\overrightarrow{A}\cdot w-\overrightarrow{A})$

$\stackrel{n}{=} 0$

$\stackrel{simplify}{=} 0$

The eliminants

$\overrightarrow{Y} \stackrel{Y}{=} -((\overrightarrow{D}\cdot w+\overrightarrow{C})\cdot w)$

$\overrightarrow{X} \stackrel{X}{=} -((\overrightarrow{C}\cdot w+\overrightarrow{B})\cdot w)$

$\overrightarrow{D} \stackrel{D}{=} \overrightarrow{C}-\overrightarrow{B}+\overrightarrow{A}$

Using complex numbers, we can also deal with some theorems involving squares easily. For two points A and B, C is a point introduced by (TRATIO C A B 1), or equivalently C satisfies $CA = AB$, $CA \perp AB$, and $S_{CAB} > 0$, iff

$$\overrightarrow{AC} = i \cdot \overrightarrow{AB}.$$

Geometrically, the above equation means that $\overrightarrow{AC}$ is obtained by rotating $\overrightarrow{AB}$ by 90° counterclockwise, or CAB is an isosceles right triangle with positive orientation. Similarly, C is a point introduced by (TRATIO C A B -1) iff

$$\overrightarrow{AC} = -i \cdot \overrightarrow{AB}.$$

We thus can introduce two new constructions

(PE-SQUARE C B A) introduces a point C such that CAB is an isosceles right triangle with positive orientation, i.e., $\overrightarrow{C} = \overrightarrow{A} + i \cdot \overrightarrow{AB}$.

(NE-SQUARE C B A) introduces a point C such that CAB is an isosceles right triangle with negative orientation, i.e, $\overrightarrow{C} = \overrightarrow{A} - i \cdot \overrightarrow{AB}$.

Example 5.63 *On the two sides AB and AC of triangle ABC, two squares $ABEF$ and $ACGH$ are drawn externally. Show that $FC \perp BH$ and $FC = BH$.*

Constructive description:

((POINTS A B C) (CONSTANT i^2-1) (PE-SQUARE F A B) (NE-SQUARE H A C) ($\overrightarrow{FC}-i\cdot\overrightarrow{BH} = 0$))

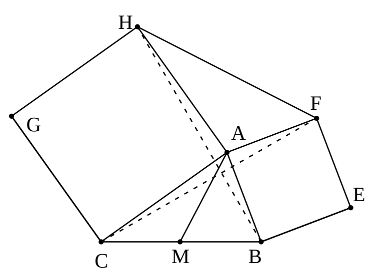


Figure 5-15

The machine proof

$$-(\overrightarrow{CF}+\overrightarrow{BH}\cdot i)$$
$$\stackrel{n}{=} -(\overrightarrow{CF}-\overrightarrow{AC}\cdot i^2-\overrightarrow{AB}\cdot i)$$
$$\stackrel{n}{=} -(-\overrightarrow{AC}\cdot i^2-\overrightarrow{AC})$$
$$\stackrel{simplify}{=} (i^2+1)\cdot\overrightarrow{AC}$$
$$\stackrel{n}{=} 0$$

The eliminants

$$\overrightarrow{BH} \stackrel{H}{=} -\left(\overrightarrow{AC}\cdot i+\overrightarrow{AB}\right)$$
$$\overrightarrow{CF} \stackrel{F}{=} -\left(\overrightarrow{AC}-\overrightarrow{AB}\cdot i\right)$$

Example 5.64 *On the two sides AB and AC of triangle ABC, two squares $ABEF$ and $ACGH$ are drawn externally. M is the midpoint of BC. Show that $FH = 2AM$ and $FH \perp AM$. (Figure 5-15)*

Constructive description

((POINTS A B C)
(PE-SQUARE F A B)
(NE-SQUARE H A C)
(MIDPOINT M B C)
($\overrightarrow{HF}-2i\cdot\overrightarrow{AM} = 0$))

The machine proof

$$-(\overrightarrow{FH}+2\overrightarrow{AM}\cdot i)$$
$$\stackrel{n}{=} -(\overrightarrow{FH}+\overrightarrow{AC}\cdot i+\overrightarrow{AB}\cdot i)$$
$$\stackrel{n}{=} -(-\overrightarrow{AF}+\overrightarrow{AB}\cdot i)$$
$$\stackrel{n}{=} 0$$

The eliminants

$$\overrightarrow{AM} \stackrel{M}{=} \frac{1}{2}\left(\overrightarrow{AC}+\overrightarrow{AB}\right)$$
$$\overrightarrow{FH} \stackrel{H}{=} -\left(\overrightarrow{AF}+\overrightarrow{AC}\cdot i\right)$$
$$\overrightarrow{AF} \stackrel{F}{=} \overrightarrow{AB}\cdot i$$

Example 5.65 [2] *Starting with any triangle ABC, construct the exterior (or interior) squares $BCDE$, $ACFG$, and $BAHK$; then construct parallelograms $FCDQ$ and $EBKP$. Show that PAQ is an isosceles right triangle.*

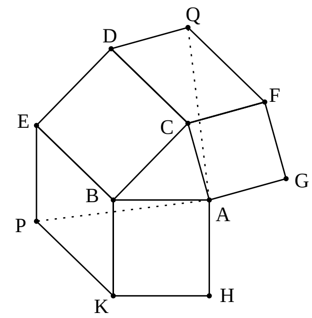


Figure 5-16

Constructive description

((POINTS A B C)
(CONSTANT i^2 -1)
(NE-SQUARE F C A)
(PE-SQUARE D C B)
(PRATIO E D C B 1)
(PE-SQUARE K B A)
(PRATIO Q F C D 1)
(PRATIO P K B E 1)
($\overrightarrow{AQ}-i\cdot\overrightarrow{AP}=0$))

The machine proof

$$
\begin{aligned}
&-(\overrightarrow{AP}\cdot i-\overrightarrow{AQ})\\
\overset{n}{=}\ &-(-\overrightarrow{AQ}+\overrightarrow{AK}\cdot i+\overrightarrow{AE}\cdot i-\overrightarrow{AB}\cdot i)\\
\overset{n}{=}\ &-\overrightarrow{AK}\cdot i-\overrightarrow{AE}\cdot i+\overrightarrow{AD}+\overrightarrow{AF}-\overrightarrow{AC}+\overrightarrow{AB}\cdot i\\
\overset{n}{=}\ &-(\overrightarrow{AE}\cdot i-\overrightarrow{AD}-\overrightarrow{AF}+\overrightarrow{AC}-\overrightarrow{AB}\cdot i^2)\\
\overset{n}{=}\ &-(\overrightarrow{AD}\cdot i-\overrightarrow{AD}-\overrightarrow{AF}-\overrightarrow{AC}\cdot i+\overrightarrow{AC}-\overrightarrow{AB}\cdot i^2+\overrightarrow{AB}\cdot i)\\
\overset{n}{=}\ &-(-\overrightarrow{BC}\cdot i^2+\overrightarrow{BC}\cdot i-\overrightarrow{AF}-\overrightarrow{AB}\cdot i^2+\overrightarrow{AB}\cdot i)\\
\overset{n}{=}\ &\overrightarrow{BC}\cdot i^2-\overrightarrow{BC}\cdot i+\overrightarrow{AC}\cdot i+\overrightarrow{AC}+\overrightarrow{AB}\cdot i^2-\overrightarrow{AB}\cdot i\\
\overset{n}{=}\ &-\overrightarrow{BC}\cdot i-\overrightarrow{BC}+\overrightarrow{AC}\cdot i+\overrightarrow{AC}-\overrightarrow{AB}\cdot i-\overrightarrow{AB}\\
\overset{simplify}{=}\ &-(i+1)\cdot(\overrightarrow{BC}-\overrightarrow{AC}+\overrightarrow{AB})\\
\overset{n}{=}\ &-(i+1)\cdot(0)\\
\overset{simplify}{=}\ &0
\end{aligned}
$$

The eliminants

$$
\begin{aligned}
&\overrightarrow{AP}\overset{P}{=}\overrightarrow{AK}+\overrightarrow{AE}-\overrightarrow{AB}\\
&\overrightarrow{AQ}\overset{Q}{=}\overrightarrow{AD}+\overrightarrow{AF}-\overrightarrow{AC}\\
&\overrightarrow{AK}\overset{K}{=}-\left((i-1)\cdot\overrightarrow{AB}\right)\\
&\overrightarrow{AE}\overset{E}{=}\overrightarrow{AD}-\overrightarrow{AC}+\overrightarrow{AB}\\
&\overrightarrow{AD}\overset{D}{=}-\left(\overrightarrow{BC}\cdot i-\overrightarrow{AC}\right)\\
&\overrightarrow{AF}\overset{F}{=}(i+1)\cdot\overrightarrow{AC}\\
&\overrightarrow{AB}=\overrightarrow{B}-\overrightarrow{A}\\
&\overrightarrow{AC}=\overrightarrow{C}-\overrightarrow{A}\\
&\overrightarrow{BC}=\overrightarrow{C}-\overrightarrow{B}
\end{aligned}
$$

Summary of Chapter 5

- The metric vector space is a vector space with inner and exterior products. The metric geometry of dimension three associated with field $\mathcal{E}$ is the nonsingular metric vector space $\mathcal{E}^3$.

- Some basic geometry quantities can be described by the inner and exterior products:

  1. $P_{ABC}=2\langle\overrightarrow{AB},\overrightarrow{CB}\rangle$; $P_{ABCD}=2\langle\overrightarrow{AC},\overrightarrow{DB}\rangle$.
  2. If four points A, B, C, and D are collinear or $AB \parallel CD$,

$$\overrightarrow{AB}=\frac{\overline{AB}}{\overline{CD}}\overrightarrow{CD},\qquad \frac{\overline{AB}}{\overline{CD}}=\frac{\langle\overrightarrow{AB},\overrightarrow{CD}\rangle}{\langle\overrightarrow{CD},\overrightarrow{CD}\rangle}.$$

[2] This example is from Amer. Math. Mon. 75(1968), p.899.

3. $V_{ABCD} = \frac{1}{6}\langle\overrightarrow{AD}, [\overrightarrow{AB}, \overrightarrow{AC}]\rangle$.
4. If six points $A, B, C, P, Q,$ and R are coplanar or $ABC \parallel PQR$,

$$[\overrightarrow{AB}, \overrightarrow{AC}] = \frac{S_{ABC}}{S_{PQR}}[\overrightarrow{PQ}, \overrightarrow{PR}], \qquad \frac{S_{ABC}}{S_{PQR}} = \frac{\langle[\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle}{\langle[\overrightarrow{PQ}, \overrightarrow{PR}], [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle}.$$

- We have the following criteria for parallel and perpendicularity.

1. $AB \perp CD \iff \langle\overrightarrow{AB}, \overrightarrow{CD}\rangle = 0$.
2. $AB \parallel CD \iff [\overrightarrow{AB}, \overrightarrow{CD}] = 0$.
3. $AB \perp PQR \iff [\overrightarrow{AB}, [\overrightarrow{PQ}, \overrightarrow{PR}]] = 0$.
4. $AB \parallel PQR \iff \langle\overrightarrow{AB}, [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle = 0$.
5. $ABC \perp PQR \iff \langle[\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}]\rangle = 0$.
6. $ABC \parallel PQR \iff [[\overrightarrow{AB}, \overrightarrow{AC}], [\overrightarrow{PQ}, \overrightarrow{PR}]] = 0$.

- A mechanical proving method is presented for constructive geometry statements in metric geometry of dimensions two and three. This method works similarly to the area-volume-Pythagoras difference method developed in Chapters 3 and 4. The basis of the method is to eliminate points from vectors and inner and exterior products of vectors.

- We have proved the following meta theorem.

  A pure constructive geometry statement is true in one metric geometry iff it is true in all the metric geometries associated with the same field.

Chapter 6

A Collection of 400 Mechanically Proved Theorems

This chapter is a collection of 400 geometry theorems in plane geometry proved automatically by a computer program[1] based on the method developed in the first part of this book, including 205 machine proofs (in LaTeX form) produced *entirely automatically* by the program. In addition, there are another 78 problems solved by our computer program in Chapters 2–5.

We include a collection like this for two reasons. First, it would show the power of the method/program: most theorems involving equalities only in geometry textbooks are in the collection. Second, many proofs produced by the program are very beautiful. Following the spirit of E. W. Dijkstra (p. 174 [19]), we believe that such beautiful proofs deserve special attention.

Among the work consulted to collect these examples, special mention must be made of the following [1, 3, 4, 13, 23, 39]. The figures and inputs for many of the examples in this collection are from [12] directly. The examples are classified according to the types of constructions needed to describe them.

6.1 Notation Convention

As an example of how to read the examples in this collection, let us consider Ceva's theorem. As shown, a typical entry in this collection includes a description of the theorem in English, a diagram, a constructive description of the theorem as the input to the program, and a machine proof produced by the program.

[1]The program is available via ftp at emcity.cs.twsu.edu: pub/geometry.

Example 6.0 (Ceva's Theorem) (0.01,1,3) *The lines joining the vertices of a triangle to a given point determine on the sides of the triangle six segments such that the product of three nonconsecutive segments is equal to the product of the remaining three segments.*

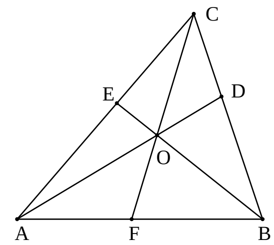


Figure 6-0

Constructive description

((points $A\ B\ C\ O$)

(inter D (l $B\ C$) (l $O\ A$))

(inter E (l $A\ C$) (l $O\ B$))

(inter F (l $A\ B$) (l $O\ C$))

($\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$))

The machine proof

$$-\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}\cdot\frac{\overline{AF}}{\overline{BF}}$$

$$\overset{F}{=} \frac{-(-S_{ACO})}{-S_{BCO}}\cdot\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{E}{=} \frac{-S_{BCO}\cdot S_{ACO}}{S_{BCO}\cdot(-S_{ABO})}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{simplify}{=} \frac{S_{ACO}}{S_{ABO}}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{D}{=} \frac{S_{ABO}\cdot S_{ACO}}{S_{ABO}\cdot S_{ACO}} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}} \overset{F}{=} \frac{S_{ACO}}{S_{BCO}}$$

$$\frac{\overline{CE}}{\overline{AE}} \overset{E}{=} \frac{S_{BCO}}{-S_{ABO}}$$

$$\frac{\overline{BD}}{\overline{CD}} \overset{D}{=} \frac{S_{ABO}}{S_{ACO}}$$

Notation Convention.

1. We use a triple *(time,maxt,lems)* to measure how difficult a machine proof is: *time* is the time needed to complete the machine proof in a NexT Turbo workstation (25 MIPS); *maxt* is the number of terms of the maximal polynomial occurring in the machine proof; and *lems* is the number of elimination lemmas used to eliminate points from geometry quantities, i.e., the number of the eliminants. For Ceva's theorem, *time = 0.01 second*, *maxterm = 1*, and *lems = 3*.

2. We use some abbreviations in the constructive description of the geometry statements in order to save printing space. For instance, LINE is represented by l; PLINE is represented by p; BLINE is represented by b; and ALINE is represented by a.

3. The ndg conditions and the predicate forms of the geometry statements are not given, since they can be generated directly from the constructive description. See page 110.

4. For explanation of the machine proof and the eliminants, see page 72.

In Part I of the book, we have introduced 28 constructions for plane geometry, which are listed here for your convenience.

1. (ARATIO $A\ O\ U\ V\ r_O\ r_U\ r_V$). Take a point A such that r_U, r_V, and r_O are the *area coordinates* of A with respect to OUV. For the exact definition of this construction, see page 136.

2. (CENTROID $G\ A\ B\ C$). G is the centroid of triangle ABC. See page 138.

3. (CIRCLE $Y_1 \cdots Y_s$), $(s \geq 3)$. Points $Y_1 \cdots Y_s$ are on the same circle. See page 144.

4. (CIRCUMCENTER O A B C). O is the circumcenter of triangle ABC. See page 138.

5. (CONSTANT $p(r)$) where $p(r)$ is an irreducible polynomial in r. This construction introduces an algebraic number r which is a root of $p(r) = 0$. See page 123.

6. (HARMONIC D C A B) means that D is a point such that A, B, C, and D form a harmonic consequence. See page 123.

7. (INCENTER C I A B) I is the center of the inscribed circle of triangle ABC. This construction is to construct point C from points A, B, and I. See page 138.

8. (INTER Y $ln1$ $ln2$). Point Y is the intersection of lines $ln1$ and $ln2$. See page 110.

9. (INTER Y ln (CIR O P)). Point Y is the intersection of line ln and circle (CIR O P) other than point P. Line ln could be (LINE P U), (PLINE P U V), and (TLINE P U V). See page 110.

10. (INTER Y (LINE U V) (CIR O r)). Point Y is one of the intersections of line (LINE U V) and circle (CIR O r). See page 148.

11. (INTER Y (CIR O_1 P) (CIR O_2 P)). Point Y is the intersection of the circle (CIR O_1 P) and the circle (CIR O_2 P) other than point P. See page 110.

12. (INTER Y (CIR O_1 r_1) (CIR O_2 r_2)). Point Y is one of the intersections of the circle (CIR O_1 r_1) and the circle (CIR O_2 r_2). See page 148.

13. (INVERSION P Q O A) means that P is the inversion of Q with regard to circle (CIR O A). See page 123.

14. (LRATIO Y U V r). Y is a point on UV such that $\frac{\overline{UY}}{\overline{UV}} = r$. See page 122.

15. (ON Y ln). Take a point Y on a line ln. Line ln could be one of the forms below.

    (LINE A B) is the line passing through two points A and B.

    (PLINE C A B) is the line passing through C and parallel to (LINE A B).

    (TLINE C A B) is the line passing through C and perpendicular to (LINE A B).

    (BLINE A B) is the perpendicular-bisector of AB.

    (ALINE P Q U W V) is the line l passing through P such that $\angle[PQ, l] = \angle[UW, WV]$.

    See pages 110 and 129.

16. (ON Y (CIR O P)). Take a point Y on a circle (CIR O P). See page 110.

17. (MIDPOINT Y U V). Y is the midpoint of UV. See page 122.

18. (MRATIO $Y\ U\ V\ r$). Y is a point on UV such that $\frac{\overline{UY}}{\overline{YV}} = r$. See page 122.

19. (NE-SQUARE $C\ B\ A$) introduces a point C such that CAB is an isosceles right triangle with negative orientation, i.e, $\overrightarrow{C} = \overrightarrow{A} - i \cdot \overrightarrow{AB}$. See page 255.

20. (NE-TRIANGLE $C\ B\ A$) introduces a point C such that ABC is an equilateral triangle with negative orientation. See page 251.

21. (ORTHOCENTER $H\ A\ B\ C$). H is the orthocenter of the triangle ABC. See page 138.

22. (PE-SQUARE $C\ B\ A$) introduces a point C such that CAB is an isosceles right triangle with positive orientation, i.e., $\overrightarrow{C} = \overrightarrow{A} + i \cdot \overrightarrow{AB}$. See page 255.

23. (PE-TRIANGLE $C\ B\ A$) introduces a point C such that ABC is an equilateral triangle with positive orientation. See page 251.

24. (POINT[S] $Y_1, \cdots, Y_l$). Take arbitrary points $Y_1, \cdots, Y_l$ in the plane. See page 110.

25. (PRATIO $Y\ W\ U\ V\ r$). Take a point Y on the line passing through W and parallel to line UV such that $\overline{WY} = r\overline{UV}$. See page 110.

26. (SIM-TRIANGLE $Q\ C\ B\ P\ A\ C$) introduces a point Q such that triangle QCB and triangle PAC are similar and have the same orientation. See page 250.

27. (SYMMETRY $Y\ U\ V$). Y is the symmetry of point V with respect to point U. See page 122.

28. (TRATIO $Y\ U\ V\ r$). Take a point Y on line (TLINE $U\ U\ V$) such that $r = \frac{4S_{UVY}}{P_{UVU}} (= \frac{\overline{UY}}{\overline{UV}})$. See page 110.

The following are the predicates accepted by the program as conclusions.

1. (COCIRCLE $A\ B\ C\ D$). Points A, B, C, and D are co-circle iff $\angle[CAD] = \angle[CBD]$, or equivalently, $S_{CAD}P_{CBD} = P_{CAD}P_{CBD}$. See page 130.

2. (COLLINEAR $A\ B\ C$). Points A, B, and C are collinear iff $S_{ABC} = 0$. Also see the comments after Example 2.36 on page 74.

3. (EQANGLE $A\ B\ C\ D\ E\ F$). $\angle[ABC] = \angle[DEF]$ iff $S_{ABC}P_{DEF} = S_{DEF}P_{ABC}$. See page 114.

4. (EQDISTANCE $A\ B\ C\ D$). AB has the same length as CD iff $P_{ABA} = P_{CDC}$. See page 114.

5. (EQ-PRODUCT $A\ B\ C\ D\ P\ Q\ R\ S$). The product of AB and CD is equal to the product of PQ and RS. See page 114.

6. (HARMONIC A B C D). A, B and C, D are harmonic points iff $\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$. See page 123.

7. (INVERSION P Q O A). P is the inversion of Q with regard to circle (CIR O A). See page 123.

8. (MIDPOINT O A B). O is the midpoint of AB iff $\frac{\overline{AO}}{\overline{OB}} = 1$. See page 114.

9. (NE-TRIANGLE C B A) ABC is an equilateral triangle with negative orientation, i.e, $\overrightarrow{C} + \omega\overrightarrow{B} + \omega^2\overrightarrow{A} = 0$. See page 251.

10. (ON-RADICAL P O_1 A O_2 B). P is on the axis of circles O_1A and O_2B iff $P_{PO_1P} - P_{AO_1A} = P_{PO_2P} - P_{BO_2B}$.

11. (PARALLEL A B C D). AB is parallel to CD iff $S_{ACD} = S_{BCD}$. See page 114.

12. (PE-TRIANGLE C B A). ABC is an equilateral triangle with positive orientation, i.e., $\overrightarrow{C} + \omega^2\overrightarrow{B} + \omega\overrightarrow{A} = 0$. See page 251.

13. (PERPENDICULAR A B C D). AB is perpendicular to CD iff $P_{ACD} = P_{BCD}$. See page 114.

14. (PERP-BISECT O P Q). O is on the perpendicular bisector of PQ iff $P_{OPO} = P_{OQO}$.

15. (TANGENT O_1 A O_2 B). Circle (CIR O_1 A) is tangent to circle (CIR O_2 B). See page 114.

6.2 Geometry of Incidence

In this section, we include those geometric statements that can be formulated and proved without the measurement or comparison of distances or of angles, i.e., geometric facts involving incidence only. Such problems include the transversal, properties of cross-ratios, projective configurations, etc. These problems belong essentially to *affine geometry*. To prove statements involving incidence only, we need only to use Algorithm 2.32. Actually, the *co-side theorem* would be enough for the proofs of most of the examples in this section.

6.2.1 Menelaus' Theorem

For the machine proof of Menelaus' theorem, see Example 2.35 on page 73.

Example 6.1 (Converse of Menelaus' Theorem) (0.033, 2, 6) *If three points are taken, one on each side of a triangle, so that these points divide the sides into six segments such that the products of the segments in each of the two sets of nonconsecutive segments are equal in magnitude and opposite in sign, the three points are collinear.*

Constructive description

((points A B C)
(mratio D B C r_1)
(mratio E C A r_2)
(inter F (l D E) (l A B))
($\frac{\overline{BF}}{\overline{FA}} = -r_1 \cdot r_2$))

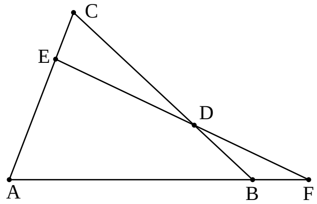


Figure 6-1

The machine proof

$$\frac{\frac{\overline{BF}}{\overline{AF}}}{r_2 \cdot r_1}$$

$$\overset{F}{=} \frac{S_{BDE}}{r_2 \cdot r_1 \cdot S_{ADE}}$$

$$\overset{E}{=} \frac{S_{ABD} \cdot r_2 \cdot (r_2+1)}{r_2 \cdot r_1 \cdot (-S_{ACD}) \cdot (r_2+1)}$$

$$\overset{simplify}{=} \frac{-S_{ABD}}{r_1 \cdot S_{ACD}}$$

$$\overset{D}{=} \frac{-S_{ABC} \cdot r_1 \cdot (r_1+1)}{r_1 \cdot (-S_{ABC}) \cdot (r_1+1)}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BF}}{\overline{AF}} \overset{F}{=} \frac{S_{BDE}}{S_{ADE}}$$

$$S_{ADE} \overset{E}{=} \frac{-S_{ACD}}{r_2+1}$$

$$S_{BDE} \overset{E}{=} \frac{S_{ABD} \cdot r_2}{r_2+1}$$

$$S_{ACD} \overset{D}{=} \frac{-S_{ABC}}{r_1+1}$$

$$S_{ABD} \overset{D}{=} \frac{S_{ABC} \cdot r_1}{r_1+1}$$

Example 6.2 (Menelaus' Theorem for a Quadrilateral) (0.033, 1, 4) *A line XY meets the sides AB, BC, CD, and DA of a quadrilateral at A_1, B_1, C_1, and D_1 respectively. Show that* $\frac{\overline{AA_1}}{\overline{BA_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{DD_1}}{\overline{AD_1}} = 1$.

Constructive description

((points A B C D X Y)
(inter A_1 (l A B) (l X Y))
(inter B_1 (l B C) (l X Y))
(inter C_1 (l C D) (l X Y))
(inter D_1 (l A D) (l X Y))
($\frac{\overline{AA_1}}{\overline{BA_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{DD_1}}{\overline{AD_1}} = 1$))

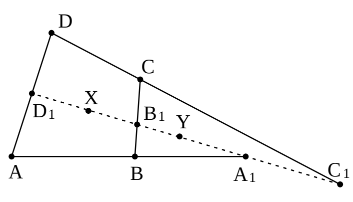


Figure 6-2

The machine proof

$$\frac{\overline{DD_1}}{\overline{AD_1}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{D_1}{=} \frac{S_{DXY}}{S_{AXY}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{C_1}{=} \frac{S_{DXY} \cdot S_{CXY}}{S_{AXY} \cdot S_{DXY}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{simplify}{=} \frac{S_{CXY}}{S_{AXY}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{B_1}{=} \frac{S_{CXY} \cdot S_{BXY}}{S_{AXY} \cdot S_{CXY}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{simplify}{=} \frac{S_{BXY}}{S_{AXY}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\overset{A_1}{=} \frac{S_{BXY} \cdot S_{AXY}}{S_{AXY} \cdot S_{BXY}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{DD_1}}{\overline{AD_1}} \overset{D_1}{=} \frac{S_{DXY}}{S_{AXY}}$$

$$\frac{\overline{CC_1}}{\overline{DC_1}} \overset{C_1}{=} \frac{S_{CXY}}{S_{DXY}}$$

$$\frac{\overline{BB_1}}{\overline{CB_1}} \overset{B_1}{=} \frac{S_{BXY}}{S_{CXY}}$$

$$\frac{\overline{AA_1}}{\overline{BA_1}} \overset{A_1}{=} \frac{S_{AXY}}{S_{BXY}}$$

Example 6.3 (0.033, 2, 7) *The converse of Example 6.2. The figure of this example is the same as Example 6.2.*

Constructive description

((points $A\ B\ C\ D$)
(mratio $B_1\ B\ C\ r_1$)
(mratio $C_1\ C\ D\ r_2$)
(inter D_1 (l $D\ A$) (l $B_1\ C_1$))
(inter A_1 (l $A\ B$) (l $B_1\ C_1$))
($\frac{\overline{BA_1}}{\overline{A_1A}} = r_1\cdot r_2\cdot\frac{\overline{DD_1}}{\overline{D_1A}}$))

The machine proof

$$\frac{\frac{\overline{BA_1}}{\overline{AA_1}}}{\frac{\overline{DD_1}}{\overline{AD_1}}\cdot r_2\cdot r_1}$$

$$\stackrel{A_1}{=} \frac{S_{BB_1C_1}}{\frac{\overline{DD_1}}{\overline{AD_1}}\cdot r_2\cdot r_1\cdot S_{AB_1C_1}}$$

$$\stackrel{D_1}{=} \frac{S_{BB_1C_1}\cdot S_{AB_1C_1}}{S_{DB_1C_1}\cdot r_2\cdot r_1\cdot S_{AB_1C_1}}$$

$$\stackrel{simplify}{=} \frac{S_{BB_1C_1}}{S_{DB_1C_1}\cdot r_2\cdot r_1}$$

$$\stackrel{C_1}{=} \frac{(-S_{BDB_1}\cdot r_2)\cdot(r_2+1)}{S_{CDB_1}\cdot r_2\cdot r_1\cdot(r_2+1)}$$

$$\stackrel{simplify}{=} \frac{-S_{BDB_1}}{S_{CDB_1}\cdot r_1}$$

$$\stackrel{B_1}{=} \frac{-(-S_{BCD}\cdot r_1)\cdot(r_1+1)}{S_{BCD}\cdot r_1\cdot(r_1+1)} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BA_1}}{\overline{AA_1}} \stackrel{A_1}{=} \frac{S_{BB_1C_1}}{S_{AB_1C_1}}$$

$$\frac{\overline{DD_1}}{\overline{AD_1}} \stackrel{D_1}{=} \frac{S_{DB_1C_1}}{S_{AB_1C_1}}$$

$$S_{DB_1C_1} \stackrel{C_1}{=} \frac{S_{CDB_1}}{r_2+1}$$

$$S_{BB_1C_1} \stackrel{C_1}{=} \frac{-S_{BDB_1}\cdot r_2}{r_2+1}$$

$$S_{CDB_1} \stackrel{B_1}{=} \frac{S_{BCD}}{r_1+1}$$

$$S_{BDB_1} \stackrel{B_1}{=} \frac{-S_{BCD}\cdot r_1}{r_1+1}$$

Definition. *Two points on a side of a triangle are said to be isotomic to each other, if they are equidistant from the midpoint of this side.*

Example 6.4 [2] (0.083, 2, 9) *The isotomic points of three collinear points are collinear.*

Constructive description

((points $A\ B\ C$)
(lratio $D\ B\ C\ r_1$)
(lratio $E\ A\ C\ r_2$)
(inter F (l $A\ B$) (l $D\ E$))
(pratio $D_1\ B\ C\ D\ -1$)
(pratio $E_1\ C\ A\ E\ -1$)
(pratio $F_1\ A\ B\ F\ -1$)
($\frac{\overline{AF_1}}{\overline{F_1B}}\frac{\overline{BD_1}}{\overline{D_1C}}\frac{\overline{CE_1}}{\overline{E_1A}} = -1$))

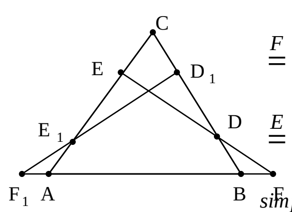


Figure 6-4

The machine proof

$$\frac{\overline{CE_1}}{\overline{AE_1}}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}\cdot\frac{\overline{AF_1}}{\overline{BF_1}}$$

$$\stackrel{F_1}{=} \frac{-1}{-\frac{\overline{AB}}{\overline{BF}}-1}\cdot\frac{\overline{CE_1}}{\overline{AE_1}}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}$$

$$\stackrel{E_1}{=} \frac{-1}{(\frac{\overline{AB}}{\overline{BF}}+1)\cdot(\frac{\overline{AC}}{\overline{AE}}-1)}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}$$

$$\stackrel{D_1}{=} \frac{-(-1)}{(\frac{\overline{AB}}{\overline{BF}}+1)\cdot(\frac{\overline{AC}}{\overline{AE}}-1)\cdot(-\frac{\overline{BC}}{\overline{CD}}-1)}$$

$$\stackrel{F}{=} \frac{-S_{BDE}}{(S_{ADBE}+S_{BDE})\cdot(\frac{\overline{AC}}{\overline{AE}}-1)\cdot(\frac{\overline{BC}}{\overline{CD}}+1)}$$

$$\stackrel{E}{=} \frac{-(-S_{ABD}\cdot r_2+S_{ABD})\cdot r_2}{(-S_{ABD}\cdot r_2+S_{ABC}\cdot r_2)\cdot(-r_2+1)\cdot(\frac{\overline{BC}}{\overline{CD}}+1)}$$

$$\stackrel{simplify}{=} \frac{S_{ABD}}{(S_{ABD}-S_{ABC})\cdot(\frac{\overline{BC}}{\overline{CD}}+1)}$$

$$\stackrel{D}{=} \frac{S_{ABC}\cdot r_1\cdot(r_1-1)}{(S_{ABC}\cdot r_1-S_{ABC})\cdot r_1}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF_1}}{\overline{BF_1}} \stackrel{F_1}{=} \frac{1}{\frac{\overline{AB}}{\overline{BF}}+1}$$

$$\frac{\overline{CE_1}}{\overline{AE_1}} \stackrel{E_1}{=} \frac{-1}{\frac{\overline{AC}}{\overline{AE}}-1}$$

$$\frac{\overline{BD_1}}{\overline{CD_1}} \stackrel{D_1}{=} \frac{1}{\frac{\overline{BC}}{\overline{CD}}+1}$$

$$\frac{\overline{AB}}{\overline{BF}} \stackrel{F}{=} \frac{S_{ADBE}}{S_{BDE}}$$

$$\frac{\overline{AC}}{\overline{AE}} \stackrel{E}{=} \frac{1}{r_2}$$

$$S_{ADBE} \stackrel{E}{=} -\left(S_{ABD}-S_{ABC}\cdot r_2\right)$$

$$S_{BDE} \stackrel{E}{=} -\left((r_2-1)\cdot S_{ABD}\right)$$

$$\frac{\overline{BC}}{\overline{CD}} \stackrel{D}{=} \frac{1}{r_1-1}$$

$$S_{ABD} \stackrel{D}{=} S_{ABC}\cdot r_1$$

[2] In this chapter, the figure indices are always the same with the indices of the examples that they belong to.

6.2.2 Ceva's Theorem

For the machine proof of Ceva's theorem, see page 72. For the generalizations of Ceva's theorem to arbitrary polygons, see Subsection 2.7.3.

Example 6.5 (The Converse of Ceva's Theorem) (0.066, 2, 8) *If three points taken on the sides of a triangle determine on these sides six segments such that the products of the segments in the two nonconsecutive sets are equal, both in magnitude and in sign, the lines joining these points to the respectively opposite vertices are concurrent.*

Constructive description

((points A B C)
(mratio D B C r_1)
(mratio E C A r_2)
(mratio F B A $r_1 \cdot r_2$)
(inter O (l B E) (l A D))
(inter Z (l B A) (l O C))
($\frac{\overline{BZ}}{\overline{AZ}} = \frac{\overline{BF}}{\overline{AF}}$))

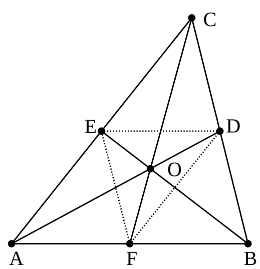


Figure 6-5

The machine proof

$$\left(\frac{\overline{BZ}}{\overline{AZ}}\right)/\left(\frac{\overline{BF}}{\overline{AF}}\right)$$

$$\overset{Z}{=} \frac{-S_{BCO}}{\frac{\overline{BF}}{\overline{AF}}\cdot(-S_{ACO})}$$

$$\overset{O}{=} \frac{(-S_{BCE}\cdot S_{ABD})\cdot S_{ABDE}}{\frac{\overline{BF}}{\overline{AF}}\cdot S_{ACD}\cdot S_{ABE}\cdot(-S_{ABDE})}$$

$$\overset{simplify}{=} \frac{S_{BCE}\cdot S_{ABD}}{\frac{\overline{BF}}{\overline{AF}}\cdot S_{ACD}\cdot S_{ABE}}$$

$$\overset{F}{=} \frac{S_{BCE}\cdot S_{ABD}\cdot(-1)}{r_2\cdot r_1\cdot S_{ACD}\cdot S_{ABE}}$$

$$\overset{E}{=} \frac{-S_{ABC}\cdot r_2\cdot S_{ABD}\cdot(r_2+1)}{r_2\cdot r_1\cdot S_{ACD}\cdot S_{ABC}\cdot(r_2+1)}$$

$$\overset{simplify}{=} \frac{-S_{ABD}}{r_1\cdot S_{ACD}}$$

$$\overset{D}{=} \frac{-S_{ABC}\cdot r_1\cdot(r_1+1)}{r_1\cdot(-S_{ABC})\cdot(r_1+1)}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BZ}}{\overline{AZ}} \overset{Z}{=} \frac{S_{BCO}}{S_{ACO}}$$

$$S_{ACO} \overset{O}{=} \frac{S_{ACD}\cdot S_{ABE}}{S_{ABDE}}$$

$$S_{BCO} \overset{O}{=} \frac{S_{BCE}\cdot S_{ABD}}{S_{ABDE}}$$

$$\frac{\overline{BF}}{\overline{AF}} \overset{F}{=} -(r_2\cdot r_1)$$

$$S_{ABE} \overset{E}{=} \frac{S_{ABC}}{r_2+1}$$

$$S_{BCE} \overset{E}{=} \frac{S_{ABC}\cdot r_2}{r_2+1}$$

$$S_{ACD} \overset{D}{=} \frac{-S_{ABC}}{r_1+1}$$

$$S_{ABD} \overset{D}{=} \frac{S_{ABC}\cdot r_1}{r_1+1}$$

Definition. *If a line is drawn through a vertex of a triangle, the segment included between the vertex and the opposite side is called a cevian. The triangle DEF in Figure 6-5 is called the cevian triangle of the point O for the triangle ABC.*

Example 6.6 (0.050, 3, 12) *If LMN is the cevian triangle of the point S for the triangle ABC, we have $SL/AL + SM/BM + SN/CN = 1$.*

Constructive description

((points A B C S)
(inter L (l B C) (l A S))
(inter N (l B A) (l C S))
(inter M (l A C) (l B S))
($\frac{\overline{SL}}{\overline{AL}}+\frac{\overline{SM}}{\overline{BM}}+\frac{\overline{SN}}{\overline{CN}} = 1$))

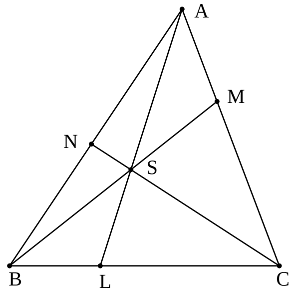


Figure 6-6

The machine proof

$$\frac{\overline{SM}}{\overline{BM}}+\frac{\overline{SN}}{\overline{CN}}+\frac{\overline{SL}}{\overline{AL}}$$

$$\overset{M}{=} \frac{-\frac{\overline{SN}}{\overline{CN}}\cdot S_{ABC}-\frac{\overline{SL}}{\overline{AL}}\cdot S_{ABC}+S_{ACS}}{-S_{ABC}}$$

$$\overset{N}{=} \frac{-\frac{\overline{SL}}{\overline{AL}}\cdot S_{ABC}^2+S_{ACS}\cdot S_{ABC}-S_{ABS}\cdot S_{ABC}}{S_{ABC}\cdot(-S_{ABC})}$$

$$\overset{simplify}{=} \frac{\frac{\overline{SL}}{\overline{AL}}\cdot S_{ABC}-S_{ACS}+S_{ABS}}{S_{ABC}}$$

$$\overset{L}{=} \frac{S_{BCS}\cdot S_{ABC}-S_{ACS}\cdot S_{ABC}+S_{ABS}\cdot S_{ABC}}{(S_{ABC})^2}$$

$$\overset{simplify}{=} \frac{S_{BCS}-S_{ACS}+S_{ABS}}{S_{ABC}}$$

$$\overset{area-co}{=} \frac{S_{ABC}}{S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{SM}}{\overline{BM}} \overset{M}{=} \frac{S_{ACS}}{-S_{ABC}}$$

$$\frac{\overline{SN}}{\overline{CN}} \overset{N}{=} \frac{S_{ABS}}{S_{ABC}}$$

$$\frac{\overline{SL}}{\overline{AL}} \overset{L}{=} \frac{S_{BCS}}{S_{ABC}}$$

$$S_{BCS}=S_{ACS}-S_{ABS}+S_{ABC}$$

Example 6.7 *If LMN is the cevian triangle of the point S for the triangle ABC (Figure 6-6), we have* $\frac{S_{AML}S_{BNM}S_{CLN}}{S_{ANL}S_{BLM}S_{CNM}} = 1$.

Constructive description: ((points A B C S) (inter L (l B C) (l S A)) (inter M (l A C) (l S B)) (inter N (l A B) (l S C)) ($S_{AML}S_{BNM}S_{CLN} = S_{ANL}S_{BLM}S_{CNM}$))

The machine proof

$$\frac{S_{CLN}\cdot S_{BMN}\cdot S_{ALM}}{S_{CMN}\cdot S_{BLM}\cdot S_{ALN}}$$

$$\overset{N}{=} \frac{(-S_{CSL}\cdot S_{ABC})\cdot(-S_{BCS}\cdot S_{ABM})\cdot S_{ALM}\cdot S_{ACBS}\cdot(-S_{ACBS})}{(-S_{CSM}\cdot S_{ABC})\cdot S_{BLM}\cdot(-S_{ACS}\cdot S_{ABL})\cdot S_{ACBS}\cdot(-S_{ACBS})}$$

$$\overset{simplify}{=} \frac{S_{CSL}\cdot S_{BCS}\cdot S_{ABM}\cdot S_{ALM}}{S_{CSM}\cdot S_{BLM}\cdot S_{ACS}\cdot S_{ABL}}$$

$$\overset{M}{=} \frac{S_{CSL}\cdot S_{BCS}\cdot S_{ABS}\cdot S_{ABC}\cdot(-S_{ACL}\cdot S_{ABS})\cdot(-S_{ABCS})\cdot S_{ABCS}}{S_{BCS}\cdot S_{ACS}\cdot S_{BSL}\cdot S_{ABC}\cdot S_{ACS}\cdot S_{ABL}\cdot(S_{ABCS})^2}$$

$$\overset{simplify}{=} \frac{S_{CSL}\cdot(S_{ABS})^2\cdot S_{ACL}}{(S_{ACS})^2\cdot S_{BSL}\cdot S_{ABL}}$$

$$\overset{L}{=} \frac{S_{BCS}\cdot S_{ACS}\cdot(S_{ABS})^2\cdot(-S_{ACS}\cdot S_{ABC})\cdot((-S_{ABSC}))^2}{(S_{ACS})^2\cdot S_{BCS}\cdot S_{ABS}\cdot(-S_{ABS}\cdot S_{ABC})\cdot((-S_{ABSC}))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{ALN} \overset{N}{=} \frac{-S_{ACS}\cdot S_{ABL}}{S_{ACBS}}$$

$$S_{CMN} \overset{N}{=} \frac{S_{CSM}\cdot S_{ABC}}{S_{ACBS}}$$

$$S_{BMN} \overset{N}{=} \frac{-S_{BCS}\cdot S_{ABM}}{S_{ACBS}}$$

$$S_{CLN} \overset{N}{=} \frac{S_{CSL}\cdot S_{ABC}}{S_{ACBS}}$$

$$S_{BLM} \overset{M}{=} \frac{S_{BSL}\cdot S_{ABC}}{-S_{ABCS}}$$

$$S_{CSM} \overset{M}{=} \frac{S_{BCS}\cdot S_{ACS}}{S_{ABCS}}$$

$$S_{ALM} \overset{M}{=} \frac{-S_{ACL}\cdot S_{ABS}}{S_{ABCS}}$$

$$S_{ABM} \overset{M}{=} \frac{S_{ABS}\cdot S_{ABC}}{S_{ABCS}}$$

$$S_{ABL} \overset{L}{=} \frac{S_{ABS}\cdot S_{ABC}}{S_{ABSC}}$$

$$S_{BSL} \overset{L}{=} \frac{S_{BCS}\cdot S_{ABS}}{-S_{ABSC}}$$

$$S_{ACL} \overset{L}{=} \frac{S_{ACS}\cdot S_{ABC}}{S_{ABSC}}$$

$$S_{CSL} \overset{L}{=} \frac{S_{BCS}\cdot S_{ACS}}{-S_{ABSC}}$$

Example 6.8 (0.033, 3, 4) *The lines AP, BQ, CR through the vertices of a triangle ABC parallel, respectively, to the lines* OA_1, OB_1, OC_1 *joining any point O to the points* A_1, B_1, C_1 *marked in any manner whatever, on the sides of BC, CA, AB meet these sides in the points P, Q, R. Show that* $OA_1/AP + OB_1/BQ + OC_1/CR = 1$.

Constructive description

((points $A\ B\ C\ O$)
(on A_1 (l $B\ C$))
(on B_1 (l $C\ A$))
(on C_1 (l $B\ A$))
(inter P (l $B\ C$) (p $A\ A_1\ O$))
(inter Q (l $A\ C$) (p $B\ B_1\ O$))
(inter R (l $A\ B$) (p $C\ C_1\ O$))
($\frac{\overline{OA_1}}{\overline{AP}}+\frac{\overline{OB_1}}{\overline{BQ}}+\frac{\overline{OC_1}}{\overline{CR}}=1$))

The machine proof

$$\frac{\overline{OC_1}}{\overline{CR}}+\frac{\overline{OB_1}}{\overline{BQ}}+\frac{\overline{OA_1}}{\overline{AP}}$$

$$\overset{R}{=}\frac{\frac{\overline{OB_1}}{\overline{BQ}}\cdot S_{ABC}+\frac{\overline{OA_1}}{\overline{AP}}\cdot S_{ABC}+S_{ABO}}{S_{ABC}}$$

$$\overset{Q}{=}\frac{-\frac{\overline{OA_1}}{\overline{AP}}\cdot S_{ABC}^2+S_{ACO}\cdot S_{ABC}-S_{ABO}\cdot S_{ABC}}{S_{ABC}\cdot(-S_{ABC})}$$

$$\overset{simplify}{=}\frac{\frac{\overline{OA_1}}{\overline{AP}}\cdot S_{ABC}-S_{ACO}+S_{ABO}}{S_{ABC}}$$

$$\overset{P}{=}\frac{S_{BCO}\cdot S_{ABC}-S_{ACO}\cdot S_{ABC}+S_{ABO}\cdot S_{ABC}}{(S_{ABC})^2}$$

$$\overset{simplify}{=}\frac{S_{BCO}-S_{ACO}+S_{ABO}}{S_{ABC}}$$

$$\overset{area-co}{=}\frac{S_{ABC}}{S_{ABC}}$$

$$\overset{simplify}{=}1$$

The eliminants

$$\frac{\overline{OC_1}}{\overline{CR}}\overset{R}{=}\frac{S_{ABO}}{S_{ABC}}$$

$$\frac{\overline{OB_1}}{\overline{BQ}}\overset{Q}{=}\frac{S_{ACO}}{-S_{ABC}},\quad \frac{\overline{OA_1}}{\overline{AP}}\overset{P}{=}\frac{S_{BCO}}{S_{ABC}}$$

$$S_{BCO}=S_{ACO}-S_{ABO}+S_{ABC}$$

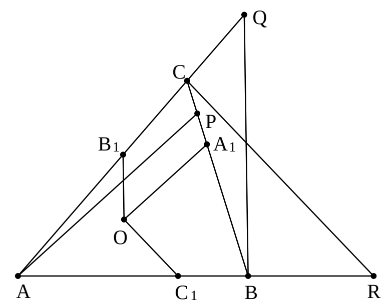


Figure 6-8

Example 6.9 (0.083, 2, 9) *If LMN is the cevian triangle of the point S for the triangle ABC, we have $AS/SL = AM/MC + AN/NB$.*

Constructive description

((points $A\ B\ C\ S$)
(inter L (l $B\ C$) (l $A\ S$))
(inter N (l $B\ A$) (l $C\ S$))
(inter M (l $A\ C$) (l $B\ S$))
($\frac{\overline{AM}}{\overline{MC}}+\frac{\overline{AN}}{\overline{NB}}=\frac{\overline{AS}}{\overline{SL}}$))

The machine proof

$$\frac{-(\frac{\overline{AM}}{\overline{CM}}+\frac{\overline{AN}}{\overline{BN}})}{\frac{\overline{AS}}{\overline{SL}}}$$

$$\overset{M}{=}\frac{-(-\frac{\overline{AN}}{\overline{BN}}\cdot S_{BCS}+S_{ABS})}{\frac{\overline{AS}}{\overline{SL}}\cdot(-S_{BCS})}$$

$$\overset{N}{=}\frac{-(S_{BCS}\cdot S_{ACS}-S_{BCS}\cdot S_{ABS})}{\frac{\overline{AS}}{\overline{SL}}\cdot(S_{BCS})^2}$$

$$\overset{simplify}{=}\frac{-(S_{ACS}-S_{ABS})}{\frac{\overline{AS}}{\overline{SL}}\cdot S_{BCS}}$$

$$\overset{L}{=}\frac{-(S_{ACS}-S_{ABS})\cdot S_{BCS}}{S_{ABSC}\cdot S_{BCS}}$$

$$\overset{simplify}{=}\frac{-(S_{ACS}-S_{ABS})}{S_{ABSC}}$$

$$\overset{area-co}{=}\frac{-(S_{ACS}-S_{ABS})}{-S_{ACS}+S_{ABS}}$$

$$\overset{simplify}{=}1$$

The eliminants

$$\frac{\overline{AM}}{\overline{CM}}\overset{M}{=}\frac{S_{ABS}}{-S_{BCS}}$$

$$\frac{\overline{AN}}{\overline{BN}}\overset{N}{=}\frac{S_{ACS}}{S_{BCS}}$$

$$\frac{\overline{AS}}{\overline{SL}}\overset{L}{=}\frac{S_{ABSC}}{S_{BCS}}$$

$$S_{ABSC}=-\left(S_{ACS}-S_{ABS}\right)$$

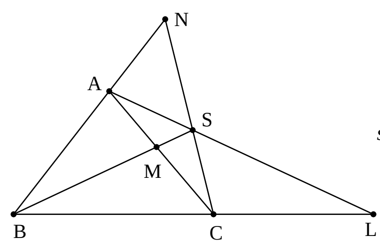


Figure 6-9

Example 6.10 (0.083, 1, 5) *The Ceva's theorem for a pentagon.*

Constructive description

((points $A\ B\ C\ D\ E\ O$)
(inter A_1 (l $O\ A$) (l $C\ D$))
(inter B_1 (l $O\ B$) (l $D\ E$))
(inter C_1 (l $O\ C$) (l $E\ A$))
(inter D_1 (l $O\ D$) (l $A\ B$))
(inter E_1 (l $O\ E$) (l $B\ C$))
($\frac{\overline{CA_1}}{\overline{A_1D}}\frac{\overline{DB_1}}{\overline{B_1E}}\frac{\overline{EC_1}}{\overline{C_1A}}\frac{\overline{AD_1}}{\overline{D_1B}}\frac{\overline{BE_1}}{\overline{E_1C}} = 1$))

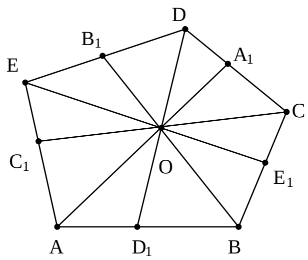


Figure 6-10

The machine proof

$$-\frac{\overline{EC_1}}{\overline{AC_1}}\cdot\frac{\overline{DB_1}}{\overline{EB_1}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}\cdot\frac{\overline{BE_1}}{\overline{CE_1}}\cdot\frac{\overline{AD_1}}{\overline{BD_1}}$$

$$\stackrel{E_1}{=} \frac{-(-S_{BEO})}{-S_{CEO}}\cdot\frac{\overline{EC_1}}{\overline{AC_1}}\cdot\frac{\overline{DB_1}}{\overline{EB_1}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}\cdot\frac{\overline{AD_1}}{\overline{BD_1}}$$

$$\stackrel{D_1}{=} \frac{-S_{BEO}\cdot(-S_{ADO})}{S_{CEO}\cdot(-S_{BDO})}\cdot\frac{\overline{EC_1}}{\overline{AC_1}}\cdot\frac{\overline{DB_1}}{\overline{EB_1}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}$$

$$\stackrel{C_1}{=} \frac{-S_{CEO}\cdot S_{BEO}\cdot S_{ADO}}{S_{CEO}\cdot S_{BDO}\cdot(-S_{ACO})}\cdot\frac{\overline{DB_1}}{\overline{EB_1}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}$$

$$\stackrel{simplify}{=} \frac{S_{BEO}\cdot S_{ADO}}{S_{BDO}\cdot S_{ACO}}\cdot\frac{\overline{DB_1}}{\overline{EB_1}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}$$

$$\stackrel{B_1}{=} \frac{S_{BDO}\cdot S_{BEO}\cdot S_{ADO}}{S_{BDO}\cdot S_{ACO}\cdot S_{BEO}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ADO}}{S_{ACO}}\cdot\frac{\overline{CA_1}}{\overline{DA_1}}$$

$$\stackrel{A_1}{=} \frac{S_{ACO}\cdot S_{ADO}}{S_{ACO}\cdot S_{ADO}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BE_1}}{\overline{CE_1}} \stackrel{E_1}{=} \frac{S_{BEO}}{S_{CEO}}$$

$$\frac{\overline{AD_1}}{\overline{BD_1}} \stackrel{D_1}{=} \frac{S_{ADO}}{S_{BDO}}$$

$$\frac{\overline{EC_1}}{\overline{AC_1}} \stackrel{C_1}{=} \frac{S_{CEO}}{-S_{ACO}}$$

$$\frac{\overline{DB_1}}{\overline{EB_1}} \stackrel{B_1}{=} \frac{S_{BDO}}{S_{BEO}}$$

$$\frac{\overline{CA_1}}{\overline{DA_1}} \stackrel{A_1}{=} \frac{S_{ACO}}{S_{ADO}}$$

Example 6.11 (0.067, 3, 16) *If the three lines joining three points marked on the sides of a triangle to the respectively opposite vertices are concurrent, the same is true of the isotomics of the given points.*

Constructive description

((points $A\ B\ C\ O$)
(inter D (l $A\ O$) (l $C\ B$))
(inter E (l $B\ O$) (l $A\ C$))
(inter F (l $C\ O$) (l $A\ B$))
(pratio $D_1\ B\ C\ D\ -1$)
(pratio $E_1\ C\ A\ E\ -1$)
(pratio $F_1\ A\ B\ F\ -1$)
($\frac{\overline{AF_1}}{\overline{F_1B}}\frac{\overline{BD_1}}{\overline{D_1C}}\frac{\overline{CE_1}}{\overline{E_1A}} = 1$))

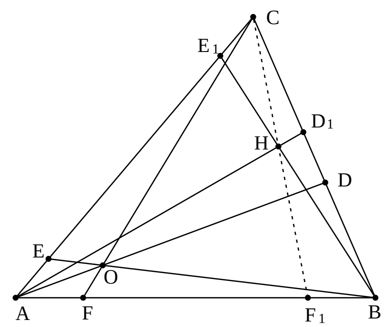


Figure 6-11

The machine proof

$$-\frac{\overline{CE_1}}{\overline{AE_1}}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}\cdot\frac{\overline{AF_1}}{\overline{BF_1}}$$

$$\stackrel{F_1}{=} \frac{-(-1)}{-\frac{\overline{AB}}{\overline{BF}}-1}\cdot\frac{\overline{CE_1}}{\overline{AE_1}}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}$$

$$\stackrel{E_1}{=} \frac{-(-1)}{(\frac{\overline{AB}}{\overline{BF}}+1)\cdot(\frac{\overline{AC}}{\overline{AE}}-1)}\cdot\frac{\overline{BD_1}}{\overline{CD_1}}$$

$$\stackrel{D_1}{=} \frac{-1}{(\frac{\overline{AB}}{\overline{BF}}+1)\cdot(\frac{\overline{AC}}{\overline{AE}}-1)\cdot(-\frac{\overline{BC}}{\overline{CD}}-1)}$$

$$\stackrel{F}{=} \frac{S_{BCO}}{(S_{ACBO}+S_{BCO})\cdot(\frac{\overline{AC}}{\overline{AE}}-1)\cdot(\frac{\overline{BC}}{\overline{CD}}+1)}$$

The eliminants

$$\frac{\overline{AF_1}}{\overline{BF_1}} \stackrel{F_1}{=} \frac{1}{\frac{\overline{AB}}{\overline{BF}}+1}$$

$$\frac{\overline{CE_1}}{\overline{AE_1}} \stackrel{E_1}{=} \frac{-1}{\frac{\overline{AC}}{\overline{AE}}-1}$$

$$\frac{\overline{BD_1}}{\overline{CD_1}} \stackrel{D_1}{=} \frac{1}{\frac{\overline{BC}}{\overline{CD}}+1}$$

$$\frac{\overline{AB}}{\overline{BF}} \stackrel{F}{=} \frac{S_{ACBO}}{S_{BCO}}$$

$$\frac{\overline{AC}}{\overline{AE}} \stackrel{E}{=} \frac{S_{ABCO}}{S_{ABO}}$$

$$\frac{\overline{BC}}{\overline{CD}} \stackrel{D}{=} \frac{S_{ABOC}}{S_{ACO}}$$

The machine proof

$\overset{E}{=} \frac{S_{BCO} \cdot S_{ABO}}{(S_{ACBO}+S_{BCO}) \cdot (S_{ABCO}-S_{ABO}) \cdot (\frac{\overline{BC}}{\overline{CD}}+1)}$

$\overset{D}{=} \frac{S_{BCO} \cdot S_{ABO} \cdot (-S_{ACO})}{(S_{ACBO}+S_{BCO}) \cdot (S_{ABCO}-S_{ABO}) \cdot (-S_{ABOC}-S_{ACO})}$

$\overset{area-co}{=} \frac{(S_{ACO}-S_{ABO}+S_{ABC}) \cdot S_{ABO} \cdot S_{ACO}}{S_{ACO} \cdot (S_{ACO}-S_{ABO}+S_{ABC}) \cdot S_{ABO}}$

$\overset{simplify}{=} 1$

The eliminants

$S_{ABOC} = -(S_{ACO}-S_{ABO})$

$S_{ABCO}=S_{ACO}+S_{ABC}$

$S_{ACBO}=S_{ABO}-S_{ABC}$

$S_{BCO}=S_{ACO}-S_{ABO}+S_{ABC}$

6.2.3 The Cross-ratio and Harmonic Points

Definition *Let A, B, C, and D be four collinear points. The cross ratio of them, denoted by $(ABCD)$, is defined to be*

$$(ABCD) = (\frac{\overline{CA}}{\overline{CB}})/(\frac{\overline{DA}}{\overline{DB}}).$$

Example 6.12 (0.066, 2, 10) *The cross ratio of four points on a line is unchanged under a projection.*

Constructive description
((points O A B C_1 D_1)
(inter C (l A B) (l O C_1))
(inter D (l A B) (l O D_1))
(inter A_1 (l C_1 D_1) (l O A))
(inter B_1 (l C_1 D_1) (l O B))
($\frac{\overline{AC}}{\overline{BC}} \frac{\overline{BD}}{\overline{AD}} = \frac{\overline{A_1C_1}}{\overline{A_1D_1}} \frac{\overline{B_1D_1}}{\overline{B_1C_1}}$))

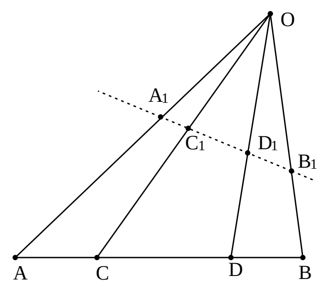


Figure 6-12

The machine proof

$\frac{\frac{\overline{BD}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{BC}}}{\frac{\overline{D_1B_1}}{\overline{C_1B_1}} \cdot \frac{\overline{C_1A_1}}{\overline{D_1A_1}}}$

$\overset{B_1}{=} \frac{S_{OBC_1}}{S_{OBD_1} \cdot \frac{\overline{C_1A_1}}{\overline{D_1A_1}}} \cdot \frac{\overline{BD}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{A_1}{=} \frac{S_{OBC_1} \cdot S_{OAD_1}}{S_{OBD_1} \cdot S_{OAC_1}} \cdot \frac{\overline{BD}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{D}{=} \frac{(-S_{OBD_1}) \cdot S_{OBC_1} \cdot S_{OAD_1}}{S_{OBD_1} \cdot S_{OAC_1} \cdot (-S_{OAD_1})} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{simplify}{=} \frac{S_{OBC_1}}{S_{OAC_1}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{C}{=} \frac{(-S_{OAC_1}) \cdot S_{OBC_1}}{S_{OAC_1} \cdot (-S_{OBC_1})}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{D_1B_1}}{\overline{C_1B_1}} \overset{B_1}{=} \frac{S_{OBD_1}}{S_{OBC_1}}$

$\frac{\overline{C_1A_1}}{\overline{D_1A_1}} \overset{A_1}{=} \frac{S_{OAC_1}}{S_{OAD_1}}$

$\frac{\overline{BD}}{\overline{AD}} \overset{D}{=} \frac{S_{OBD_1}}{S_{OAD_1}}$

$\frac{\overline{AC}}{\overline{BC}} \overset{C}{=} \frac{S_{OAC_1}}{S_{OBC_1}}$

Definition. *For any five different points O, A, B, C, and D, the cross-ratio of any four collinear points on lines OA, OB, OC, and OD is a constant and is denote by $O(ABCD)$.*

Example 6.13 (0.050, 2, 10) *Two lines $OABC$ and $OA_1B_1C_1$ intersect at point O. If $(OABC) = (OA_1B_1C_1)$ then AA_1, BB_1, and CC_1 are concurrent.*

Constructive description

((points O A A_1)
(mratio B A O r_1)
(mratio C O A r_2)
(mratio B_1 O A_1 r_3)
(mratio C_1 O A_1 $r_1 \cdot r_2 \cdot r_3$)
(inter X (l A A_1) (l B B_1))
(inter Y (l A A_1) (l C C_1))
($\frac{\overline{AX}}{\overline{XA_1}} = \frac{\overline{AY}}{\overline{YA_1}}$))

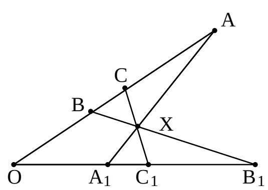


Figure 6-13

The machine proof

$$\left(\frac{\overline{AX}}{\overline{A_1X}}\right)/\left(\frac{\overline{AY}}{\overline{A_1Y}}\right)$$

$$\overset{Y}{=} \frac{S_{A_1CC_1}}{S_{ACC_1}} \cdot \frac{\overline{AX}}{\overline{A_1X}}$$

$$\overset{X}{=} \frac{S_{ABB_1} \cdot S_{A_1CC_1}}{S_{ACC_1} \cdot S_{A_1BB_1}}$$

$$\overset{C_1}{=} \frac{S_{ABB_1} \cdot S_{OA_1C} \cdot (r_3 \cdot r_2 \cdot r_1 + 1)}{(-S_{AA_1C} \cdot r_3 \cdot r_2 \cdot r_1) \cdot S_{A_1BB_1} \cdot (r_3 \cdot r_2 \cdot r_1 + 1)}$$

$$\overset{simplify}{=} \frac{S_{ABB_1} \cdot S_{OA_1C}}{-S_{AA_1C} \cdot r_3 \cdot r_2 \cdot r_1 \cdot S_{A_1BB_1}}$$

$$\overset{B_1}{=} \frac{(-S_{AA_1B} \cdot r_3) \cdot S_{OA_1C} \cdot (r_3 + 1)}{-S_{AA_1C} \cdot r_3 \cdot r_2 \cdot r_1 \cdot S_{OA_1B} \cdot (r_3 + 1)}$$

$$\overset{simplify}{=} \frac{S_{AA_1B} \cdot S_{OA_1C}}{S_{AA_1C} \cdot r_2 \cdot r_1 \cdot S_{OA_1B}}$$

$$\overset{C}{=} \frac{S_{AA_1B} \cdot (-S_{OAA_1} \cdot r_2) \cdot (r_2 + 1)}{S_{OAA_1} \cdot r_2 \cdot r_1 \cdot S_{OA_1B} \cdot (r_2 + 1)}$$

$$\overset{simplify}{=} \frac{-S_{AA_1B}}{r_1 \cdot S_{OA_1B}}$$

$$\overset{B}{=} \frac{-S_{OAA_1} \cdot r_1 \cdot (r_1 + 1)}{r_1 \cdot (-S_{OAA_1}) \cdot (r_1 + 1)}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AY}}{\overline{A_1Y}} \overset{Y}{=} \frac{S_{ACC_1}}{S_{A_1CC_1}}$$

$$\frac{\overline{AX}}{\overline{A_1X}} \overset{X}{=} \frac{S_{ABB_1}}{S_{A_1BB_1}}$$

$$S_{ACC_1} \overset{C_1}{=} \frac{-S_{AA_1C} \cdot r_3 \cdot r_2 \cdot r_1}{r_3 \cdot r_2 \cdot r_1 + 1}$$

$$S_{A_1CC_1} \overset{C_1}{=} \frac{S_{OA_1C}}{r_3 \cdot r_2 \cdot r_1 + 1}$$

$$S_{A_1BB_1} \overset{B_1}{=} \frac{S_{OA_1B}}{r_3 + 1}$$

$$S_{ABB_1} \overset{B_1}{=} \frac{-S_{AA_1B} \cdot r_3}{r_3 + 1}$$

$$S_{AA_1C} \overset{C}{=} \frac{S_{OAA_1}}{r_2 + 1}$$

$$S_{OA_1C} \overset{C}{=} \frac{-S_{OAA_1} \cdot r_2}{r_2 + 1}$$

$$S_{OA_1B} \overset{B}{=} \frac{-S_{OAA_1}}{r_1 + 1}$$

$$S_{AA_1B} \overset{B}{=} \frac{S_{OAA_1} \cdot r_1}{r_1 + 1}$$

Definition. *If $(ABCD) = -1$, we call A, B, C, D a harmonic sequence, or C, D divide the segment AB harmonically. Four lines passing through a point O is said to form a harmonic pencil if another line meets them in a harmonic sequence A, B, C, D. Each of the line is called a ray of the the pencil and O is called the center of the pencil. OA and OB, OC and OD are said to be a pair of conjugate rays, or OC and OD are harmonically separated by OA and OB.*

Example 6.14 (0.016, 2, 4) *Let A, B, C, D be four harmonic points and O be the midpoint of AB. Then $\overline{OC} \cdot \overline{OD} = \overline{OA}^2$.*

Constructive description

((points A B)
(mratio C A B r)
(mratio D A B $-r$)
(midpoint O A B)
($\frac{\overline{OC}}{\overline{OA}} = \frac{\overline{OA}}{\overline{OD}}$))

The machine proof

$$\left(\frac{\overline{CO}}{\overline{AO}}\right)/\left(\frac{\overline{AO}}{\overline{DO}}\right)$$
$$\overset{O}{=} \frac{(-\frac{\overline{AC}}{\overline{AB}}+\frac{1}{2})\cdot(-\frac{\overline{AD}}{\overline{AB}}+\frac{1}{2})}{((\frac{1}{2}))^2}$$
$$\overset{D}{=} \frac{(2\frac{\overline{AC}}{\overline{AB}}-1)\cdot(-r-1)}{-r+1}$$
$$\overset{C}{=} \frac{(r-1)\cdot(r+1)}{(r-1)\cdot(r+1)}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AO}}{\overline{DO}} \overset{O}{=} \frac{1}{-(2\frac{\overline{AD}}{\overline{AB}}-1)}$$
$$\frac{\overline{CO}}{\overline{AO}} \overset{O}{=} -\left(2\frac{\overline{AC}}{\overline{AB}}-1\right)$$
$$\frac{\overline{AD}}{\overline{AB}} \overset{D}{=} \frac{r}{r-1}$$
$$\frac{\overline{AC}}{\overline{AB}} \overset{C}{=} \frac{r}{r+1}$$

Example 6.15 (0.050, 2, 3) *The converse theorem. If O is the midpoint of AB, and if C, D are points of the line AB such that $\overline{OC}\cdot\overline{OD} = \overline{OA}^2$, then A, B, C, D are harmonic points.*

Constructive description

((points A B)
(midpoint O A B)
(lratio C O A r)
(lratio D O A $\frac{1}{r}$)
(harmonic A B C D))

The machine proof

$$\left(-\frac{\overline{AC}}{\overline{BC}}\right)/\left(\frac{\overline{AD}}{\overline{BD}}\right)$$
$$\overset{D}{=} \frac{-\frac{\overline{BO}}{\overline{AO}}\cdot r+1}{-r+1}\cdot -\frac{\overline{AC}}{\overline{BC}}$$
$$\overset{C}{=} \frac{-(r-1)\cdot(\frac{\overline{BO}}{\overline{AO}}\cdot r-1)}{(r-1)\cdot(-\frac{\overline{BO}}{\overline{AO}}+r)}$$
$$\overset{simplify}{=} \frac{\frac{\overline{BO}}{\overline{AO}}\cdot r-1}{\frac{\overline{BO}}{\overline{AO}}-r}$$
$$\overset{O}{=} \frac{(-\frac{1}{2}r-\frac{1}{2})\cdot(\frac{1}{2})}{(-\frac{1}{2}r-\frac{1}{2})\cdot(\frac{1}{2})}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AD}}{\overline{BD}} \overset{D}{=} \frac{r-1}{\frac{\overline{BO}}{\overline{AO}}\cdot r-1}$$
$$\frac{\overline{AC}}{\overline{BC}} \overset{C}{=} \frac{r-1}{-(\frac{\overline{BO}}{\overline{AO}}-r)}$$
$$\frac{\overline{BO}}{\overline{AO}} \overset{O}{=} -(1)$$

Example 6.16 (0.266, 8, 6) *The sum of the squares of two harmonic segments is equal to four times the square of the distance between the midpoints of these segments.*

Constructive description

((points A B)
(mratio C A B r)
(mratio D A B $-r$)
(midpoint O A B)
(midpoint M C D)
($(\frac{\overline{AB}}{\overline{MO}})^2+(\frac{\overline{CD}}{\overline{MO}})^2 = 4$))

The eliminants

$$\frac{\overline{AB}}{\overline{OM}} \overset{M}{=} \frac{\frac{\overline{AB}}{\overline{CD}}}{(-\frac{1}{2})\cdot(2\frac{\overline{CO}}{\overline{CD}}-1)}$$
$$\frac{\overline{CD}}{\overline{OM}} \overset{M}{=} \frac{1}{(-\frac{1}{2})\cdot(2\frac{\overline{CO}}{\overline{CD}}-1)}$$
$$\frac{\overline{CO}}{\overline{CD}} \overset{O}{=} \frac{(-\frac{1}{2})\cdot(2\frac{\overline{AC}}{\overline{AB}}-1)}{\frac{\overline{CD}}{\overline{AB}}}$$
$$\frac{\overline{CD}}{\overline{AB}} \overset{D}{=} \frac{\frac{\overline{AC}}{\overline{AB}}\cdot r-\frac{\overline{AC}}{\overline{AB}}-r}{-(r-1)}$$
$$\frac{\overline{AB}}{\overline{CD}} \overset{D}{=} \frac{-(r-1)}{\frac{\overline{AC}}{\overline{AB}}\cdot r-\frac{\overline{AC}}{\overline{AB}}-r}$$
$$\frac{\overline{AC}}{\overline{AB}} \overset{C}{=} \frac{r}{r+1}$$

The machine proof

$$\frac{1}{4}\left(\frac{\overline{CD}}{\overline{OM}}^{2}+\frac{\overline{AB}}{\overline{OM}}^{2}\right)$$

$$\stackrel{M}{=} \frac{\frac{\overline{CO}}{\overline{CD}}^{2}\cdot\frac{\overline{AB}}{\overline{CD}}^{2}+\frac{\overline{CO}}{\overline{CD}}^{2}-\frac{\overline{CO}}{\overline{CD}}\cdot\frac{\overline{AB}}{\overline{CD}}^{2}-\frac{\overline{CO}}{\overline{CD}}+\frac{1}{4}\frac{\overline{AB}}{\overline{CD}}^{2}+\frac{1}{4}}{(4)\cdot(-\frac{\overline{CO}}{\overline{CD}}+\frac{1}{2})^{4}}$$

$$\stackrel{simplify}{=} \frac{\frac{\overline{AB}}{\overline{CD}}^{2}+1}{(2\frac{\overline{CO}}{\overline{CD}}-1)^{2}}$$

$$\stackrel{O}{=} \frac{\frac{\overline{AB}}{\overline{CD}}^{2}+1}{(-\frac{\overline{CD}}{\overline{AB}}-2\frac{\overline{AC}}{\overline{AB}}+1)^{2}}\cdot(\frac{\overline{CD}}{\overline{AB}})^{2}$$

$$\stackrel{D}{=} \frac{(\frac{\overline{AC}}{\overline{AB}}^{2}\cdot r^{2}-2\frac{\overline{AC}}{\overline{AB}}^{2}\cdot r+\frac{\overline{AC}}{\overline{AB}}^{2}-2\frac{\overline{AC}}{\overline{AB}}\cdot r^{2}+2\frac{\overline{AC}}{\overline{AB}}\cdot r+2r^{2}-2r+1)\cdot(\frac{\overline{AC}}{\overline{AB}}\cdot r-\frac{\overline{AC}}{\overline{AB}}-r)^{2}\cdot(-r+1)^{2}}{(-\frac{\overline{AC}}{\overline{AB}}\cdot r+\frac{\overline{AC}}{\overline{AB}}-1)^{2}\cdot(-r+1)^{2}\cdot(\frac{\overline{AC}}{\overline{AB}}\cdot r-\frac{\overline{AC}}{\overline{AB}}-r)^{2}}$$

$$\stackrel{simplify}{=} \frac{\frac{\overline{AC}}{\overline{AB}}^{2}\cdot r^{2}-2\frac{\overline{AC}}{\overline{AB}}^{2}\cdot r+\frac{\overline{AC}}{\overline{AB}}^{2}-2\frac{\overline{AC}}{\overline{AB}}\cdot r^{2}+2\frac{\overline{AC}}{\overline{AB}}\cdot r+2r^{2}-2r+1}{(\frac{\overline{AC}}{\overline{AB}}\cdot r-\frac{\overline{AC}}{\overline{AB}}+1)^{2}}$$

$$\stackrel{C}{=} \frac{(r^{4}+2r^{2}+1)\cdot(r+1)^{2}}{(r^{2}+1)^{2}\cdot(r+1)^{2}}$$

$$\stackrel{simplify}{=} 1$$

Example 6.17 (0.083, 1, 10) *Let A, B, R be three points on a plane, Q and P be points on line AR and BR respectively. S is the intersection of QB and AP. C is the intersection of RS and AB. F is the intersection of QP and AB. Show that $(ABCF) = -1$.*

Constructive description

((points A B R)
(on P (l B R))
(on Q (l A R))
(inter S (l A P) (l B Q))
(inter C (l R S) (l A B))
(inter F (l P Q) (l A B))
($\frac{\overline{AC}}{\overline{BC}} = -\frac{\overline{AF}}{\overline{BF}}$))

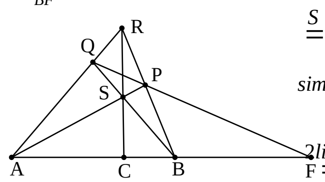


Figure 6-17

The machine proof

$$\frac{\frac{\overline{AC}}{\overline{BC}}}{-\frac{\overline{AF}}{\overline{BF}}}$$

$$\stackrel{F}{=} \frac{S_{BPQ}}{-S_{APQ}}\cdot\frac{\overline{AC}}{\overline{BC}}$$

$$\stackrel{C}{=} \frac{S_{ARS}\cdot S_{BPQ}}{-S_{APQ}\cdot S_{BRS}}$$

$$\stackrel{S}{=} \frac{-S_{ARP}\cdot S_{ABQ}\cdot S_{BPQ}\cdot(-S_{ABPQ})}{S_{APQ}\cdot(-S_{BRQ}\cdot S_{ABP})\cdot S_{ABPQ}}$$

$$\stackrel{simplify}{=} \frac{-S_{ARP}\cdot S_{ABQ}\cdot S_{BPQ}}{S_{APQ}\cdot S_{BRQ}\cdot S_{ABP}}$$

$$\stackrel{2lines}{=} \frac{-\overline{RP}\cdot\overline{AR}\cdot\beta\cdot\overline{BR}\cdot\overline{AQ}\cdot\beta\cdot(-\overline{RQ}\cdot\overline{BP}\cdot\beta)\cdot((2))^{3}}{(-\overline{RP}\cdot\overline{AQ}\cdot\beta)\cdot(-\overline{RQ}\cdot\overline{BR}\cdot\beta)\cdot\overline{BP}\cdot\overline{AR}\cdot\beta\cdot((2))^{3}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}} \stackrel{F}{=} \frac{S_{APQ}}{S_{BPQ}}$$

$$\frac{\overline{AC}}{\overline{BC}} \stackrel{C}{=} \frac{S_{ARS}}{S_{BRS}}$$

$$S_{BRS} \stackrel{S}{=} \frac{S_{BRQ}\cdot S_{ABP}}{S_{ABPQ}}$$

$$S_{ARS} \stackrel{S}{=} \frac{S_{ARP}\cdot S_{ABQ}}{S_{ABPQ}}$$

$$S_{ABP} = \frac{1}{2}(\overline{BP}\cdot\overline{AR}\cdot\beta)$$

$$S_{BRQ} = -\frac{1}{2}(\overline{RQ}\cdot\overline{BR}\cdot\beta)$$

$$S_{APQ} = -\frac{1}{2}(\overline{RP}\cdot\overline{AQ}\cdot\beta)$$

$$S_{BPQ} = -\frac{1}{2}(\overline{RQ}\cdot\overline{BP}\cdot\beta)$$

$$S_{ABQ} = \frac{1}{2}(\overline{BR}\cdot\overline{AQ}\cdot\beta)$$

$$S_{ARP} = \frac{1}{2}(\overline{RP}\cdot\overline{AR}\cdot\beta)$$

Example 6.18 (0.033, 7, 6) *Given $(ABCD) = -1$ and a point O outside the line AB, if a parallel through B to OA meets OC, OD in P, Q, we then have $PB = BQ$.*

Constructive description
((points A B O)
(mratio C A B r)
(mratio D A B $-r$)
(inter P (l O C) (p B O A))
(inter Q (l O D) (p B O A))
(midpoint B P Q))

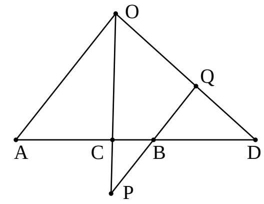


Figure 6-18

The machine proof

$-\frac{\overline{BP}}{\overline{BQ}}$

$\overset{Q}{=} \frac{-S_{BOPD}}{S_{BOD}}$

$\overset{P}{=} \frac{-(-S_{OCD}\cdot S_{ABO}+S_{BOD}\cdot S_{AOC})}{S_{BOD}\cdot S_{AOC}}$

$\overset{D}{=} \frac{(S_{BOC}\cdot S_{ABO}\cdot r^2-S_{BOC}\cdot S_{ABO}\cdot r)\cdot(-r+1)}{S_{ABO}\cdot S_{AOC}\cdot(-r+1)^2}$

$\overset{simplify}{=} \frac{-S_{BOC}\cdot r}{S_{AOC}}$

$\overset{C}{=} \frac{-S_{ABO}\cdot r\cdot(r+1)}{(-S_{ABO}\cdot r)\cdot(r+1)} \overset{simplify}{=} 1$

The eliminants

$\frac{\overline{BP}}{\overline{BQ}} \overset{Q}{=} \frac{S_{BOPD}}{S_{BOD}}$

$S_{BOPD} \overset{P}{=} \frac{-(S_{OCD}\cdot S_{ABO}-S_{BOD}\cdot S_{AOC})}{S_{AOC}}$

$S_{BOD} \overset{D}{=} \frac{S_{ABO}}{-(r-1)}$

$S_{OCD} \overset{D}{=} \frac{S_{BOC}\cdot r-S_{AOC}}{r-1}$

$S_{AOC} \overset{C}{=} \frac{-S_{ABO}\cdot r}{r+1}$

$S_{BOC} \overset{C}{=} \frac{S_{ABO}}{r+1}$

Example 6.19 (0.033, 2, 8) *The converse theorem of Example 6.18.*

Constructive description
((points A B O)
(on C (l A B))
(inter P (l O C) (p B O A))
(lratio Q B P -1)
(inter D (l O Q) (l A B))
(harmonic A B C D))

The machine proof

$(-\frac{\overline{AC}}{\overline{BC}})/(\frac{\overline{AD}}{\overline{BD}})$

$\overset{D}{=} \frac{S_{BOQ}}{S_{AOQ}} \cdot -\frac{\overline{AC}}{\overline{BC}}$

$\overset{Q}{=} \frac{-(-S_{BOP})}{-S_{AOP}-2S_{ABO}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{P}{=} \frac{-(-S_{BOC})\cdot S_{ABO}}{S_{ABO}\cdot S_{AOC}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{simplify}{=} \frac{S_{BOC}}{S_{AOC}} \cdot \frac{\overline{AC}}{\overline{BC}}$

$\overset{C}{=} \frac{\frac{\overline{AC}}{\overline{AB}}\cdot(-S_{ABO}\cdot\frac{\overline{AC}}{\overline{AB}}+S_{ABO})}{(-S_{ABO}\cdot\frac{\overline{AC}}{\overline{AB}})\cdot(\frac{\overline{AC}}{\overline{AB}}-1)}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{AD}}{\overline{BD}} \overset{D}{=} \frac{S_{AOQ}}{S_{BOQ}}$

$S_{AOQ} \overset{Q}{=} -\left(S_{AOP}+2S_{ABO}\right)$

$S_{BOQ} \overset{Q}{=} -\left(S_{BOP}\right)$

$S_{AOP} \overset{P}{=} -\left(S_{ABO}\right)$

$S_{BOP} \overset{P}{=} \frac{-S_{BOC}\cdot S_{ABO}}{S_{AOC}}$

$S_{AOC} \overset{C}{=} -\left(S_{ABO}\cdot\frac{\overline{AC}}{\overline{AB}}\right)$

$S_{BOC} \overset{C}{=} -\left((\frac{\overline{AC}}{\overline{AB}}-1)\cdot S_{ABO}\right)$

$\frac{\overline{AC}}{\overline{BC}} \overset{C}{=} \frac{\frac{\overline{AC}}{\overline{AB}}}{\frac{\overline{AC}}{\overline{AB}}-1}$

6.2.4 Pappus' Theorem and Desargues' Theorem

For the background about n_3 configurations, see Subsection 2.7.2.

Example 6.20 (Pappus' Theorem) (0.166, 1, 14) *Let ABC and $A_1B_1C_1$ be two lines, and $P = AB_1 \cap A_1B$, $Q = AC_1 \cap A_1C$, $S = BC_1 \cap B_1C$. Then P, Q and S are collinear.*

For a machine proof of this example, see Example 2.42. The following proof is shorter.

Constructive description

((points A A_1 B B_1)
(on C (l A B))
(on C_1 (l A_1 B_1))
(inter P (l A_1 B) (l A B_1))
(inter Q (l A C_1) (l A_1 C))
(inter S (l B_1 C) (l B C_1))
(inter T (l B_1 C) (l P Q))
($\frac{\overline{B_1S}}{\overline{CS}} = \frac{\overline{B_1T}}{\overline{CT}}$))

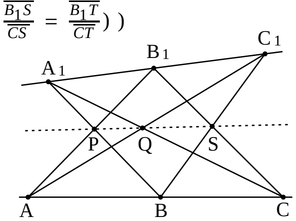


Figure 6-20

The machine proof

$$\left(\frac{\overline{B_1S}}{\overline{CS}}\right)/\left(\frac{\overline{B_1T}}{\overline{CT}}\right)$$
$$\overset{T}{=} \frac{S_{QCP}}{-S_{B_1QP}} \cdot \frac{\overline{B_1S}}{\overline{CS}}$$
$$\overset{S}{=} \frac{(-S_{BB_1C_1})\cdot S_{QCP}}{-S_{B_1QP}\cdot(-S_{BCC_1})}$$
$$\overset{P}{=} \frac{-S_{BB_1C_1}\cdot(-S_{BQC}\cdot S_{AA_1B_1})\cdot S_{AA_1B_1B}}{(-S_{A_1BB_1}\cdot S_{AB_1Q})\cdot S_{BCC_1}\cdot(-S_{AA_1B_1B})}$$
$$\overset{simplify}{=} \frac{S_{BB_1C_1}\cdot S_{BQC}\cdot S_{AA_1B_1}}{S_{A_1BB_1}\cdot S_{AB_1Q}\cdot S_{BCC_1}}$$
$$\overset{C_1}{=} \frac{S_{A_1BB_1}\cdot S_{AB_1Q}\cdot S_{BQC}\cdot S_{AA_1B_1}\cdot S_{AA_1QB_1}}{S_{A_1BB_1}\cdot S_{AB_1Q}\cdot(-S_{BQC}\cdot S_{AA_1B_1})\cdot(-S_{AA_1QB_1})}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{B_1T}}{\overline{CT}} \overset{T}{=} \frac{-S_{B_1QP}}{S_{QCP}}$$
$$\frac{\overline{B_1S}}{\overline{CS}} \overset{S}{=} \frac{S_{BB_1C_1}}{S_{BCC_1}}$$
$$S_{B_1QP} \overset{P}{=} \frac{-S_{A_1BB_1}\cdot S_{AB_1Q}}{S_{AA_1B_1B}}$$
$$S_{QCP} \overset{P}{=} \frac{S_{BQC}\cdot S_{AA_1B_1}}{S_{AA_1B_1B}}$$
$$S_{BCC_1} \overset{C_1}{=} \frac{-S_{BQC}\cdot S_{AA_1B_1}}{S_{AA_1QB_1}}$$
$$S_{BB_1C_1} \overset{C_1}{=} \frac{S_{A_1BB_1}\cdot S_{AB_1Q}}{-S_{AA_1QB_1}}$$

Line PQ is called the *Pappus' line* and is denoted by [123].

Example 6.21 (0.116, 1, 6) *The dual of Pappus' theorem.*

Constructive description

((points A B C O O_1)
(inter A_1 (l O_1 B) (l O C))
(inter B_1 (l O A) (l O_1 C))
(inter C_1 (l O B) (l O_1 A))
(inter I (l B B_1) (l A A_1))
(inter Z_I (l C_1 C) (l A A_1))
($\frac{\overline{AI}}{\overline{A_1I}} = \frac{\overline{AZ_I}}{\overline{A_1Z_I}}$))

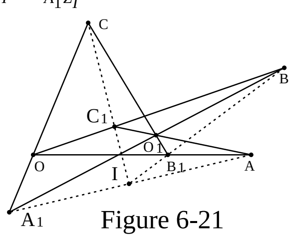


Figure 6-21

The machine proof

$$\left(\frac{\overline{AI}}{\overline{A_1I}}\right)/\left(\frac{\overline{AZ_I}}{\overline{A_1Z_I}}\right)$$
$$\overset{Z_I}{=} \frac{S_{CA_1C_1}}{-S_{ACC_1}} \cdot \frac{\overline{AI}}{\overline{A_1I}}$$
$$\overset{I}{=} \frac{S_{ABB_1}\cdot S_{CA_1C_1}}{-S_{ACC_1}\cdot(-S_{BA_1B_1})}$$
$$\overset{C_1}{=} \frac{S_{ABB_1}\cdot S_{BCA_1}\cdot S_{AOO_1}\cdot S_{ABO_1O}}{S_{ACO_1}\cdot S_{ABO}\cdot S_{BA_1B_1}\cdot(-S_{ABO_1O})}$$
$$\overset{simplify}{=} \frac{S_{ABB_1}\cdot S_{BCA_1}\cdot S_{AOO_1}}{-S_{ACO_1}\cdot S_{ABO}\cdot S_{BA_1B_1}}$$
$$\overset{B_1}{=} \frac{S_{ACO_1}\cdot S_{ABO}\cdot S_{BCA_1}\cdot S_{AOO_1}\cdot(-S_{ACOO_1})}{-S_{ACO_1}\cdot S_{ABO}\cdot S_{BCA_1}\cdot S_{AOO_1}\cdot S_{ACOO_1}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_I}}{\overline{A_1Z_I}} \overset{Z_I}{=} \frac{-S_{ACC_1}}{S_{CA_1C_1}}$$
$$\frac{\overline{AI}}{\overline{A_1I}} \overset{I}{=} \frac{S_{ABB_1}}{-S_{BA_1B_1}}$$
$$S_{ACC_1} \overset{C_1}{=} \frac{S_{ACO_1}\cdot S_{ABO}}{S_{ABO_1O}}$$
$$S_{CA_1C_1} \overset{C_1}{=} \frac{S_{BCA_1}\cdot S_{AOO_1}}{-S_{ABO_1O}}$$
$$S_{BA_1B_1} \overset{B_1}{=} \frac{S_{BCA_1}\cdot S_{AOO_1}}{-S_{ACOO_1}}$$
$$S_{ABB_1} \overset{B_1}{=} \frac{S_{ACO_1}\cdot S_{ABO}}{S_{ACOO_1}}$$

Example 6.22 (Pappus Point Theorem) (0.550, 2, 26) *The three Pappus lines* [123], [312] *and* [231] *are concurrent. So are the Pappus lines* [213], [321] *and* [132].

Constructive description
((points A A_1 B C_1)
(on B_1 (l A_1 C_1))
(on C (l A B))
(inter E (l A_1 B) (l A B_1))
(inter F (l A_1 C) (l A C_1))
(inter G (l B C_1) (l A A_1))
(inter H (l B B_1) (l C A_1))
(inter I (l B B_1) (l A C_1))
(inter J (l B A_1) (l C C_1))
(inter K (l G H) (l E F)) (collinear I J K))

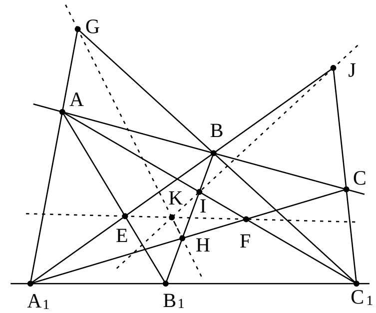


Figure 6-22

Example 6.23 (Leisening's Theorem) (1.233, 4, 32) *Continuing from Example 6.20, let* $O = AB \cap A_1B_1$*, and* L_1, L_2*, and* L_3 *be the intersections of lines* OP *and* CC_1*, lines* OQ *and* BB_1*, and lines* OS *and* AA_1 *respectively. Show that* L_1, L_2, L_3 *are collinear.*

Constructive description
((points A B A_1 B_1)
(on C (l A B)), (on C_1 (l A_1 B_1))
(inter P (l A B_1) (l B A_1))
(inter Q (l A_1 C) (l A C_1))
(inter S (l B C_1) (l B_1 C))
(inter O (l A B) (l A_1 B_1))
(inter L_1 (l O P) (l C C_1))
(inter L_2 (l O Q) (l B B_1))
(inter Z_1 (l L_1 L_2) (l A A_1))
(inter Z_2 (l L_1 L_2) (l O S))
($\frac{\overline{L_1Z_1}}{\overline{L_2Z_1}} = \frac{\overline{L_1Z_2}}{\overline{L_2Z_2}}$))

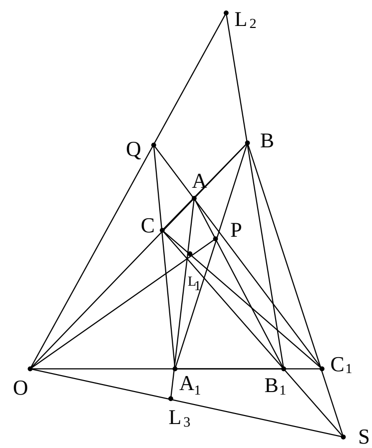


Figure 6-23

Example 6.24 (Desargues' Theorem) (0.250, 2, 18) *Given two triangle* ABC, $A_1B_1C_1$*, if the three lines* AA_1, BB_1, CC_1 *meet in a point,* S*, the three points* $P = BC \cap B_1C_1$, $Q = CA \cap C_1A_1$, $R = AB \cap A_1B_1$ *lie on a line.*

Constructive description
((points A B C S)
(on A_1 (l S A))
(on B_1 (l S B))
(on C_1 (l S C))
(inter P (l B C) (l B_1 C_1))
(inter Q (l A C) (l A_1 C_1))
(inter R (l A B) (l A_1 B_1))
(inter Z_2 (l A_1 B_1) (l P Q))
(inter Z_1 (l A B) (l P Q)) ($\frac{\overline{PZ_1}}{\overline{QZ_1}} \cdot \frac{\overline{QZ_2}}{\overline{PZ_2}} = 1$))

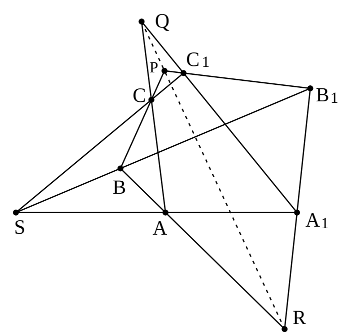


Figure 6-24

The machine proof

$$\frac{\overline{PZ_1}}{\overline{QZ_1}} \cdot \frac{\overline{QZ_2}}{\overline{PZ_2}}$$

$$\overset{Z_1}{=} \frac{S_{ABP}}{S_{ABQ}} \cdot \frac{\overline{QZ_2}}{\overline{PZ_2}}$$

$$\overset{Z_2}{=} \frac{S_{ABP} \cdot S_{A_1B_1Q}}{S_{ABQ} \cdot S_{A_1B_1P}}$$

$$\overset{Q}{=} \frac{S_{ABP} \cdot S_{A_1B_1C_1} \cdot S_{ACA_1} \cdot S_{AA_1CC_1}}{S_{AA_1C_1} \cdot S_{ABC} \cdot S_{A_1B_1P} \cdot (-S_{AA_1CC_1})}$$

$$\overset{simplify}{=} \frac{-S_{ABP} \cdot S_{A_1B_1C_1} \cdot S_{ACA_1}}{S_{AA_1C_1} \cdot S_{ABC} \cdot S_{A_1B_1P}}$$

$$\overset{P}{=} \frac{-S_{BB_1C_1} \cdot S_{ABC} \cdot S_{A_1B_1C_1} \cdot S_{ACA_1} \cdot (-S_{BB_1CC_1})}{S_{AA_1C_1} \cdot S_{ABC} \cdot S_{A_1B_1C_1} \cdot S_{BCB_1} \cdot S_{BB_1CC_1}}$$

$$\overset{simplify}{=} \frac{S_{BB_1C_1} \cdot S_{ACA_1}}{S_{AA_1C_1} \cdot S_{BCB_1}}$$

$$\overset{C_1}{=} \frac{(-S_{BCB_1} \cdot \frac{\overline{SC_1}}{\overline{SC}}) \cdot S_{ACA_1}}{(-S_{ACA_1} \cdot \frac{\overline{SC_1}}{\overline{SC}}) \cdot S_{BCB_1}} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{PZ_1}}{\overline{QZ_1}} \overset{Z_1}{=} \frac{S_{ABP}}{S_{ABQ}}$$

$$\frac{\overline{QZ_2}}{\overline{PZ_2}} \overset{Z_2}{=} \frac{S_{A_1B_1Q}}{S_{A_1B_1P}}$$

$$S_{ABQ} \overset{Q}{=} \frac{S_{AA_1C_1} \cdot S_{ABC}}{S_{AA_1CC_1}}$$

$$S_{A_1B_1Q} \overset{Q}{=} \frac{S_{A_1B_1C_1} \cdot S_{ACA_1}}{-S_{AA_1CC_1}}$$

$$S_{A_1B_1P} \overset{P}{=} \frac{S_{A_1B_1C_1} \cdot S_{BCB_1}}{-S_{BB_1CC_1}}$$

$$S_{ABP} \overset{P}{=} \frac{S_{BB_1C_1} \cdot S_{ABC}}{S_{BB_1CC_1}}$$

$$S_{AA_1C_1} \overset{C_1}{=} -\left(S_{ACA_1} \cdot \frac{\overline{SC_1}}{\overline{SC}}\right)$$

$$S_{BB_1C_1} \overset{C_1}{=} -\left(S_{BCB_1} \cdot \frac{\overline{SC_1}}{\overline{SC}}\right)$$

Example 6.25 (0.216, 2, 12) *The converse of Desargues' theorem.*

Constructive description

((points A B C A_1) (on P (l B C)) (on Q (l A C)) (inter R (l A B) (l P Q)) (on B_1 (l R A_1)) (inter C_1 (l A_1 Q) (l B_1 P)) (inter S (l A A_1) (l B B_1)) (inter Z (l A A_1) (l C C_1)) ($\frac{\overline{AS}}{\overline{SA_1}} = \frac{\overline{AZ}}{\overline{ZA_1}}$))

The machine proof

$$\left(\frac{\overline{AS}}{\overline{A_1S}}\right) / \left(\frac{\overline{AZ}}{\overline{A_1Z}}\right)$$

$$\overset{Z}{=} \frac{-S_{CA_1C_1}}{S_{ACC_1}} \cdot \frac{\overline{AS}}{\overline{A_1S}}$$

$$\overset{S}{=} \frac{-S_{ABB_1} \cdot S_{CA_1C_1}}{S_{ACC_1} \cdot (-S_{BA_1B_1})}$$

$$\overset{C_1}{=} \frac{S_{ABB_1} \cdot S_{A_1PB_1} \cdot S_{CA_1Q} \cdot S_{A_1PQB_1}}{S_{PQB_1} \cdot S_{ACA_1} \cdot S_{BA_1B_1} \cdot S_{A_1PQB_1}}$$

$$\overset{simplify}{=} \frac{S_{ABB_1} \cdot S_{A_1PB_1} \cdot S_{CA_1Q}}{S_{PQB_1} \cdot S_{ACA_1} \cdot S_{BA_1B_1}}$$

$$\overset{B_1}{=} \frac{S_{ABA_1} \cdot \frac{\overline{RB_1}}{\overline{RA_1}} \cdot (-S_{A_1PR} \cdot \frac{\overline{RB_1}}{\overline{RA_1}} + S_{A_1PR}) \cdot S_{CA_1Q}}{S_{A_1PQ} \cdot \frac{\overline{RB_1}}{\overline{RA_1}} \cdot S_{ACA_1} \cdot (-S_{BA_1R} \cdot \frac{\overline{RB_1}}{\overline{RA_1}} + S_{BA_1R})}$$

$$\overset{simplify}{=} \frac{S_{ABA_1} \cdot S_{A_1PR} \cdot S_{CA_1Q}}{S_{A_1PQ} \cdot S_{ACA_1} \cdot S_{BA_1R}}$$

$$\overset{R}{=} \frac{S_{ABA_1} \cdot S_{A_1PQ} \cdot S_{ABP} \cdot S_{CA_1Q} \cdot S_{APBQ}}{S_{A_1PQ} \cdot S_{ACA_1} \cdot (-S_{BPQ} \cdot S_{ABA_1}) \cdot (-S_{APBQ})}$$

$$\overset{simplify}{=} \frac{S_{ABP} \cdot S_{CA_1Q}}{S_{ACA_1} \cdot S_{BPQ}}$$

$$\overset{Q}{=} \frac{S_{ABP} \cdot (-S_{ACA_1} \cdot \frac{\overline{AQ}}{\overline{AC}} + S_{ACA_1})}{S_{ACA_1} \cdot (-S_{ABP} \cdot \frac{\overline{AQ}}{\overline{AC}} + S_{ABP})} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ}}{\overline{A_1Z}} \overset{Z}{=} \frac{S_{ACC_1}}{-S_{CA_1C_1}}$$

$$\frac{\overline{AS}}{\overline{A_1S}} \overset{S}{=} \frac{S_{ABB_1}}{-S_{BA_1B_1}}$$

$$S_{ACC_1} \overset{C_1}{=} \frac{S_{PQB_1} \cdot S_{ACA_1}}{S_{A_1PQB_1}}$$

$$S_{CA_1C_1} \overset{C_1}{=} \frac{S_{A_1PB_1} \cdot S_{CA_1Q}}{S_{A_1PQB_1}}$$

$$S_{BA_1B_1} \overset{B_1}{=} -\left(\left(\frac{\overline{RB_1}}{\overline{RA_1}} - 1\right) \cdot S_{BA_1R}\right)$$

$$S_{PQB_1} \overset{B_1}{=} S_{A_1PQ} \cdot \frac{\overline{RB_1}}{\overline{RA_1}}$$

$$S_{A_1PB_1} \overset{B_1}{=} -\left(\left(\frac{\overline{RB_1}}{\overline{RA_1}} - 1\right) \cdot S_{A_1PR}\right)$$

$$S_{ABB_1} \overset{B_1}{=} S_{ABA_1} \cdot \frac{\overline{RB_1}}{\overline{RA_1}}$$

$$S_{BA_1R} \overset{R}{=} \frac{-S_{BPQ} \cdot S_{ABA_1}}{S_{APBQ}}$$

$$S_{A_1PR} \overset{R}{=} \frac{S_{A_1PQ} \cdot S_{ABP}}{-S_{APBQ}}$$

$$S_{BPQ} \overset{Q}{=} -\left(\left(\frac{\overline{AQ}}{\overline{AC}} - 1\right) \cdot S_{ABP}\right)$$

$$S_{CA_1Q} \overset{Q}{=} -\left(\left(\frac{\overline{AQ}}{\overline{AC}} - 1\right) \cdot S_{ACA_1}\right)$$

6.2.5 Miscellaneous

Example 6.26 (0.133, 1, 8) *In a hexagon $AC_1BA_1CB_1$, BB_1, C_1A, A_1C are concurrent and CC_1, A_1B, B_1A are concurrent. Prove that AA_1, B_1C, C_1B are also concurrent.*

Constructive description

((points A A_1 B C C_1)
(inter O (l A C_1) (l A_1 C))
(inter H (l A_1 B) (l C C_1))
(inter B_1 (l A H) (l B O))
(inter I (l B_1 C) (l A A_1))
(inter Z_I (l C_1 B) (l A A_1))
($\frac{\overline{AI}}{\overline{A_1I}} = \frac{\overline{AZ_I}}{\overline{A_1Z_I}}$))

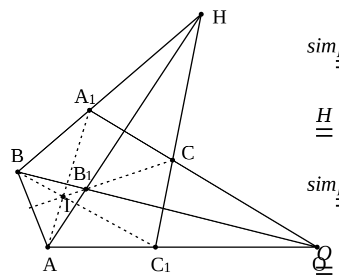


Figure 6-26

The machine proof

$$\left(\frac{\overline{AI}}{\overline{A_1I}}\right)/\left(\frac{\overline{AZ_I}}{\overline{A_1Z_I}}\right)$$

$$\overset{Z_I}{=} \frac{-S_{A_1BC_1}}{-S_{ABC_1}} \cdot \frac{\overline{AI}}{\overline{A_1I}}$$

$$\overset{I}{=} \frac{(-S_{ACB_1})\cdot S_{A_1BC_1}}{S_{ABC_1}\cdot(-S_{A_1CB_1})}$$

$$\overset{B_1}{=} \frac{S_{ACH}\cdot S_{ABO}\cdot S_{A_1BC_1}\cdot(-S_{ABHO})}{S_{ABC_1}\cdot(-S_{A_1BC}\cdot S_{AOH})\cdot S_{ABHO}}$$

$$\overset{simplify}{=} \frac{S_{ACH}\cdot S_{ABO}\cdot S_{A_1BC_1}}{S_{ABC_1}\cdot S_{A_1BC}\cdot S_{AOH}}$$

$$\overset{H}{=} \frac{S_{A_1BC}\cdot S_{ACC_1}\cdot S_{ABO}\cdot S_{A_1BC_1}\cdot(-S_{A_1CBC_1})}{S_{ABC_1}\cdot S_{A_1BC}\cdot S_{A_1BC_1}\cdot S_{ACO}\cdot(-S_{A_1CBC_1})}$$

$$\overset{simplify}{=} \frac{S_{ACC_1}\cdot S_{ABO}}{S_{ABC_1}\cdot S_{ACO}}$$

$$\overset{O}{=} \frac{S_{ACC_1}\cdot S_{ABC_1}\cdot S_{AA_1C}\cdot(-S_{AA_1C_1C})}{S_{ABC_1}\cdot(-S_{ACC_1}\cdot S_{AA_1C})\cdot S_{AA_1C_1C}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_I}}{\overline{A_1Z_I}} \overset{Z_I}{=} \frac{S_{ABC_1}}{S_{A_1BC_1}}$$

$$\frac{\overline{AI}}{\overline{A_1I}} \overset{I}{=} \frac{S_{ACB_1}}{S_{A_1CB_1}}$$

$$S_{A_1CB_1} \overset{B_1}{=} \frac{S_{A_1BC}\cdot S_{AOH}}{S_{ABHO}}$$

$$S_{ACB_1} \overset{B_1}{=} \frac{S_{ACH}\cdot S_{ABO}}{S_{ABHO}}$$

$$S_{AOH} \overset{H}{=} \frac{S_{A_1BC_1}\cdot S_{ACO}}{-S_{A_1CBC_1}}$$

$$S_{ACH} \overset{H}{=} \frac{S_{A_1BC}\cdot S_{ACC_1}}{-S_{A_1CBC_1}}$$

$$S_{ACO} \overset{O}{=} \frac{S_{ACC_1}\cdot S_{AA_1C}}{S_{AA_1C_1C}}$$

$$S_{ABO} \overset{O}{=} \frac{S_{ABC_1}\cdot S_{AA_1C}}{S_{AA_1C_1C}}$$

Example 6.27 (Nehring's Theorem (1942)) (0.533, 2, 14) *Let AA_1, BB_1, CC_1 be three concurrent cevian lines for triangle ABC. Let X_1 be a point on BC, $X_2 = X_1B_1 \cap BA$, $X_3 = X_2A_1 \cap AC$, $X_4 = X_3C_1 \cap CB$, $X_5 = X_4B_1 \cap BA$, $X_6 = X_5A_1 \cap AC$, $X_7 = X_6C_1 \cap CB$. Show $X_7 = X_1$.*

Constructive description

((points B C A O)
(inter A_1 (l A O) (l B C))
(inter B_1 (l B O) (l C A))
(inter C_1 (l C O) (l B A))
(on X_1 (l B C))
(inter X_2 (l B_1 X_1) (l B A))
(inter X_3 (l A_1 X_2) (l C A))
(inter X_4 (l C_1 X_3) (l B C))
(inter X_5 (l B_1 X_4) (l B A))
(inter X_6 (l A_1 X_5) (l C A))
(inter Z_2 (l C A) (l C_1 X_1))
(inter Z_1 (l A_1 X_5) (l C_1 X_1))

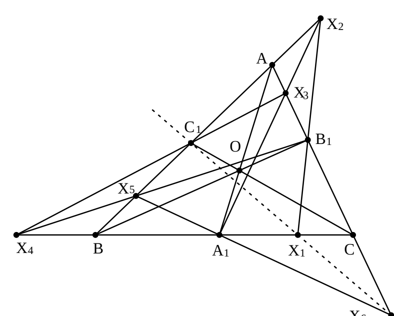


Figure 6-27

$$\left(\frac{\overline{C_1Z_1}}{\overline{X_1Z_1}} \cdot \frac{\overline{X_1Z_2}}{\overline{C_1Z_2}} = 1\right))$$

The machine proof

$$\frac{\overline{C_1Z_1}}{\overline{X_1Z_1}} \cdot \frac{\overline{X_1Z_2}}{\overline{C_1Z_2}}$$

$$\overset{Z_1}{=} \frac{-S_{A_1C_1X_5}}{-S_{A_1X_1X_5}} \cdot \frac{\overline{X_1Z_2}}{\overline{C_1Z_2}}$$

$$\overset{Z_2}{=} \frac{S_{A_1C_1X_5} \cdot S_{CAX_1}}{S_{A_1X_1X_5} \cdot S_{CAC_1}}$$

$$\overset{X_5}{=} \frac{(-S_{B_1C_1X_4} \cdot S_{BAA_1}) \cdot S_{CAX_1} \cdot S_{BB_1AX_4}}{S_{AA_1X_1} \cdot S_{BB_1X_4} \cdot S_{CAC_1} \cdot S_{BB_1AX_4}}$$

$$\overset{simplify}{=} \frac{-S_{B_1C_1X_4} \cdot S_{BAA_1} \cdot S_{CAX_1}}{S_{AA_1X_1} \cdot S_{BB_1X_4} \cdot S_{CAC_1}}$$

$$\overset{X_4}{=} \frac{-S_{B_1C_1X_3} \cdot S_{BCC_1} \cdot S_{BAA_1} \cdot S_{CAX_1} \cdot S_{BC_1CX_3}}{S_{AA_1X_1} \cdot (-S_{BC_1X_3} \cdot S_{BCB_1}) \cdot S_{CAC_1} \cdot (-S_{BC_1CX_3})}$$

$$\overset{simplify}{=} \frac{-S_{B_1C_1X_3} \cdot S_{BCC_1} \cdot S_{BAA_1} \cdot S_{CAX_1}}{S_{AA_1X_1} \cdot S_{BC_1X_3} \cdot S_{BCB_1} \cdot S_{CAC_1}}$$

$$\overset{X_3}{=} \frac{-S_{A_1B_1X_2} \cdot S_{CAC_1} \cdot S_{BCC_1} \cdot S_{BAA_1} \cdot S_{CAX_1} \cdot S_{CA_1AX_2}}{S_{AA_1X_1} \cdot S_{AA_1X_2} \cdot S_{BCC_1} \cdot S_{BCB_1} \cdot S_{CAC_1} \cdot S_{CA_1AX_2}}$$

$$\overset{simplify}{=} \frac{-S_{A_1B_1X_2} \cdot S_{BAA_1} \cdot S_{CAX_1}}{S_{AA_1X_1} \cdot S_{AA_1X_2} \cdot S_{BCB_1}}$$

$$\overset{X_2}{=} \frac{-S_{A_1B_1X_1} \cdot S_{BAB_1} \cdot S_{BAA_1} \cdot S_{CAX_1} \cdot S_{BB_1AX_1}}{S_{AA_1X_1} \cdot (-S_{AB_1X_1} \cdot S_{BAA_1}) \cdot S_{BCB_1} \cdot (-S_{BB_1AX_1})}$$

$$\overset{simplify}{=} \frac{-S_{A_1B_1X_1} \cdot S_{BAB_1} \cdot S_{CAX_1}}{S_{AA_1X_1} \cdot S_{AB_1X_1} \cdot S_{BCB_1}}$$

$$\overset{X_1}{=} \frac{-(\frac{\overline{BA_1}}{\overline{BC}} \cdot S_{BCB_1} - S_{BCB_1} \cdot \frac{\overline{BX_1}}{\overline{BC}}) \cdot S_{BAB_1} \cdot (-S_{BCA} \cdot \frac{\overline{BX_1}}{\overline{BC}} + S_{BCA})}{(-\frac{\overline{BA_1}}{\overline{BC}} \cdot S_{BCA} + S_{BCA} \cdot \frac{\overline{BX_1}}{\overline{BC}}) \cdot (-S_{BAB_1} \cdot \frac{\overline{BX_1}}{\overline{BC}} + S_{BAB_1}) \cdot S_{BCB_1}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{C_1Z_1}}{\overline{X_1Z_1}} \overset{Z_1}{=} \frac{S_{A_1C_1X_5}}{S_{A_1X_1X_5}}$$

$$\frac{\overline{X_1Z_2}}{\overline{C_1Z_2}} \overset{Z_2}{=} \frac{S_{CAX_1}}{S_{CAC_1}}$$

$$S_{A_1X_1X_5} \overset{X_5}{=} \frac{S_{AA_1X_1} \cdot S_{BB_1X_4}}{S_{BB_1AX_4}}$$

$$S_{A_1C_1X_5} \overset{X_5}{=} \frac{-S_{B_1C_1X_4} \cdot S_{BAA_1}}{S_{BB_1AX_4}}$$

$$S_{BB_1X_4} \overset{X_4}{=} \frac{-S_{BC_1X_3} \cdot S_{BCB_1}}{S_{BC_1CX_3}}$$

$$S_{B_1C_1X_4} \overset{X_4}{=} \frac{S_{B_1C_1X_3} \cdot S_{BCC_1}}{-S_{BC_1CX_3}}$$

$$S_{BC_1X_3} \overset{X_3}{=} \frac{S_{AA_1X_2} \cdot S_{BCC_1}}{S_{CA_1AX_2}}$$

$$S_{B_1C_1X_3} \overset{X_3}{=} \frac{S_{A_1B_1X_2} \cdot S_{CAC_1}}{S_{CA_1AX_2}}$$

$$S_{AA_1X_2} \overset{X_2}{=} \frac{-S_{AB_1X_1} \cdot S_{BAA_1}}{S_{BB_1AX_1}}$$

$$S_{A_1B_1X_2} \overset{X_2}{=} \frac{S_{A_1B_1X_1} \cdot S_{BAB_1}}{-S_{BB_1AX_1}}$$

$$S_{AB_1X_1} \overset{X_1}{=} -\left((\frac{\overline{BX_1}}{\overline{BC}} - 1) \cdot S_{BAB_1}\right)$$

$$S_{AA_1X_1} \overset{X_1}{=} -\left((\frac{\overline{BA_1}}{\overline{BC}} - \frac{\overline{BX_1}}{\overline{BC}}) \cdot S_{BCA}\right)$$

$$S_{CAX_1} \overset{X_1}{=} -\left((\frac{\overline{BX_1}}{\overline{BC}} - 1) \cdot S_{BCA}\right)$$

$$S_{A_1B_1X_1} \overset{X_1}{=} (\frac{\overline{BA_1}}{\overline{BC}} - \frac{\overline{BX_1}}{\overline{BC}}) \cdot S_{BCB_1}$$

Example 6.28 (0.733, 2, 12) *Let three triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ be given such that lines AB, A_1B_1, A_2B_2 intersect in a point P, lines AC, A_1C_1, A_2C_2 intersect in a point Q, lines BC, B_1C_1, B_2C_2 intersect in a point R, and P, Q, R are collinear. In view of Desargues' theorem, the lines in each of the triads AA_1, BB_1, CC_1; AA_2, BB_2, CC_2; A_1A_2, B_1B_2, C_1C_2; intersect in a point. Prove that these three points are collinear.*

Constructive description

((points A A_1 B B_1 B_2 C)
(inter P (l A_1 B_1) (l A B))
(on Q (l A C))
(inter R (l P Q) (l B C))
(inter C_1 (l B_1 R) (l A_1 Q))
(on A_2 (l B_2 P))
(inter C_2 (l B_2 R) (l A_2 Q))
(inter I (l B B_1) (l A A_1))

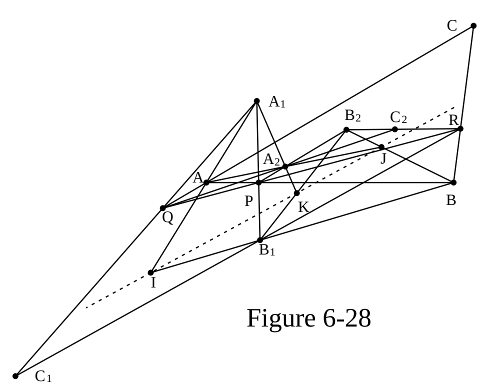


Figure 6-28

(inter J (l B B_2) (l A A_2))
(inter K (l B_1 B_2) (l A_1 A_2))
(inter Z_K (l J I) (l B_1 B_2)) $(\frac{\overline{B_1K}}{\overline{B_2K}} = \frac{\overline{B_1Z_K}}{\overline{B_2Z_K}})$)

The machine proof

$$(\frac{\overline{B_1K}}{\overline{B_2K}})/(\frac{\overline{B_1Z_K}}{\overline{B_2Z_K}})$$

$$\stackrel{Z_K}{=} \frac{-S_{B_2IJ}}{-S_{B_1IJ}} \cdot \frac{\overline{B_1K}}{\overline{B_2K}}$$

$$\stackrel{K}{=} \frac{(-S_{A_1B_1A_2}) \cdot S_{B_2IJ}}{S_{B_1IJ} \cdot (-S_{A_1B_2A_2})}$$

$$\stackrel{J}{=} \frac{S_{A_1B_1A_2} \cdot S_{BB_2I} \cdot S_{AB_2A_2} \cdot (-S_{ABA_2B_2})}{S_{B_1B_2I} \cdot S_{ABA_2} \cdot S_{A_1B_2A_2} \cdot (-S_{ABA_2B_2})}$$

$$\stackrel{simplify}{=} \frac{S_{A_1B_1A_2} \cdot S_{BB_2I} \cdot S_{AB_2A_2}}{S_{B_1B_2I} \cdot S_{ABA_2} \cdot S_{A_1B_2A_2}}$$

$$\stackrel{I}{=} \frac{S_{A_1B_1A_2} \cdot (-S_{BB_1B_2} \cdot S_{AA_1B}) \cdot S_{AB_2A_2} \cdot (-S_{ABA_1B_1})}{(-S_{BB_1B_2} \cdot S_{AA_1B_1}) \cdot S_{ABA_2} \cdot S_{A_1B_2A_2} \cdot (-S_{ABA_1B_1})}$$

$$\stackrel{simplify}{=} \frac{S_{A_1B_1A_2} \cdot S_{AA_1B} \cdot S_{AB_2A_2}}{S_{AA_1B_1} \cdot S_{ABA_2} \cdot S_{A_1B_2A_2}}$$

$$\stackrel{A_2}{=} \frac{(-S_{A_1B_1B_2} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}} + S_{A_1B_1B_2}) \cdot S_{AA_1B} \cdot S_{AB_2P} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}}}{S_{AA_1B_1} \cdot (-S_{ABB_2} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}} + S_{ABB_2}) \cdot S_{A_1B_2P} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}}}$$

$$\stackrel{simplify}{=} \frac{S_{A_1B_1B_2} \cdot S_{AA_1B} \cdot S_{AB_2P}}{S_{AA_1B_1} \cdot S_{ABB_2} \cdot S_{A_1B_2P}}$$

$$\stackrel{P}{=} \frac{S_{A_1B_1B_2} \cdot S_{AA_1B} \cdot (-S_{ABB_2} \cdot S_{AA_1B_1}) \cdot (-S_{AA_1BB_1})}{S_{AA_1B_1} \cdot S_{ABB_2} \cdot S_{A_1B_1B_2} \cdot S_{AA_1B} \cdot S_{AA_1BB_1}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{B_1Z_K}}{\overline{B_2Z_K}} \stackrel{Z_K}{=} \frac{S_{B_1IJ}}{S_{B_2IJ}}$$

$$\frac{\overline{B_1K}}{\overline{B_2K}} \stackrel{K}{=} \frac{S_{A_1B_1A_2}}{S_{A_1B_2A_2}}$$

$$S_{B_1IJ} \stackrel{J}{=} \frac{S_{B_1B_2I} \cdot S_{ABA_2}}{-S_{ABA_2B_2}}$$

$$S_{B_2IJ} \stackrel{J}{=} \frac{S_{BB_2I} \cdot S_{AB_2A_2}}{-S_{ABA_2B_2}}$$

$$S_{B_1B_2I} \stackrel{I}{=} \frac{S_{BB_1B_2} \cdot S_{AA_1B_1}}{S_{ABA_1B_1}}$$

$$S_{BB_2I} \stackrel{I}{=} \frac{S_{BB_1B_2} \cdot S_{AA_1B}}{S_{ABA_1B_1}}$$

$$S_{A_1B_2A_2} \stackrel{A_2}{=} S_{A_1B_2P} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}}$$

$$S_{ABA_2} \stackrel{A_2}{=} -\left((\frac{\overline{B_2A_2}}{\overline{B_2P}} - 1) \cdot S_{ABB_2}\right)$$

$$S_{AB_2A_2} \stackrel{A_2}{=} S_{AB_2P} \cdot \frac{\overline{B_2A_2}}{\overline{B_2P}}$$

$$S_{A_1B_1A_2} \stackrel{A_2}{=} -\left((\frac{\overline{B_2A_2}}{\overline{B_2P}} - 1) \cdot S_{A_1B_1B_2}\right)$$

$$S_{A_1B_2P} \stackrel{P}{=} \frac{S_{A_1B_1B_2} \cdot S_{AA_1B}}{-S_{AA_1BB_1}}$$

$$S_{AB_2P} \stackrel{P}{=} \frac{-S_{ABB_2} \cdot S_{AA_1B_1}}{S_{AA_1BB_1}}$$

Example 6.29 (0.250, 1, 18) *If* (Q) *is the cevian triangle of a point* M *for the triangle* (P), *show that the triangle formed by the parallels through the vertices of* (P) *to the corresponding sides of* (Q) *is perspective to* (P).

Constructive description
((points A B C M)
(inter A_1 (l B C) (l A M))
(inter B_1 (l A C) (l B M))
(inter C_1 (l A B) (l C M))
(inter C_2 (p B A_1 C_1) (p A B_1 C_1))
(inter A_2 (l C_2 B) (p C A_1 B_1))
(inter B_2 (l C_2 A) (l A_2 C))
(inter P (l B B_2) (l A A_2))
(inter Z_P (l C_2 C) (l A A_2))
$(\frac{\overline{AP}}{\overline{A_2P}} = \frac{\overline{AZ_P}}{\overline{A_2Z_P}})$)

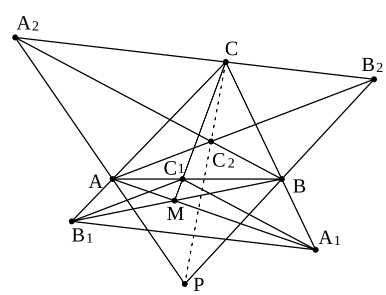


Figure 6-29

The machine proof

$$\left(\frac{\overline{AP}}{\overline{A_2P}}\right)/\left(\frac{\overline{AZ_P}}{\overline{A_2Z_P}}\right)$$

$$\stackrel{Z_P}{=} \frac{-S_{CC_2A_2}}{-S_{ACC_2}} \cdot \frac{\overline{AP}}{\overline{A_2P}}$$

$$\stackrel{P}{=} \frac{S_{ABB_2} \cdot S_{CC_2A_2}}{S_{ACC_2} \cdot (-S_{BA_2B_2})}$$

$$\stackrel{B_2}{=} \frac{-S_{ACA_2} \cdot S_{ABC_2} \cdot S_{CC_2A_2} \cdot (-S_{ACC_2A_2})}{S_{ACC_2} \cdot S_{BCA_2} \cdot S_{AC_2A_2} \cdot S_{ACC_2A_2}}$$

$$\stackrel{simplify}{=} \frac{S_{ACA_2} \cdot S_{ABC_2} \cdot S_{CC_2A_2}}{S_{ACC_2} \cdot S_{BCA_2} \cdot S_{AC_2A_2}}$$

$$\stackrel{A_2}{=} \frac{S_{BCC_2} \cdot S_{ACA_1} \cdot S_{ABC_2} \cdot S_{CA_1C_2B_1} \cdot S_{BCC_2} \cdot (S_{BA_1C_2B_1})^2}{S_{ACC_2} \cdot S_{BCC_2} \cdot S_{BCB_1} \cdot (-S_{CA_1C_2B_1} \cdot S_{ABC_2}) \cdot S_{BA_1C_2B_1} \cdot (-S_{BA_1C_2B_1})}$$

$$\stackrel{simplify}{=} \frac{S_{ACA_1} \cdot S_{BCC_2}}{S_{ACC_2} \cdot S_{BCB_1}}$$

$$\stackrel{C_2}{=} \frac{S_{ACA_1} \cdot S_{BCC_1} \cdot S_{ABB_1} \cdot (-S_{A_1B_1C_1})}{(-S_{ACC_1} \cdot S_{ABA_1}) \cdot S_{BCB_1} \cdot S_{A_1B_1C_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ACA_1} \cdot S_{BCC_1} \cdot S_{ABB_1}}{S_{ACC_1} \cdot S_{ABA_1} \cdot S_{BCB_1}}$$

$$\stackrel{C_1}{=} \frac{S_{ACA_1} \cdot (-S_{BCM} \cdot S_{ABC}) \cdot S_{ABB_1} \cdot S_{ACBM}}{(-S_{ACM} \cdot S_{ABC}) \cdot S_{ABA_1} \cdot S_{BCB_1} \cdot S_{ACBM}}$$

$$\stackrel{simplify}{=} \frac{S_{ACA_1} \cdot S_{BCM} \cdot S_{ABB_1}}{S_{ACM} \cdot S_{ABA_1} \cdot S_{BCB_1}}$$

$$\stackrel{B_1}{=} \frac{S_{ACA_1} \cdot S_{BCM} \cdot S_{ABM} \cdot S_{ABC} \cdot S_{ABCM}}{S_{ACM} \cdot S_{ABA_1} \cdot S_{BCM} \cdot S_{ABC} \cdot S_{ABCM}}$$

$$\stackrel{simplify}{=} \frac{S_{ACA_1} \cdot S_{ABM}}{S_{ACM} \cdot S_{ABA_1}}$$

$$\stackrel{A_1}{=} \frac{(-S_{ACM} \cdot S_{ABC}) \cdot S_{ABM} \cdot (-S_{ABMC})}{S_{ACM} \cdot (-S_{ABM} \cdot S_{ABC}) \cdot (-S_{ABMC})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_P}}{\overline{A_2Z_P}} \stackrel{Z_P}{=} \frac{S_{ACC_2}}{S_{CC_2A_2}}$$

$$\frac{\overline{AP}}{\overline{A_2P}} \stackrel{P}{=} \frac{S_{ABB_2}}{-S_{BA_2B_2}}$$

$$S_{BA_2B_2} \stackrel{B_2}{=} \frac{S_{BCA_2} \cdot S_{AC_2A_2}}{-S_{ACC_2A_2}}$$

$$S_{ABB_2} \stackrel{B_2}{=} \frac{S_{ACA_2} \cdot S_{ABC_2}}{S_{ACC_2A_2}}$$

$$S_{AC_2A_2} \stackrel{A_2}{=} \frac{-S_{CA_1C_2B_1} \cdot S_{ABC_2}}{S_{BA_1C_2B_1}}$$

$$S_{BCA_2} \stackrel{A_2}{=} \frac{S_{BCC_2} \cdot S_{BCB_1}}{S_{BA_1C_2B_1}}$$

$$S_{CC_2A_2} \stackrel{A_2}{=} \frac{S_{CA_1C_2B_1} \cdot S_{BCC_2}}{S_{BA_1C_2B_1}}$$

$$S_{ACA_2} \stackrel{A_2}{=} \frac{S_{BCC_2} \cdot S_{ACA_1}}{-S_{BA_1C_2B_1}}$$

$$S_{ACC_2} \stackrel{C_2}{=} \frac{S_{ACC_1} \cdot S_{ABA_1}}{S_{A_1B_1C_1}}$$

$$S_{BCC_2} \stackrel{C_2}{=} \frac{S_{BCC_1} \cdot S_{ABB_1}}{S_{A_1B_1C_1}}$$

$$S_{ACC_1} \stackrel{C_1}{=} \frac{-S_{ACM} \cdot S_{ABC}}{S_{ACBM}}$$

$$S_{BCC_1} \stackrel{C_1}{=} \frac{-S_{BCM} \cdot S_{ABC}}{S_{ACBM}}$$

$$S_{BCB_1} \stackrel{B_1}{=} \frac{S_{BCM} \cdot S_{ABC}}{S_{ABCM}}$$

$$S_{ABB_1} \stackrel{B_1}{=} \frac{S_{ABM} \cdot S_{ABC}}{S_{ABCM}}$$

$$S_{ABA_1} \stackrel{A_1}{=} \frac{S_{ABM} \cdot S_{ABC}}{S_{ABMC}}$$

$$S_{ACA_1} \stackrel{A_1}{=} \frac{S_{ACM} \cdot S_{ABC}}{S_{ABMC}}$$

Example 6.30 (0.066, 2, 7) *If a hexagon $ABCDEF$ has two opposite sides BC and EF parallel to the diagonal AD and two opposite sides CD and FA parallel to the diagonal BE, while the remaining sides DE and AB also are parallel, then the third diagonal CF is parallel to AB.*

Constructive description
((points B C E)
(on D (p C B E))
(inter A (p B D E) (p D B C))
(inter F (p A C D) (p E B C))
(midpoint N_1 A C)
(midpoint N_2 C E)
(inter G (l A N_2) (l E N_1))
(inter H (l D F) (l G B))

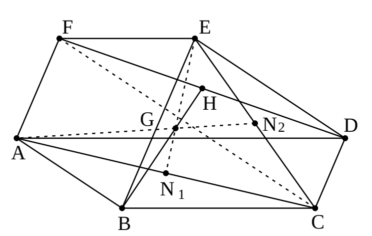


Figure 6-30

(parallel $C\ F\ A\ B$))

The machine proof

$$\frac{S_{BCA}}{-S_{BAF}}$$

$$\stackrel{F}{=} \frac{S_{BCA}\cdot(-S_{BCD})}{-S_{BECA}\cdot S_{BCAD}}$$

$$\stackrel{A}{=} \frac{(S_{BCD})^2\cdot(-S_{BECD})}{(S_{BCD}-S_{BCE})\cdot(-S_{BECD}\cdot S_{BCD}+S_{BED}\cdot S_{BCD})}$$

$$\stackrel{simplify}{=} \frac{S_{BCD}\cdot S_{BECD}}{(S_{BCD}-S_{BCE})\cdot(S_{BECD}-S_{BED})}$$

$$\stackrel{D}{=} \frac{S_{BCE}\cdot\frac{\overline{CD}}{\overline{BE}}\cdot(S_{BCE}\cdot\frac{\overline{CD}}{\overline{BE}}-S_{BCE})}{(S_{BCE}\cdot\frac{\overline{CD}}{\overline{BE}}-S_{BCE})\cdot S_{BCE}\cdot\frac{\overline{CD}}{\overline{BE}}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{BAF}\stackrel{F}{=}\frac{S_{BECA}\cdot S_{BCAD}}{-S_{BCD}}$$

$$S_{BCAD}\stackrel{A}{=}\frac{(S_{BECD}-S_{BED})\cdot S_{BCD}}{S_{BECD}}$$

$$S_{BECA}\stackrel{A}{=}S_{BCD}-S_{BCE}$$

$$S_{BCA}\stackrel{A}{=}S_{BCD}$$

$$S_{BED}\stackrel{D}{=}-(S_{BCE})$$

$$S_{BECD}\stackrel{D}{=}(\frac{\overline{CD}}{\overline{BE}}-1)\cdot S_{BCE}$$

$$S_{BCD}\stackrel{D}{=}S_{BCE}\cdot\frac{\overline{CD}}{\overline{BE}}$$

Example 6.31 (0.683, 4, 37) *Prove that the lines joining the midpoints of three concurrent cevians to the midpoints of the corresponding sides of the given triangle are concurrent.*

Constructive description
((points $A\ B\ C\ O$)
(inter D (l $B\ C$) (l $A\ O$))
(inter E (l $A\ C$) (l $B\ O$))
(inter F (l $A\ B$) (l $C\ O$))
(midpoint $D_1\ A\ D$)
(midpoint $E_1\ B\ E$)
(midpoint $F_1\ C\ F$)
(midpoint $A_1\ B\ C$)
(midpoint $B_1\ C\ A$)
(midpoint $C_1\ A\ B$) (inter I (l $B_1\ E_1$) (l $A_1\ D_1$)) (collinear $I\ C_1\ F_1$))

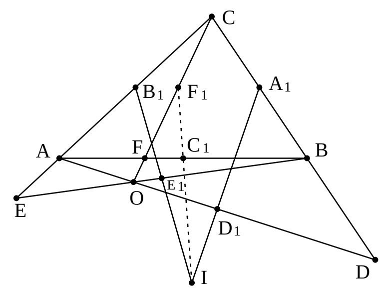


Figure 6-31

Example 6.32 (1.583, 8, 28) *Let O and U be two points in the plane of triangle ABC. Let AO, BO, CO intersect the opposite sides BC, CA, AB in P, Q, R. Let PU, QU, RU intersect QR, RP, PQ respectively in X, Y, Z. Show that AX, BY, CZ are concurrent.*

Constructive description
((points $A\ B\ C\ O\ U$)
(inter P (l $B\ C$) (l $A\ O$))
(inter Q (l $C\ A$) (l $B\ O$))
(inter R (l $A\ B$) (l $C\ O$))
(inter X (l $Q\ R$) (l $P\ U$))
(inter Y (l $R\ P$) (l $Q\ U$))
(inter Z (l $P\ Q$) (l $R\ U$))
(inter I (l $B\ Y$) (l $A\ X$))
(collinear $C\ Z\ I$))

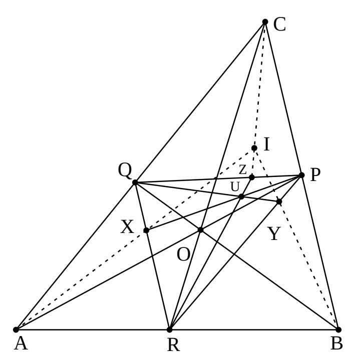


Figure 6-32

Example 6.33 (0.667, 4, 27) *Let O be a point in the plane of a triangle ABC, and let A_1, B_1, C_1 be the points of intersection of the lines AO, BO, CO with the sides of the triangle opposite A, B, C. Prove that if the points A_2, B_2, C_2 on sides B_1C_1, C_1A_1, A_1B_1 of $\triangle A_1B_1C_1$ are collinear, then the points of intersection of the liens AA_2, BB_2, CC_2 with the opposite sides of $\triangle ABC$ are collinear.*

Constructive description
((points A B C O)
(inter A_1 (l B C) (l A O))
(inter B_1 (l A C) (l B O))
(inter C_1 (l A B) (l C O))
(on A_2 (l B_1 C_1)) (on B_2 (l A_1 C_1))
(inter C_2 (l A_2 B_2) (l A_1 B_1))
(inter A_3 (l B C) (l A A_2))
(inter B_3 (l A C) (l B B_2)) (inter C_3 (l A B) (l C C_2)) (collinear A_3 B_3 C_3))

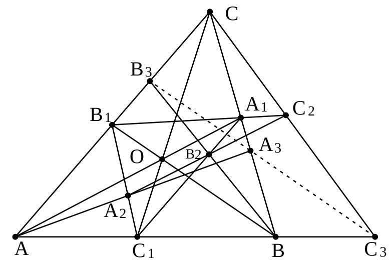


Figure 6-33

Example 6.34 (0.533, 5, 20) *The sides BC, CA, AB of a triangle ABC are met by two transversal PQR, $P_1Q_1R_1$ in the pairs of points P, P_1; Q, Q_1; R, R_1. Show that the points $X = BC \cap QR_1$, $Y = CA \cap RP_1$, $Z = AB \cap PQ_1$ are collinear.*

Constructive description
((points A B C)
(on P (l B C))
(on Q (l C A))
(inter R (l P Q) (l A B))
(on P_1 (l B C))
(on Q_1 (l C A))
(inter R_1 (l P_1 Q_1) (l A B))
(inter X (l Q R_1) (l B C))
(inter Y (l R P_1) (l C A)) (inter Z (l P Q_1) (l A B)) (collinear X Y Z))

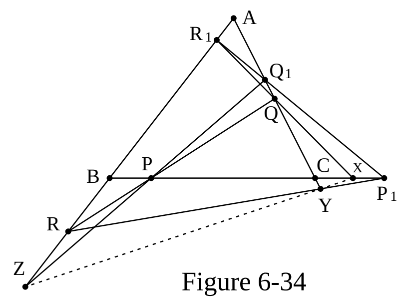


Figure 6-34

Example 6.35 (0.516, 4, 28) *Through the vertices of a triangle ABC lines are drawn intersecting in O and meeting the opposite sides in D, E, F. Prove that the lines joining A, B, C to the midpoints of EF, FD, DE are concurrent.*

Constructive description
((points A B C O)
(inter D (l B C) (l A O))
(inter E (l A C) (l B O))
(inter F (l A B) (l C O))
(midpoint A_1 E F)
(midpoint B_1 F D)
(midpoint C_1 D E)
(inter I (l B B_1) (l A A_1)) (collinear I C C_1))

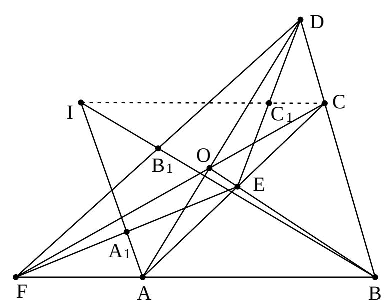


Figure 6-35

Example 6.36 (0.500, 4, 25) *A transversal cuts the sides BC, CA, AB of triangle ABC in D, E, F. P, Q, R are the midpoints of EF, FD, DE, and AP, BQ, CR intersect BC, CA, AB in X, Y, Z. Show that X, Y, Z are collinear.*

Constructive description
((points A B C)
(on D (l B C))
(on E (l A C))
(inter F (l E D) (l A B))
(midpoint P E F)
(midpoint Q F D)
(midpoint R D E)
(inter X (l A P) (l B C))
(inter Y (l B Q) (l A C)) (inter Z (l C R) (l A B)) (collinear X Y Z))

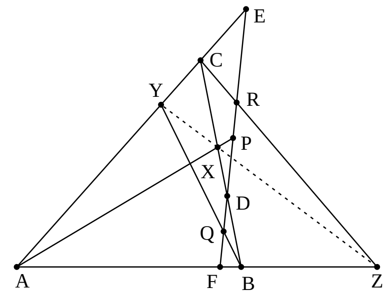


Figure 6-36

Example 6.37 (0.200, 2, 19) *The lines AL, BL, CL joining the vertices of a triangle ABC to a point L meet the respectively opposite sides in A_1, B_1, C_1. The parallels through A_1 to BB_1 CC_1 meet AC, AB in P, Q, and the parallels through A_1 to AC, AB meet BB_1, CC_1 in R, S. Show that the four points P, Q, R, S are collinear.*

Constructive description
((points A B C L)
(inter A_1 (l B C) (l A L))
(inter B_1 (l A C) (l B L))
(inter C_1 (l A B) (l C L))
(inter P (l A C) (p A_1 B B_1))
(inter Q (l A B) (p A_1 C C_1))
(inter R (l B B_1) (p A_1 A C))
(inter Z_R (l Q P) (l B_1 B)) ($\frac{\overline{B_1R}}{\overline{BR}} = \frac{\overline{B_1Z_R}}{\overline{BZ_R}}$))

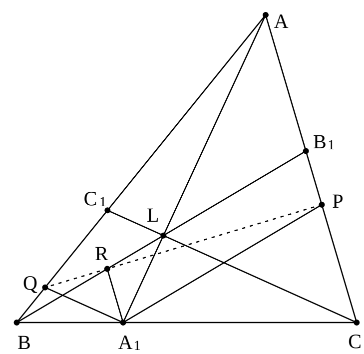


Figure 6-37

Example 6.38 (0.533, 3, 18) *Starting from five points A, B, C, D and E with A, B, C collinear, new lines and points of intersection are formed as in Figure 6-38. Show that AB, GJ and HI are collinear.*

Constructive description for (1)
((points A B D E) (on C (l A B))
(inter F (l E D) (l A B)) (inter I (l D B) (l A E))
(inter J (l C D) (l A E))
(inter L (l C D) (l B E))
(inter G (l B E) (l A D))
(inter K (l C E) (l D B))
(inter H (l E C) (l A D)) (inter O (l G J) (l A B))
(inter Z_O (l H I) (l F C)) ($\frac{\overline{FO}}{\overline{CO}} = \frac{\overline{FZ_O}}{\overline{CZ_O}}$))

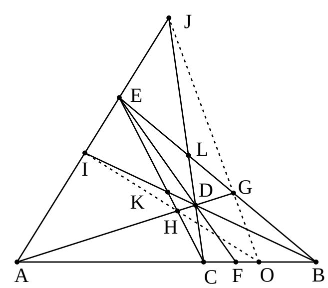


Figure 6-38

6.3 Triangles

6.3.1 Medians and Centroids

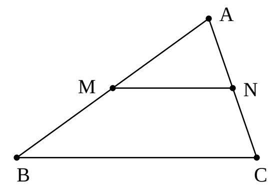


Figure 6-39

Example 6.39 (0.001, 1, 2) *The line joining the midpoints of two sides of a triangle is parallel to the third side and is equal to one-half its length.*

We have to prove two results.

Constructive description

((points $A\ B\ C$)
(midpoint $M\ A\ B$)
(midpoint $N\ A\ C$)
(parallel $M\ N\ B\ C$))

The machine proof

$$\frac{S_{BCM}}{S_{BCN}} \overset{N}{=} \frac{S_{BCM}}{\frac{1}{2}S_{ABC}} \overset{M}{=} \frac{(2)\cdot(\frac{1}{2}S_{ABC})}{S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$S_{BCN} \overset{N}{=} \frac{1}{2}(S_{ABC})$
$S_{BCM} \overset{M}{=} \frac{1}{2}(S_{ABC})$

Constructive description

((points $A\ B\ C$)
(midpoint $M\ A\ B$)
(midpoint $N\ A\ C$)
($\frac{\overline{MN}}{\overline{BC}} = 1/2$))

The machine proof

$$2(\frac{\overline{MN}}{\overline{BC}}) \overset{N}{=} \frac{S_{ACM}}{(\frac{1}{2})\cdot(-S_{ABC})} \overset{M}{=} \frac{(-2)\cdot(-\frac{1}{2}S_{ABC})}{S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{MN}}{\overline{BC}} \overset{N}{=} \frac{S_{ACM}}{-S_{ABC}}$
$S_{ACM} \overset{M}{=} -\frac{1}{2}(S_{ABC})$

Definition. *The line joining a vertex of a triangle and the midpoint of the opposite side is called a median of the triangle.*

Example 6.40 (Theorem of Centroid) (0.016, 1, 4) *The three medians of a triangle meet in a point, and each median is trisected by this point.*

Constructive description

((points $A\ B\ C$)
(midpoint $F\ A\ C$)
(midpoint $E\ B\ C$)
(inter O (l $A\ E$) (l $B\ F$))
($\frac{\overline{AO}}{\overline{OE}} = 2$))

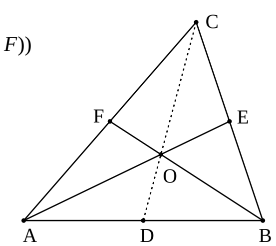


Figure 6-40

The machine proof

$$\frac{1}{2}(-\frac{\overline{AO}}{\overline{EO}}) \overset{O}{=} \frac{-S_{ABF}}{(2)\cdot S_{BFE}} \overset{E}{=} \frac{-S_{ABF}}{(2)\cdot(-\frac{1}{2}S_{BCF})} \overset{F}{=} \frac{\frac{1}{2}S_{ABC}}{\frac{1}{2}S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{AO}}{\overline{EO}} \overset{O}{=} \frac{S_{ABF}}{S_{BFE}}$
$S_{BFE} \overset{E}{=} -\frac{1}{2}(S_{BCF})$
$S_{BCF} \overset{F}{=} \frac{1}{2}(S_{ABC})$
$S_{ABF} \overset{F}{=} \frac{1}{2}(S_{ABC})$

For other forms of the centroid theorem, see Examples 5.46 and 5.47 on page 243.

Example 6.41 (0.083, 2, 9) *With the medians of a triangle a new triangle is constructed. The medians of the second triangle are equal to the three-fourth of the respective sides of the given triangle.*

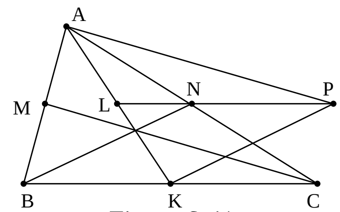


Figure 6-41

Constructive description

((points A B C)
(midpoint M A B)
(midpoint N A C)
(midpoint K B C)
(inter P (p A C M)
(p K B N))
(midpoint L A K)
($\frac{\overline{LP}}{\overline{BC}} = 3/4$))

The machine proof

$$\frac{1}{3}\left((-4)\cdot\frac{\overline{PL}}{\overline{BC}}\right)$$
$$\overset{L}{=} \frac{(-4)\cdot S_{AKP}}{(3)\cdot(-S_{ABC})}$$
$$\overset{P}{=} \frac{(4)\cdot S_{ACKM}\cdot S_{ABKN}}{(3)\cdot S_{ABC}\cdot(-S_{BCNM})}$$
$$\overset{K}{=} \frac{(-4)\cdot(-\frac{1}{2}S_{BCM}+S_{ACM})\cdot(\frac{1}{2}S_{BCN}+S_{ABN})}{(3)\cdot S_{ABC}\cdot S_{BCNM}}$$
$$\overset{N}{=} \frac{(S_{BCM}-2S_{ACM})\cdot(\frac{3}{2}S_{ABC})}{(3)\cdot S_{ABC}\cdot(S_{BCM}-\frac{1}{2}S_{ACM})}$$
$$\overset{simplify}{=} \frac{S_{BCM}-2S_{ACM}}{2S_{BCM}-S_{ACM}}$$
$$\overset{M}{=} \frac{\frac{3}{2}S_{ABC}}{\frac{3}{2}S_{ABC}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{PL}}{\overline{BC}} \overset{L}{=} \frac{S_{AKP}}{-S_{ABC}}$$
$$S_{AKP} \overset{P}{=} \frac{S_{ACKM}\cdot S_{ABKN}}{-S_{BCNM}}$$
$$S_{ABKN} \overset{K}{=} \frac{1}{2}(S_{BCN}+2S_{ABN})$$
$$S_{ACKM} \overset{K}{=} -\frac{1}{2}(S_{BCM}-2S_{ACM})$$
$$S_{BCNM} \overset{N}{=} \frac{1}{2}(2S_{BCM}-S_{ACM})$$
$$S_{ABN} \overset{N}{=} \frac{1}{2}(S_{ABC})$$
$$S_{BCN} \overset{N}{=} \frac{1}{2}(S_{ABC})$$
$$S_{ACM} \overset{M}{=} -\frac{1}{2}(S_{ABC})$$
$$S_{BCM} \overset{M}{=} \frac{1}{2}(S_{ABC})$$

Example 6.42 (0.050, 2, 8) *The area of the triangle having for sides the medians of a given triangle is equal to three-fourth of the given triangle (Figure 6-41).*

Constructive description

((points A B C)
(midpoint M A B)
(midpoint N A C)
(midpoint K B C)
(inter P (p A C M)
(p K B N))
(midpoint L A K)
($4S_{AKP} = 3S_{ABC}$))

The machine proof

$$\frac{(4)\cdot S_{AKP}}{(3)\cdot S_{ABC}}$$
$$\overset{P}{=} \frac{(4)\cdot S_{ACKM}\cdot S_{ABKN}}{(3)\cdot S_{ABC}\cdot(-S_{BCNM})}$$
$$\overset{K}{=} \frac{(-4)\cdot(-\frac{1}{2}S_{BCM}+S_{ACM})\cdot(\frac{1}{2}S_{BCN}+S_{ABN})}{(3)\cdot S_{ABC}\cdot S_{BCNM}}$$
$$\overset{N}{=} \frac{(S_{BCM}-2S_{ACM})\cdot(\frac{3}{2}S_{ABC})}{(3)\cdot S_{ABC}\cdot(S_{BCM}-\frac{1}{2}S_{ACM})}$$
$$\overset{simplify}{=} \frac{S_{BCM}-2S_{ACM}}{2S_{BCM}-S_{ACM}}$$
$$\overset{M}{=} \frac{\frac{3}{2}S_{ABC}}{\frac{3}{2}S_{ABC}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$S_{AKP} \overset{P}{=} \frac{S_{ACKM}\cdot S_{ABKN}}{-S_{BCNM}}$$
$$S_{ABKN} \overset{K}{=} \frac{1}{2}(S_{BCN}+2S_{ABN})$$
$$S_{ACKM} \overset{K}{=} -\frac{1}{2}(S_{BCM}-2S_{ACM})$$
$$S_{BCNM} \overset{N}{=} \frac{1}{2}(2S_{BCM}-S_{ACM})$$
$$S_{ABN} \overset{N}{=} \frac{1}{2}(S_{ABC})$$
$$S_{BCN} \overset{N}{=} \frac{1}{2}(S_{ABC})$$
$$S_{ACM} \overset{M}{=} -\frac{1}{2}(S_{ABC})$$
$$S_{BCM} \overset{M}{=} \frac{1}{2}(S_{ABC})$$

Example 6.43 (0.016, 1, 4) *Show that the line joining the midpoint of a median to a vertex of the triangle trisects the side opposite the vertex considered.*

Constructive description

((points A B C)
(midpoint F A B)
(midpoint N C F)
(inter K (l B C) (l A N))
($\frac{\overline{BK}}{\overline{KC}} = 2$))

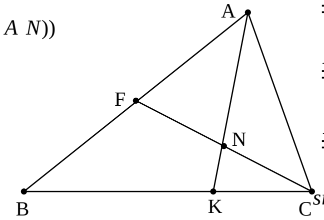


Figure 6-43

The machine proof

$\frac{1}{2}(-\frac{\overline{BK}}{\overline{CK}})$

$\overset{K}{=} \frac{-(-S_{ABN})}{(2)\cdot(-S_{ACN})}$

$\overset{N}{=} \frac{-(\frac{1}{2}S_{ABC})}{(2)\cdot(\frac{1}{2}S_{ACF})}$

$\overset{F}{=} \frac{-S_{ABC}}{(2)\cdot(-\frac{1}{2}S_{ABC})}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{BK}}{\overline{CK}} \overset{K}{=} \frac{S_{ABN}}{S_{ACN}}$

$S_{ACN} \overset{N}{=} \frac{1}{2}(S_{ACF})$

$S_{ABN} \overset{N}{=} \frac{1}{2}(S_{ABC})$

$S_{ACF} \overset{F}{=} -\frac{1}{2}(S_{ABC})$

Example 6.44 (0.033, 2, 6) *Show that a parallel to a side of a triangle through the centroid divides the area of the triangle into two parts, in the ratio 4:5.*

Constructive description

((points A B C)
(centroid G A B C)
(inter P (l A B) (p G B C))
(inter Q (l A C) (p G B C))
($4S_{PQCB} = 5S_{AQP}$))

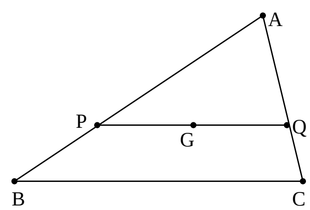


Figure 6-44

The machine proof

$\frac{(4)\cdot S_{BCQP}}{(5)\cdot S_{APQ}}$

$\overset{Q}{=} \frac{(4)\cdot(S_{BCP}\cdot S_{ABC}-S_{BCG}\cdot S_{ACP})\cdot S_{ABC}}{(5)\cdot(-S_{ABGC}\cdot S_{ACP})\cdot S_{ABC}}$

$\overset{simplify}{=} \frac{(-4)\cdot(S_{BCP}\cdot S_{ABC}-S_{BCG}\cdot S_{ACP})}{(5)\cdot S_{ABGC}\cdot S_{ACP}}$

$\overset{P}{=} \frac{(-4)\cdot(S_{ABGC}\cdot S_{BCG}\cdot S_{ABC}+S_{BCG}\cdot S_{ABC}^2)\cdot S_{ABC}}{(5)\cdot S_{ABGC}\cdot(-S_{ABGC}\cdot S_{ABC})\cdot S_{ABC}}$

$\overset{simplify}{=} \frac{(4)\cdot(S_{ABGC}+S_{ABC})\cdot S_{BCG}}{(5)\cdot(S_{ABGC})^2}$

$\overset{G}{=} \frac{(4)\cdot(5S_{ABC})\cdot S_{ABC}\cdot((3))^2}{(5)\cdot((2S_{ABC}))^2\cdot((3))^2}$

$\overset{simplify}{=} 1$

The eliminants

$S_{APQ} \overset{Q}{=} \frac{-S_{ABGC}\cdot S_{ACP}}{S_{ABC}}$

$S_{BCQP} \overset{Q}{=} \frac{S_{BCP}\cdot S_{ABC}-S_{BCG}\cdot S_{ACP}}{S_{ABC}}$

$S_{ACP} \overset{P}{=} -S_{ABGC}$

$S_{BCP} \overset{P}{=} S_{BCG}$

$S_{BCG} \overset{G}{=} \frac{1}{3}(S_{ABC})$

$S_{ABGC} \overset{G}{=} \frac{2}{3}(S_{ABC})$

Example 6.45 (0.001, 2, 5) *If L is the harmonic conjugate of the centroid G of a triangle ABC with respect to the ends A, D of the median AD, show that LD = AD.*

Constructive description

((points A B C)
(midpoint D B C)
(centroid G A B C)
(harmonic L G D A)
(midpoint D L A))

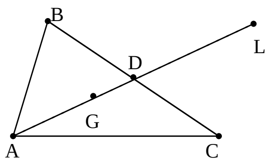


Figure 6-45

The machine proof

$$\frac{\overline{DL}}{\overline{AD}}$$

$$\stackrel{L}{=} \frac{1}{-\frac{\overline{DG}}{\overline{AG}}-1} \cdot \frac{\overline{DG}}{\overline{AG}}$$

$$\stackrel{G}{=} \frac{-S_{ACD}\cdot S_{ABC}}{(S_{ACD}+S_{ABC})\cdot S_{ABC}}$$

$$\stackrel{simplify}{=} \frac{-S_{ACD}}{S_{ACD}+S_{ABC}}$$

$$\stackrel{D}{=} \frac{-(-\frac{1}{2}S_{ABC})}{\frac{1}{2}S_{ABC}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{DL}}{\overline{AD}} \stackrel{L}{=} \frac{\frac{\overline{DG}}{\overline{AG}}}{-(\frac{\overline{DG}}{\overline{AG}}+1)}$$

$$\frac{\overline{DG}}{\overline{AG}} \stackrel{G}{=} \frac{S_{ACD}}{S_{ABC}}$$

$$S_{ABD} \stackrel{D}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACD} \stackrel{D}{=} -\frac{1}{2}(S_{ABC})$$

Example 6.46 (0.033, 1, 6) *Show that the distances of a point on a median of triangle from the sides including the median are inversely proportional to these sides.*

Constructive description

((points A B C)
(midpoint F A B)
(on N (l C F))
(foot K N A C)
(foot J N B C)
(eq-product N K A C N J B C))

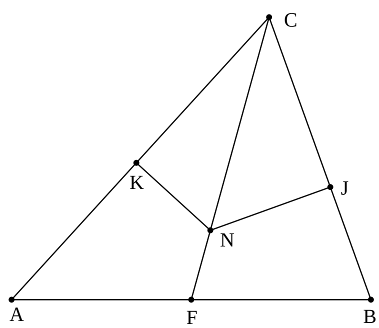


Figure 6-46

The machine proof

$$\frac{P_{NKN}\cdot P_{ACA}}{P_{NJN}\cdot P_{BCB}}$$

$$\stackrel{J}{=} \frac{P_{NKN}\cdot P_{ACA}\cdot P_{BCB}}{(16S_{BCN}^2)\cdot P_{BCB}}$$

$$\stackrel{simplify}{=} \frac{P_{NKN}\cdot P_{ACA}}{(16)\cdot(S_{BCN})^2}$$

$$\stackrel{K}{=} \frac{(16S_{ACN}^2)\cdot P_{ACA}}{(16)\cdot(S_{BCN})^2\cdot P_{ACA}}$$

$$\stackrel{simplify}{=} \frac{(S_{ACN})^2}{(S_{BCN})^2}$$

$$\stackrel{N}{=} \frac{(S_{ACF}\cdot\frac{\overline{CN}}{\overline{CF}})^2}{(S_{BCF}\cdot\frac{\overline{CN}}{\overline{CF}})^2}$$

$$\stackrel{simplify}{=} \frac{(S_{ACF})^2}{(S_{BCF})^2}$$

$$\stackrel{F}{=} \frac{((-\frac{1}{2}S_{ABC}))^2}{((\frac{1}{2}S_{ABC}))^2}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{NJN} \stackrel{J}{=} \frac{(16)\cdot(S_{BCN})^2}{P_{BCB}}$$

$$P_{NKN} \stackrel{K}{=} \frac{(16)\cdot(S_{ACN})^2}{P_{ACA}}$$

$$S_{BCN} \stackrel{N}{=} S_{BCF}\cdot\frac{\overline{CN}}{\overline{CF}}$$

$$S_{ACN} \stackrel{N}{=} S_{ACF}\cdot\frac{\overline{CN}}{\overline{CF}}$$

$$S_{BCF} \stackrel{F}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACF} \stackrel{F}{=} -\frac{1}{2}(S_{ABC})$$

Example 6.47 (0.066, 5, 7) *Show that, if a line through the centroid G of the triangle ABC meets AB in M and AC in N, we have, both in magnitude and in sign, $AN\cdot MB+AM\cdot NC = AM\cdot AN$.*

Constructive description

((points $A\ B\ C$)
(centroid $G\ A\ B\ C$)
(on M (l $A\ B$))
(inter N (l $G\ M$) (l $A\ C$))
$(\frac{\overline{MB}}{\overline{AM}}+\frac{\overline{NC}}{\overline{AN}} = 1)$)

The eliminants

$$\frac{\overline{CN}}{\overline{AN}} \stackrel{N}{=} \frac{S_{CGM}}{S_{AGM}}$$
$$S_{CGM} \stackrel{M}{=} S_{BCG}\cdot\frac{\overline{AM}}{\overline{AB}} - S_{ACG}\cdot\frac{\overline{AM}}{\overline{AB}} + S_{ACG}$$
$$S_{AGM} \stackrel{M}{=} -\left(S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}}\right)$$
$$\frac{\overline{BM}}{\overline{AM}} \stackrel{M}{=} \frac{\frac{\overline{AM}}{\overline{AB}}-1}{\frac{\overline{AM}}{\overline{AB}}}$$
$$S_{ABG} \stackrel{G}{=} \frac{1}{3}(S_{ABC})$$
$$S_{ACG} \stackrel{G}{=} -\frac{1}{3}(S_{ABC})$$
$$S_{BCG} \stackrel{G}{=} \frac{1}{3}(S_{ABC})$$

The machine proof

$$-\left(\frac{\overline{CN}}{\overline{AN}}+\frac{\overline{BM}}{\overline{AM}}\right)$$
$$\stackrel{N}{=} \frac{-\left(\frac{\overline{BM}}{\overline{AM}}\cdot S_{AGM}+S_{CGM}\right)}{S_{AGM}}$$
$$\stackrel{M}{=} \frac{-\left(S_{BCG}\cdot\frac{\overline{AM}}{\overline{AB}}^2 - S_{ACG}\cdot\frac{\overline{AM}}{\overline{AB}}^2 + S_{ACG}\cdot\frac{\overline{AM}}{\overline{AB}} - S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}}^2 + S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}}\right)}{\left(-S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}}\right)\cdot\frac{\overline{AM}}{\overline{AB}}}$$
$$\stackrel{simplify}{=} \frac{S_{BCG}\cdot\frac{\overline{AM}}{\overline{AB}} - S_{ACG}\cdot\frac{\overline{AM}}{\overline{AB}} + S_{ACG} - S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}} + S_{ABG}}{S_{ABG}\cdot\frac{\overline{AM}}{\overline{AB}}}$$
$$\stackrel{G}{=} \frac{(9S_{ABC}\cdot\frac{\overline{AM}}{\overline{AB}})\cdot(3)}{S_{ABC}\cdot\frac{\overline{AM}}{\overline{AB}}\cdot((3))^3} \stackrel{simplify}{=} 1$$

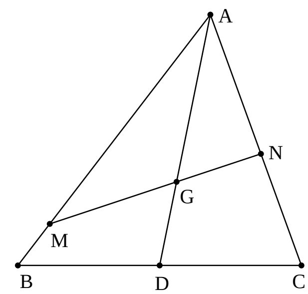


Figure 6-47

Example 6.48 (0.033, 1, 8) *Two equal segments AE, AF are taken on the sides AB, AC of the triangle ABC. Show that median issued from A divides EF in the ratio of the sides AC, AB.*

Constructive description

((points $F\ E$)
(on A (b $F\ E$))
(on C (l $A\ F$))
(on B (l $A\ E$))
(midpoint $A_1\ B\ C$)
(inter D (l $E\ F$) (l $A\ A_1$))
$(\frac{\overline{ED}}{\overline{FD}}\frac{\overline{ED}}{\overline{FD}}P_{ABA} = P_{ACA})$)

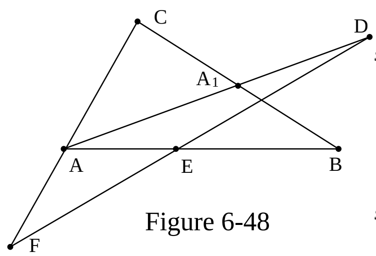


Figure 6-48

The machine proof

$$\frac{P_{ABA}}{P_{ACA}}\cdot\left(\frac{\overline{ED}}{\overline{FD}}\right)^2$$
$$\stackrel{D}{=} \frac{(S_{EAA_1})^2\cdot P_{ABA}}{P_{ACA}\cdot(S_{FAA_1})^2}$$
$$\stackrel{A_1}{=} \frac{((\frac{1}{2}S_{EAC}))^2\cdot P_{ABA}}{P_{ACA}\cdot((\frac{1}{2}S_{FAB}))^2}$$
$$\stackrel{B}{=} \frac{(S_{EAC})^2\cdot P_{EAE}\cdot\frac{\overline{AB}}{\overline{AE}}^2}{P_{ACA}\cdot((-S_{FEA}\cdot\frac{\overline{AB}}{\overline{AE}}))^2}$$
$$\stackrel{simplify}{=} \frac{(S_{EAC})^2\cdot P_{EAE}}{P_{ACA}\cdot(S_{FEA})^2}$$
$$\stackrel{C}{=} \frac{(S_{FEA}\cdot\frac{\overline{AC}}{\overline{AF}})^2\cdot P_{EAE}}{P_{FAF}\cdot\frac{\overline{AC}}{\overline{AF}}^2\cdot(S_{FEA})^2}$$
$$\stackrel{simplify}{=} \frac{P_{EAE}}{P_{FAF}} \stackrel{A}{=} \frac{P_{FAF}}{P_{FAF}} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{ED}}{\overline{FD}} \stackrel{D}{=} \frac{S_{EAA_1}}{S_{FAA_1}}$$
$$S_{FAA_1} \stackrel{A_1}{=} \frac{1}{2}(S_{FAB})$$
$$S_{EAA_1} \stackrel{A_1}{=} \frac{1}{2}(S_{EAC})$$
$$S_{FAB} \stackrel{B}{=} -\left(S_{FEA}\cdot\frac{\overline{AB}}{\overline{AE}}\right)$$
$$P_{ABA} \stackrel{B}{=} P_{EAE}\cdot\left(\frac{\overline{AB}}{\overline{AE}}\right)^2$$
$$P_{ACA} \stackrel{C}{=} P_{FAF}\cdot\left(\frac{\overline{AC}}{\overline{AF}}\right)^2$$
$$S_{EAC} \stackrel{C}{=} S_{FEA}\cdot\frac{\overline{AC}}{\overline{AF}}$$
$$P_{EAE} \stackrel{A}{=} P_{FAF}$$

Example 6.49 (0.250, 4, 8) *Show that the (algebraic) sum of the distances of the vertices of a triangle from any line in the plane is equal to the sum of the distances of the midpoints of the sides of the triangle from this line.*

Constructive description

((points $A\ B\ C\ X\ Y$)
(midpoint $D\ B\ C$)
(midpoint $E\ A\ C$)
(midpoint $F\ A\ B$)
(foot $A_1\ A\ X\ Y$) (foot $B_1\ B\ X\ Y$)
(foot $C_1\ C\ X\ Y$) (foot $D_1\ D\ X\ Y$)
(foot $E_1\ E\ X\ Y$)
(foot $F_1\ F\ X\ Y$)
$(1+\frac{\overline{BB_1}}{\overline{AA_1}}+\frac{\overline{CC_1}}{\overline{AA_1}} = \frac{\overline{DD_1}}{\overline{AA_1}}+\frac{\overline{EE_1}}{\overline{AA_1}}+\frac{\overline{FF_1}}{\overline{AA_1}})$)

The eliminants

$\frac{\overline{FF_1}}{\overline{AA_1}} \overset{F_1}{=} \frac{S_{XYF}}{S_{AXY}}$ $\frac{\overline{EE_1}}{\overline{AA_1}} \overset{E_1}{=} \frac{S_{XYE}}{S_{AXY}}$

$\frac{\overline{DD_1}}{\overline{AA_1}} \overset{D_1}{=} \frac{S_{XYD}}{S_{AXY}}$ $\frac{\overline{CC_1}}{\overline{AA_1}} \overset{C_1}{=} \frac{S_{CXY}}{S_{AXY}}$

$\frac{\overline{BB_1}}{\overline{AA_1}} \overset{B_1}{=} \frac{S_{BXY}}{S_{AXY}}$

$S_{XYF} \overset{F}{=} \frac{1}{2}(S_{BXY}+S_{AXY})$

$S_{XYE} \overset{E}{=} \frac{1}{2}(S_{CXY}+S_{AXY})$

$S_{XYD} \overset{D}{=} \frac{1}{2}(S_{CXY}+S_{BXY})$

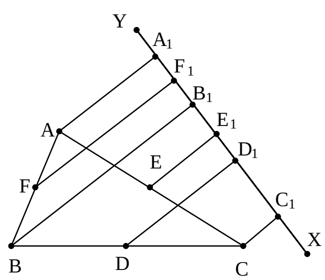


Figure 6.49

The machine proof

$$\frac{\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1}{\frac{\overline{FF_1}}{\overline{AA_1}}+\frac{\overline{EE_1}}{\overline{AA_1}}+\frac{\overline{DD_1}}{\overline{AA_1}}} \overset{F_1}{=} \frac{(\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1)\cdot S_{AXY}}{\frac{\overline{EE_1}}{\overline{AA_1}}\cdot S_{AXY}+\frac{\overline{DD_1}}{\overline{AA_1}}\cdot S_{AXY}+S_{XYF}}$$

$$\overset{E_1}{=} \frac{(\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1)\cdot (S_{AXY})^2}{\frac{\overline{DD_1}}{\overline{AA_1}}\cdot S_{AXY}^2+S_{XYF}\cdot S_{AXY}+S_{XYE}\cdot S_{AXY}}$$

$$\overset{simplify}{=} \frac{(\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1)\cdot S_{AXY}}{\frac{\overline{DD_1}}{\overline{AA_1}}\cdot S_{AXY}+S_{XYF}+S_{XYE}}$$

$$\overset{D_1}{=} \frac{(\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1)\cdot (S_{AXY})^2}{S_{XYF}\cdot S_{AXY}+S_{XYE}\cdot S_{AXY}+S_{XYD}\cdot S_{AXY}}$$

$$\overset{simplify}{=} \frac{(\frac{\overline{CC_1}}{\overline{AA_1}}+\frac{\overline{BB_1}}{\overline{AA_1}}+1)\cdot S_{AXY}}{S_{XYF}+S_{XYE}+S_{XYD}}$$

$$\overset{C_1}{=} \frac{(\frac{\overline{BB_1}}{\overline{AA_1}}\cdot S_{AXY}+S_{CXY}+S_{AXY})\cdot S_{AXY}}{(S_{XYF}+S_{XYE}+S_{XYD})\cdot S_{AXY}}$$

$$\overset{simplify}{=} \frac{\frac{\overline{BB_1}}{\overline{AA_1}}\cdot S_{AXY}+S_{CXY}+S_{AXY}}{S_{XYF}+S_{XYE}+S_{XYD}}$$

$$\overset{B_1}{=} \frac{S_{CXY}\cdot S_{AXY}+S_{BXY}\cdot S_{AXY}+S_{AXY}^2}{(S_{XYF}+S_{XYE}+S_{XYD})\cdot S_{AXY}}$$

$$\overset{simplify}{=} \frac{S_{CXY}+S_{BXY}+S_{AXY}}{S_{XYF}+S_{XYE}+S_{XYD}}$$

$$\overset{F}{=} \frac{S_{CXY}+S_{BXY}+S_{AXY}}{S_{XYE}+S_{XYD}+\frac{1}{2}S_{BXY}+\frac{1}{2}S_{AXY}} \overset{E}{=} \frac{(2)\cdot(S_{CXY}+S_{BXY}+S_{AXY})}{2S_{XYD}+S_{CXY}+S_{BXY}+2S_{AXY}}$$

$$\overset{D}{=} \frac{(2)\cdot(S_{CXY}+S_{BXY}+S_{AXY})}{2S_{CXY}+2S_{BXY}+2S_{AXY}} \overset{simplify}{=} 1$$

Example 6.50 (0.016, 3, 1) *Compute the square of the lengths of the medians.*

Constructive description

((points $A\ B\ C$)
(midpoint $M\ B\ C$)
$(\overline{MA}^2)$)

The machine proof

$$\frac{1}{2}(P_{AMA})$$

$$\overset{M}{=} \frac{-\frac{1}{4}P_{BCB}+\frac{1}{2}P_{ACA}+\frac{1}{2}P_{ABA}}{2}$$

The eliminants

$P_{AMA} \overset{M}{=} -\frac{1}{4}(P_{BCB}-2P_{ACA}-2P_{ABA})$

Example 6.51 (0.066, 2, 14) *If K, K_1 are two isotomic points on the side BC of the triangle ABC, and the line AK meets the line NM of in K_2, where N and M are the midpoints of AB and AC. Show that line K_1K_2 passes through the centroid G of ABC.*

Constructive description

((points $A\ B\ C$)
(midpoint $N\ A\ B$)
(midpoint $M\ A\ C$)
(inter G (l $C\ N$) (l $B\ M$))
(lratio $K\ B\ C\ r$)
(lratio $K_1\ C\ B\ r$)
(inter K_2 (l $M\ N$) (l $A\ K$))
(inter Z (l $B\ C$) (l $K_2\ G$))
($\frac{\overline{BZ}}{\overline{ZC}} = \frac{\overline{BK_1}}{\overline{K_1C}}$))

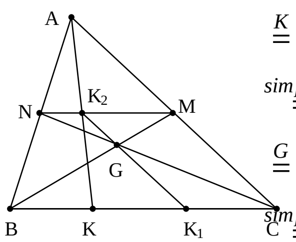


Figure 6-51

The machine proof

$$\left(\frac{\overline{BZ}}{\overline{CZ}}\right)/\left(\frac{\overline{BK_1}}{\overline{CK_1}}\right)$$

$$\overset{Z}{=} \frac{-S_{BGK_2}}{\frac{\overline{BK_1}}{\overline{CK_1}}\cdot(-S_{CGK_2})}$$

$$\overset{K_2}{=} \frac{(-S_{BNG}\cdot S_{AMK})\cdot(-S_{ANKM})}{\frac{\overline{BK_1}}{\overline{CK_1}}\cdot S_{CMG}\cdot S_{ANK}\cdot(-S_{ANKM})}$$

$$\overset{simplify}{=} \frac{-S_{BNG}\cdot S_{AMK}}{\frac{\overline{BK_1}}{\overline{CK_1}}\cdot S_{CMG}\cdot S_{ANK}}$$

$$\overset{K_1}{=} \frac{-S_{BNG}\cdot S_{AMK}\cdot r}{(r-1)\cdot S_{CMG}\cdot S_{ANK}}$$

$$\overset{K}{=} \frac{-S_{BNG}\cdot(S_{ABM}\cdot r-S_{ABM})\cdot r}{(r-1)\cdot S_{CMG}\cdot(-S_{ACN}\cdot r)}$$

$$\overset{simplify}{=} \frac{S_{BNG}\cdot S_{ABM}}{S_{CMG}\cdot S_{ACN}}$$

$$\overset{G}{=} \frac{(-S_{BNM}\cdot S_{BCN})\cdot S_{ABM}\cdot(-S_{BCMN})}{S_{CNM}\cdot S_{BCM}\cdot S_{ACN}\cdot(-S_{BCMN})}$$

$$\overset{simplify}{=} \frac{-S_{BNM}\cdot S_{BCN}\cdot S_{ABM}}{S_{CNM}\cdot S_{BCM}\cdot S_{ACN}}$$

$$\overset{M}{=} \frac{-(-\frac{1}{2}S_{BCN})\cdot S_{BCN}\cdot(\frac{1}{2}S_{ABC})}{(\frac{1}{2}S_{ACN})\cdot(\frac{1}{2}S_{ABC})\cdot S_{ACN}}$$

$$\overset{simplify}{=} \frac{(S_{BCN})^2}{(S_{ACN})^2}$$

$$\overset{N}{=} \frac{((\frac{1}{2}S_{ABC}))^2}{((-\frac{1}{2}S_{ABC}))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BZ}}{\overline{CZ}} \overset{Z}{=} \frac{S_{BGK_2}}{S_{CGK_2}}$$
$$S_{CGK_2} \overset{K_2}{=} \frac{S_{CMG}\cdot S_{ANK}}{-S_{ANKM}}$$
$$S_{BGK_2} \overset{K_2}{=} \frac{S_{BNG}\cdot S_{AMK}}{S_{ANKM}}$$
$$\frac{\overline{BK_1}}{\overline{CK_1}} \overset{K_1}{=} \frac{r-1}{r}$$
$$S_{ANK} \overset{K}{=} -(S_{ACN}\cdot r)$$
$$S_{AMK} \overset{K}{=} (r-1)\cdot S_{ABM}$$
$$S_{CMG} \overset{G}{=} \frac{S_{CNM}\cdot S_{BCM}}{-S_{BCMN}}$$
$$S_{BNG} \overset{G}{=} \frac{S_{BNM}\cdot S_{BCN}}{S_{BCMN}}$$
$$S_{BCM} \overset{M}{=} \frac{1}{2}(S_{ABC})$$
$$S_{CNM} \overset{M}{=} \frac{1}{2}(S_{ACN})$$
$$S_{ABM} \overset{M}{=} \frac{1}{2}(S_{ABC})$$
$$S_{BNM} \overset{M}{=} -\frac{1}{2}(S_{BCN})$$
$$S_{ACN} \overset{N}{=} -\frac{1}{2}(S_{ABC})$$
$$S_{BCN} \overset{N}{=} \frac{1}{2}(S_{ABC})$$

Example 6.52 (0.083, 5, 3) *The sum of the squares of the medians of a triangle is equal to three-fourth the sum of the squares of the sides.*

Constructive description

((points $A\ B\ C$)
(midpoint $E\ A\ C$)
(midpoint $F\ A\ B$)
(midpoint $D\ B\ C$)
($4\overline{AD}^2+4\overline{FC}^2+4\overline{BE}^2$
$= 3\overline{AC}^2+3\overline{AB}^2+3\overline{BC}^2$))

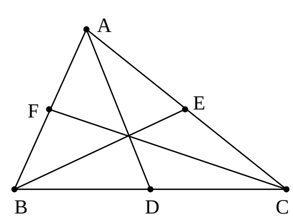


Figure 6-52

The machine proof

$$\frac{(4)\cdot(P_{CFC}+P_{BEB}+P_{ADA})}{(3)\cdot(P_{BCB}+P_{ACA}+P_{ABA})}$$
$$\stackrel{D}{=} \frac{(4)\cdot(P_{CFC}+P_{BEB}-\frac{1}{4}P_{BCB}+\frac{1}{2}P_{ACA}+\frac{1}{2}P_{ABA})}{(3)\cdot(P_{BCB}+P_{ACA}+P_{ABA})}$$
$$\stackrel{F}{=} \frac{4P_{BEB}+P_{BCB}+4P_{ACA}+P_{ABA}}{(3)\cdot(P_{BCB}+P_{ACA}+P_{ABA})}$$
$$\stackrel{E}{=} \frac{3P_{BCB}+3P_{ACA}+3P_{ABA}}{(3)\cdot(P_{BCB}+P_{ACA}+P_{ABA})}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{ADA} \stackrel{D}{=} -\frac{1}{4}(P_{BCB}-2P_{ACA}-2P_{ABA})$$
$$P_{CFC} \stackrel{F}{=} \frac{1}{4}(2P_{BCB}+2P_{ACA}-P_{ABA})$$
$$P_{BEB} \stackrel{E}{=} \frac{1}{4}(2P_{BCB}-P_{ACA}+2P_{ABA})$$

Example 6.53 (0.050, 4, 6) *If two points are equidistant from the centroid of a triangle, show that the sums of the squares of their distances from the vertices of a triangle are equal.*

Constructive description
((points A B N M)
(on G (b N M))
(midpoint F A B)
(lratio C F G 3)
$(\overline{AN}^2+\overline{NB}^2+\overline{NC}^2 = \overline{AM}^2+\overline{MB}^2+\overline{MC}^2)$)

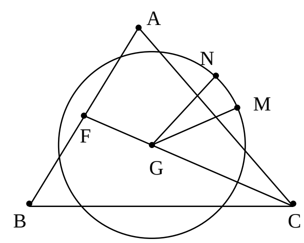


Figure 6-53

The machine proof.

$$\frac{P_{NCN}+P_{BNB}+P_{ANA}}{P_{MCM}+P_{BMB}+P_{AMA}}$$
$$\stackrel{C}{=} \frac{6P_{GFG}-2P_{NFN}+3P_{NGN}+P_{BNB}+P_{ANA}}{6P_{GFG}-2P_{MFM}+3P_{MGM}+P_{BMB}+P_{AMA}}$$
$$\stackrel{F}{=} \frac{3P_{NGN}+3P_{BGB}+3P_{AGA}-P_{ABA}}{3P_{MGM}+3P_{BGB}+3P_{AGA}-P_{ABA}}$$
$$\stackrel{G}{=} \frac{3P_{NGN}+3P_{BGB}+3P_{AGA}-P_{ABA}}{3P_{NGN}+3P_{BGB}+3P_{AGA}-P_{ABA}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{MCM} \stackrel{C}{=} 6P_{GFG}-2P_{MFM}+3P_{MGM}$$
$$P_{NCN} \stackrel{C}{=} 6P_{GFG}-2P_{NFN}+3P_{NGN}$$
$$P_{MFM} \stackrel{F}{=} \frac{1}{4}(2P_{BMB}+2P_{AMA}-P_{ABA})$$
$$P_{NFN} \stackrel{F}{=} \frac{1}{4}(2P_{BNB}+2P_{ANA}-P_{ABA})$$
$$P_{GFG} \stackrel{F}{=} \frac{1}{4}(2P_{BGB}+2P_{AGA}-P_{ABA})$$
$$P_{MGM} \stackrel{G}{=} P_{NGN}$$

Definition. *The triangle having for its vertices the midpoints of the sides of a given triangle is called the medial triangle of the given triangle. The triangle formed by the parallels to the sides of a given triangle through the opposite vertices is called the anticomplementary triangle of the given triangle.*

Example 6.54 (0.050, 2, 6) *The median AA_1 of the triangle ABC meets the side B_1C_1 of the medial triangle $A_1B_1C_1$ in P, and CP meets AB in Q. Show that $AB = 3AQ$.*

Constructive description
((points $A\ B\ C$)
(midpoint $A_1\ B\ C$)
(midpoint $C_1\ A\ B$)
(inter P (l $A\ A_1$) (p $C_1\ B\ C$))
(inter Q (l $A\ B$) (l $P\ C$))
($\frac{\overline{QA}}{\overline{AB}} = -1/3$))

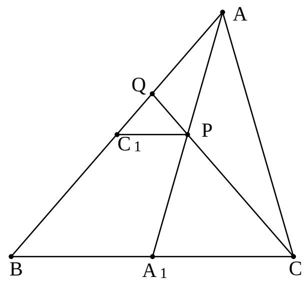


Figure 6-54

The machine proof

$$(3)\cdot\frac{\overline{AQ}}{\overline{AB}}$$

$$\overset{Q}{=} \frac{(3)\cdot(-S_{ACP})}{-S_{ACBP}}$$

$$\overset{P}{=} \frac{(3)\cdot(-S_{ACC_1}\cdot S_{ACA_1})\cdot S_{ABC}}{(-S_{ACC_1}\cdot S_{ABA_1}-S^2_{ABC})\cdot S_{ABC}}$$

$$\overset{simplify}{=} \frac{(3)\cdot S_{ACC_1}\cdot S_{ACA_1}}{S_{ACC_1}\cdot S_{ABA_1}+S^2_{ABC}}$$

$$\overset{C_1}{=} \frac{(3)\cdot(-\frac{1}{2}S_{ABC})\cdot S_{ACA_1}}{-\frac{1}{2}S_{ABA_1}\cdot S_{ABC}+S^2_{ABC}}$$

$$\overset{simplify}{=} \frac{(3)\cdot S_{ACA_1}}{S_{ABA_1}-2S_{ABC}}$$

$$\overset{A_1}{=} \frac{(3)\cdot(-\frac{1}{2}S_{ABC})}{-\frac{3}{2}S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AQ}}{\overline{AB}} \overset{Q}{=} \frac{S_{ACP}}{S_{ACBP}}$$

$$S_{ACBP} \overset{P}{=} \frac{-(S_{ACC_1}\cdot S_{ABA_1}+S^2_{ABC})}{S_{ABC}}$$

$$S_{ACP} \overset{P}{=} \frac{-S_{ACC_1}\cdot S_{ACA_1}}{S_{ABC}}$$

$$S_{ACC_1} \overset{C_1}{=} -\frac{1}{2}(S_{ABC})$$

$$S_{ABA_1} \overset{A_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ABC})$$

Example 6.55 (0.066, 4, 5) *The sum of the squares of the distances of the centroid of a triangle from the vertices is equal to one-third the sum of the squares of the sides.*

Constructive description
((points $A\ B\ C$)
(midpoint $E\ A\ C$)
(lratio $G\ B\ E\ 2/3$)
($3\overline{AG}^2+3\overline{GC}^2+3\overline{BG}^2 = \overline{AC}^2+\overline{AB}^2+\overline{BC}^2$))

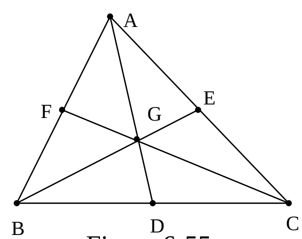


Figure 6-55

The machine proof

$$\frac{(3)\cdot(P_{CGC}+P_{BGB}+P_{AGA})}{P_{BCB}+P_{ACA}+P_{ABA}}$$

$$\overset{G}{=} \frac{(3)\cdot(\frac{2}{3}P_{CEC}+\frac{1}{3}P_{BCB}+\frac{2}{3}P_{AEA}+\frac{1}{3}P_{ABA})}{P_{BCB}+P_{ACA}+P_{ABA}}$$

$$\overset{E}{=} \frac{P_{BCB}+P_{ACA}+P_{ABA}}{P_{BCB}+P_{ACA}+P_{ABA}} \overset{simplify}{=} 1$$

The eliminants

$$P_{AGA} \overset{G}{=} -\frac{1}{9}(2P_{BEB}-6P_{AEA}-3P_{ABA})$$

$$P_{BGB} \overset{G}{=} \frac{4}{9}(P_{BEB})$$

$$P_{CGC} \overset{G}{=} \frac{1}{9}(6P_{CEC}-2P_{BEB}+3P_{BCB})$$

$$P_{AEA} \overset{E}{=} \frac{1}{4}(P_{ACA}),\quad P_{CEC} \overset{E}{=} \frac{1}{4}(P_{ACA})$$

Example 6.56 (0.050, 7, 8) *If M is any point in the plane of the triangle ABC, and G is the centroid of ABC, we have $\overline{MA}^2+\overline{MB}^2+\overline{MC}^2 = \overline{GA}^2+\overline{GB}^2+\overline{GC}^2+3\overline{MG}^2$.*

Constructive description
((points $A\ B\ C\ M$)
(midpoint $E\ A\ C$)
(lratio $G\ B\ E\ 2/3$)

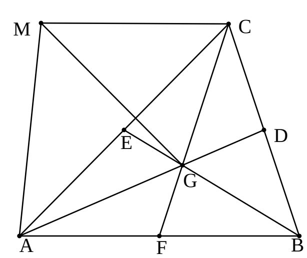


Figure 6-56

$(3\overline{MG}^2+\overline{AG}^2+\overline{GC}^2+\overline{BG}^2 = \overline{AM}^2+\overline{MB}^2+\overline{MC}^2)$)

The machine proof

$$\frac{3P_{MGM}+P_{CGC}+P_{BGB}+P_{AGA}}{P_{CMC}+P_{BMB}+P_{AMA}}$$
$$\stackrel{G}{=} \frac{2P_{MEM}+\frac{2}{3}P_{CEC}-\frac{2}{3}P_{BEB}+P_{BMB}+\frac{1}{3}P_{BCB}+\frac{2}{3}P_{AEA}+\frac{1}{3}P_{ABA}}{P_{CMC}+P_{BMB}+P_{AMA}}$$
$$\stackrel{E}{=} \frac{3P_{CMC}+3P_{BMB}+3P_{AMA}}{(3)\cdot(P_{CMC}+P_{BMB}+P_{AMA})}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{AGA} \stackrel{G}{=} -\frac{1}{9}\left(2P_{BEB}-6P_{AEA}-3P_{ABA}\right)$$
$$P_{BGB} \stackrel{G}{=} \frac{4}{9}\left(P_{BEB}\right)$$
$$P_{CGC} \stackrel{G}{=} \frac{1}{9}\left(6P_{CEC}-2P_{BEB}+3P_{BCB}\right)$$
$$P_{MGM} \stackrel{G}{=} \frac{1}{9}\left(6P_{MEM}-2P_{BEB}+3P_{BMB}\right)$$
$$P_{AEA} \stackrel{E}{=} \frac{1}{4}\left(P_{ACA}\right)$$
$$P_{BEB} \stackrel{E}{=} \frac{1}{4}\left(2P_{BCB}-P_{ACA}+2P_{ABA}\right)$$
$$P_{CEC} \stackrel{E}{=} \frac{1}{4}\left(P_{ACA}\right)$$
$$P_{MEM} \stackrel{E}{=} \frac{1}{4}\left(2P_{CMC}+2P_{AMA}-P_{ACA}\right)$$

Example 6.57 (0.067, 7, 10) *If the pairs of points D, D_1; E, E_1; F, F_1 are isotomic on the sides BC, CA, AB of the triangle ABC, the areas of two triangles DEF, $D_1E_1F_1$ are the same.*

Constructive description

((points A B C)
(lratio D B C r_1)
(lratio E C A r_2)
(lratio F A B r_3)
(lratio D_1 C B r_1)
(lratio E_1 A C r_2)
(lratio F_1 B A r_3)
($S_{D_1E_1F_1} = S_{DEF}$))

The eliminants

$$S_{D_1E_1F_1} \stackrel{F_1}{=} -\left(S_{BD_1E_1}\cdot r_3-S_{BD_1E_1}-S_{AD_1E_1}\cdot r_3\right)$$
$$S_{AD_1E_1} \stackrel{E_1}{=} -\left(S_{ACD_1}\cdot r_2\right)$$
$$S_{BD_1E_1} \stackrel{E_1}{=} -\left((r_2-1)\cdot S_{ABD_1}\right)$$
$$S_{ABD_1} \stackrel{D_1}{=} -\left((r_1-1)\cdot S_{ABC}\right)$$
$$S_{ACD_1} \stackrel{D_1}{=} -\left(S_{ABC}\cdot r_1\right)$$
$$S_{DEF} \stackrel{F}{=} S_{BDE}\cdot r_3-S_{ADE}\cdot r_3+S_{ADE}$$
$$S_{ADE} \stackrel{E}{=} (r_2-1)\cdot S_{ACD}$$
$$S_{BDE} \stackrel{E}{=} S_{ABD}\cdot r_2$$
$$S_{ABD} \stackrel{D}{=} S_{ABC}\cdot r_1$$
$$S_{ACD} \stackrel{D}{=} (r_1-1)\cdot S_{ABC}$$

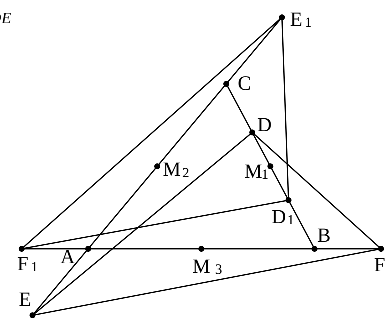


Figure 6-57

The machine proof

$$\frac{S_{D_1E_1F_1}}{S_{DEF}}$$
$$\stackrel{F_1}{=} \frac{-S_{BD_1E_1}\cdot r_3+S_{BD_1E_1}+S_{AD_1E_1}\cdot r_3}{S_{DEF}}$$
$$\stackrel{E_1}{=} \frac{-(S_{ACD_1}\cdot r_3\cdot r_2-S_{ABD_1}\cdot r_3\cdot r_2+S_{ABD_1}\cdot r_3+S_{ABD_1}\cdot r_2-S_{ABD_1})}{S_{DEF}}$$
$$\stackrel{D_1}{=} \frac{-(-S_{ABC}\cdot r_3\cdot r_2-S_{ABC}\cdot r_3\cdot r_1+S_{ABC}\cdot r_3-S_{ABC}\cdot r_2\cdot r_1+S_{ABC}\cdot r_2+S_{ABC}\cdot r_1-S_{ABC})}{S_{DEF}}$$
$$\stackrel{simplify}{=} \frac{(r_3\cdot r_2+r_3\cdot r_1-r_3+r_2\cdot r_1-r_2-r_1+1)\cdot S_{ABC}}{S_{DEF}}$$
$$\stackrel{F}{=} \frac{(r_3\cdot r_2+r_3\cdot r_1-r_3+r_2\cdot r_1-r_2-r_1+1)\cdot S_{ABC}}{S_{BDE}\cdot r_3-S_{ADE}\cdot r_3+S_{ADE}}$$
$$\stackrel{E}{=} \frac{(r_3\cdot r_2+r_3\cdot r_1-r_3+r_2\cdot r_1-r_2-r_1+1)\cdot S_{ABC}}{-S_{ACD}\cdot r_3\cdot r_2+S_{ACD}\cdot r_3+S_{ACD}\cdot r_2-S_{ACD}+S_{ABD}\cdot r_3\cdot r_2}$$
$$\stackrel{D}{=} \frac{-(r_3\cdot r_2+r_3\cdot r_1-r_3+r_2\cdot r_1-r_2-r_1+1)\cdot S_{ABC}}{-S_{ABC}\cdot r_3\cdot r_2-S_{ABC}\cdot r_3\cdot r_1+S_{ABC}\cdot r_3-S_{ABC}\cdot r_2\cdot r_1+S_{ABC}\cdot r_2+S_{ABC}\cdot r_1-S_{ABC}} \stackrel{simplify}{=} 1$$

Example 6.58 (0.200, 3, 11) *Show that the parallels through the vertices A, B, C of the triangle ABC to the medians of this triangle issued from the vertices B, C, A, respectively, form a triangle whose area is three times the area of the given triangle.*

Constructive description

((points A B C)
(midpoint A_1 C B)
(midpoint B_1 A C)
(midpoint C_1 B A)
(inter C_2 (p B C C_1) (p A B B_1))
(inter B_2 (p A B B_1) (p C A A_1))
(inter A_2 (p C A A_1) (p B C C_1))
($S_{A_2B_2C_2} = 3S_{ABC}$))

The eliminants

$$S_{C_2B_2A_2} \overset{A_2}{=} \frac{S_{BCB_2C_1} \cdot S_{ACA_1C_2}}{-S_{ACA_1C_1}}$$

$$S_{BCB_2C_1} \overset{B_2}{=} \frac{-(S_{ACA_1C_1} \cdot S_{ABC} - S_{ABA_1B_1} \cdot S_{BCC_1})}{S_{ABA_1B_1}}$$

$$S_{ACA_1C_2} \overset{C_2}{=} \frac{S_{BCB_1C_1} \cdot S_{ACA_1} - S_{ABA_1B_1} \cdot S_{ABC}}{S_{BCB_1C_1}}$$

$$S_{BCB_1C_1} \overset{C_1}{=} \frac{1}{2}(2S_{BCB_1} + S_{ABB_1})$$

$$S_{BCC_1} \overset{C_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACA_1C_1} \overset{C_1}{=} \frac{1}{2}(2S_{ACA_1} - S_{ABA_1})$$

$$S_{ABB_1} \overset{B_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{BCB_1} \overset{B_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ABA_1B_1} \overset{B_1}{=} -\frac{1}{2}(S_{ACA_1} - 2S_{ABA_1})$$

$$S_{ABA_1} \overset{A_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ABC})$$

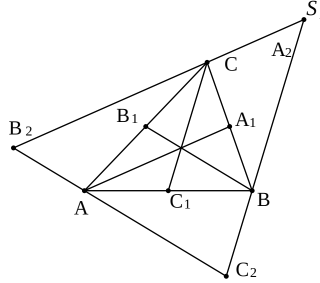


Figure 6-58

The machine proof

$$\frac{-S_{C_2B_2A_2}}{(3) \cdot S_{ABC}}$$

$$\overset{A_2}{=} \frac{-S_{BCB_2C_1} \cdot S_{ACA_1C_2}}{(3) \cdot S_{ABC} \cdot (-S_{ACA_1C_1})}$$

$$\overset{B_2}{=} \frac{(-S_{ACA_1C_1} \cdot S_{ABC} + S_{ABA_1B_1} \cdot S_{BCC_1}) \cdot S_{ACA_1C_2}}{(3) \cdot S_{ABC} \cdot S_{ACA_1C_1} \cdot S_{ABA_1B_1}}$$

$$\overset{C_2}{=} \frac{-(S_{ACA_1C_1} \cdot S_{ABC} - S_{ABA_1B_1} \cdot S_{BCC_1}) \cdot (S_{BCB_1C_1} \cdot S_{ACA_1} - S_{ABA_1B_1} \cdot S_{ABC})}{(3) \cdot S_{ABC} \cdot S_{ACA_1C_1} \cdot S_{ABA_1B_1} \cdot S_{BCB_1C_1}}$$

$$\overset{C_1}{=} \frac{-(-\frac{1}{2}S_{ABA_1B_1} \cdot S_{ABC} + S_{ACA_1} \cdot S_{ABC} - \frac{1}{2}S_{ABA_1} \cdot S_{ABC}) \cdot (-S_{ABA_1B_1} \cdot S_{ABC} + S_{BCB_1} \cdot S_{ACA_1} + \frac{1}{2}S_{ACA_1} \cdot S_{ABB_1})}{(3) \cdot S_{ABC} \cdot (S_{ACA_1} - \frac{1}{2}S_{ABA_1}) \cdot S_{ABA_1B_1} \cdot (S_{BCB_1} + \frac{1}{2}S_{ABB_1})}$$

$$\overset{simplify}{=} \frac{-(S_{ABA_1B_1} - 2S_{ACA_1} + S_{ABA_1}) \cdot (2S_{ABA_1B_1} \cdot S_{ABC} - 2S_{BCB_1} \cdot S_{ACA_1} - S_{ACA_1} \cdot S_{ABB_1})}{(3) \cdot (2S_{ACA_1} - S_{ABA_1}) \cdot S_{ABA_1B_1} \cdot (2S_{BCB_1} + S_{ABB_1})}$$

$$\overset{B_1}{=} \frac{-(-\frac{5}{2}S_{ACA_1} + 2S_{ABA_1}) \cdot (-\frac{5}{2}S_{ACA_1} \cdot S_{ABC} + 2S_{ABA_1} \cdot S_{ABC})}{(3) \cdot (2S_{ACA_1} - S_{ABA_1}) \cdot (-\frac{1}{2}S_{ACA_1} + S_{ABA_1}) \cdot (\frac{3}{2}S_{ABC})}$$

$$\overset{simplify}{=} \frac{(5S_{ACA_1} - 4S_{ABA_1})^2}{(9) \cdot (2S_{ACA_1} - S_{ABA_1}) \cdot (S_{ACA_1} - 2S_{ABA_1})}$$

$$\overset{A_1}{=} \frac{((-\frac{9}{2}S_{ABC}))^2}{(9) \cdot ((-\frac{3}{2}S_{ABC}))^2}$$

$$\overset{simplify}{=} 1$$

Definition. *If the corresponding sides of two similar polygons are parallel, the two polygons are said to be homothetic. The lines joining the corresponding vertices of two homothetic polygons are concurrent at the homothetic center.*

Example 6.59 (0.500, 5, 29) *Let L, L_1 and M, M_1 be two pairs of isotomic points on the two sides AC, AB of the triangle ABC, and L_2, M_2 the traces of the lines BL, CM on the sides A_1C_1, A_1B_1 of the medial triangle $A_1B_1C_1$ of ABC. Show that the triangles AL_1M_1, $A_1L_2M_2$ are homothetic.*

Constructive description
((points A B C) (midpoint B_1 A C)
(inter A_1 (l B C) (p B_1 A B))
(midpoint C_1 A B)
(on M (l A B)) (on L (l A C))
(lratio L_1 B_1 L -1)
(lratio M_1 C_1 M -1)
(inter L_2 (l A_1 C_1) (l B L))
(inter M_2 (l A_1 B_1) (l C M))
(inter O (l L_1 L_2) (l A A_1)) (collinear O M_1 M_2))

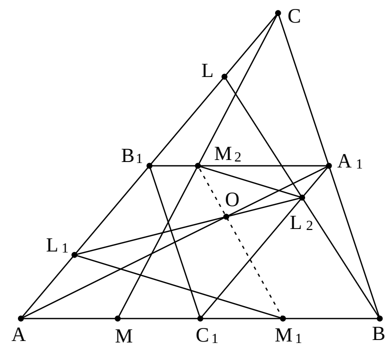


Figure 6-59

Example 6.60 (0.283, 6, 16) *A parallel to the median AA_1 of the triangle ABC meets BC, CA, AB in the points H, N, D. Prove that the symmetries of H with respect to the midpoints of NC, BD are symmetrical with respect to the vertex A.*

Constructive description
((points A B C) (on H (l B C)) (midpoint A_1 B C)
(inter N (l C A) (p H A A_1))
(inter D (l H N) (l A B))
(midpoint K N C)
(midpoint L B D) (lratio H_1 K H -1)
(lratio H_2 L H -1) (midpoint A H_1 H_2))

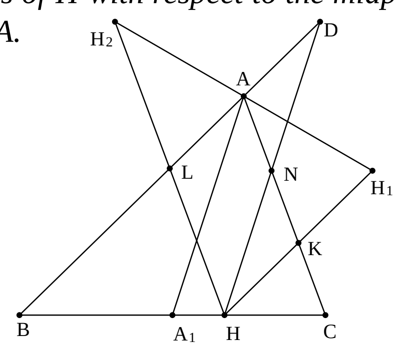


Figure 6-60

Example 6.61 (0.366, 4, 16) *The parallels to the sides of a triangle ABC through the same point, M, meet the respective medians in the points P, Q, R. Prove that we have, both in magnitude and in sign,* $(GP/GA) + (GQ/GB) + (GR/GC) = 0$.

Constructive description
((points A B C M)
(midpoint D B C)
(midpoint E C A) (midpoint F A B)
(inter P (l A D) (p M B C))

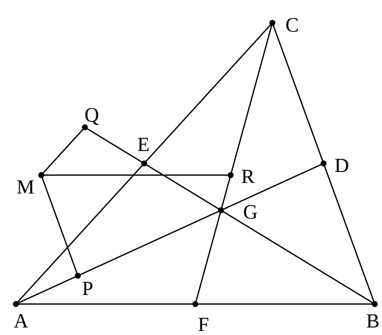


Figure 6-61

(inter Q (l B E) (p M A C))
(inter R (l C F) (p M A B))
(inter G (l A D) (l C F))
(collinear G B E) ($\frac{\overline{GQ}}{\overline{GB}} + \frac{\overline{GR}}{\overline{GC}} = \frac{\overline{GP}}{\overline{AG}}$))

6.3.2 Altitudes and Orthocenters

For machine proofs of the orthocenter theorem, see Example 3.36 on page 120 and Example 5.53 on page 247.

Example 6.62 (0.033, 1, 6) *The dual of the orthocenter theorem.*

Constructive description
((points A B C O)
(inter D (l B C) (t O O A))
(inter E (l C A) (t O O B))
(inter F (l A B) (t O O C))
(inter Z_F (l E D) (l A B))
($\frac{\overline{AF}}{\overline{BF}} = \frac{\overline{AZ_F}}{\overline{BZ_F}}$))

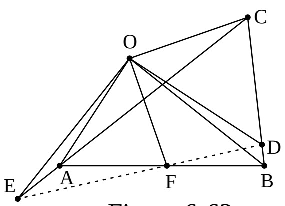


Figure 6-62

The machine proof

$$\left(\frac{\overline{AF}}{\overline{BF}}\right)/\left(\frac{\overline{AZ_F}}{\overline{BZ_F}}\right)$$

$$\stackrel{Z_F}{=} \frac{-S_{BDE}}{-S_{ADE}} \cdot \frac{\overline{AF}}{\overline{BF}}$$

$$\stackrel{F}{=} \frac{P_{AOC} \cdot S_{BDE}}{S_{ADE} \cdot P_{BOC}}$$

$$\stackrel{E}{=} \frac{P_{AOC} \cdot P_{BOC} \cdot S_{ABD} \cdot (-P_{CBAO})}{P_{AOB} \cdot S_{ACD} \cdot P_{BOC} \cdot (-P_{CBAO})}$$

$$\stackrel{simplify}{=} \frac{P_{AOC} \cdot S_{ABD}}{P_{AOB} \cdot S_{ACD}}$$

$$\stackrel{D}{=} \frac{P_{AOC} \cdot P_{AOB} \cdot S_{ABC} \cdot P_{CABO}}{P_{AOB} \cdot P_{AOC} \cdot S_{ABC} \cdot P_{CABO}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_F}}{\overline{BZ_F}} \stackrel{Z_F}{=} \frac{S_{ADE}}{S_{BDE}}$$

$$\frac{\overline{AF}}{\overline{BF}} \stackrel{F}{=} \frac{P_{AOC}}{P_{BOC}}$$

$$S_{ADE} \stackrel{E}{=} \frac{P_{AOB} \cdot S_{ACD}}{-P_{CBAO}}$$

$$S_{BDE} \stackrel{E}{=} \frac{P_{BOC} \cdot S_{ABD}}{-P_{CBAO}}$$

$$S_{ACD} \stackrel{D}{=} \frac{P_{AOC} \cdot S_{ABC}}{P_{CABO}}$$

$$S_{ABD} \stackrel{D}{=} \frac{P_{AOB} \cdot S_{ABC}}{P_{CABO}}$$

Example 6.63 (0.050, 1, 8) *In a given triangle the three products of the segments into which the orthocenter divides the altitudes are equal.*

Constructive description
((points A B C)
(foot D C A B)
(foot E B A C)
(inter H (l C D) (l B E))
(eq-product C H H D B H H E))

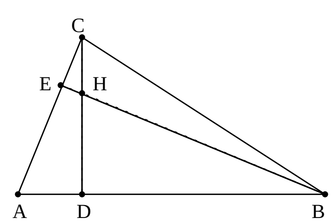


Figure 6-63

The machine proof

$$\frac{-P_{CHD}}{-P_{BHE}}$$

$$\stackrel{H}{=} \frac{(-P_{CDC}\cdot S_{BDE}\cdot S_{BCE})\cdot S_{BCED}^2}{(-P_{BEB}\cdot S_{CDE}\cdot S_{BCD})\cdot S_{BCED}^2}$$

$$\stackrel{simplify}{=} \frac{P_{CDC}\cdot S_{BDE}\cdot S_{BCE}}{P_{BEB}\cdot S_{CDE}\cdot S_{BCD}}$$

$$\stackrel{E}{=} \frac{P_{CDC}\cdot(-P_{BAC}\cdot S_{BCD})\cdot P_{ACB}\cdot S_{ABC}\cdot(P_{ACA})^2}{(16S_{ABC}^2)\cdot P_{ACB}\cdot S_{ACD}\cdot S_{BCD}\cdot(P_{ACA})^2}$$

$$\stackrel{simplify}{=} \frac{-P_{CDC}\cdot P_{BAC}}{(16)\cdot S_{ABC}\cdot S_{ACD}}$$

$$\stackrel{D}{=} \frac{-(16S_{ABC}^2)\cdot P_{BAC}\cdot P_{ABA}}{(16)\cdot S_{ABC}\cdot(-P_{BAC}\cdot S_{ABC})\cdot P_{ABA}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{BHE} \stackrel{H}{=} \frac{P_{BEB}\cdot S_{CDE}\cdot S_{BCD}}{(S_{BCED})^2}$$

$$P_{CHD} \stackrel{H}{=} \frac{P_{CDC}\cdot S_{BDE}\cdot S_{BCE}}{(S_{BCED})^2}$$

$$S_{CDE} \stackrel{E}{=} \frac{P_{ACB}\cdot S_{ACD}}{P_{ACA}}$$

$$P_{BEB} \stackrel{E}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{ACA}}$$

$$S_{BCE} \stackrel{E}{=} \frac{P_{ACB}\cdot S_{ABC}}{P_{ACA}}$$

$$S_{BDE} \stackrel{E}{=} \frac{-P_{BAC}\cdot S_{BCD}}{P_{ACA}}$$

$$S_{ACD} \stackrel{D}{=} \frac{-P_{BAC}\cdot S_{ABC}}{P_{ABA}}$$

$$P_{CDC} \stackrel{D}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{ABA}}$$

Example 6.64 (0.066, 2, 8) *The product of the segments into which a side of a triangle is divided by the foot of the altitude is equal to this altitude multiplied by the distance of the side from the orthocenter.*

Constructive description
((points A B C)
(foot F C A B)
(foot E B A C)
(inter H (l C F) (l B E))
(eq-product A F F B C F H F))

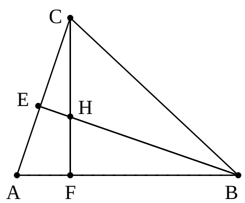


Figure 6-64

The eliminants

$$P_{CFH} \stackrel{H}{=} \frac{P_{CFE}\cdot S_{BCF}}{S_{BCEF}}$$

$$P_{CFE} \stackrel{E}{=} \frac{P_{CFC}\cdot P_{BAC}}{P_{ACA}}$$

$$S_{BCEF} \stackrel{E}{=} \frac{-(P_{ACB}\cdot S_{ACF}-P_{ACA}\cdot S_{BCF})}{P_{ACA}}$$

$$P_{CFC} \stackrel{F}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{ABA}}$$

$$S_{BCF} \stackrel{F}{=} \frac{P_{ABC}\cdot S_{ABC}}{P_{ABA}}$$

$$S_{ACF} \stackrel{F}{=} \frac{-P_{BAC}\cdot S_{ABC}}{P_{ABA}}$$

$$P_{AFB} \stackrel{F}{=} \frac{-P_{BAC}\cdot P_{ABC}}{P_{ABA}}$$

$$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB} + P_{ACA}\cdot P_{ABC}$$

The machine proof

$$\frac{-P_{AFB}}{P_{CFH}}$$

$$\stackrel{H}{=} \frac{(-P_{AFB})\cdot S_{BCEF}}{P_{CFE}\cdot S_{BCF}}$$

$$\stackrel{E}{=} \frac{-P_{AFB}\cdot(-P_{ACB}\cdot S_{ACF}+P_{ACA}\cdot S_{BCF})\cdot P_{ACA}}{P_{CFC}\cdot P_{BAC}\cdot S_{BCF}\cdot P_{ACA}}$$

$$\stackrel{simplify}{=} \frac{P_{AFB}\cdot(P_{ACB}\cdot S_{ACF}-P_{ACA}\cdot S_{BCF})}{P_{CFC}\cdot P_{BAC}\cdot S_{BCF}}$$

$$\stackrel{F}{=} \frac{(-P_{BAC}\cdot P_{ABC})\cdot(-P_{BAC}\cdot P_{ACB}\cdot P_{ABA}\cdot S_{ABC}-P_{ACA}\cdot P_{ABC}\cdot P_{ABA}\cdot S_{ABC})\cdot(P_{ABA})^2}{(16S_{ABC}^2)\cdot P_{BAC}\cdot P_{ABC}\cdot S_{ABC}\cdot(P_{ABA})^3}$$

$$\overset{simplify}{=} \frac{P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}}{(16)\cdot(S_{ABC})^2}$$

$$\overset{herron}{=} \frac{(P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC})\cdot(16)}{16P_{BAC}\cdot P_{ACB}+16P_{ACA}\cdot P_{ABC}}$$ (This step uses Herron-Qin's formula.)

$$\overset{simplify}{=} 1$$

Example 6.65 (0.050, 3, 4) *If p, q, r are the distances of a point inside a triangle ABC from the sides of the triangle, show that* $(p/h_a)+(q/h_b)+(r/h_c)=1$.

Constructive description
((points A B C O)
(foot D A B C)
(foot E B A C)
(foot F C A B) (foot D_1 O B C)
(foot E_1 O A C) (foot F_1 O A B)
$(\frac{\overline{OD_1}}{\overline{AD}}+\frac{\overline{OE_1}}{\overline{BE}}+\frac{\overline{OF_1}}{\overline{CF}}=1)$)

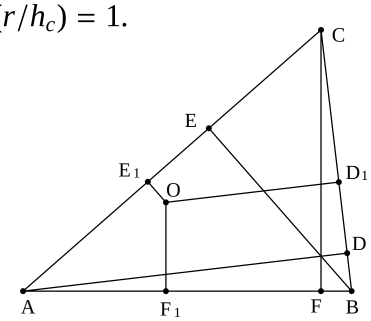


Figure 6-65

The machine proof

$$\frac{\overline{OF_1}}{\overline{CF}}+\frac{\overline{OE_1}}{\overline{BE}}+\frac{\overline{OD_1}}{\overline{AD}}$$

$$\overset{F_1}{=} \frac{\frac{\overline{OE_1}}{\overline{BE}}\cdot S_{ABC}+\frac{\overline{OD_1}}{\overline{AD}}\cdot S_{ABC}+S_{ABO}}{S_{ABC}}$$

$$\overset{E_1}{=} \frac{-\frac{\overline{OD_1}}{\overline{AD}}\cdot S_{ABC}^2+S_{ACO}\cdot S_{ABC}-S_{ABO}\cdot S_{ABC}}{S_{ABC}\cdot(-S_{ABC})}$$

$$\overset{simplify}{=} \frac{\frac{\overline{OD_1}}{\overline{AD}}\cdot S_{ABC}-S_{ACO}+S_{ABO}}{S_{ABC}}$$

$$\overset{D_1}{=} \frac{S_{BCO}\cdot S_{ABC}-S_{ACO}\cdot S_{ABC}+S_{ABO}\cdot S_{ABC}}{(S_{ABC})^2}$$

$$\overset{simplify}{=} \frac{S_{BCO}-S_{ACO}+S_{ABO}}{S_{ABC}}$$

$$\overset{area-co}{=} \frac{S_{ABC}}{S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{OF_1}}{\overline{CF}} \overset{F_1}{=} \frac{S_{ABO}}{S_{ABC}}$$

$$\frac{\overline{OE_1}}{\overline{BE}} \overset{E_1}{=} \frac{S_{ACO}}{-S_{ABC}}$$

$$\frac{\overline{OD_1}}{\overline{AD}} \overset{D_1}{=} \frac{S_{BCO}}{S_{ABC}}$$

$$S_{BCO}=S_{ACO}-S_{ABO}+S_{ABC}$$

Example 6.66 (0.050, 2, 3) *Show that the sum of distances of a point on the base of an isosceles triangle to its two sides is equal to the altitudes on the sides.*

The eliminants: $\frac{\overline{DK}}{\overline{AH}} \overset{K}{=} \frac{S_{BCD}}{-S_{BAC}}$, $\frac{\overline{DF}}{\overline{BG}} \overset{F}{=} \frac{S_{ACD}}{S_{BAC}}$, $S_{ACD}-S_{BCD}=S_{BAC}$.

Constructive description
((points B A)
(on C (b A B))
(on D (l A B))
(foot G B A C)
(foot F D A C)
(foot H A B C)
(foot K D B C)
($\frac{\overline{DK}}{\overline{AH}}+\frac{\overline{DF}}{\overline{BG}} = 1$))

The machine proof

$$\frac{\overline{DK}}{\overline{AH}}+\frac{\overline{DF}}{\overline{BG}}$$

$$\overset{K}{=} \frac{-\frac{\overline{DF}}{\overline{BG}}\cdot S_{BAC}+S_{BCD}}{-S_{BAC}}$$

$$\overset{F}{=} \frac{S_{ACD}\cdot S_{BAC}-S_{BCD}\cdot S_{BAC}}{(S_{BAC})^2}$$

$$\overset{simplify}{=} \frac{S_{ACD}-S_{BCD}}{S_{BAC}}$$

$$\overset{G}{=} \frac{S_{BAC}}{S_{BAC}} \overset{simplify}{=} 1$$

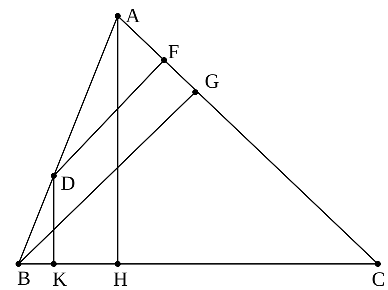


Figure 6-66

Definition. *The triangle having for its vertices the feet of the altitudes of a given triangle is called the orthic triangle of the given triangle.*

Example 6.67 (0.016, 1, 2) *The three triangles cut off from a given triangle by the sides of its orthic triangles are similar to the given triangle.*

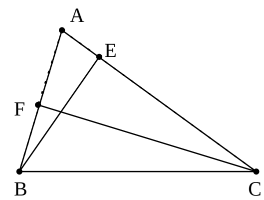


Figure 6-67

Constructive description
((points A B C)
(foot F C A B)
(foot E B A C)
(eq-product A F A B A E A C))

The machine proof

$$\frac{P_{BAF}}{P_{CAE}} \overset{E}{=} \frac{P_{BAF}}{P_{BAC}}$$

$$\overset{F}{=} \frac{P_{BAC}}{P_{BAC}} \overset{simplify}{=} 1$$

The eliminants

$$P_{CAE} \overset{E}{=} P_{BAC}$$

$$P_{BAF} \overset{F}{=} P_{BAC}$$

Example 6.68 (0.083, 1, 12) *The sides of the orthic triangle meet the sides of the given triangle in three collinear points.*

Constructive description
((points A B C)
(foot D A B C)
(foot E B A C)
(foot F C A B)
(inter A_1 (l B C) (l E F))
(inter B_1 (l A C) (l D F))
(inter C_1 (l A B) (l D E))
(inter Z_{C_1} (l B_1 A_1) (l D E))
($\frac{\overline{DC_1}}{\overline{EC_1}} = \frac{\overline{DZ_{C_1}}}{\overline{EZ_{C_1}}}$))

The machine proof

$$\left(\frac{\overline{DC_1}}{\overline{EC_1}}\right)/\left(\frac{\overline{DZ_{C_1}}}{\overline{EZ_{C_1}}}\right)$$

$$\stackrel{Z_{C_1}}{=} \frac{-S_{EA_1B_1}}{-S_{DA_1B_1}} \cdot \frac{\overline{DC_1}}{\overline{EC_1}}$$

$$\stackrel{C_1}{=} \frac{S_{ABD} \cdot S_{EA_1B_1}}{S_{DA_1B_1} \cdot S_{ABE}}$$

$$\stackrel{B_1}{=} \frac{S_{ABD} \cdot S_{DEF} \cdot S_{ACA_1} \cdot (-S_{ADCF})}{(-S_{DFA_1} \cdot S_{ACD}) \cdot S_{ABE} \cdot S_{ADCF}}$$

$$\stackrel{simplify}{=} \frac{S_{ABD} \cdot S_{DEF} \cdot S_{ACA_1}}{S_{DFA_1} \cdot S_{ACD} \cdot S_{ABE}}$$

$$\stackrel{A_1}{=} \frac{S_{ABD} \cdot S_{DEF} \cdot S_{CEF} \cdot S_{ABC} \cdot S_{BECF}}{(-S_{DEF} \cdot S_{BCF}) \cdot S_{ACD} \cdot S_{ABE} \cdot S_{BECF}}$$

$$\stackrel{simplify}{=} \frac{S_{ABD} \cdot S_{CEF} \cdot S_{ABC}}{-S_{BCF} \cdot S_{ACD} \cdot S_{ABE}}$$

$$\stackrel{F}{=} \frac{S_{ABD} \cdot P_{BAC} \cdot S_{BCE} \cdot S_{ABC} \cdot P_{ABA}}{-P_{ABC} \cdot S_{ABC} \cdot S_{ACD} \cdot S_{ABE} \cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{S_{ABD} \cdot P_{BAC} \cdot S_{BCE}}{-P_{ABC} \cdot S_{ACD} \cdot S_{ABE}}$$

$$\stackrel{E}{=} \frac{S_{ABD} \cdot P_{BAC} \cdot P_{ACB} \cdot S_{ABC} \cdot P_{ACA}}{-P_{ABC} \cdot S_{ACD} \cdot P_{BAC} \cdot S_{ABC} \cdot P_{ACA}}$$

$$\stackrel{simplify}{=} \frac{S_{ABD} \cdot P_{ACB}}{-P_{ABC} \cdot S_{ACD}}$$

$$\stackrel{D}{=} \frac{P_{ABC} \cdot S_{ABC} \cdot P_{ACB} \cdot P_{BCB}}{-P_{ABC} \cdot (-P_{ACB} \cdot S_{ABC}) \cdot P_{BCB}} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{DZ_{C_1}}}{\overline{EZ_{C_1}}} \stackrel{Z_{C_1}}{=} \frac{S_{DA_1B_1}}{S_{EA_1B_1}}$$

$$\frac{\overline{DC_1}}{\overline{EC_1}} \stackrel{C_1}{=} \frac{S_{ABD}}{S_{ABE}}$$

$$S_{DA_1B_1} \stackrel{B_1}{=} \frac{S_{DFA_1} \cdot S_{ACD}}{S_{ADCF}}$$

$$S_{EA_1B_1} \stackrel{B_1}{=} \frac{S_{DEF} \cdot S_{ACA_1}}{S_{ADCF}}$$

$$S_{DFA_1} \stackrel{A_1}{=} \frac{-S_{DEF} \cdot S_{BCF}}{S_{BECF}}$$

$$S_{ACA_1} \stackrel{A_1}{=} \frac{S_{CEF} \cdot S_{ABC}}{S_{BECF}}$$

$$S_{BCF} \stackrel{F}{=} \frac{P_{ABC} \cdot S_{ABC}}{P_{ABA}}$$

$$S_{CEF} \stackrel{F}{=} \frac{P_{BAC} \cdot S_{BCE}}{P_{ABA}}$$

$$S_{ABE} \stackrel{E}{=} \frac{P_{BAC} \cdot S_{ABC}}{P_{ACA}}$$

$$S_{BCE} \stackrel{E}{=} \frac{P_{ACB} \cdot S_{ABC}}{P_{ACA}}$$

$$S_{ACD} \stackrel{D}{=} \frac{-P_{ACB} \cdot S_{ABC}}{P_{BCB}}$$

$$S_{ABD} \stackrel{D}{=} \frac{P_{ABC} \cdot S_{ABC}}{P_{BCB}}$$

Example 6.69 (0.067, 4, 11) *The altitudes of a triangle bisect the internal angles of its orthic triangle.*

Constructive description
((points A B C)
(foot F C A B)
(foot E B A C)
(foot D A B C)
(eqangle E D C B D F))

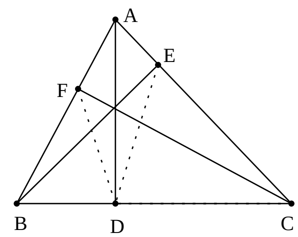


Figure 6-69

The machine proof.

$$\frac{S_{CED}\cdot P_{BDF}}{(-S_{BFD})\cdot P_{CDE}}$$

$$\stackrel{D}{=} \frac{P_{ACB}\cdot S_{BCE}\cdot(P_{BCF}\cdot P_{ABC}-P_{ACB}\cdot P_{ABC})\cdot(P_{BCB})^2}{P_{ABC}\cdot S_{BCF}\cdot(P_{CBE}\cdot P_{ACB}-P_{ACB}\cdot P_{ABC})\cdot(P_{BCB})^2}$$

$$\stackrel{simplify}{=} \frac{S_{BCE}\cdot(P_{BCF}-P_{ACB})}{S_{BCF}\cdot(P_{CBE}-P_{ABC})}$$

$$\stackrel{E}{=} \frac{P_{ACB}\cdot S_{ABC}\cdot(P_{BCF}-P_{ACB})\cdot P_{ACA}}{S_{BCF}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC})\cdot P_{ACA}}$$

$$\stackrel{simplify}{=} \frac{P_{ACB}\cdot S_{ABC}\cdot(P_{BCF}-P_{ACB})}{S_{BCF}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC})}$$

$$\stackrel{F}{=} \frac{P_{ACB}\cdot S_{ABC}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACB}\cdot P_{ABA})\cdot P_{ABA}}{P_{ABC}\cdot S_{ABC}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC})\cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{P_{ACB}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACB}\cdot P_{ABA})}{P_{ABC}\cdot(P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC})}$$

$$\stackrel{py}{=} \frac{(P_{BCB}+P_{ACA}-P_{ABA})\cdot(-2P_{BCB}^2+4P_{BCB}\cdot P_{ACA}-2P_{ACA}^2+2P_{ABA}^2)\cdot((2))^4}{(P_{BCB}-P_{ACA}+P_{ABA})\cdot(-2P_{BCB}^2+4P_{BCB}\cdot P_{ABA}+2P_{ACA}^2-2P_{ABA}^2)\cdot((2))^4}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{CDE} \stackrel{D}{=} \frac{(P_{CBE}-P_{ABC})\cdot P_{ACB}}{P_{BCB}}$$
$$S_{BFD} \stackrel{D}{=} \frac{-P_{ABC}\cdot S_{BCF}}{P_{BCB}}$$
$$P_{BDF} \stackrel{D}{=} \frac{(P_{BCF}-P_{ACB})\cdot P_{ABC}}{P_{BCB}}$$
$$S_{CED} \stackrel{D}{=} \frac{P_{ACB}\cdot S_{BCE}}{P_{BCB}}$$
$$P_{CBE} \stackrel{E}{=} \frac{P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}}{P_{ACA}}$$
$$S_{BCE} \stackrel{E}{=} \frac{P_{ACB}\cdot S_{ABC}}{P_{ACA}}$$
$$S_{BCF} \stackrel{F}{=} \frac{P_{ABC}\cdot S_{ABC}}{P_{ABA}}$$
$$P_{BCF} \stackrel{F}{=} \frac{P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}}{P_{ABA}}$$
$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$
$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$
$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$

Definition. *Let H be the orthocenter of triangle ABC. Then the four points A,B,C,H are such that each is the orthocenter of the triangle formed by the remaining three. Four such points will be revered to as an orthocentric group of points, or an orthocentric quadrilateral.*

Example 6.70 (0.067 4 10) *Let H be the orthocenter of triangle ABC. Then the circumcenters of the four triangles ABC, ABH, ACH, and HBC form a triangle congruent to ABC; the sides of the two triangles are parallel.*

Constructive description

((points A B C)
(orthocenter H A B C)
(circumcenter O_a B C H)
(circumcenter O_b A C H)
(circumcenter O_c A B H)
(circumcenter O A B C)
($\frac{\overline{O_bO_c}}{\overline{CB}}$ = 1))

The eliminants

$$\frac{\overline{O_bO_c}}{\overline{BC}} \stackrel{O_c}{=} \frac{P_{BAO_b}-\frac{1}{2}P_{ABA}}{P_{ABC}}$$
$$P_{BAO_b} \stackrel{O_b}{=} \frac{P_{BAH}\cdot P_{AHC}\cdot P_{ACA}+P_{BAC}\cdot P_{AHA}\cdot P_{ACH}}{(32)\cdot(S_{ACH})^2}$$
$$S_{ACH} \stackrel{H}{=} \frac{P_{BAC}\cdot P_{ACB}}{(-16)\cdot S_{ABC}}$$
$$P_{ACH} \stackrel{H}{=} P_{ACB}$$
$$P_{AHA} \stackrel{H}{=} \frac{P_{BCB}\cdot(P_{BAC})^2}{(16)\cdot(S_{ABC})^2}$$
$$P_{AHC} \stackrel{H}{=} \frac{P_{BAC}\cdot P_{ACB}\cdot P_{ABC}}{(-16)\cdot(S_{ABC})^2}$$
$$P_{BAH} \stackrel{H}{=} P_{BAC}$$
$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$
$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$
$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

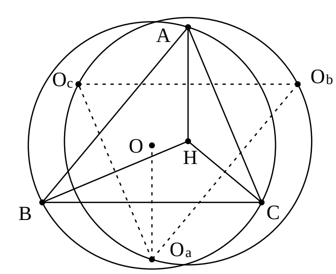


Figure 6-70

The machine proof

$$-\frac{\overline{O_bO_c}}{\overline{BC}} \overset{O_c}{=} \frac{-(P_{BAO_b}-\frac{1}{2}P_{ABA})}{P_{ABC}}$$

$$\overset{O_b}{=} \frac{-(P_{BAH}\cdot P_{AHC}\cdot P_{ACA}+P_{BAC}\cdot P_{AHA}\cdot P_{ACH}-16P_{ABA}\cdot S_{ACH}^2)}{P_{ABC}\cdot(32S_{ACH}^2)}$$

$$\overset{H}{=} \Big(-\ (-4096P_{BCB}\cdot P_{BAC}^3\cdot P_{ACB}\cdot S_{ABC}^6+4096P_{BAC}^2\cdot P_{ACB}^2\cdot P_{ABA}\cdot S_{ABC}^6+4096P_{BAC}^2\cdot P_{ACB}\cdot P_{ACA}\cdot P_{ABC}\cdot S_{ABC}^6)\cdot((16S_{ABC}^2))^2\Big) \Big/ \Big((32)\cdot P_{ABC}\cdot((-P_{BAC}\cdot P_{ACB}\cdot S_{ABC}))^2\cdot((16S_{ABC}^2))^3\cdot(-16S_{ABC}^2)\Big)$$

$$\overset{simplify}{=} \frac{-(P_{BCB}\cdot P_{BAC}-P_{ACB}\cdot P_{ABA}-P_{ACA}\cdot P_{ABC})}{(2)\cdot P_{ABC}\cdot P_{ACB}}$$

$$\overset{py}{=} \frac{-(-4P_{BCB}^2+4P_{ACA}^2-8P_{ACA}\cdot P_{ABA}+4P_{ABA}^2)\cdot((2))^2}{((2))^4\cdot(P_{BCB}-P_{ACA}+P_{ABA})\cdot(P_{BCB}+P_{ACA}-P_{ABA})} \overset{simplify}{=} 1$$

Example 6.71 (0.067, 1, 6) *Continuing from Example 6.70, show that the point H is the circumcenter of the triangle $O_aO_bO_c$.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O_a\ B\ C\ H$)
(circumcenter $O_b\ A\ C\ H$)
(circumcenter $O_c\ A\ B\ H$)
(circumcenter $O\ A\ B\ C$)
(eqdistance $H\ O_a\ H\ O_b$))

The machine proof

$$\frac{P_{HO_aH}}{P_{HO_bH}}$$

$$\overset{O_b}{=} \frac{P_{HO_aH}\cdot(64S_{ACH}^2)}{P_{CHC}\cdot P_{AHA}\cdot P_{ACA}}$$

$$\overset{O_a}{=} \frac{(64)\cdot P_{CHC}\cdot P_{BHB}\cdot P_{BCB}\cdot(S_{ACH})^2}{P_{CHC}\cdot P_{AHA}\cdot P_{ACA}\cdot(64S_{BCH}^2)}$$

$$\overset{simplify}{=} \frac{P_{BHB}\cdot P_{BCB}\cdot(S_{ACH})^2}{P_{AHA}\cdot P_{ACA}\cdot(S_{BCH})^2}$$

$$\overset{H}{=} \frac{P_{ACA}\cdot P_{ABC}^2\cdot P_{BCB}\cdot((-P_{BAC}\cdot P_{ACB}\cdot S_{ABC}))^2\cdot((16S_{ABC}^2))^3}{P_{BCB}\cdot P_{BAC}^2\cdot P_{ACA}\cdot(P_{ACB}\cdot P_{ABC}\cdot S_{ABC})^2\cdot((16S_{ABC}^2))^3}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{HO_bH} \overset{O_b}{=} \frac{P_{CHC}\cdot P_{AHA}\cdot P_{ACA}}{(64)\cdot(S_{ACH})^2}$$

$$P_{HO_aH} \overset{O_a}{=} \frac{P_{CHC}\cdot P_{BHB}\cdot P_{BCB}}{(64)\cdot(S_{BCH})^2}$$

$$S_{BCH} \overset{H}{=} \frac{P_{ACB}\cdot P_{ABC}}{(16)\cdot S_{ABC}}$$

$$P_{AHA} \overset{H}{=} \frac{P_{BCB}\cdot(P_{BAC})^2}{(16)\cdot(S_{ABC})^2}$$

$$S_{ACH} \overset{H}{=} \frac{P_{BAC}\cdot P_{ACB}}{(-16)\cdot S_{ABC}}$$

$$P_{BHB} \overset{H}{=} \frac{P_{ACA}\cdot(P_{ABC})^2}{(16)\cdot(S_{ABC})^2}$$

Example 6.72 (0.350, 10, 17) *Show that the three perpendiculars to the sides of a triangle at the points isotomic to the feet of the respective altitudes are concurrent.*

Constructive description
((points $A\ B\ C$) (foot $D\ A\ B\ C$)
(foot $E\ B\ A\ C$) (foot $F\ C\ A\ B$)
(pratio $D_1\ B\ C\ D\ -1$)
(pratio $E_1\ A\ C\ E\ -1$)
(pratio $F_1\ A\ B\ F\ -1$)
(inter I (t $E_1\ A\ C$) (t $D_1\ B\ C$))
(perpendicular $I\ F_1\ A\ B$))

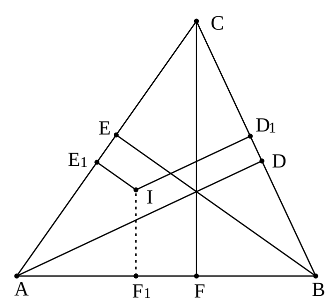


Figure 6-72

Example 6.73 (0.100, 4, 15) *Show that the symmetries of the foot of the altitude to the base of a triangle with respect to the other two sides lie on the side of the orthic triangle relative to the base.*

Constructive description
((points *A B C*)
(foot *D A B C*)
(foot *E B A C*)
(foot *F C A B*)
(foot *P D B A*)
(inter *J* (l *E F*) (l *D P*))
(midpoint *P D J*))

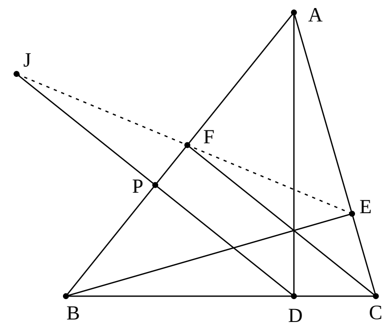


Figure 6-73

Example 6.74 (0.183, 3, 11) *Show that the product of the segments into which a side of a triangle is divided by the corresponding vertex of the orthic triangle is equal to the product of the sides of the orthic triangle passing through the vertex considered.*

Constructive description
((points *A B C*)
(foot *F C A B*)
(foot *E B A C*)
(foot *D A B C*)
(eq-product *B D D C E D F D*))

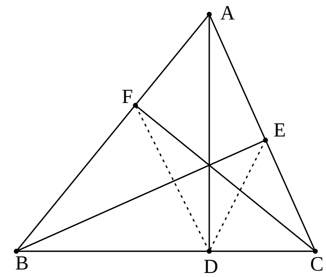


Figure 6-74

Example 6.75 (0.100, 3, 14) *If P, Q are the feet of the perpendiculars from the vertices B, C of the triangle ABC upon the sides DF, DE, respectively, of the orthic triangle DEF, show that $EQ = FP$.*

Constructive description
((points *A B C*)
(foot *F C A B*)
(foot *E B A C*)
(foot *D A B C*)
(foot *Q C D E*) (foot *P B D F*)
(eqdistance *E Q F P*))

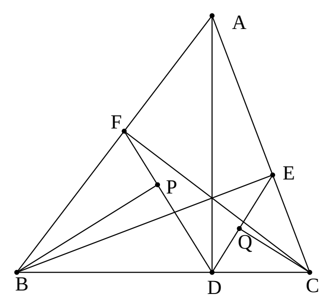


Figure 6-75

Example 6.76 (0.167, 6, 22) *The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.*

Constructive description
((points *A B C*)
(orthocenter *H A B C*)
(foot *F C A B*)
(foot *P F A C*)
(foot *T F B C*)
(foot *Q F A H*)
(collinear *P Q T*))

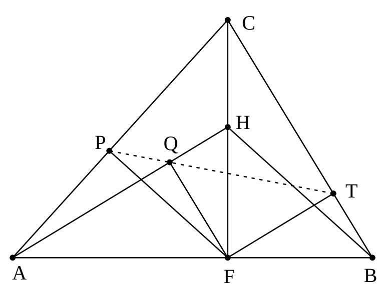


Figure 6-76

Example 6.77 (0.083, 4, 9) *DP, DQ are the perpendiculars from the foot D of the altitude AD of the triangle ABC upon the sides AC, AB. Prove that the points B, C, P, Q are cyclic.*

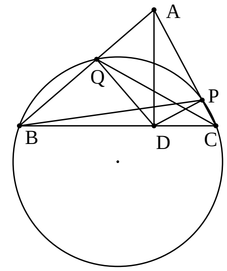


Figure 6-77

Constructive description
((points A B C)
(foot D A B C)
(foot Q D A B)
(foot P D A C)
(cocircle B C P Q))

The machine proof

$$\frac{(-S_{BCP})\cdot P_{BQC}}{(-S_{BCQ})\cdot P_{BPC}}$$
$$\overset{P}{=} \frac{(-P_{ACD}\cdot S_{ABC})\cdot P_{BQC}\cdot P_{ACA}}{(-S_{BCQ})\cdot(-P_{CAD}\cdot P_{ACD}+P_{BAC}\cdot P_{ACD})\cdot P_{ACA}}$$
$$\overset{simplify}{=} \frac{-S_{ABC}\cdot P_{BQC}}{S_{BCQ}\cdot(P_{CAD}-P_{BAC})}$$
$$\overset{Q}{=} \frac{-S_{ABC}\cdot(-P_{BAD}\cdot P_{ABD}+P_{BAC}\cdot P_{ABD})\cdot P_{ABA}}{P_{ABD}\cdot S_{ABC}\cdot(P_{CAD}-P_{BAC})\cdot P_{ABA}}$$
$$\overset{simplify}{=} \frac{P_{BAD}-P_{BAC}}{P_{CAD}-P_{BAC}}$$
$$\overset{D}{=} \frac{(-P_{BCB}\cdot P_{BAC}+P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})\cdot P_{BCB}}{(-P_{BCB}\cdot P_{BAC}+P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC})\cdot P_{BCB}}$$
$$\overset{simplify}{=} \frac{P_{BCB}\cdot P_{BAC}-P_{BAC}\cdot P_{ABC}-P_{ACB}\cdot P_{ABA}}{P_{BCB}\cdot P_{BAC}-P_{BAC}\cdot P_{ACB}-P_{ACA}\cdot P_{ABC}}$$
$$\overset{py}{=} \frac{(-2P_{BCB}^2+2P_{ACA}^2-4P_{ACA}\cdot P_{ABA}+2P_{ABA}^2)\cdot((2))^3}{(-2P_{BCB}^2+2P_{ACA}^2-4P_{ACA}\cdot P_{ABA}+2P_{ABA}^2)\cdot((2))^3} \overset{simplify}{=} 1$$

The eliminants

$$P_{BPC} \overset{P}{=} \frac{-(P_{CAD}-P_{BAC})\cdot P_{ACD}}{P_{ACA}}$$
$$S_{BCP} \overset{P}{=} \frac{P_{ACD}\cdot S_{ABC}}{P_{ACA}}$$
$$S_{BCQ} \overset{Q}{=} \frac{P_{ABD}\cdot S_{ABC}}{P_{ABA}}$$
$$P_{BQC} \overset{Q}{=} \frac{-(P_{BAD}-P_{BAC})\cdot P_{ABD}}{P_{ABA}}$$
$$P_{CAD} \overset{D}{=} \frac{P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}}{P_{BCB}}$$
$$P_{BAD} \overset{D}{=} \frac{P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA}}{P_{BCB}}$$
$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$
$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$
$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

Example 6.78 (0.450, 6, 29) *The perpendiculars DP, DQ dropped from the foot D of the altitude AD of the triangle ABC upon the sides AB, AC meet the perpendiculars BP, CQ erected to BC at B, C in the points P, Q respectively. Prove that the line PQ passes through the orthocenter H of ABC.*

Constructive description
((points A B C)
(foot D A B C)
(inter H (l A D) (t B A C))
(inter P (t D A B) (t B B C))
(inter Q (t D A C) (t C B C))
(collinear H Q P))

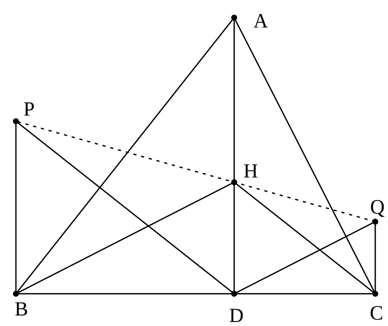


Figure 6-78

Example 6.79 (2.067, 68, 61) *The algebraic sum of the distances of the points of an orthocentric group from any line passing through the nine-point center of the group is equal to zero.*

Constructive description
((points A B C)
(orthocenter H A B C) (midpoint M_1 B C)
(midpoint M_2 C A) (midpoint M_3 A B)
(midpoint N_1 M_1 M_2) (midpoint N_2 M_1 M_3)
(inter N (t N_1 N_1 M_1) (t N_2 N_2 M_1))
(on X (l B C))
(foot H_1 H X N)
(inter A_1 (l X N) (p A H H_1))
(inter B_1 (l X N) (p B H H_1)) (inter C_1 (l X N) (p C H H_1)) ($\frac{\overline{AA_1}}{\overline{HH_1}}+\frac{\overline{BB_1}}{\overline{HH_1}} = \frac{\overline{C_1C}}{\overline{HH_1}}-1$))

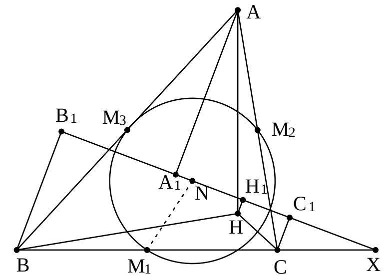


Figure 6-79

Example 6.80 (0.350, 6, 10) *If through the midpoints of the sides of a triangle having its vertices on the altitudes of a given triangle, perpendiculars are dropped to the respective sides of the given triangle, show that the three perpendiculars are concurrent.*

Constructive description
((points A B C) (orthocenter H A B C)
(on R (l A H)) (on P (l B H))
(on Q (l C H)) (midpoint R_1 P Q)
(midpoint Q_1 R P)
(midpoint P_1 Q R)
(inter I (p R_1 A H) (p Q_1 C H))
(perpendicular I P_1 A C))

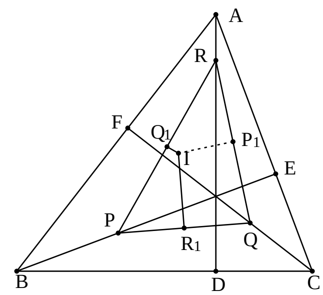


Figure 6-80

Example 6.81 (2.833, 7, 43) *Show that the perpendiculars dropped from the orthocenter of a triangle upon the lines joining the vertices to a given points meet the respectively opposite sides of the triangle in three collinear points.*

Constructive description
((points A B C O)
(orthocenter H A B C)
(inter A_1 (l B C) (t H O A))
(inter B_1 (l A C) (t H O B))
(inter C_1 (l A B) (t H O C))
(inter C_2 (l A B) (l A_1 B_1))
($\frac{\overline{AC_1}}{\overline{BC_1}} = \frac{\overline{AC_2}}{\overline{BC_2}}$))

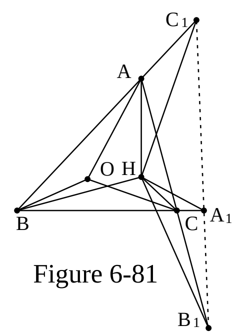


Figure 6-81

Definition. *Two lines passing through the vertex of a given angle and marking equal angles with the bisector of the given angle are said to be isogoal or isogonal conjugates.*

Example 6.82 (0.133, 5, 22) *Show that the line joining a given point to the vertex of a given angle has for its isogonal line the mediator of the segment determined by the symmetries of the given point with respect to the sides of the angle.*

Constructive description
((points $A\ B\ C\ M$)
(foot $P\ M\ A\ C$)
(foot $Q\ M\ A\ B$)
(lratio $P_1\ P\ M\ -1$)
(lratio $Q_1\ Q\ M\ -1$)
(midpoint $N\ P_1\ Q_1$)
(perpendicular $A\ N\ P_1\ Q_1$))

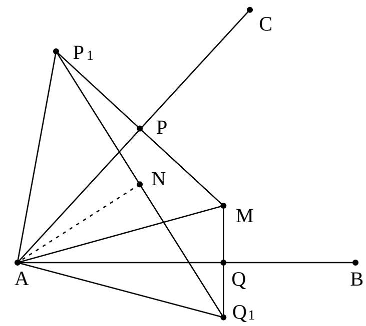


Figure 6-82

Example 6.83 (1.550, 5, 25) *If circles are constructed on two cevians as diameters, their radical axis passes through the orthocenter H of the triangle.*

Constructive description
((points $A\ B\ C\ P$)
(orthocenter $H\ A\ B\ C$)
(inter D (l $B\ C$) (l $A\ P$))
(inter E (l $A\ C$) (l $B\ P$))
(midpoint $M\ A\ D$)
(midpoint $N\ B\ E$)
(on-radical $H\ M\ A\ N\ B$))

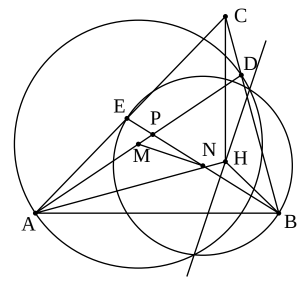


Figure 6-83

Example 6.84 (0.183, 3, 12) *In triangle ABC, let F the midpoint of the side BC, D and E the feet of the altitudes on AB and AC, respectively. FG is perpendicular to DE at G. Show that G is the midpoint of DE.*

Constructive description
((points $A\ B\ C$)
(foot $D\ C\ A\ B$)
(foot $E\ B\ A\ C$)
(midpoint $F\ B\ C$)
(midpoint $G\ D\ E$)
(perpendicular $G\ F\ D\ E$))

The eliminants

$P_{EDG} \overset{G}{=} \frac{1}{2}(P_{DED})\quad P_{EDF} \overset{F}{=} \frac{1}{2}(P_{CDE}+P_{BDE})$

$P_{BDE} \overset{E}{=} \frac{P_{ADB}\cdot P_{ACB}}{P_{ACA}}\quad P_{CDE} \overset{E}{=} \frac{P_{CDC}\cdot P_{BAC}}{P_{ACA}}$

$P_{DED} \overset{E}{=} \frac{P_{CDC}\cdot P_{BAC}-P_{BAC}\cdot P_{ACB}+P_{ADA}\cdot P_{ACB}}{P_{ACA}}$

$P_{ADB} \overset{D}{=} \frac{-P_{BAC}\cdot P_{ABC}}{P_{ABA}}\quad P_{ADA} \overset{D}{=} \frac{(P_{BAC})^2}{P_{ABA}}$

$P_{CDC} \overset{D}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{ABA}}$

$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$

$P_{ABC} = \frac{1}{2}(P_{BCB}-P_{ACA}+P_{ABA})$

$P_{ACB} = \frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$

$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$

The machine proof

$$\frac{P_{EDG}}{P_{EDF}} \overset{G}{=} \frac{\frac{1}{2}P_{DED}}{P_{EDF}} \overset{F}{=} \frac{P_{DED}}{(2)\cdot(\frac{1}{2}P_{CDE}+\frac{1}{2}P_{BDE})}$$

$$\overset{E}{=} \frac{(P_{CDC}\cdot P_{BAC}-P_{BAC}\cdot P_{ACB}+P_{ADA}\cdot P_{ACB})\cdot(P_{ACA})^2}{(P_{CDC}\cdot P_{BAC}\cdot P_{ACA}+P_{ADB}\cdot P_{ACB}\cdot P_{ACA})\cdot P_{ACA}}$$

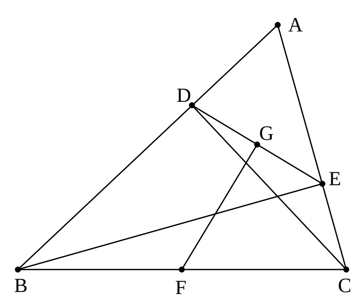


Figure 6-84

$$\overset{simplify}{=} \frac{P_{CDC}\cdot P_{BAC}-P_{BAC}\cdot P_{ACB}+P_{ADA}\cdot P_{ACB}}{P_{CDC}\cdot P_{BAC}+P_{ADB}\cdot P_{ACB}}$$

$$\overset{D}{=} \frac{(P_{BAC}^2\cdot P_{ACB}\cdot P_{ABA}-P_{BAC}\cdot P_{ACB}\cdot P_{ABA}^2+16P_{BAC}\cdot P_{ABA}\cdot S_{ABC}^2)\cdot(P_{ABA})^2}{(-P_{BAC}\cdot P_{ACB}\cdot P_{ABC}\cdot P_{ABA}+16P_{BAC}\cdot P_{ABA}\cdot S_{ABC}^2)\cdot(P_{ABA})^2}$$

$$\overset{simplify}{=} \frac{P_{BAC}\cdot P_{ACB}-P_{ACB}\cdot P_{ABA}+16S_{ABC}^2}{-(P_{ACB}\cdot P_{ABC}-16S_{ABC}^2)}$$

$$\overset{herron}{=} \frac{(32P_{BAC}\cdot P_{ACB}-16P_{ACB}\cdot P_{ABA}+16P_{ACA}\cdot P_{ABC})\cdot(16)}{(16P_{BAC}\cdot P_{ACB}-16P_{ACB}\cdot P_{ABC}+16P_{ACA}\cdot P_{ABC})\cdot(16)}$$

$$\overset{py}{=} \frac{((2))^4\cdot(-2P_{BCB}^2+2P_{BCB}\cdot P_{ACA}+2P_{BCB}\cdot P_{ABA})}{(-4P_{BCB}^2+4P_{BCB}\cdot P_{ACA}+4P_{BCB}\cdot P_{ABA})\cdot((2))^3} \overset{simplify}{=} 1$$

Example 6.85 (0.083, 3, 10) *Let B_1 and C_1 be the midpoints of AC and AB, D the foot from A to BC. Show that the triangle DB_1C_1 is congruent to the Euler triangle of triangle ABC.*

Constructive description

((points A B C)
(foot D A B C)
(foot F_H B A C)
(inter H (l A D) (l B F_H))
(midpoint P A H)
(midpoint Q B H)
(midpoint C_1 B A)
(eqdistance C_1 D P Q))

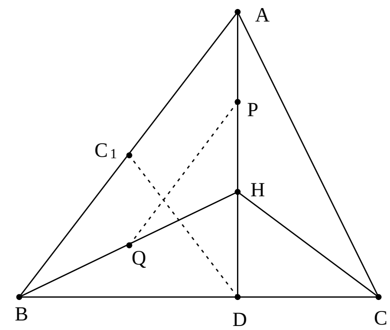


Figure 6-85

The machine proof

$$\frac{P_{DC_1D}}{P_{PQP}}$$

$$\overset{C_1}{=} \frac{\frac{1}{2}P_{BDB}+\frac{1}{2}P_{ADA}-\frac{1}{4}P_{ABA}}{P_{PQP}}$$

$$\overset{Q}{=} \frac{2P_{BDB}+2P_{ADA}-P_{ABA}}{(4)\cdot(\frac{1}{2}P_{HPH}+\frac{1}{2}P_{BPB}-\frac{1}{4}P_{BHB})}$$

$$\overset{P}{=} \frac{2P_{BDB}+2P_{ADA}-P_{ABA}}{P_{ABA}}$$

$$\overset{D}{=} \frac{-P_{BCB}^2\cdot P_{ABA}+2P_{BCB}\cdot P_{ABC}^2+32P_{BCB}\cdot S_{ABC}^2}{P_{ABA}\cdot(P_{BCB})^2}$$

$$\overset{simplify}{=} \frac{-(P_{BCB}\cdot P_{ABA}-2P_{ABC}^2-32S_{ABC}^2)}{P_{ABA}\cdot P_{BCB}}$$

$$\overset{herron}{=} \frac{-16P_{BCB}\cdot P_{ABA}+32P_{BAC}\cdot P_{ACB}+32P_{ACA}\cdot P_{ABC}+32P_{ABC}^2}{P_{BCB}\cdot P_{ABA}\cdot(16)}$$

$$\overset{py}{=} \frac{-(-16P_{BCB}\cdot P_{ABA})}{P_{BCB}\cdot P_{ABA}\cdot((2))^4} \overset{simplify}{=} 1$$

The eliminants

$$P_{DC_1D} \overset{C_1}{=} \frac{1}{4}\left(2P_{BDB}+2P_{ADA}-P_{ABA}\right)$$

$$P_{PQP} \overset{Q}{=} \frac{1}{4}\left(2P_{HPH}+2P_{BPB}-P_{BHB}\right)$$

$$P_{BPB} \overset{P}{=} \frac{1}{4}\left(2P_{BHB}-P_{AHA}+2P_{ABA}\right)$$

$$P_{HPH} \overset{P}{=} \frac{1}{4}\left(P_{AHA}\right)$$

$$P_{ADA} \overset{D}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{BCB}},\ P_{BDB} \overset{D}{=} \frac{(P_{ABC})^2}{P_{BCB}}$$

$$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$$

$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$

$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$

$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

6.3.3 The Circumcircle

Definition *The circle passing through the three vertices of a triangle is called the circumcircle of the triangle. The center of the circumcircle is called the circumcenter of the given triangle.*

Example 6.86 (0.033, 6, 8) *The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of the angle of the triangle at the vertex considered.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(eqangle B A D O A C))

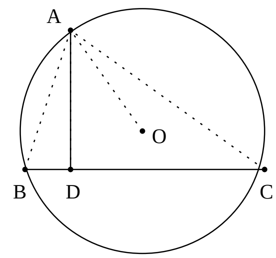


Figure 6-86

The machine proof

$$\frac{(-S_{ABD})\cdot P_{CAO}}{S_{ACO}\cdot P_{BAD}}$$

$$\overset{D}{=} \frac{(-P_{ABC}\cdot S_{ABC})\cdot P_{CAO}\cdot P_{BCB}}{S_{ACO}\cdot(P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})\cdot P_{BCB}}$$

$$\overset{simplify}{=} \frac{-P_{ABC}\cdot S_{ABC}\cdot P_{CAO}}{S_{ACO}\cdot(P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})}$$

$$\overset{O}{=} \frac{-P_{ABC}\cdot S_{ABC}\cdot P_{ACA}\cdot(-32S_{ABC})}{P_{ACA}\cdot P_{ABC}\cdot(P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})\cdot(2)}$$

$$\overset{simplify}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA}}$$

$$\overset{herron}{=} \frac{16P_{BAC}\cdot P_{ACB}+16P_{ACA}\cdot P_{ABC}}{(P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})\cdot(16)}$$

$$\overset{py}{=} \frac{(-2P_{BCB}^2+4P_{BCB}\cdot P_{ACA}+4P_{BCB}\cdot P_{ABA}-2P_{ACA}^2+4P_{ACA}\cdot P_{ABA}-2P_{ABA}^2)\cdot((2))^3}{(-2P_{BCB}^2+4P_{BCB}\cdot P_{ACA}+4P_{BCB}\cdot P_{ABA}-2P_{ACA}^2+4P_{ACA}\cdot P_{ABA}-2P_{ABA}^2)\cdot((2))^3}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{BAD}\overset{D}{=}\frac{P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA}}{P_{BCB}}$$
$$S_{ABD}\overset{D}{=}\frac{P_{ABC}\cdot S_{ABC}}{P_{BCB}}$$
$$S_{ACO}\overset{O}{=}\frac{P_{ACA}\cdot P_{ABC}}{(-32)\cdot S_{ABC}}$$
$$P_{CAO}\overset{O}{=}\frac{1}{2}\left(P_{ACA}\right)$$
$$16S_{ABC}^2=P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$$
$$P_{ABC}=\frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$
$$P_{ACB}=\frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$
$$P_{BAC}=-\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

Example 6.87 (0.001, 1, 2) *The product of two sides of a triangle is equal to the altitude to the third side multiplied by the circumdiameter.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot F C A B)
($\overline{AC}^2\overline{BC}^2 = 4\overline{OA}^2\cdot\overline{CF}^2$))

The eliminants

$$P_{CFC}\overset{F}{=}\frac{(16)\cdot(S_{ABC})^2}{P_{ABA}}$$
$$P_{AOA}\overset{O}{=}\frac{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}{(64)\cdot(S_{ABC})^2}$$

The machine proof

$$\frac{(\frac{1}{4})\cdot P_{BCB}\cdot P_{ACA}}{P_{CFC}\cdot P_{AOA}}$$

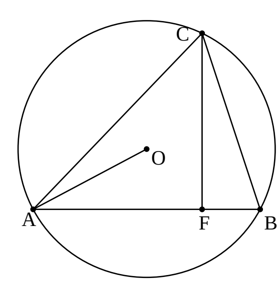


Figure 6-87

$$\overset{F}{=} \frac{(\frac{1}{4})\cdot P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}{(16S_{ABC}^{2})\cdot P_{AOA}}$$

$$\overset{O}{=} \frac{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}\cdot(64S_{ABC}^{2})}{(64)\cdot(S_{ABC})^{2}\cdot P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}$$

$$\overset{simplify}{=} 1$$

Example 6.88 (0.050, 5, 14) *Prove that the circumcenter of a triangle is the orthocenter of its medial triangle.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $A_1\ B\ C$)
(midpoint $B_1\ A\ C$)
(midpoint $C_1\ A\ B$)
(perpendicular $O\ A_1\ B_1\ C_1$))

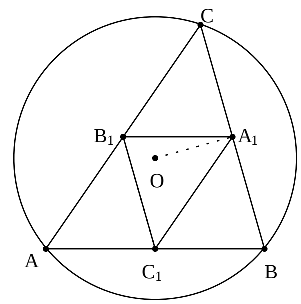


Figure 6-88

The machine proof

$$\frac{P_{OB_1C_1}}{P_{A_1B_1C_1}}$$

$$\overset{C_1}{=} \frac{\frac{1}{2}P_{BB_1O}+\frac{1}{2}P_{AB_1O}}{\frac{1}{2}P_{BB_1A_1}+\frac{1}{2}P_{AB_1A_1}}$$

$$\overset{B_1}{=} \frac{\frac{1}{2}P_{BCO}+\frac{1}{2}P_{BAO}+\frac{1}{2}P_{ACO}-\frac{1}{2}P_{ACA}}{\frac{1}{2}P_{BCA_1}+\frac{1}{2}P_{BAA_1}+\frac{1}{2}P_{ACA_1}-\frac{1}{2}P_{ACA}}$$

$$\overset{A_1}{=} \frac{P_{BCO}+P_{BAO}+P_{ACO}-P_{ACA}}{\frac{1}{2}P_{BCB}+\frac{1}{2}P_{BAC}+\frac{1}{2}P_{ACB}-P_{ACA}+\frac{1}{2}P_{ABA}}$$

$$\overset{O}{=} \frac{(2)\cdot(4P_{BCB}-4P_{ACA}+4P_{ABA})}{(P_{BCB}+P_{BAC}+P_{ACB}-2P_{ACA}+P_{ABA})\cdot((2))^{3}}$$

$$\overset{py}{=} \frac{(P_{BCB}-P_{ACA}+P_{ABA})\cdot((2))^{2}}{4P_{BCB}-4P_{ACA}+4P_{ABA}}$$

$$\overset{simplify}{=} 1$$

The eliminants

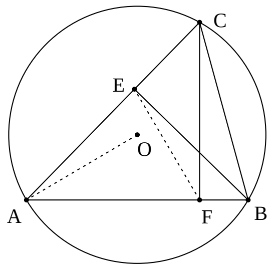
$P_{A_1B_1C_1} \overset{C_1}{=} \frac{1}{2}\left(P_{BB_1A_1}+P_{AB_1A_1}\right)$
$P_{OB_1C_1} \overset{C_1}{=} \frac{1}{2}\left(P_{BB_1O}+P_{AB_1O}\right)$
$P_{AB_1A_1} \overset{B_1}{=} \frac{1}{4}\left(2P_{ACA_1}-P_{ACA}\right)$
$P_{BB_1A_1} \overset{B_1}{=} \frac{1}{4}\left(2P_{BCA_1}+2P_{BAA_1}-P_{ACA}\right)$
$P_{AB_1O} \overset{B_1}{=} \frac{1}{4}\left(2P_{ACO}-P_{ACA}\right)$
$P_{BB_1O} \overset{B_1}{=} \frac{1}{4}\left(2P_{BCO}+2P_{BAO}-P_{ACA}\right)$
$P_{ACA_1} \overset{A_1}{=} \frac{1}{2}\left(P_{ACB}\right)$ $P_{BAA_1} \overset{A_1}{=} \frac{1}{2}\left(P_{BAC}+P_{ABA}\right)$
$P_{BCA_1} \overset{A_1}{=} \frac{1}{2}\left(P_{BCB}\right)$, $P_{ACO} \overset{O}{=} \frac{1}{2}\left(P_{ACA}\right)$
$P_{BAO} \overset{O}{=} \frac{1}{2}\left(P_{ABA}\right)$, $P_{BCO} \overset{O}{=} \frac{1}{2}\left(P_{BCB}\right)$
$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$
$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$

Example 6.89 (0.150, 4, 11) *The area of a triangle is equal to the product of its three sides divided by the double circumdiameter of the triangle.*

Constructive description
((points $A\ B\ C$) (midpoint $N\ A\ C$) (midpoint $L\ A\ B$)
(midpoint $M\ B\ C$) (inter O (t $L\ L\ B$) (t $M\ M\ B$)) ($\overline{AB}^2\overline{AC}^2\overline{CB}^2 = 16S_{ABC}^2\cdot\overline{OA}^2$))

Example 6.90 (0.016, 1, 4) *The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle.*

Constructive description:
((points $A\ B\ C$) (circumcenter $O\ A\ B\ C$)
(foot $F\ C\ A\ B$) (foot $E\ B\ A\ C$) (perpendicular $E\ F\ A\ O$))



The machine proof

$$\frac{P_{OAE}}{P_{OAF}} \stackrel{E}{=} \frac{P_{CAO}\cdot P_{BAC}}{P_{OAF}\cdot P_{ACA}}$$

$$\stackrel{F}{=} \frac{P_{CAO}\cdot P_{BAC}\cdot P_{ABA}}{P_{BAO}\cdot P_{BAC}\cdot P_{ACA}}$$

$$\stackrel{simplify}{=} \frac{P_{CAO}\cdot P_{ABA}}{P_{BAO}\cdot P_{ACA}}$$

$$\stackrel{O}{=} \frac{P_{ACA}\cdot P_{ABA}\cdot(2)}{P_{ABA}\cdot P_{ACA}\cdot(2)} \stackrel{simplify}{=} 1$$

The eliminants

$$P_{OAE} \stackrel{E}{=} \frac{P_{CAO}\cdot P_{BAC}}{P_{ACA}}$$

$$P_{OAF} \stackrel{F}{=} \frac{P_{BAO}\cdot P_{BAC}}{P_{ABA}}$$

$$P_{BAO} \stackrel{O}{=} \frac{1}{2}(P_{ABA})$$

$$P_{CAO} \stackrel{O}{=} \frac{1}{2}(P_{ACA})$$

Figure 6-90

Example 6.91 (0.016, 1, 3) *Let ABC be a triangle with $AC = AB$. D is a point on BC. Line AD meets the circumcircle of ABC at E. Show that $AB^2 = AD \cdot AE$.*

The eliminants: $P_{DAE} \stackrel{E}{=} (2)\cdot P_{OAD}$. $P_{OAD} \stackrel{D}{=} P_{BAO}$. $P_{BAO} \stackrel{O}{=} \frac{1}{2}(P_{BAB})$.

Constructive description
((points B M)
(on A (t M M B))
(inter O (l A M) (b A B))
(lratio C M B -1)
(on D (l B M))
(inter E (l A D) (cir O A))
(eq-product A B A B A D A E))

The machine proof

$$\frac{P_{BAB}}{P_{DAE}} \stackrel{E}{=} \frac{P_{BAB}\cdot P_{ADA}}{2P_{OAD}\cdot P_{ADA}}$$

$$\stackrel{simplify}{=} \frac{P_{BAB}}{(2)\cdot P_{OAD}}$$

$$\stackrel{D}{=} \frac{P_{BAB}}{(2)\cdot P_{BAO}}$$

$$\stackrel{O}{=} \frac{P_{BAB}\cdot(2)}{(2)\cdot P_{BAB}} \stackrel{simplify}{=} 1$$

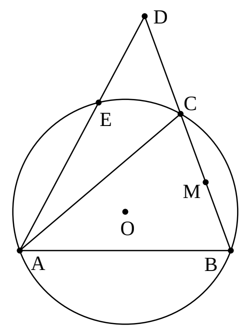


Figure 6-91

Example 6.92 (0.001, 1, 3) *Let C be the midpoint of the arc AB of circle (O). D is a point on the circle. $E = AB \cap CD$. Show that $CA^2 = CE \cdot CD$.*

The eliminants: $P_{ECD} \stackrel{D}{=} (2)\cdot P_{OCE}$. $P_{OCE} \stackrel{E}{=} P_{ACO}$. $P_{ACO} \stackrel{O}{=} \frac{1}{2}(P_{ACA})$.

Constructive description
((points A M)
(on C (t M M A))
(inter O (l C M) (b A C))
(on E (l A M))
(inter D (l C E) (cir O C))
(eq-product C A C A C E C D))

The machine proof

$$\frac{P_{ACA}}{P_{ECD}} \stackrel{D}{=} \frac{P_{ACA}\cdot P_{CEC}}{2P_{OCE}\cdot P_{CEC}}$$

$$\stackrel{simplify}{=} \frac{P_{ACA}}{(2)\cdot P_{OCE}}$$

$$\stackrel{E}{=} \frac{P_{ACA}}{(2)\cdot P_{ACO}}$$

$$\stackrel{O}{=} \frac{P_{ACA}\cdot(2)}{(2)\cdot P_{ACA}} \stackrel{simplify}{=} 1$$

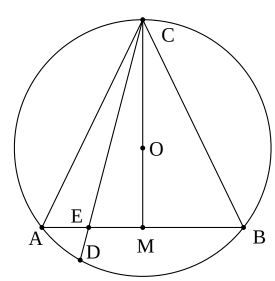


Figure 6-92

Example 6.93 (0.050, 3, 7) *The distance of a side of a triangle from the circumcenter is equal to half the distance of the opposite vertex from the orthocenter.*

Constructive description

((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(foot $D\ A\ B\ C$)
(inter A_1 (l $B\ C$)
(p $O\ A\ D$))
($\frac{\overline{AH}}{\overline{OA_1}}$ = 2))

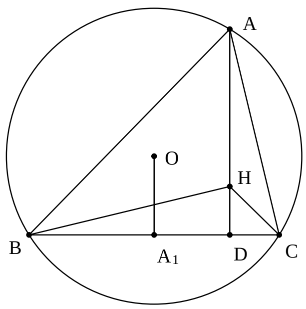


Figure 6-93

The machine proof

$\frac{1}{2}\left(\frac{\overline{AH}}{\overline{OA_1}}\right)$

$\stackrel{A_1}{=} \frac{S_{ABHC}}{(2)\cdot S_{BCO}}$

$\stackrel{H}{=} \frac{-P_{ACB}\cdot P_{ABC}\cdot S_{ABC}+16S_{ABC}^3}{(2)\cdot S_{BCO}\cdot(16S_{ABC}^2)}$

$\stackrel{simplify}{=} \frac{-(P_{ACB}\cdot P_{ABC}-16S_{ABC}^2)}{(32)\cdot S_{BCO}\cdot S_{ABC}}$

$\stackrel{O}{=} \frac{-(P_{ACB}\cdot P_{ABC}-16S_{ABC}^2)\cdot(32S_{ABC})}{(32)\cdot P_{BCB}\cdot P_{BAC}\cdot S_{ABC}}$

$\stackrel{simplify}{=} \frac{-(P_{ACB}\cdot P_{ABC}-16S_{ABC}^2)}{P_{BCB}\cdot P_{BAC}}$

$\stackrel{herron}{=} \frac{16P_{BAC}\cdot P_{ACB}-16P_{ACB}\cdot P_{ABC}+16P_{ACA}\cdot P_{ABC}}{P_{BCB}\cdot P_{BAC}\cdot(16)}$

$\stackrel{py}{=} \frac{(-4P_{BCB}^2+4P_{BCB}\cdot P_{ACA}+4P_{BCB}\cdot P_{ABA})\cdot(2)}{P_{BCB}\cdot(-P_{BCB}+P_{ACA}+P_{ABA})\cdot((2))^3}$

$\stackrel{simplify}{=} 1$

The eliminants

$\frac{\overline{AH}}{\overline{OA_1}} \stackrel{A_1}{=} \frac{S_{ABHC}}{S_{BCO}}$

$S_{ABHC} \stackrel{H}{=} \frac{P_{ACB}\cdot P_{ABC}-16S_{ABC}^2}{(-16)\cdot S_{ABC}}$

$S_{BCO} \stackrel{O}{=} \frac{P_{BCB}\cdot P_{BAC}}{(32)\cdot S_{ABC}}$

$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$

$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$

$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$

$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$

Example 6.94 (0.067, 3, 9) *The ratio of a side of a triangle to the corresponding side of the orthic triangle is equal to the ratio of the circumradius to the distance of the side considered from the circumcenter.*

Constructive description

((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $Q\ O\ A\ B$)
(foot $E\ B\ A\ C$)
(foot $D\ A\ B\ C$)
(eq-product $A\ B\ O\ Q\ O\ A\ E\ D$))

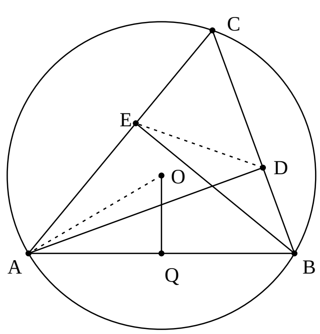


Figure 6-94

The eliminants

$P_{EDE} \stackrel{D}{=} \frac{P_{CEC}\cdot P_{ABC}+P_{BEB}\cdot P_{ACB}-P_{ACB}\cdot P_{ABC}}{P_{BCB}}$

$P_{BEB} \stackrel{E}{=} \frac{(16)\cdot(S_{ABC})^2}{P_{ACA}}$

$P_{CEC} \stackrel{E}{=} \frac{(P_{ACB})^2}{P_{ACA}}$

$P_{OQO} \stackrel{Q}{=} \frac{(16)\cdot(S_{ABO})^2}{P_{ABA}}$

$P_{AOA} \stackrel{O}{=} \frac{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}{(64)\cdot(S_{ABC})^2}$

$S_{ABO} \stackrel{O}{=} \frac{P_{ACB}\cdot P_{ABA}}{(32)\cdot S_{ABC}}$

$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$

$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$

$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$

The machine proof

$$\frac{P_{ABA}\cdot P_{OQO}}{P_{AOA}\cdot P_{EDE}}$$

$$\stackrel{D}{=} \frac{P_{ABA}\cdot P_{OQO}\cdot P_{BCB}}{P_{AOA}\cdot(P_{CEC}\cdot P_{ABC}+P_{BEB}\cdot P_{ACB}-P_{ACB}\cdot P_{ABC})}$$

$$\stackrel{E}{=} \frac{P_{ABA}\cdot P_{OQO}\cdot P_{BCB}\cdot(P_{ACA})^2}{P_{AOA}\cdot(P_{ACB}^2\cdot P_{ACA}\cdot P_{ABC}-P_{ACB}\cdot P_{ACA}^2\cdot P_{ABC}+16P_{ACB}\cdot P_{ACA}\cdot S_{ABC}^2)}$$

$$\stackrel{simplify}{=} \frac{P_{ABA}\cdot P_{OQO}\cdot P_{BCB}\cdot P_{ACA}}{P_{AOA}\cdot(P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC}+16S_{ABC}^2)\cdot P_{ACB}}$$

$$\stackrel{Q}{=} \frac{P_{ABA}\cdot(16S_{ABO}^2)\cdot P_{BCB}\cdot P_{ACA}}{P_{AOA}\cdot(P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC}+16S_{ABC}^2)\cdot P_{ACB}\cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{(16)\cdot(S_{ABO})^2\cdot P_{BCB}\cdot P_{ACA}}{P_{AOA}\cdot(P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC}+16S_{ABC}^2)\cdot P_{ACB}}$$

$$\stackrel{O}{=} \frac{(16)\cdot(P_{ACB}\cdot P_{ABA})^2\cdot P_{BCB}\cdot P_{ACA}\cdot(64S_{ABC}^2)}{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}\cdot(P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC}+16S_{ABC}^2)\cdot P_{ACB}\cdot((32S_{ABC}))^2}$$

$$\stackrel{simplify}{=} \frac{P_{ACB}\cdot P_{ABA}}{P_{ACB}\cdot P_{ABC}-P_{ACA}\cdot P_{ABC}+16S_{ABC}^2}$$

$$\stackrel{herron}{=} \frac{P_{ACB}\cdot P_{ABA}\cdot(16)}{16P_{BAC}\cdot P_{ACB}+16P_{ACB}\cdot P_{ABC}}$$

$$\stackrel{simplify}{=} \frac{P_{ABA}}{P_{BAC}+P_{ABC}}$$

$$\stackrel{py}{=} \frac{P_{ABA}\cdot((2))^2}{4P_{ABA}} \stackrel{simplify}{=} 1$$

Example 6.95 (0.083, 2, 6) $\overline{AH}^2+\overline{BC}^2=4\overline{OA}^2$.

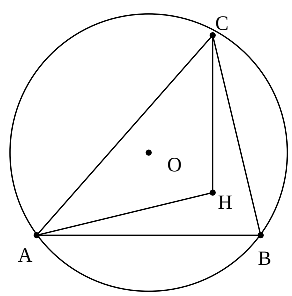


Figure 6-95

Constructive description

((points A B C)
(orthocenter H A B C)
(circumcenter O A B C)
($\overline{AH}^2+\overline{BC}^2 = 4\overline{AO}^2$))

The machine proof

$$\frac{P_{BCB}+P_{AHA}}{(4)\cdot P_{AOA}}$$

$$\stackrel{O}{=} \frac{(P_{BCB}+P_{AHA})\cdot(64S_{ABC}^2)}{(4)\cdot P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}$$

$$\stackrel{H}{=} \frac{(16)\cdot(P_{BCB}\cdot P_{BAC}^2+16P_{BCB}\cdot S_{ABC}^2)\cdot(S_{ABC})^2}{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}\cdot(16S_{ABC}^2)}$$

$$\stackrel{simplify}{=} \frac{P_{BAC}^2+16S_{ABC}^2}{P_{ACA}\cdot P_{ABA}}$$

$$\stackrel{herron}{=} \frac{16P_{BAC}^2+16P_{BAC}\cdot P_{ACB}+16P_{ACA}\cdot P_{ABC}}{P_{ACA}\cdot P_{ABA}\cdot(16)}$$

$$\stackrel{py}{=} \frac{16P_{ACA}\cdot P_{ABA}}{P_{ACA}\cdot P_{ABA}\cdot((2))^4}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{AOA} \stackrel{O}{=} \frac{P_{BCB}\cdot P_{ACA}\cdot P_{ABA}}{(64)\cdot(S_{ABC})^2}$$

$$P_{AHA} \stackrel{H}{=} \frac{P_{BCB}\cdot(P_{BAC})^2}{(16)\cdot(S_{ABC})^2}$$

$$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$$

$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$

$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$

$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

Example 6.96 (0.116, 4, 7) *The circumdiameters AP, BQ, CR of a triangle ABC meet the sides BC, CA, AB in the points K, L, M. Show that* $(KP/AK)+(LQ/BL)+(MR/CM)=1$.

Constructive description
((points A B C) (circumcenter O A B C) (lratio P O A -1) (lratio Q O B -1) (lratio R O C -1) (inter K (l B C) (l A O)) (inter L (l C A) (l B O)) (inter M (l A B) (l C O)) ($\frac{\overline{KP}}{\overline{AK}}+\frac{\overline{LQ}}{\overline{BL}}+\frac{\overline{MR}}{\overline{CM}}=1$))

The machine proof

$$-\left(\frac{\overline{RM}}{\overline{CM}}+\frac{\overline{QL}}{\overline{BL}}+\frac{\overline{PK}}{\overline{AK}}\right)$$

$$\overset{M}{=} \frac{-(\frac{\overline{QL}}{\overline{BL}}\cdot S_{ABC}+\frac{\overline{PK}}{\overline{AK}}\cdot S_{ABC}+S_{ABR})}{S_{ABC}}$$

$$\overset{L}{=} \frac{-(\frac{\overline{PK}}{\overline{AK}}\cdot S_{ABC}^2-S_{ACQ}\cdot S_{ABC}+S_{ABR}\cdot S_{ABC})}{(S_{ABC})^2}$$

$$\overset{simplify}{=} \frac{-(\frac{\overline{PK}}{\overline{AK}}\cdot S_{ABC}-S_{ACQ}+S_{ABR})}{S_{ABC}}$$

$$\overset{K}{=} \frac{-(S_{BCP}\cdot S_{ABC}-S_{ACQ}\cdot S_{ABC}+S_{ABR}\cdot S_{ABC})}{(S_{ABC})^2}$$

$$\overset{simplify}{=} \frac{-(S_{BCP}-S_{ACQ}+S_{ABR})}{S_{ABC}}$$

$$\overset{R}{=} \frac{-(S_{BCP}-S_{ACQ}+2S_{ABO}-S_{ABC})}{S_{ABC}}$$

$$\overset{Q}{=} \frac{-(S_{BCP}-2S_{ACO}+2S_{ABO}-2S_{ABC})}{S_{ABC}}$$

$$\overset{P}{=} \frac{-(2S_{BCO}-2S_{ACO}+2S_{ABO}-3S_{ABC})}{S_{ABC}}$$

$$\overset{O}{=} \frac{(-2)\cdot(-\frac{1}{2}S_{ABC})}{S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{RM}}{\overline{CM}} \overset{M}{=} \frac{S_{ABR}}{S_{ABC}}$$

$$\frac{\overline{QL}}{\overline{BL}} \overset{L}{=} \frac{-S_{ACQ}}{S_{ABC}}$$

$$\frac{\overline{PK}}{\overline{AK}} \overset{K}{=} \frac{S_{BCP}}{S_{ABC}}$$

$$S_{ABR} \overset{R}{=} 2S_{ABO}-S_{ABC}$$

$$S_{ACQ} \overset{Q}{=} 2S_{ACO}+S_{ABC}$$

$$S_{BCP} \overset{P}{=} 2S_{BCO}-S_{ABC}$$

$$S_{BCO}-S_{ACO}+S_{ABO}-\frac{3}{2}S_{ABC}=-\frac{1}{2}S_{ABC}$$

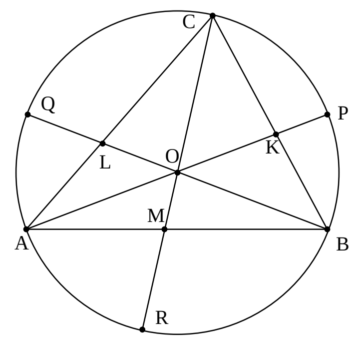


Figure 6-96

Example 6.97 (0.066, 2, 8) *The mediators of the sides AC, AB of the triangle ABC meet the sides AB, AC in P and Q. Prove that the points B, C, P, Q lie on a circle.*

Constructive description
((points A B C)
(inter P (l A C) (b A B))
(inter Q (l A B) (b A C))
(cocircle B C P Q))

The machine proof

$$\frac{(-S_{BPQ})\cdot P_{PCQ}}{(-S_{CPQ})\cdot P_{PBQ}}$$

$$\overset{Q}{=} \frac{(P_{BAC}\cdot S_{ABP}-\frac{1}{2}P_{ACA}\cdot S_{ABP})\cdot P_{ACP}\cdot((-P_{BAC}))^2}{(\frac{1}{2}P_{ACA}\cdot S_{BCP})\cdot(-P_{BAC}\cdot P_{ABP}+\frac{1}{2}P_{ACA}\cdot P_{ABP})\cdot(2)\cdot(-P_{BAC})}$$

$$\overset{simplify}{=} \frac{S_{ABP}\cdot P_{ACP}\cdot P_{BAC}}{P_{ACA}\cdot S_{BCP}\cdot P_{ABP}}$$

$$\overset{P}{=} \frac{(-\frac{1}{2}P_{ABA}\cdot S_{ABC})\cdot(-P_{BAC}\cdot P_{ACA}+\frac{1}{2}P_{ACA}\cdot P_{ABA})\cdot P_{BAC}\cdot(2)\cdot(-P_{BAC})}{P_{ACA}\cdot(-P_{BAC}\cdot S_{ABC}+\frac{1}{2}P_{ABA}\cdot S_{ABC})\cdot P_{ABA}\cdot((-P_{BAC}))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$P_{PBQ} \overset{Q}{=} \frac{(2P_{BAC}-P_{ACA})\cdot P_{ABP}}{(2)\cdot P_{BAC}}$

$S_{CPQ} \overset{Q}{=} \frac{P_{ACA}\cdot S_{BCP}}{(2)\cdot P_{BAC}}$

$P_{PCQ} \overset{Q}{=} \frac{1}{2}(P_{ACP})$

$S_{BPQ} \overset{Q}{=} \frac{(2P_{BAC}-P_{ACA})\cdot S_{ABP}}{(2)\cdot P_{BAC}}$

$P_{ABP} \overset{P}{=} \frac{1}{2}(P_{ABA})$, $S_{BCP} \overset{P}{=} \frac{(2P_{BAC}-P_{ABA})\cdot S_{ABC}}{(2)\cdot P_{BAC}}$

$P_{ACP} \overset{P}{=} \frac{(2P_{BAC}-P_{ABA})\cdot P_{ACA}}{(2)\cdot P_{BAC}}$, $S_{ABP} \overset{P}{=} \frac{P_{ABA}\cdot S_{ABC}}{(2)\cdot P_{BAC}}$

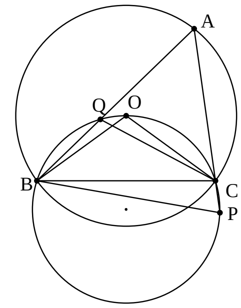


Figure 6-97

Example 6.98 (0.083, 4, 13) *The two tangents to the circumcircle of ABC at A and C meet at E. The mediator of BC meet AB at D. Show that DE || BC.*

Constructive description

((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $H\ B\ C$)
(inter D (l $A\ B$) (l $O\ H$))
(inter E (t $C\ C\ O$) (t $A\ A\ O$))
(parallel $D\ E\ B\ C$))

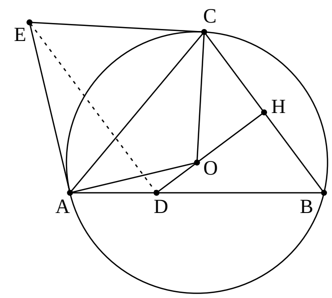


Figure 6-98

The machine proof

$$\frac{S_{BCD}}{S_{BCE}}$$

$$\overset{E}{=} \frac{S_{BCD}\cdot(-4S_{ACO})}{\frac{1}{4}P_{CAO}\cdot P_{BCO}}$$

$$\overset{D}{=} \frac{(-16)\cdot(-S_{BOH}\cdot S_{ABC})\cdot S_{ACO}}{P_{CAO}\cdot P_{BCO}\cdot S_{AOBH}}$$

$$\overset{H}{=} \frac{(16)\cdot(-\frac{1}{2}S_{BCO})\cdot S_{ABC}\cdot S_{ACO}}{P_{CAO}\cdot P_{BCO}\cdot(-S_{ABO}+\frac{1}{2}S_{ABC})}$$

$$\overset{O}{=} \frac{(16)\cdot P_{BCB}\cdot P_{BAC}\cdot S_{ABC}\cdot P_{ACA}\cdot P_{ABC}\cdot(32S_{ABC})\cdot((2))^2}{P_{ACA}\cdot P_{BCB}\cdot(2P_{ACB}\cdot P_{ABA}-32S_{ABC}^2)\cdot(-32S_{ABC})\cdot(32S_{ABC})}$$

$$\overset{simplify}{=} \frac{-P_{BAC}\cdot P_{ABC}}{P_{ACB}\cdot P_{ABA}-16S_{ABC}^2}$$

$$\overset{herron}{=} \frac{(-P_{BAC}\cdot P_{ABC})\cdot(16)}{-16P_{BAC}\cdot P_{ACB}+16P_{ACB}\cdot P_{ABA}-16P_{ACA}\cdot P_{ABC}}$$

$$\overset{py}{=} \frac{(-P_{BCB}+P_{ACA}+P_{ABA})\cdot(P_{BCB}-P_{ACA}+P_{ABA})\cdot((2))^3}{(-2P_{BCB}^2+4P_{BCB}\cdot P_{ACA}-2P_{ACA}^2+2P_{ABA}^2)\cdot((2))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$S_{BCE} \overset{E}{=} \frac{P_{CAO}\cdot P_{BCO}}{(-16)\cdot S_{ACO}}$

$S_{BCD} \overset{D}{=} \frac{-S_{BOH}\cdot S_{ABC}}{S_{AOBH}}$

$S_{AOBH} \overset{H}{=} -\frac{1}{2}(2S_{ABO}-S_{ABC})$

$S_{BOH} \overset{H}{=} -\frac{1}{2}(S_{BCO})$

$S_{ABO} \overset{O}{=} \frac{P_{ACB}\cdot P_{ABA}}{(32)\cdot S_{ABC}}$

$P_{BCO} \overset{O}{=} \frac{1}{2}(P_{BCB})$

$P_{CAO} \overset{O}{=} \frac{1}{2}(P_{ACA})$

$S_{ACO} \overset{O}{=} \frac{P_{ACA}\cdot P_{ABC}}{(-32)\cdot S_{ABC}}$

$S_{BCO} \overset{O}{=} \frac{P_{BCB}\cdot P_{BAC}}{(32)\cdot S_{ABC}}$

$16S_{ABC}^2 = P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}$

$P_{ACB} = \frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$

$P_{ABC} = \frac{1}{2}(P_{BCB}-P_{ACA}+P_{ABA})$

$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$

Example 6.99 (0.016, 1, 12) *The lines tangent to the circumcircle of a triangle at the vertices meet opposite sides in three collinear points. (The Lemoine axis of the given triangle.)*

Constructive description

((points A B C)
(circumcenter O A B C)
(inter A_1 (l B C) (t A A O))
(inter B_1 (l A C) (t B B O))
(inter C_1 (l A B) (t C C O))
(inter Z_{C_1} (l B_1 A_1) (l A B))
($\frac{\overline{AC_1}}{\overline{BC_1}} = \frac{\overline{AZ_{C_1}}}{\overline{BZ_{C_1}}}$))

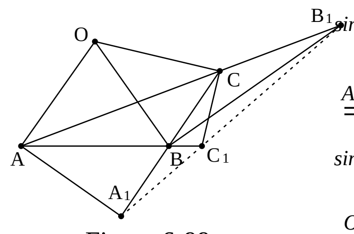


Figure 6-99

The machine proof

$$\left(\frac{\overline{AC_1}}{\overline{BC_1}}\right)/\left(\frac{\overline{AZ_{C_1}}}{\overline{BZ_{C_1}}}\right)$$

$$\overset{Z_{C_1}}{=} \frac{-S_{BA_1B_1}}{-S_{AA_1B_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}}$$

$$\overset{C_1}{=} \frac{P_{ACO} \cdot S_{BA_1B_1}}{S_{AA_1B_1} \cdot P_{BCO}}$$

$$\overset{B_1}{=} \frac{P_{ACO} \cdot (-P_{CBO} \cdot S_{ABA_1}) \cdot (-P_{CBAO})}{(-P_{ABO} \cdot S_{ACA_1}) \cdot P_{BCO} \cdot (-P_{CBAO})}$$

$$\overset{simplify}{=} \frac{P_{ACO} \cdot P_{CBO} \cdot S_{ABA_1}}{P_{ABO} \cdot S_{ACA_1} \cdot P_{BCO}}$$

$$\overset{A_1}{=} \frac{P_{ACO} \cdot P_{CBO} \cdot P_{BAO} \cdot S_{ABC} \cdot (-P_{CABO})}{P_{ABO} \cdot P_{CAO} \cdot S_{ABC} \cdot P_{BCO} \cdot (-P_{CABO})}$$

$$\overset{simplify}{=} \frac{P_{ACO} \cdot P_{CBO} \cdot P_{BAO}}{P_{ABO} \cdot P_{CAO} \cdot P_{BCO}}$$

$$\overset{O}{=} \frac{P_{ACA} \cdot P_{BCB} \cdot P_{ABA} \cdot ((2))^3}{P_{ABA} \cdot P_{ACA} \cdot P_{BCB} \cdot ((2))^3}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_{C_1}}}{\overline{BZ_{C_1}}} \overset{Z_{C_1}}{=} \frac{S_{AA_1B_1}}{S_{BA_1B_1}}$$

$$\frac{\overline{AC_1}}{\overline{BC_1}} \overset{C_1}{=} \frac{P_{ACO}}{P_{BCO}}$$

$$S_{AA_1B_1} \overset{B_1}{=} \frac{P_{ABO} \cdot S_{ACA_1}}{P_{CBAO}}$$

$$S_{BA_1B_1} \overset{B_1}{=} \frac{P_{CBO} \cdot S_{ABA_1}}{P_{CBAO}}$$

$$S_{ACA_1} \overset{A_1}{=} \frac{P_{CAO} \cdot S_{ABC}}{-P_{CABO}}$$

$$S_{ABA_1} \overset{A_1}{=} \frac{P_{BAO} \cdot S_{ABC}}{-P_{CABO}}$$

$$P_{BCO} \overset{O}{=} \frac{1}{2}(P_{BCB})$$

$$P_{CAO} \overset{O}{=} \frac{1}{2}(P_{ACA})$$

$$P_{ABO} \overset{O}{=} \frac{1}{2}(P_{ABA})$$

$$P_{BAO} \overset{O}{=} \frac{1}{2}(P_{ABA})$$

$$P_{CBO} \overset{O}{=} \frac{1}{2}(P_{BCB})$$

$$P_{ACO} \overset{O}{=} \frac{1}{2}(P_{ACA})$$

Definition. *The symmetric of a median of a triangle with respect to the internal bisector issued from the same vertex is called a symmedian.*

Example 6.100 (0.050, 1, 6) *The distances from a point on a symmedian of a triangle to the two including sides are proportional to these sides.*

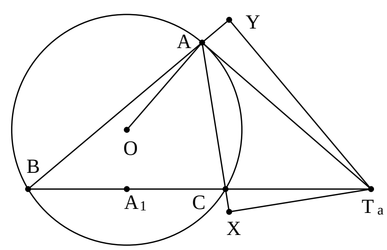


Figure 6-100

Constructive description

((points A B C)
(circumcenter O A B C)
(midpoint A_1 B C)
(inter T_a (l B C) (t A A O))
(foot X T_a A C)
(foot Y T_a A B)
(eq-product T_a Y A C T_a X A B))

The machine proof

$$\frac{P_{T_aYT_a} \cdot P_{ACA}}{P_{T_aXT_a} \cdot P_{ABA}}$$

$$\overset{Y}{=} \frac{(16S_{ABT_a}^2) \cdot P_{ACA}}{P_{T_aXT_a} \cdot (P_{ABA})^2}$$

$$\overset{X}{=} \frac{(16) \cdot (S_{ABT_a})^2 \cdot (P_{ACA})^2}{(16S_{ACT_a}^2) \cdot (P_{ABA})^2}$$

$$\overset{T_a}{=} \frac{(P_{BAO} \cdot S_{ABC})^2 \cdot (P_{ACA})^2 \cdot ((-P_{CABO}))^2}{(P_{CAO} \cdot S_{ABC})^2 \cdot (P_{ABA})^2 \cdot ((-P_{CABO}))^2}$$

$$\overset{simplify}{=} \frac{(P_{BAO})^2 \cdot (P_{ACA})^2}{(P_{CAO})^2 \cdot (P_{ABA})^2}$$

$$\overset{O}{=} \frac{(P_{ABA})^2 \cdot (P_{ACA})^2 \cdot ((2))^2}{(P_{ACA})^2 \cdot (P_{ABA})^2 \cdot ((2))^2} \overset{simplify}{=} 1$$

The eliminants

$$P_{T_aYT_a} \overset{Y}{=} \frac{(16) \cdot (S_{ABT_a})^2}{P_{ABA}}$$

$$P_{T_aXT_a} \overset{X}{=} \frac{(16) \cdot (S_{ACT_a})^2}{P_{ACA}}$$

$$S_{ACT_a} \overset{T_a}{=} \frac{P_{CAO} \cdot S_{ABC}}{-P_{CABO}}$$

$$S_{ABT_a} \overset{T_a}{=} \frac{P_{BAO} \cdot S_{ABC}}{-P_{CABO}}$$

$$P_{CAO} \overset{O}{=} \frac{1}{2}(P_{ACA})$$

$$P_{BAO} \overset{O}{=} \frac{1}{2}(P_{ABA})$$

Example 6.101 (0.033, 1, 8) *Show that the distances of the vertices of a triangle from the Lemoine axis are proportional to the squares of the respective altitudes.*

Constructive description
((points A B C) (circumcenter O A B C)
(foot E B A C) (foot D A B C)
(inter B_1 (l A C) (t B B O)) (inter C_1 (l A B) (t C C O))
(foot K A B_1 C_1) (foot J B B_1 C_1)
($\overline{AK}^2\overline{BE}^2\overline{BE}^2 = \overline{BJ}^2\overline{AD}^2\overline{AD}^2$))

The machine proof

$$\frac{(P_{BEB})^2 \cdot P_{AKA}}{P_{BJB} \cdot (P_{ADA})^2}$$

$$\overset{J}{=} \frac{(P_{BEB})^2 \cdot P_{AKA} \cdot P_{B_1C_1B_1}}{(16S^2_{BB_1C_1}) \cdot (P_{ADA})^2}$$

$$\overset{K}{=} \frac{(P_{BEB})^2 \cdot (16S^2_{AB_1C_1}) \cdot P_{B_1C_1B_1}}{(16) \cdot (S_{BB_1C_1})^2 \cdot (P_{ADA})^2 \cdot P_{B_1C_1B_1}}$$

$$\overset{simplify}{=} \frac{(P_{BEB})^2 \cdot (S_{AB_1C_1})^2}{(S_{BB_1C_1})^2 \cdot (P_{ADA})^2}$$

$$\overset{C_1}{=} \frac{(P_{BEB})^2 \cdot ((-P_{ACO} \cdot S_{ABB_1}))^2 \cdot ((-P_{BCAO}))^2}{((-P_{BCO} \cdot S_{ABB_1}))^2 \cdot (P_{ADA})^2 \cdot ((-P_{BCAO}))^2}$$

$$\overset{simplify}{=} \frac{(P_{BEB})^2 \cdot (P_{ACO})^2}{(P_{BCO})^2 \cdot (P_{ADA})^2}$$

$$\overset{D}{=} \frac{(P_{BEB})^2 \cdot (P_{ACO})^2 \cdot (P_{BCB})^2}{(P_{BCO})^2 \cdot ((16S^2_{ABC}))^2}$$

$$\overset{E}{=} \frac{((16S^2_{ABC}))^2 \cdot (P_{ACO})^2 \cdot (P_{BCB})^2}{(256) \cdot (P_{BCO})^2 \cdot (S_{ABC})^4 \cdot (P_{ACA})^2}$$

$$\overset{simplify}{=} \frac{(P_{ACO})^2 \cdot (P_{BCB})^2}{(P_{BCO})^2 \cdot (P_{ACA})^2}$$

$$\overset{O}{=} \frac{(P_{ACA})^2 \cdot (P_{BCB})^2 \cdot ((2))^2}{(P_{BCB})^2 \cdot (P_{ACA})^2 \cdot ((2))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{BJB} \overset{J}{=} \frac{(16) \cdot (S_{BB_1C_1})^2}{P_{B_1C_1B_1}}$$

$$P_{AKA} \overset{K}{=} \frac{(16) \cdot (S_{AB_1C_1})^2}{P_{B_1C_1B_1}}$$

$$S_{BB_1C_1} \overset{C_1}{=} \frac{P_{BCO} \cdot S_{ABB_1}}{P_{BCAO}}$$

$$S_{AB_1C_1} \overset{C_1}{=} \frac{P_{ACO} \cdot S_{ABB_1}}{P_{BCAO}}$$

$$P_{ADA} \overset{D}{=} \frac{(16) \cdot (S_{ABC})^2}{P_{BCB}}$$

$$P_{BEB} \overset{E}{=} \frac{(16) \cdot (S_{ABC})^2}{P_{ACA}}$$

$$P_{BCO} \overset{O}{=} \frac{1}{2}\left(P_{BCB}\right)$$

$$P_{ACO} \overset{O}{=} \frac{1}{2}\left(P_{ACA}\right)$$

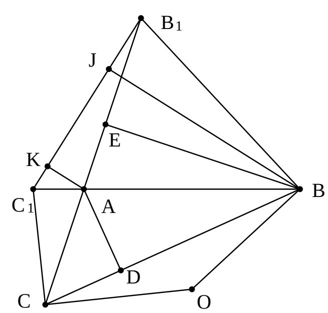


Figure 6-101

Definition *The tangents to the circumcircle at the vertices of a given triangle form a triangle called the tangential triangle of the given triangle.*

Example 6.102 (0.067, 3, 10) *Show that the vertices of the tangential triangle of ABC are the isogonal conjugates of the vertices of the anticomplementary triangle of ABC.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(pratio $A_1\ B\ A\ C\ 1$)
(inter A_2 (t $B\ B\ O$) (t $C\ C\ O$))
(eqangle $B\ A\ A_2\ A_1\ A\ C$))

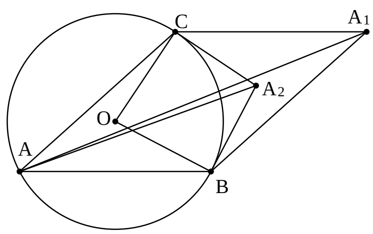


Figure 6-102

The eliminants

$$P_{BAA_2} \overset{A_2}{=} \frac{P_{BCO}\cdot S_{ABO}+P_{ABA}\cdot S_{BCO}}{S_{BCO}}$$
$$S_{ABA_2} \overset{A_2}{=} \frac{P_{BCO}\cdot P_{ABO}}{(16)\cdot S_{BCO}}$$
$$S_{ACA_1} \overset{A_1}{=} -(S_{ABC})$$
$$P_{CAA_1} \overset{A_1}{=} P_{BAC}+P_{ACA}$$
$$S_{BCO} \overset{O}{=} \frac{P_{BCB}\cdot P_{BAC}}{(32)\cdot S_{ABC}}$$
$$S_{ABO} \overset{O}{=} \frac{P_{ACB}\cdot P_{ABA}}{(32)\cdot S_{ABC}}$$
$$P_{ABO} \overset{O}{=} \frac{1}{2}(P_{ABA})$$
$$P_{BCO} \overset{O}{=} \frac{1}{2}(P_{BCB})$$
$$P_{ACB} = \frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$$
$$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$$

The machine proof

$$\frac{(-S_{ABA_2})\cdot P_{CAA_1}}{S_{ACA_1}\cdot P_{BAA_2}}$$
$$\overset{A_2}{=} \frac{(-\frac{1}{4}P_{BCO}\cdot P_{ABO})\cdot P_{CAA_1}\cdot(4S_{BCO})}{S_{ACA_1}\cdot(4P_{BCO}\cdot S_{ABO}+4P_{ABA}\cdot S_{BCO})\cdot(4S_{BCO})}$$
$$\overset{simplify}{=} \frac{-P_{BCO}\cdot P_{ABO}\cdot P_{CAA_1}}{(16)\cdot S_{ACA_1}\cdot(P_{BCO}\cdot S_{ABO}+P_{ABA}\cdot S_{BCO})}$$
$$\overset{A_1}{=} \frac{-P_{BCO}\cdot P_{ABO}\cdot(P_{BAC}+P_{ACA})}{(16)\cdot(-S_{ABC})\cdot(P_{BCO}\cdot S_{ABO}+P_{ABA}\cdot S_{BCO})}$$
$$\overset{O}{=} \frac{P_{BCB}\cdot P_{ABA}\cdot(P_{BAC}+P_{ACA})\cdot((32S_{ABC}))^2\cdot(2)}{(16)\cdot S_{ABC}\cdot(64P_{BCB}\cdot P_{BAC}\cdot P_{ABA}\cdot S_{ABC}+32P_{BCB}\cdot P_{ACB}\cdot P_{ABA}\cdot S_{ABC})\cdot((2))^2}$$
$$\overset{simplify}{=} \frac{P_{BAC}+P_{ACA}}{2P_{BAC}+P_{ACB}}$$
$$\overset{py}{=} \frac{(-P_{BCB}+3P_{ACA}+P_{ABA})\cdot((2))^2}{(-2P_{BCB}+6P_{ACA}+2P_{ABA})\cdot(2)}$$
$$\overset{simplify}{=} 1$$

Example 6.103 (1.300 3 38) *If two lines are antiparallel with respect to an angle, the perpendiculars dropped upon them from the vertex are isogonal in the angle considered.*

Constructive description
((circle $B\ C\ R\ S$)
(circumcenter $O\ B\ C\ R$)
(inter A (l $B\ R$) (l $C\ S$))
(foot $N\ A\ B\ C$)
(foot $M\ A\ R\ S$)
(eqangle $B\ A\ M\ N\ A\ C$))

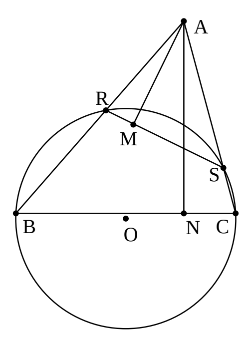


Figure 6-103

Example 6.104 (3.083, 9, 60) *Show that the four perpendiculars to the sides of an angle at four cyclic points form a parallelogram whose opposite vertices lie on isogonal conjugate lines with respect to the given angle.*

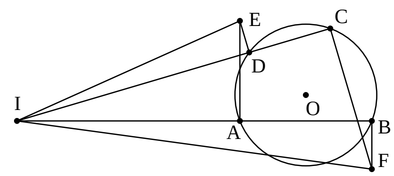


Figure 6-104

Constructive description
((circle $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ C$)
(inter I (l $A\ B$) (l $C\ D$))
(midpoint $X\ C\ D$)
(midpoint $Y\ A\ B$)
(inter E (p $D\ O\ X$) (p $A\ O\ Y$)) (inter F (p $C\ O\ X$) (p $B\ O\ Y$)) (eqangle $A\ I\ E\ F\ I\ C$))

Example 6.105 (0.001, 1, 3) *The perpendicular at the orthocenter H to the altitude HC of the triangle ABC meets the circumcircle of HBC in P. Show that $ABPH$ is a parallelogram.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ B\ C\ H$)
(pratio $P_P\ H\ A\ B\ 1$)
(inter P (l $H\ P_P$) (cir $O\ H$)) ($\frac{\overline{AH}}{\overline{BP}} = 1$))

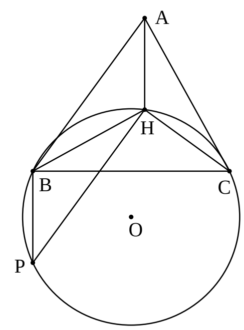


Figure 6-105

The machine proof: $\frac{\overline{AH}}{\overline{BP}} \stackrel{P}{=} \frac{S_{AHP_P}}{S_{BHP_P}} \stackrel{P_P}{=} \frac{-S_{ABH}}{-S_{ABH}} \stackrel{simplify}{=} 1.$

The eliminants: $\frac{\overline{AH}}{\overline{BP}} \stackrel{P}{=} \frac{S_{AHP_P}}{S_{BHP_P}}$. $S_{BHP_P} \stackrel{P_P}{=} -(S_{ABH})$. $S_{AHP_P} \stackrel{P_P}{=} -(S_{ABH})$.

Example 6.106 (1.533, 10, 38) *The tangential and the orthic triangles of a given triangle are homothetic. (also see Example 6.90)*

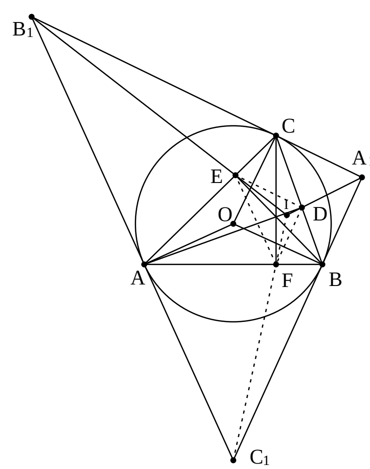


Figure 6-106

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $D\ A\ B\ C$)
(foot $E\ B\ A\ C$)
(foot $F\ C\ A\ B$)
(inter C_1 (t $B\ O\ B$) (t $A\ O\ A$))
(inter A_1 (l $C_1\ B$) (t $C\ O\ C$))
(inter B_1 (l $C\ A_1$) (l $A\ C_1$))
(inter I (l $B_1\ E$) (l $A_1\ D$))
(inter J (l $A_1\ D$) (l $C_1\ F$))
($\frac{\overline{A_1 I}}{\overline{DI}} = \frac{\overline{A_1 J}}{\overline{DJ}}$))

Example 6.107 (0.083, 4, 11) *Let P be the midpoint AH. Show that the segment OP is bisected by the median AA_1.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint A_1 B C)
(orthocenter H A B C)
(midpoint P A H)
(inter I (l O P) (l A A_1))
(midpoint I O P))

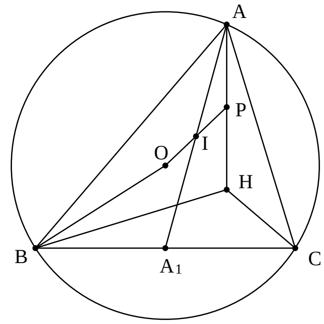


Figure 6-107

The machine proof

$-\frac{\overline{OI}}{\overline{PI}}$

$\overset{I}{=} \frac{S_{AOA_1}}{S_{AA_1P}}$

$\overset{P}{=} \frac{S_{AOA_1}}{\frac{1}{2}S_{AA_1H}}$

$\overset{H}{=} \frac{(2)\cdot S_{AOA_1}\cdot(16S_{ABC}^2)}{-P_{BAC}\cdot P_{ACB}\cdot S_{ABA_1}-P_{BAC}\cdot P_{ABC}\cdot S_{ACA_1}}$

$\overset{simplify}{=} \frac{(-32)\cdot S_{AOA_1}\cdot(S_{ABC})^2}{(P_{ACB}\cdot S_{ABA_1}+P_{ABC}\cdot S_{ACA_1})\cdot P_{BAC}}$

$\overset{A_1}{=} \frac{(-32)\cdot(-\frac{1}{2}S_{ACO}-\frac{1}{2}S_{ABO})\cdot(S_{ABC})^2}{(\frac{1}{2}P_{ACB}\cdot S_{ABC}-\frac{1}{2}P_{ABC}\cdot S_{ABC})\cdot P_{BAC}}$

$\overset{simplify}{=} \frac{(32)\cdot(S_{ACO}+S_{ABO})\cdot S_{ABC}}{(P_{ACB}-P_{ABC})\cdot P_{BAC}}$

$\overset{O}{=} \frac{(32)\cdot(-32P_{ACB}\cdot P_{ABA}\cdot S_{ABC}+32P_{ACA}\cdot P_{ABC}\cdot S_{ABC})\cdot S_{ABC}}{(P_{ACB}-P_{ABC})\cdot P_{BAC}\cdot(32S_{ABC})\cdot(-32S_{ABC})}$

$\overset{simplify}{=} \frac{P_{ACB}\cdot P_{ABA}-P_{ACA}\cdot P_{ABC}}{(P_{ACB}-P_{ABC})\cdot P_{BAC}}$

$\overset{py}{=} \frac{(-2P_{BCB}\cdot P_{ACA}+2P_{BCB}\cdot P_{ABA}+2P_{ACA}^2-2P_{ABA}^2)\cdot((2))^3}{(4P_{ACA}-4P_{ABA})\cdot(-P_{BCB}+P_{ACA}+P_{ABA})\cdot((2))^2}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{OI}}{\overline{PI}} \overset{I}{=} \frac{-S_{AOA_1}}{S_{AA_1P}}$

$S_{AA_1P} \overset{P}{=} \frac{1}{2}(S_{AA_1H})$

$S_{AA_1H} \overset{H}{=} \frac{(P_{ACB}\cdot S_{ABA_1}+P_{ABC}\cdot S_{ACA_1})\cdot P_{BAC}}{(-16)\cdot(S_{ABC})^2}$

$S_{ACA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ABC})$

$S_{ABA_1} \overset{A_1}{=} \frac{1}{2}(S_{ABC})$

$S_{AOA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ACO}+S_{ABO})$

$S_{ABO} \overset{O}{=} \frac{P_{ACB}\cdot P_{ABA}}{(32)\cdot S_{ABC}}$

$S_{ACO} \overset{O}{=} \frac{P_{ACA}\cdot P_{ABC}}{(-32)\cdot S_{ABC}}$

$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$

$P_{ABC} = \frac{1}{2}(P_{BCB}-P_{ACA}+P_{ABA})$

$P_{ACB} = \frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$

Example 6.108 (4.866, 12, 15) *Prove that HA_1 passes through the diametric opposite of A on the circumcircle.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint A_1 B C)
(orthocenter H A B C)
(inter I (l A O) (l H A_1))
(perp-biesct O B I))

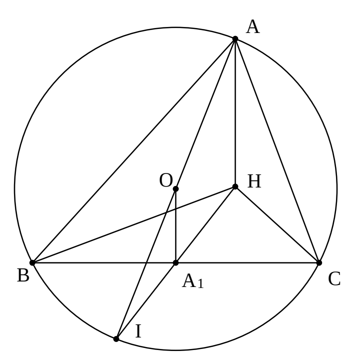


Figure 6-108

Example 6.109 (0.250, 2, 31) *Show that the product of the distances of a point of the circumcircle of a triangle from the sides of the triangle is equal to the product of the distances of the same point from the sides of the tangential triangle of the given triangle.*

Constructive description
((circle A B C D)
(circumcenter O A B C)
(foot E D B C)
(foot F D A C)
(foot G D A B)
(inter E_1 (p D O A) (t A O A))
(inter F_1 (p D O B) (t B O B))
(inter G_1 (p D O C) (t C O C))
($\overline{DE}^2\overline{DF}^2\overline{DG}^2 = \overline{DE_1}^2\overline{DF_1}^2\overline{DG_1}^2$))

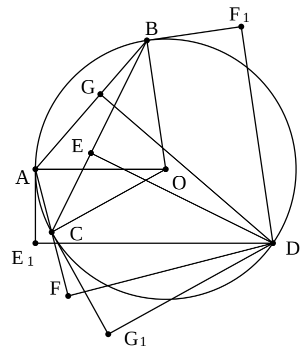


Figure 6-109

Example 6.110 (0.200, 12, 16) *If O is the circumcenter the triangle ABC, and G is the centroid of ABC, we have*

$$3\overline{OA}^2 = \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + 3\overline{OG}^2.$$

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint E A C)
(midpoint F A B)
(midpoint D B C)
(inter G (l A D) (l C F))
($3\overline{OG}^2+\overline{AG}^2+\overline{GC}^2+\overline{BG}^2 = 3\overline{OA}^2$))

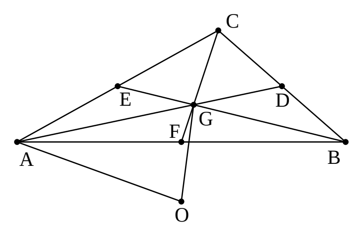


Figure 6-110

Example 6.111 (1.033, 28, 24) *The internal and external bisectors of an angle of a triangle pass through the ends of the circumdiameters which is perpendicular to the side opposite the vertex considered.*

Constructive description
((points B E P)
(on L (t E E B))
(lratio C E B -1)
(inter O (l L E) (b B L))
(inter A (l B P) (cir O B))
(eqangle B A L L A C))

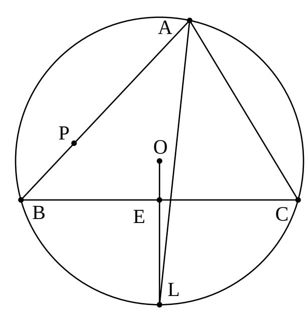


Figure 6-111

Example 6.112 (0.866, 12, 15) *The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $D\ C\ A\ B$)
(foot $E\ B\ A\ C$)
(inter H (l $C\ D$) (l $B\ E$))
(inter K (l $C\ D$) (cir $O\ C$))
(midpoint $D\ H\ K$))

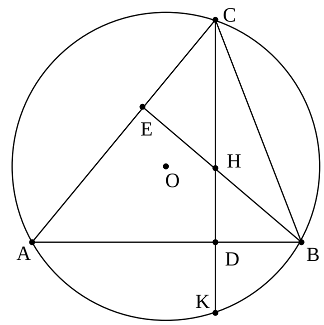


Figure 6-112

Example 6.113 (1.650, 5, 18) *The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $D\ C\ A\ B$)
(foot $E\ B\ A\ C$)
(inter H (l $C\ D$) (l $B\ E$))
(circumcenter $O_1\ A\ B\ H$)
(eqdistance $O\ A\ O_1\ H$))

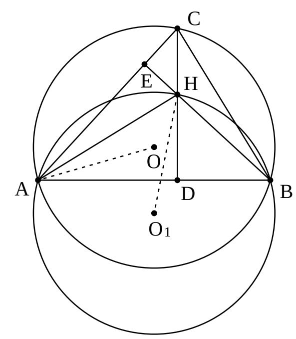


Figure 6-113

Example 6.114 (0.933, 17, 18) *A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the other two vertices.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(inter C_1 (l $C\ H$) (cir $O\ C$))
(inter A_1 (l $A\ H$) (cir $O\ A$))
(perp-biesct $B\ C_1\ A_1$))

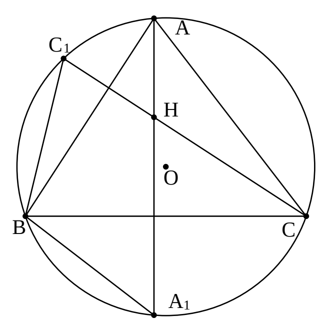


Figure 6-114

Example 6.115 (0.833, 17, 27) *If O is the circumcenter and H the orthocenter of a triangle ABC, and AH, BH, CH meet the circumcircle in D_1, E_1, F_1, prove that parallels through D_1, E_1, F_1 to OA, OB, OC, respectively, meet in a point.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(inter C_1 (l $C\ H$) (cir $O\ C$))
(inter A_1 (l $A\ H$) (cir $O\ A$))
(inter B_1 (l $B\ H$) (cir $O\ B$))
(inter I (p $A_1\ O\ A$) (p $B_1\ O\ B$))
(parallel $I\ C_1\ O\ C$))

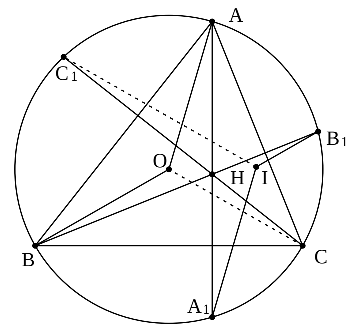


Figure 6-115

Example 6.116 (0.133, 7, 15) *Show that the foot of the altitude to the base of a triangle and the projections of the ends of the base upon the circumdiameter passing through the opposite vertex of the triangle determine a circle having for center the midpoint the base.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $M\ A\ B$)
(foot $D\ C\ A\ B$)
(foot $E\ A\ O\ C$)
(perp-biesct $M\ E\ D$))

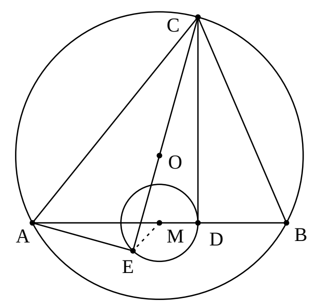


Figure 6-116

Example 6.117 (0.333, 12, 16) *Show that the symmetric of the orthocenter of a triangle with respect to a vertex, and the symmetric of that vertex with respect to the midpoint of the opposite side, are collinear with the circumcenter of the triangle.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $A_1\ B\ C$)
(lratio $S\ A\ H\ -1$)
(lratio $L\ A_1\ A\ -1$)
(collinear $O\ L\ S$))

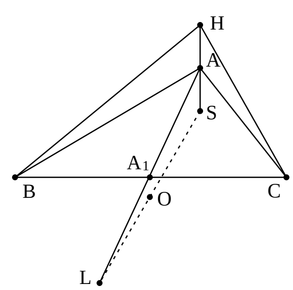


Figure 6-117

Example 6.118 (0.817, 26, 25) *If D_1 is the second point of intersection of the altitude ADD_1 of the triangle ABC with the circumcircle, center O, and P is the trace on BC of the perpendicular from D_1 to AC, show that the lines AP, AO make equal angles with the bisector of the angle DAC.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(inter D_1 (l A D) (cir O A))
(inter P (l B C) (t D_1 A C))
(eqangle O A D C A P))

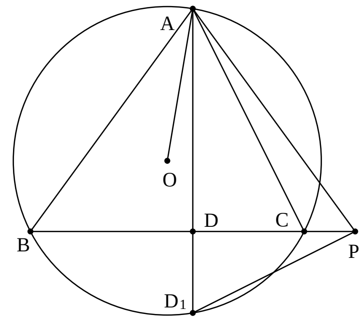


Figure 6-118

Example 6.119 (3.900, 14, 40) *Show that the triangle formed by the foot of the altitude to the base of a triangle and the midpoints of the altitudes to the lateral sides is similar to the given triangle; its circumcircle passes through the orthocenter of the given triangle and through the midpoint of its base.*

Constructive description
((points A B C)
(midpoint A_1 B C)
(foot D A B C)
(foot E B A C)
(foot F C A B)
(inter H (l B E) (l A D))
(midpoint P B E)
(midpoint Q C F)
(cocircle H P Q D))

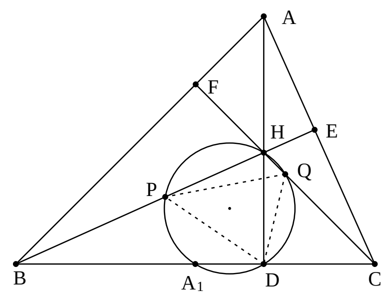


Figure 6-119

Example 6.120 (0.717, 44, 38) *The sides of the anticomplementary triangle of the triangle ABC meet the circumcircle of ABC in the points P, Q, R. Show that the area of the triangle PQR is equal to four times the area of the orthic triangle of ABC.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(foot E B A C)
(foot F C A B)
(inter P (p A B C) (cir O A))
(inter Q (p B A C) (cir O B))
(inter R (p C A B) (cir O C))
($4S_{DEF} = S_{PQR}$))

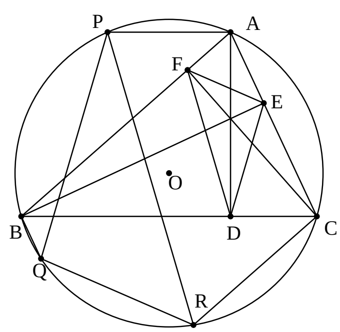


Figure 6-120

Example 6.121 (0.450, 5, 17) *Through the orthocenter of the triangle ABC parallels are drawn to the sides AB, AC, meeting BC in D, E. The perpendiculars to BC at D, E meet AB, AC in two points D_1, E_1 which are collinear with the diametric opposites of B, C on the circumcircle of ABC.*

Constructive description
((points A B C)
(circumcenter O A B C)
(orthocenter H A B C)
(inter D (l B C) (p H A B))
(inter D_1 (l A B) (p D A H))
(inter P (l B O) (cir O B))
(inter Q (l C O) (cir O C))
(collinear P Q D_1))

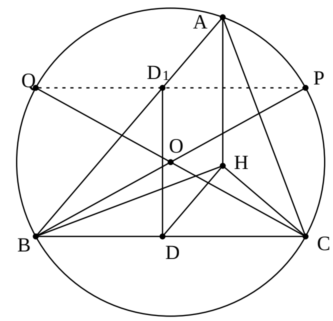


Figure 6-121

Example 6.122 (0.150, 6, 19) *If the altitudes AD, BE, CF of the triangle ABC meet the circumcircle of ABC again in P, Q, R, show that we have* $(AP/AD)+(BQ/BE)+(CR/CF) = 4$.

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(foot E B A C)
(foot F C A B)
(inter H (l A D) (l C F))
(inter P (l A D) (cir O A))
(inter Q (l B E) (cir O B))
(inter R (l C F) (cir O C))
($\frac{\overline{AP}}{\overline{AD}}+\frac{\overline{BQ}}{\overline{BE}}+\frac{\overline{CR}}{\overline{CF}} = 4$))

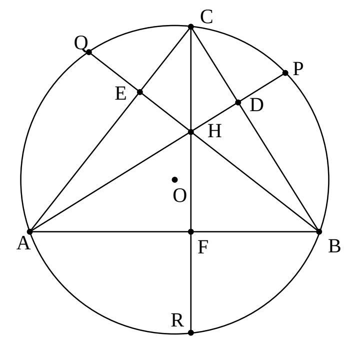


Figure 6-122

Example 6.123 (0.433, 9, 21) *Through the point of intersection of the tangents DB, DC to the circumcircle (O) of the triangle ABC a parallel is drawn to the line touching (O) at A. If this parallel meets AB, AC in E, F, show that D bisects EF.*

Constructive description
((points A B C)
(circumcenter O A B C)
(inter D (t C O C) (t B O B))
(inter F (l A B) (t D O A))
(inter E (l A C) (l F D))
(midpoint D E F))

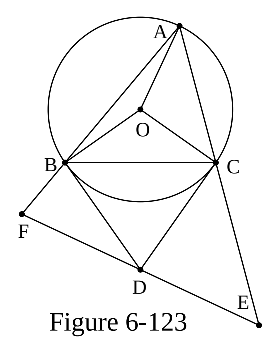


Figure 6-123

Example 6.124 (0.466, 13, 28) *In a triangle ABC, let p and q be the radii of two circles through A, touching side BC at B and C, respectively. Then* $pq = R^2$.

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $M\ B\ C$)
(inter Q (p $B\ O\ M$) (b $A\ B$))
(inter P (p $C\ O\ M$) (b $A\ C$))
(eq-product $Q\ B\ P\ C\ O\ B\ O\ B$))

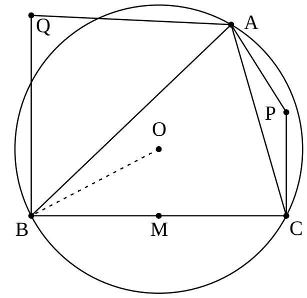


Figure 6-124

Example 6.125 (0.400, 7, 18) *The parallel to the side AC through the vertex B of the triangle ABC meets the tangent to the circumcircle* (O) *of ABC at C in* B_1*, and the parallel through C to AB meets the tangent to* (O) *at B in* C_1*. Prove that* $BC^2 = BC_1 \cdot B_1C$*.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(inter B_1 (p $B\ A\ C$) (t $C\ O\ C$))
(inter C_1 (p $C\ A\ B$) (t $B\ O\ B$))
(eq-product $B\ C\ B\ C\ B\ C_1\ B_1\ C$))

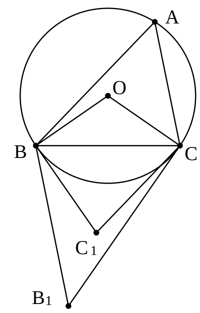


Figure 6-125

Example 6.126 (5.733, 22, 23) *Show that the foot of the perpendicular from the orthocenter of a triangle upon the line joining a vertex to the point of intersection of the opposite side with the corresponding side of the orthic triangle lies on the circumcircle of the triangle.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $D\ A\ B\ C$) (foot $E\ B\ A\ C$)
(inter H (l $A\ D$) (l $B\ E$))
(inter X (l $E\ D$) (l $A\ B$))
(foot $S\ H\ C\ X$)
(perp-biesct $O\ S\ A$))

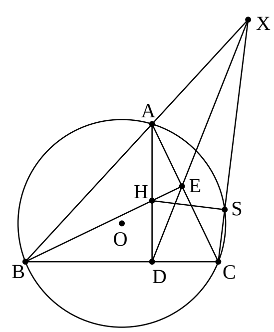


Figure 6-126

6.3.4 The Euler Line

Definition. *The circumcenter O, orthocenter H, and the centroid G of a given triangle are collinear and this line is called the Euler line of the triangle.*

For the machine proof of Euler's theorem, see Example 3.71 on page 141.

Example 6.127 (0.033, 4, 2) *Let O and H be the circumcenter and the orthocenter of a triangle ABC. Show that* $OH^2 = 9R^2 - a^2 - b^2 - c^2$*.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
($\overline{OH}^2 = 9\overline{OB}^2 - \overline{AB}^2 - \overline{AC}^2 - \overline{BC}^2$))

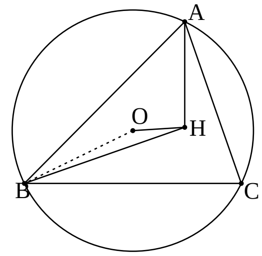


Figure 6-127

Example 6.128 (0.167, 6, 9) *With the usual notations for the triangle ABC, we have* $4\overline{AO}^2 = 4\overline{AB}^2 + 4\overline{AC}^2 + 4\overline{BC}^2 + \overline{GH}^2$.

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(centroid $G\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
($4\overline{AO}^2 = 4\overline{AB}^2 + 4\overline{AC}^2 + 4\overline{BC}^2 + \overline{GH}^2$))

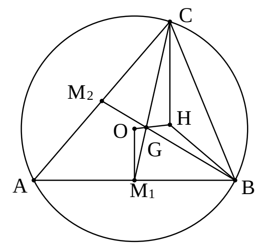


Figure 6-128

Example 6.129 (0.816, 9, 9) *With the usual notations for the triangle ABC, we have* $9\overline{AO}^2 = \overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 + \overline{OH}^2$.

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
($9\overline{AO}^2 = \overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 + \overline{OH}^2$))

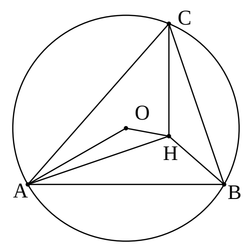


Figure 6-129

Example 6.130 (0.250, 6, 8) *With the usual notations for the triangle ABC, we have* $12\overline{AO}^2 = \overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 + \overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2$.

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
($12\overline{AO}^2 = \overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 + \overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2$))

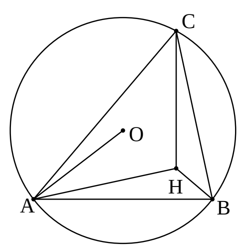


Figure 6-130

Example 6.131 (5.050, 8, 40) *The homothetic center of the orthic and the tangential triangles of a given triangle lies on the Euler line of the given triangle.*

Constructive description
((points A B C)
(orthocenter H A B C)
(circumcenter O A B C)
(inter D (l B C) (l A H))
(inter E (l A C) (l B H))
(inter F (l A B) (l C H))
(inter B_1 (t C O C) (t A O A))
(inter C_1 (l A B_1) (t B B O))
(inter A_1 (l C_1 B) (l B_1 C))
(inter I (l F C_1) (l B_1 E)) (collinear O H I))

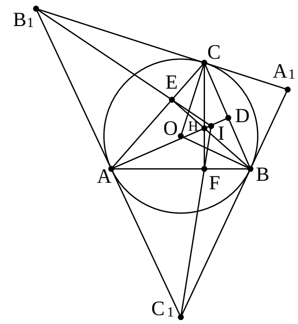


Figure 6-131

Example 6.132 (0.833, 12, 27) *The Euler lines of the four triangles of an orthocentric group are concurrent.*

Constructive description
((points A B C)
(orthocenter H A B C)
(circumcenter O A B C)
(circumcenter A_1 B C H)
(circumcenter B_1 A C H)
(inter I (l B B_1) (l A A_1))
(collinear O H I))

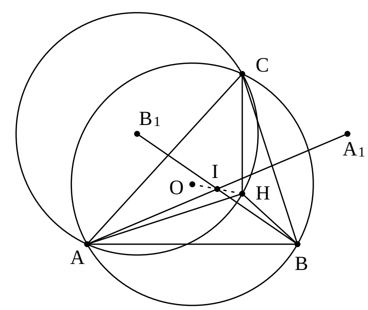


Figure 6-132

Example 6.133 (0.950, 4, 40) *Show that the perpendiculars from the vertices of a triangle to the lines joining the midpoints of the respectively opposite sides to the orthocenter of the triangle meet these sides in three points of a straight line perpendicular to the Euler line of the triangle.*

Constructive description
((points A B C) (circumcenter O A B C)
(orthocenter H A B C) (midpoint A_1 B C)
(inter A_2 (l B C) (t A A_1 H))
(midpoint B_1 C A)
(inter B_2 (l A C) (t B H B_1))
(midpoint C_1 A B)
(inter C_2 (l A B) (t C C_1 H))
(inter C_3 (l A B) (l A_2 B_2))
($\frac{\overline{AC_2}}{\overline{BC_2}} = \frac{\overline{AC_3}}{\overline{BC_3}}$))

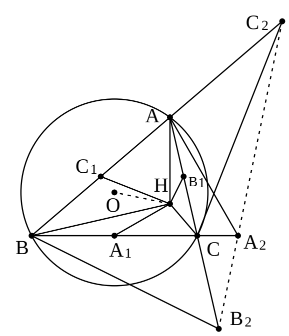


Figure 6-133

6.3.5 The Nine-Point Circle

Definition. *The midpoints of the segments joining the orthocenter of a triangle to its vertices* *are called the Euler points of the triangle. The three Euler points determine the Euler triangle.*

Example 6.134 (The Nine-Point Circle Theorem) (0.050, 2, 5) *In a triangle the midpoints of the sides, the feet of altitudes, and the Euler points lie on the same circle.*

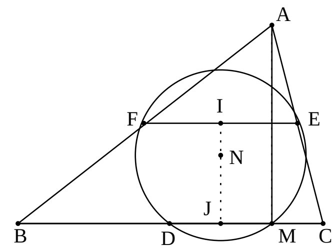


Figure 6-134

This circle is called the *nine point circle* of the given triangle.

Constructive description

((points $A\ B\ C$)
(foot $M\ A\ B\ C$)
(midpoint $D\ B\ C$)
(midpoint $E\ A\ C$)
(midpoint $F\ A\ B$)
(midpoint $I\ E\ F$)
(midpoint $J\ M\ D$)
(parallel $I\ J\ A\ M$))

The machine proof

$$\frac{S_{AMI}}{S_{AMJ}} \stackrel{J}{=} \frac{S_{AMI}}{\frac{1}{2}S_{AMD}} \stackrel{I}{=} \frac{(2)\cdot(\frac{1}{2}S_{AMF}+\frac{1}{2}S_{AME})}{S_{AMD}} \stackrel{F}{=} \frac{S_{AME}-\frac{1}{2}S_{ABM}}{S_{AMD}} \stackrel{E}{=} \frac{-S_{ACM}-S_{ABM}}{(2)\cdot S_{AMD}} \stackrel{D}{=} \frac{-(S_{ACM}+S_{ABM})}{(2)\cdot(-\frac{1}{2}S_{ACM}-\frac{1}{2}S_{ABM})} \stackrel{simplify}{=} 1$$

The eliminants

$$S_{AMJ} \stackrel{J}{=} \frac{1}{2}(S_{AMD})$$
$$S_{AMI} \stackrel{I}{=} \frac{1}{2}(S_{AMF}+S_{AME})$$
$$S_{AMF} \stackrel{F}{=} -\frac{1}{2}(S_{ABM})$$
$$S_{AME} \stackrel{E}{=} -\frac{1}{2}(S_{ACM})$$
$$S_{AMD} \stackrel{D}{=} -\frac{1}{2}(S_{ACM}+S_{ABM})$$

Since the orthocenter and the three feet of the triangle form an orthocentric group, we conclude from the above machine proof that the circle also passes the three Euler points.

Example 6.135 (Feuerbach's Theorem) (55.500 102 46) *The nine-point circle of a triangle touches each of the four tritangent circles of the triangle.*

Constructive description

((points $I\ A\ B$) (incenter $C\ I\ A\ B$)
(midpoint $M_1\ A\ B$) (midpoint $M_2\ A\ C$)
(midpoint $M_3\ B\ C$)
(midpoint $P\ M_1\ M_3$)
(midpoint $Q\ M_1\ M_2$)
(inter N (t $P\ P\ M_1$) (t $Q\ Q\ M_1$))
(foot $D\ I\ A\ B$)
(cc-tangent $I\ D\ N\ M_1$))

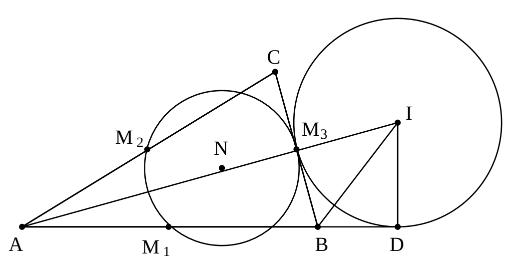


Figure 6-135

Example 6.136 (0.683, 3, 19) *The radius of the nine-point circle is equal to half the circum-radius of the triangle.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $D\ A\ B\ C$)
(foot $E\ B\ A\ C$)
(foot $F\ C\ A\ B$)
(circumcenter $N\ D\ E\ F$)
($4\overline{NF}^2 = \overline{OA}^2$))

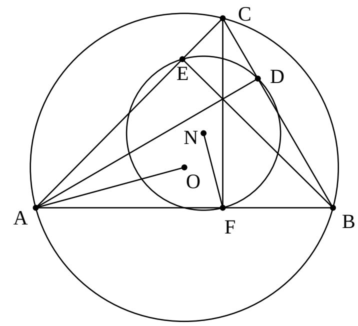


Figure 6-136

Example 6.137 (0.067, 5, 18) *The nine-point center lies on the Euler line, midway between the circumcenter and the orthocenter.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(midpoint $M_1\ B\ C$)
(midpoint $M_2\ A\ B$)
(midpoint $M_3\ A\ C$)
(circumcenter $N\ M_1\ M_2\ M_3$)
(midpoint $N\ H\ O$))

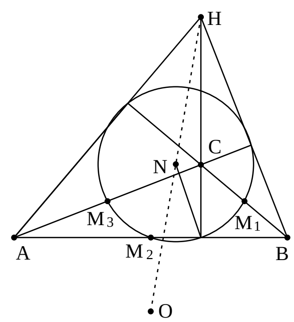


Figure 6-137

The machine proof

$$-\frac{\overline{HN}}{\overline{ON}}$$
$$\overset{N}{=} \frac{\frac{1}{2}P_{M_1M_2M_1}-P_{HM_1M_2}}{-\frac{1}{2}P_{M_1M_2M_1}+P_{OM_1M_2}}$$
$$\overset{M_2}{=} \frac{-(-P_{BM_1H}+\frac{1}{2}P_{BM_1B}-P_{AM_1H}+\frac{1}{2}P_{AM_1A}-\frac{1}{4}P_{ABA})}{-P_{BM_1O}+\frac{1}{2}P_{BM_1B}-P_{AM_1O}+\frac{1}{2}P_{AM_1A}-\frac{1}{4}P_{ABA}}$$
$$\overset{M_1}{=} \frac{-(2P_{BCH}-2P_{BCB}+2P_{ACH}-P_{ACA}+2P_{ABH})}{2P_{BCO}-2P_{BCB}+2P_{ACO}-P_{ACA}+2P_{ABO}}$$
$$\overset{H}{=} \frac{-(-2P_{BCB}+4P_{ACB}-P_{ACA}+2P_{ABC})}{2P_{BCO}-2P_{BCB}+2P_{ACO}-P_{ACA}+2P_{ABO}}$$
$$\overset{O}{=} \frac{(2P_{BCB}-4P_{ACB}+P_{ACA}-2P_{ABC})\cdot((2))^3}{-8P_{BCB}+8P_{ABA}}$$
$$\overset{py}{=} \frac{-(-4P_{BCB}+4P_{ABA})}{(P_{BCB}-P_{ABA})\cdot((2))^2}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{HN}}{\overline{ON}} \overset{N}{=} \frac{P_{M_1M_2M_1}-2P_{HM_1M_2}}{P_{M_1M_2M_1}-2P_{OM_1M_2}}$$
$$P_{OM_1M_2} \overset{M_2}{=} \frac{1}{2}\left(P_{BM_1O}+P_{AM_1O}\right)$$
$$P_{HM_1M_2} \overset{M_2}{=} \frac{1}{2}\left(P_{BM_1H}+P_{AM_1H}\right)$$
$$P_{M_1M_2M_1} \overset{M_2}{=} \frac{1}{4}\left(2P_{BM_1B}+2P_{AM_1A}-P_{ABA}\right)$$
$$P_{AM_1O} \overset{M_1}{=} -\frac{1}{4}\left(P_{BCB}-2P_{ACO}-2P_{ABO}\right)$$
$$P_{BM_1O} \overset{M_1}{=} \frac{1}{4}\left(2P_{BCO}-P_{BCB}\right)$$
$$P_{AM_1A} \overset{M_1}{=} -\frac{1}{4}\left(P_{BCB}-2P_{ACA}-2P_{ABA}\right)$$
$$P_{AM_1H} \overset{M_1}{=} -\frac{1}{4}\left(P_{BCB}-2P_{ACH}-2P_{ABH}\right)$$
$$P_{BM_1B} \overset{M_1}{=} \frac{1}{4}\left(P_{BCB}\right)$$
$$P_{BM_1H} \overset{M_1}{=} \frac{1}{4}\left(2P_{BCH}-P_{BCB}\right)$$
$$P_{ABH} \overset{H}{=} P_{ABC} \quad P_{ACH} \overset{H}{=} P_{ACB} \quad P_{BCH} \overset{H}{=} P_{ACB}$$
$$P_{ABO} \overset{O}{=} \frac{1}{2}\left(P_{ABA}\right)$$
$$P_{ACO} \overset{O}{=} \frac{1}{2}\left(P_{ACA}\right)$$
$$P_{BCO} \overset{O}{=} \frac{1}{2}\left(P_{BCB}\right)$$
$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$
$$P_{ACB} = \frac{1}{2}\left(P_{BCB}+P_{ACA}-P_{ABA}\right)$$

Example 6.138 (4.383, 51, 42) *Show that the foot of the altitude of a triangle on a side, the midpoint of the segment of the circumdiameter between this side and the opposite vertex, and the nine-point center are collinear.*

Constructive description
((points $A\ B\ C$)
(foot $D\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(inter E (l $B\ C$) (l $A\ O$))
(midpoint $M\ A\ E$) (midpoint $A_1\ C\ B$)
(midpoint $B_1\ A\ C$)
(midpoint $C_1\ B\ A$)
(circumcenter $N\ A_1\ B_1\ C_1$)
(collinear $M\ D\ N$))

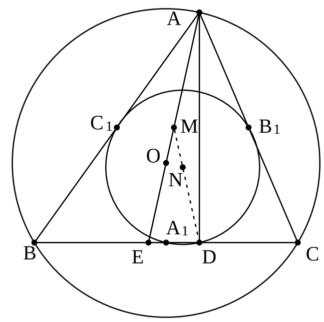


Figure 6-138

Example 6.139 (1.667, 44, 38) *The center of the nine-point circle is the midpoint of a Euler point and the midpoint of the opposite side.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(midpoint $M_1\ B\ C$)
(midpoint $M_2\ A\ C$)
(midpoint $M_3\ A\ B$)
(circumcenter $N\ M_1\ M_2\ M_3$)
(inter H_1 (l $C\ H$) (l $M_3\ N$))
(midpoint $H_1\ C\ H$))

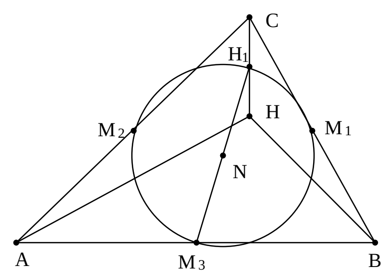


Figure 6-139

Example 6.140 (1.183, 24, 49) *The two pairs of points O and N, G and H separate each other harmonically.*

Constructive description
((points $A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(midpoint $M_2\ A\ C$)
(midpoint $M_3\ A\ B$)
(midpoint $M_1\ B\ C$)
(circumcenter $N\ M_1\ M_2\ M_3$)
(inter G (l $B\ M_2$) (l $C\ M_3$))
(harmonic $O\ N\ G\ H$))

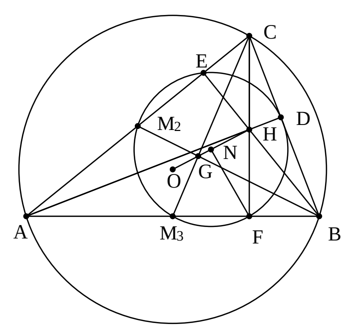


Figure 6-140

Example 6.141 (0.850, 23, 38) *If P is the symmetric of the vertex A with respect to the opposite side BC, show that HP is equal to four times the distance of the nine-point center from BC.*

Constructive description
((points A B C)
(orthocenter H A B C)
(foot D A B C)
(lratio P D A -1)
(midpoint A_1 B C)
(midpoint B_1 A C)
(midpoint C_1 B A)
(circumcenter N A_1 B_1 C_1)
(foot K N B C)
($\overline{PH}^2 = 16\overline{NK}^2$))

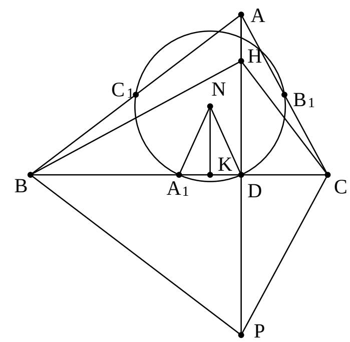


Figure 6-141

Example 6.142 (2.450, 56, 42) *Show that the square of the tangent from a vertex of a triangle to the nine-point circle is equal to the altitude issued from that vertex multiplied by the distance of the opposite side from the circumcenter.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint A_1 B C)
(midpoint B_1 A C)
(midpoint C_1 B A)
(midpoint M_1 A_1 B_1)
(midpoint M_2 A_1 C_1)
(inter N (t M_1 M_1 A_1) (t M_2 M_2 A_1))
(foot D A B C)
($P_{ANA} - P_{A_1NA_1} = P_{A_1AD} - P_{OAD}$))

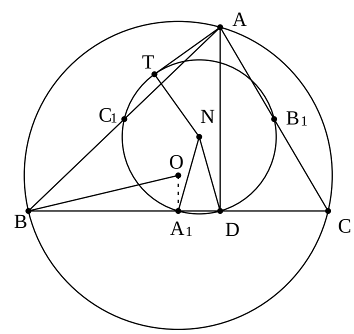


Figure 6-142

Example 6.143 (0.950, 50, 37) *Show that the symmetric of the circumcenter of a triangle with respect to a side coincides with the symmetric of the vertex opposite the side considered with respect to the nine-point center of the triangle.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint C_1 A B)
(midpoint B_1 A C)
(midpoint A_1 B C)
(circumcenter N A_1 B_1 C_1)
(inter O_1 (l O C_1) (l C N))
(midpoint N O_1 C))

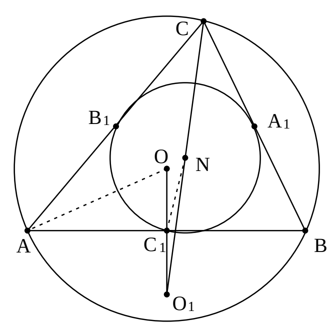


Figure 6-143

6.3.6 Incircles and the Excircles

Example 6.144 (Theorem of Incenter) (0.750, 14, 29) *The three internal bisectors of the angels of a triangle meet in a point, the incenter I of the triangle.*

Constructive description
((points $A\ B\ I$)
(foot $A_1\ B\ I\ A$)
(lratio $A_2\ A_1\ B\ -1$)
(foot $B_1\ A\ I\ B$)
(lratio $B_2\ B_1\ A\ -1$)
(inter C (l $A\ A_2$) (l $B\ B_2$))
(eqangle $A\ C\ I\ I\ C\ B$))

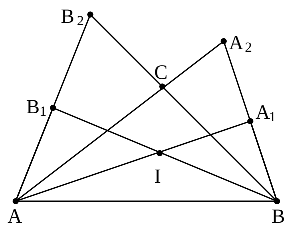


Figure 6-144

Similarly the internal bisector at vertex A (B, C) and the two external bisectors at vertices B (C, A) and C (A, B) meet in a point I_a (I_b, I_c).

Definition. Each of I, I_a, I_b, I_c is the center of a circle tangent to the three sides of the triangle. The four circles with I, I_a, I_b, I_c as centers are called the *inscribed circle* and the *exscribed circles* or the four *tritangent circles* of the given triangle.

Example 6.145 (0.067, 3, 10) *Two tritangent centers of a triangle are the ends of a diameter of a circle passing through the two vertices of the triangle which are not collinear with the centers considered.*

Constructive description
((points $I\ B\ C$)
(incenter $A\ I\ C\ B$)
(inter I_a (l $A\ I$) (t $B\ B\ I$))
(midpoint $O\ I\ I_a$)
(eqdistance $O\ B\ O\ I$))

The eliminants

$P_{IOI} \overset{O}{=} \frac{1}{4}(P_{II_aI})$

$P_{BOB} \overset{O}{=} \frac{1}{4}(2P_{BI_aB} - P_{II_aI} + 2P_{IBI})$

$P_{II_aI} \overset{I_a}{=} \frac{P_{IAI}\cdot(P_{IBI})^2}{(P_{BIA})^2}$

$P_{BI_aB} \overset{I_a}{=} \frac{-(P_{BIA}^2 - P_{BIA}\cdot P_{IAI} - P_{IAI}\cdot P_{IBA})\cdot P_{IBI}}{(P_{BIA})^2}$

$P_{IBA} \overset{A}{=} \frac{-P_{ICB}\cdot P_{IBC}\cdot P_{IBI}}{P_{BCB}\cdot P_{BIC}}$

$P_{BIA} \overset{A}{=} \frac{(16)\cdot P_{IBI}\cdot(S_{IBC})^2}{P_{BCB}\cdot P_{BIC}}$

$16S_{IBC}^2 = P_{BIC}\cdot P_{ICB} + P_{ICI}\cdot P_{IBC}$

$P_{IBC} = \frac{1}{2}(P_{BCB} - P_{ICI} + P_{IBI})$

$P_{ICB} = \frac{1}{2}(P_{BCB} + P_{ICI} - P_{IBI})$

$P_{BIC} = -\frac{1}{2}(P_{BCB} - P_{ICI} - P_{IBI})$

The machine proof

$$\frac{P_{BOB}}{P_{IOI}}$$

$$\overset{O}{=} \frac{\frac{1}{2}P_{BI_aB} - \frac{1}{4}P_{II_aI} + \frac{1}{2}P_{IBI}}{\frac{1}{4}P_{II_aI}}$$

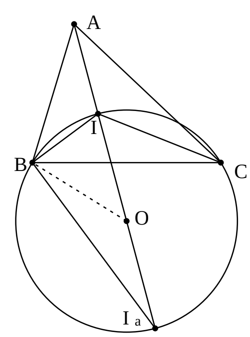


Figure 145

$$\overset{I_a}{=} \frac{(2P_{BIA}^3\cdot P_{IAI}\cdot P_{IBI}+2P_{BIA}^2\cdot P_{IAI}\cdot P_{IBA}\cdot P_{IBI}-P_{BIA}^2\cdot P_{IAI}\cdot P_{IBI}^2)\cdot P_{BIA}^2}{P_{IAI}\cdot P_{IBI}^2\cdot (P_{BIA}^2)^2}$$

$$\overset{simplify}{=} \frac{2P_{BIA}+2P_{IBA}-P_{IBI}}{P_{IBI}}$$

$$\overset{A}{=} \frac{-P_{BCB}^2\cdot P_{BIC}^2\cdot P_{IBI}-2P_{BCB}\cdot P_{BIC}\cdot P_{ICB}\cdot P_{IBC}\cdot P_{IBI}+32P_{BCB}\cdot P_{BIC}\cdot P_{IBI}\cdot S_{IBC}^2}{P_{IBI}\cdot(P_{BCB}\cdot P_{BIC})^2}$$

$$\overset{simplify}{=} \frac{-(P_{BCB}\cdot P_{BIC}+2P_{ICB}\cdot P_{IBC}-32S_{IBC}^2)}{P_{BCB}\cdot P_{BIC}}$$

$$\overset{herron}{=} \frac{-16P_{BCB}\cdot P_{BIC}+32P_{BIC}\cdot P_{ICB}-32P_{ICB}\cdot P_{IBC}+32P_{ICI}\cdot P_{IBC}}{P_{BCB}\cdot P_{BIC}\cdot(16)}$$

$$\overset{py}{=} \frac{-(4P_{BCB}^2-4P_{BCB}\cdot P_{ICI}-4P_{BCB}\cdot P_{IBI})\cdot(2)}{P_{BCB}\cdot(-P_{BCB}+P_{ICI}+P_{IBI})\cdot((2))^3}$$

$$\overset{simplify}{=} 1$$

Example 6.146 (0.300, 8, 9) *The four tritangent centers of a triangle lie on six circles which pass through the pairs of vertices of the triangle and have for their centers the midpoints of the arcs subtended by the respective sides of the triangle on its circumcircle.*

Constructive description
((points B C I)
(incenter A I C B)
(inter I_a (t B B I) (l A I))
(inter K (l A I) (b C B))
(midpoint K I I_a))

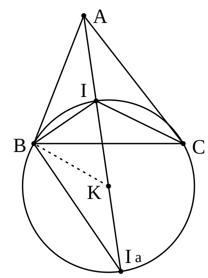


Figure 6-146

Example 6.147 (0.067, 4, 11) *Let incircle (with center I) of $\triangle ABC$ touch the side BC at X and let A_1 be the midpoint of this side. Then the line A_1I (extended) bisectors AX.*

Constructive description
((points B C I)
(incenter A I C B)
(midpoint A_1 B C)
(foot X I B C)
(inter O (l A X) (l A_1 I))
(midpoint O A X))

The eliminants

$\frac{\overline{AO}}{\overline{XO}} \overset{O}{=} \frac{S_{IAA_1}}{-S_{IA_1X}}$

$S_{IA_1X} \overset{X}{=} \frac{P_{BA_1I}\cdot S_{BIA_1}}{P_{BA_1B}}$

$S_{BIA_1} \overset{A_1}{=} -\frac{1}{2}(S_{BCI})$

$P_{BA_1I} \overset{A_1}{=} \frac{1}{4}(2P_{BCI}-P_{BCB})$

$P_{BA_1B} \overset{A_1}{=} \frac{1}{4}(P_{BCB})$

$S_{IAA_1} \overset{A_1}{=} \frac{1}{2}(S_{CIA}+S_{BIA})$

$S_{BIA} \overset{A}{=} \frac{-P_{BIB}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$

$S_{CIA} \overset{A}{=} \frac{P_{CIC}\cdot P_{CBI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$

$P_{BIC} = \frac{1}{2}(P_{CIC}+P_{BIB}-P_{BCB})$

$P_{BCI} = \frac{1}{2}(P_{CIC}-P_{BIB}+P_{BCB})$

$P_{CBI} = -\frac{1}{2}(P_{CIC}-P_{BIB}-P_{BCB})$

The machine proof

$-\frac{\overline{AO}}{\overline{XO}}$

$\stackrel{O}{=} \frac{-S_{IAA_1}}{-S_{IA_1X}}$

$\stackrel{X}{=} \frac{S_{IAA_1} \cdot P_{BA_1B}}{P_{BA_1I} \cdot S_{BIA_1}}$

$\stackrel{A_1}{=} \frac{(\frac{1}{2}S_{CIA}+\frac{1}{2}S_{BIA}) \cdot (\frac{1}{4}P_{BCB})}{(\frac{1}{2}P_{BCI}-\frac{1}{4}P_{BCB}) \cdot (-\frac{1}{2}S_{BCI})}$

$\stackrel{A}{=} \frac{-(P_{CIC} \cdot P_{CBI} \cdot P_{BIC} \cdot P_{BCB} \cdot S_{BCI}-P_{BIC} \cdot P_{BIB} \cdot P_{BCI} \cdot P_{BCB} \cdot S_{BCI}) \cdot P_{BCB}}{(2P_{BCI}-P_{BCB}) \cdot S_{BCI} \cdot (P_{BIC} \cdot P_{BCB})^2}$

$\stackrel{simplify}{=} \frac{-(P_{CIC} \cdot P_{CBI}-P_{BIB} \cdot P_{BCI})}{(2P_{BCI}-P_{BCB}) \cdot P_{BIC}}$

$\stackrel{py}{=} \frac{-(-2P_{CIC}^2+2P_{CIC} \cdot P_{BCB}+2P_{BIB}^2-2P_{BIB} \cdot P_{BCB}) \cdot ((2))^2}{(2P_{CIC}-2P_{BIB}) \cdot (P_{CIC}+P_{BIB}-P_{BCB}) \cdot ((2))^2}$

$\stackrel{simplify}{=} 1$

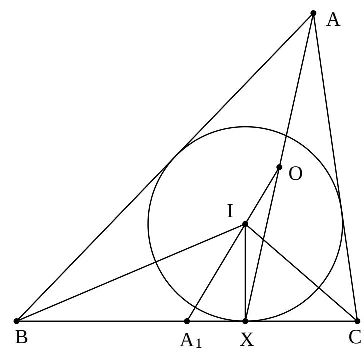


Figure 6-147

Example 6.148 (0.033, 1, 6) *The product of the distances of two tritangent centers of a triangle from the vertex of the triangle collinear with them is equal to the product of the two sides of the triangle passing through the vertex considered.*

Constructive description

((points B C I)
(incenter A I C B)
(inter I_a (t B B I) (l A I))
(eq-product A I A I_a A B A C))

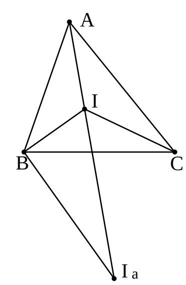


Figure 6-148

Example 6.149 (0.550, 8, 17) *Show that an external bisector of an angle of a triangle is parallel to the line joining the points where the circumcircle is met by the external (internal) bisectors of the other two angles of the triangle.*

Constructive description

((points B C I)
(incenter A I C B)
(inter M (l I C) (b B A))
(inter L_1 (t B B I) (b C A))
(parallel M L_1 A I))

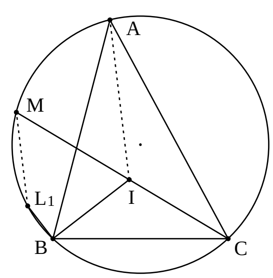


Figure 6-149

Example 6.150 (3.933, 4, 20) *The product of the four tritangent radii of a triangle is equal to the square of its area.*

Constructive description
((points $B\ C\ I$) (incenter $A\ I\ B\ C$)
(inter I_a (l $A\ I$) (t $B\ B\ I$))
(inter I_c (l $C\ I$) (l $B\ I_a$))
(inter I_b (l $B\ I$) (l $A\ I_c$))
(foot $X\ I\ B\ C$) (foot $X_a\ I_a\ B\ C$)
(foot $X_b\ I_b\ B\ C$) (foot $X_c\ I_c\ B\ C$)
($\overline{IX}^2\,\overline{I_aX_a}^2\,\overline{I_bX_b}^2\,\overline{I_cX_c}^2 = S_{ABC}S_{ABC}S_{ABC}S_{ABC}$))

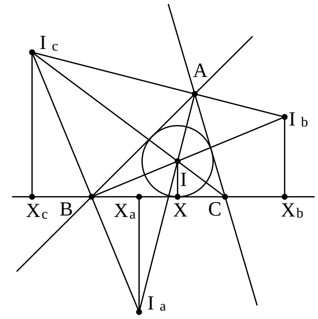


Figure 6-150

Example 6.151 [3] (0.717, 17, 28) *In triangle ABC, the bisector of angle A meets BC at L and the circumcircle of triangle ABC at N. The feet of the perpendiculars from L to AB and AC are K and M. Show that* $S_{ABC} = S_{AKNM}$.

Constructive description: ((points $I\ B\ A$) (incenter $C\ I\ B\ A$) (circumcenter $O\ A\ B\ C$) (inter L (l $B\ C$) (l $A\ I$)) (foot $M\ L\ A\ C$) (foot $K\ L\ A\ B$) (inter N (l $A\ I$) (cir $O\ A$)) ($S_{ABC} = S_{AKNM}$))

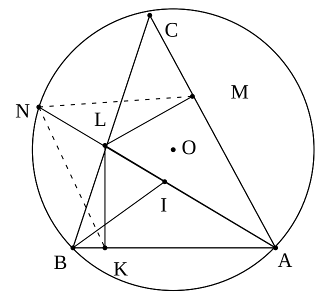


Figure 6-151

Example 6.152 (0.333, 6, 10) *The points of contact of a side of a triangle with the incircle and the excircle relative to this side are two isotomic points.*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(inter I_a (t $B\ B\ I$) (l $A\ I$))
(foot $X\ I\ B\ C$)
(foot $X_a\ I_a\ B\ C$)
(eqdistance $B\ X\ C\ X_a$))

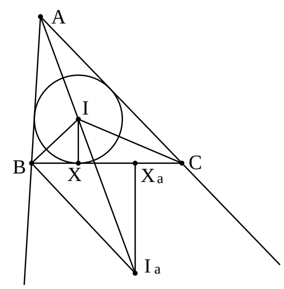


Figure 6-152

Example 6.153 (0.683, 6, 11) *In Figure 6-153,* $ZZ_a = a$.

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(inter I_a (t $B\ B\ I$) (l $A\ I$))
(foot $Z\ I\ A\ B$)
(foot $Z_a\ I_a\ A\ B$)
(eqdistance $Z\ Z_a\ B\ C$))

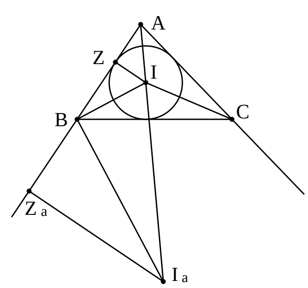


Figure 6-153

[3] This is a problem from the 1987 International Mathematical Olympiad.

Example 6.154 (0.583, 3, 20) *In Figure 6-154, $Y_bY_c = a$.*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(inter I_c (t $A\ A\ I$) (l $I\ C$))
(inter I_b (l $A\ I_c$) (l $I\ B$))
(foot $Y_c\ I_c\ A\ C$)
(foot $Y_b\ I_b\ A\ C$)
(eqdistance $Y_c\ Y_b\ B\ C$))

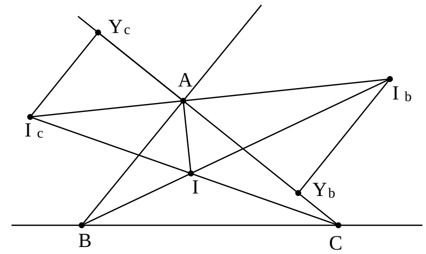


Figure 6-154

Example 6.155 (0.266, 2, 14) *The ratio of the area of a triangle to the area of the triangle determined by the points of contact of the sides with the incircle is equal to the ratio of the circumdiameter of the given triangle to its inradius.*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(circumcenter $O\ A\ B\ C$)
(foot $X\ I\ B\ C$)
(foot $Y\ I\ A\ C$)
(foot $Z\ I\ A\ B$)
($4S_{XYZ}^2\cdot\overline{OB}^2 = S_{ABC}S_{ABC}\overline{IX}^2$))

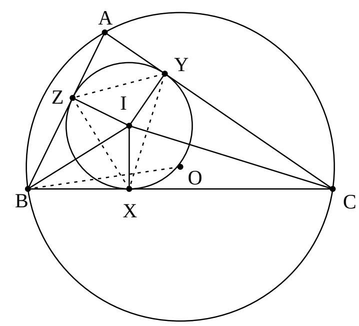


Figure 6-155

Example 6.156 (0.033, 1, 6) *Show that a parallel through a tritangent center to a side of a triangle is equal to the sum, or difference, of the two segments on the other two sides of the triangle between the two parallel lines considered.*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(inter M (l $A\ C$) (p $I\ A\ B$))
(inter N (l $I\ M$) (l $B\ C$))
(eqdistance $M\ A\ M\ I$))

The eliminants

$$P_{IMI} \overset{M}{=} \frac{P_{BAB}\cdot(S_{CIA})^2}{(S_{BCA})^2}$$
$$P_{AMA} \overset{M}{=} \frac{P_{CAC}\cdot(S_{BIA})^2}{(S_{BCA})^2}$$
$$S_{CIA} \overset{A}{=} \frac{P_{CIC}\cdot P_{CBI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$$
$$P_{BAB} \overset{A}{=} \frac{(P_{BIB})^2\cdot(P_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$
$$S_{BIA} \overset{A}{=} \frac{-P_{BIB}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$$
$$P_{CAC} \overset{A}{=} \frac{(P_{CIC})^2\cdot(P_{CBI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$

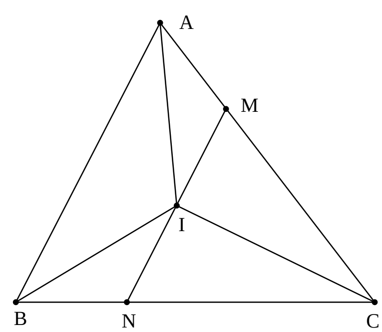


Figure 6-156

The machine proof

$$\frac{P_{AMA}}{P_{IMI}} \overset{M}{=} \frac{P_{CAC}\cdot S_{BIA}^2\cdot S_{BCA}^2}{P_{BAB}\cdot S_{CIA}^2\cdot S_{BCA}^2} \overset{simplify}{=} \frac{P_{CAC}\cdot(S_{BIA})^2}{P_{BAB}\cdot(S_{CIA})^2}$$

$$\overset{A}{=} \frac{P_{CIC}^2\cdot P_{CBI}^2\cdot((-P_{BIB}\cdot P_{BCI}\cdot S_{BCI}))^2\cdot(P_{BIC}\cdot P_{BCB})^2\cdot P_{BIC}^2\cdot P_{BCB}}{P_{BIB}^2\cdot P_{BCI}^2\cdot(P_{CIC}\cdot P_{CBI}\cdot S_{BCI})^2\cdot(P_{BIC}\cdot P_{BCB})^2\cdot P_{BIC}^2\cdot P_{BCB}} \overset{simplify}{=} 1$$

Example 6.157 (0.100, 1, 9) *Prove the formula:* $AZ \cdot BX \cdot CY = r \bigtriangledown ABC$.

Constructive description
((points B C I)
(incenter A I B C)
(foot X I B C)
(foot Y I A C)
(foot Z I A B)
($\overline{AZ}^2\overline{BX}^2\overline{CY}^2 = \overline{XI}^2 S_{ABC} S_{ABC}$))

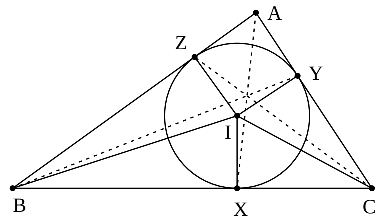


Figure 6-157

Example 6.158 (0.666, 10, 18) *The projection of the vertex B of the triangle ABC upon the internal bisector of the angle A lies on the line joining the points of contact of the incircle with the sides BC and AC.*

Constructive description
((points B C I)
(incenter A I B C)
(foot X I B C)
(foot Y I A C)
(foot L B A I)
(inter J (l A I) (l X Y))
($\frac{\overline{AL}}{\overline{IL}} = \frac{\overline{AJ}}{\overline{IJ}}$))

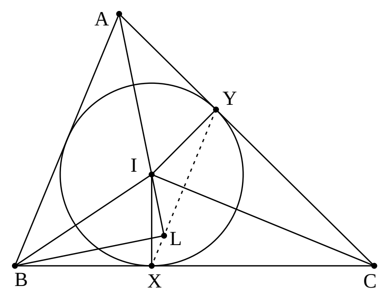


Figure 6-158

Example 6.159 (8.250, 21, 34) *The midpoint of a side of a triangle, the foot of the altitude on this side, and the projections of the ends of this side upon the internal bisector of the opposite angle are four cyclic points.*

Constructive description
((points B C I)
(incenter A I B C)
(foot X B A I)
(foot Y C A I)
(midpoint A_1 B C)
(foot D A B C)
(cocircle X Y A_1 D))

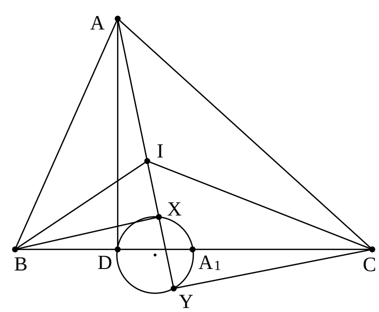


Figure 6-159

Example 6.160 (0.466, 8, 17) *Show that the midpoint of an altitude of a triangle, the point of contact of the corresponding side with the excircle relative to that side, and the incenter of the triangle are collinear.*

Constructive description
((points B C I)
(incenter A I B C)
(foot D A B C)
(inter I_a (t B B I) (l I A))
(foot X_a I_a B C)
(inter M (l A D) (l I X_a))
(midpoint M A D))

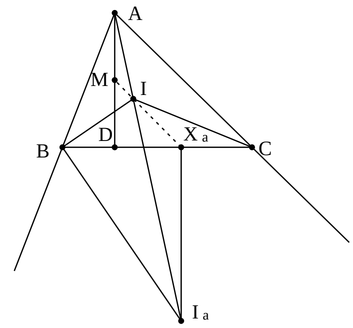


Figure 6-160

Example 6.161 (1.633, 10, 18) *The internal bisectors of the angles B, C of the triangle ABC meet the line AX_a joining A to the point of contact of BC with the excircle relative to this side in the points L, M. Prove that $AL/AM = AB/AC$.*

Constructive description
((points B C I)
(incenter A I B C)
(inter I_a (t B B I) (l I A))
(foot X_a I_a B C)
(inter L (l B I) (l A X_a))
(inter M (l C I) (l A X_a))
(eq-product A M A B A L A C))

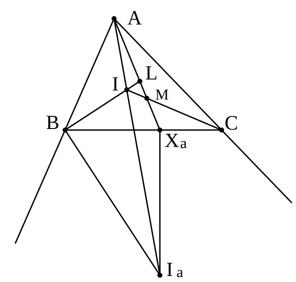


Figure 6-161

Example 6.162 (0.050, 1, 6) *Show that the product of the distances of the incenter of a triangle from the three vertices of the triangle is equal to $4Rr^2$.*

Constructive description
((points B C I)
(incenter A I B C)
(circumcenter O A B C)
(foot X I B C)
($16\overline{OB}^2\cdot\overline{IX}^4 = \overline{IA}^2\overline{IB}^2\overline{IC}^2$))

The eliminants

$$P_{IXI} \stackrel{X}{=} \frac{(16)\cdot(S_{BCI})^2}{P_{BCB}}$$
$$P_{BOB} \stackrel{O}{=} \frac{P_{CAC}\cdot P_{BAB}\cdot P_{BCB}}{(64)\cdot(S_{BCA})^2}$$
$$S_{BCA} \stackrel{A}{=} \frac{(-2)\cdot P_{CBI}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$$
$$P_{IAI} \stackrel{A}{=} \frac{(16)\cdot P_{CIC}\cdot P_{BIB}\cdot(S_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$
$$P_{BAB} \stackrel{A}{=} \frac{(P_{BIB})^2\cdot(P_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$
$$P_{CAC} \stackrel{A}{=} \frac{(P_{CIC})^2\cdot(P_{CBI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$

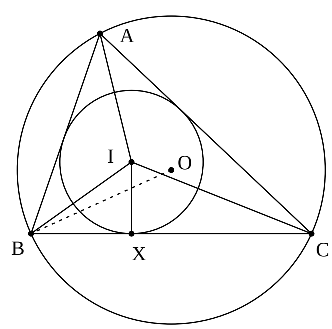


Figure 6-162

The machine proof

$$\frac{(16)\cdot(P_{IXI})^2\cdot P_{BOB}}{P_{IAI}\cdot P_{CIC}\cdot P_{BIB}}$$
$$\stackrel{X}{=} \frac{(16)\cdot((16S_{BCI}^2))^2\cdot P_{BOB}}{P_{IAI}\cdot P_{CIC}\cdot P_{BIB}\cdot(P_{BCB})^2}$$
$$\stackrel{O}{=} \frac{(4096)\cdot(S_{BCI})^4\cdot P_{CAC}\cdot P_{BAB}\cdot P_{BCB}}{P_{IAI}\cdot P_{CIC}\cdot P_{BIB}\cdot(P_{BCB})^2\cdot(64S_{BCA}^2)}$$

$$\overset{simplify}{=} \frac{(64)\cdot(S_{BCI})^4\cdot P_{CAC}\cdot P_{BAB}}{P_{IAI}\cdot P_{CIC}\cdot P_{BIB}\cdot P_{BCB}\cdot(S_{BCA})^2}$$
$$\overset{A}{=} \frac{(64)\cdot(S_{BCI})^4\cdot P_{CIC}^2\cdot P_{CBI}^2\cdot P_{BIB}^2\cdot P_{BCI}^2\cdot(P_{BIC}\cdot P_{BCB})^2\cdot P_{BIC}^2\cdot P_{BCB}}{(16P_{CIC}\cdot P_{BIB}\cdot S_{BCI}^2)\cdot P_{CIC}\cdot P_{BIB}\cdot P_{BCB}\cdot((-2P_{CBI}\cdot P_{BCI}\cdot S_{BCI}))^2\cdot(P_{BIC}^2\cdot P_{BCB})^2}$$
$$\overset{simplify}{=} 1$$

Example 6.163 (0.433, 6, 10) *If the lines AX, BY, CZ joining the vertices of a triangle ABC to the points of contact X, Y, Z of the sides BC, CA, AB with the incircle meet that circle again in the points X_1, Y_1, Z_1, show that: $AX \cdot XX_1 \cdot BC = 4rS$, where r, S are the inradius and the area of ABC.*

Constructive description
((points B C I)
(incenter A I B C)
(foot X I B C)
(foot Y I A C)
(foot Z I A B)
(inter X_1 (l A X) (cir I X))
(inter Y_1 (l B Y) (cir I Y))
(inter Z_1 (l C Z) (cir I Z))
($16\overline{IX}^2\cdot S_{ABC}^2 = \overline{XA}^2\overline{XX_1}^2\overline{BC}^2$))

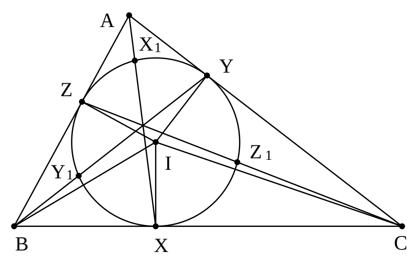


Figure 6-163

Example 6.164 (2.783, 16, 13) *If h, m, t are the altitude, the median, and the internal bisector issued from the same vertex of a triangle whose circumradius is R, show that $4R^2h^2(t^2 - h^2) = t^4(m^2 - h^2)$.*

Constructive description
((points B C I)
(incenter A I B C)
(circumcenter O A B C)
(foot D A B C)
(midpoint A_1 B C)
(inter T (l I A) (l B C))
($4\overline{TA}^2\cdot\overline{OB}^2\cdot\overline{AD}^2 - 4\overline{AD}^2\cdot\overline{OB}^2\cdot\overline{AD}^2 = \overline{AA_1}^2\cdot\overline{TA}^4 - \overline{AD}^2\cdot\overline{TA}^4$))

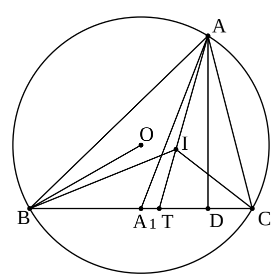


Figure 6-164

Example 6.165 (0.050, 1, 10) *The external bisectors of the angles of a triangle meet the opposite sides in three collinear points.*

Constructive description
((points B C I)
(incenter A I B C)

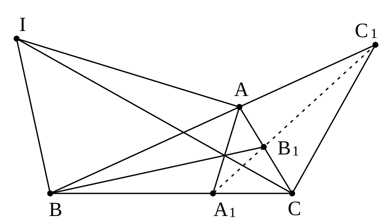


Figure 6-165

(inter A_1 (l B C) (t A A I))
(inter B_1 (l A C) (t B B I))
(inter C_1 (l A B) (t C C I))
(inter C_2 (l A_1 B_1) (l A B))
($\frac{\overline{AC_1}}{\overline{BC_1}} = \frac{\overline{AC_2}}{\overline{BC_2}}$))

The machine proof

$$\left(\frac{\overline{AC_1}}{\overline{BC_1}}\right)/\left(\frac{\overline{AC_2}}{\overline{BC_2}}\right)$$

$$\stackrel{C_2}{=} \frac{S_{BA_1B_1}}{S_{AA_1B_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}}$$

$$\stackrel{C_1}{=} \frac{P_{ICA} \cdot S_{BA_1B_1}}{S_{AA_1B_1} \cdot P_{BCI}}$$

$$\stackrel{B_1}{=} \frac{P_{ICA} \cdot P_{CBI} \cdot S_{BAA_1} \cdot P_{ICBA}}{P_{IBA} \cdot S_{CAA_1} \cdot P_{BCI} \cdot P_{ICBA}}$$

$$\stackrel{simplify}{=} \frac{P_{ICA} \cdot P_{CBI} \cdot S_{BAA_1}}{P_{IBA} \cdot S_{CAA_1} \cdot P_{BCI}}$$

$$\stackrel{A_1}{=} \frac{P_{ICA} \cdot P_{CBI} \cdot (-P_{BAI} \cdot S_{BCA}) \cdot P_{CIBA}}{P_{IBA} \cdot (-P_{CAI} \cdot S_{BCA}) \cdot P_{BCI} \cdot P_{CIBA}}$$

$$\stackrel{simplify}{=} \frac{P_{ICA} \cdot P_{CBI} \cdot P_{BAI}}{P_{IBA} \cdot P_{CAI} \cdot P_{BCI}}$$

$$\stackrel{A}{=} \frac{(-P_{CIC} \cdot P_{CBI} \cdot P_{BCI}) \cdot P_{CBI} \cdot (16 P_{BIB} \cdot P_{BCI} \cdot S_{BCI}^2) \cdot P_{BIC}^2 \cdot P_{BCB} \cdot P_{BIC} \cdot P_{BCB}}{(-P_{CBI} \cdot P_{BIB} \cdot P_{BCI}) \cdot (16 P_{CIC} \cdot P_{CBI} \cdot S_{BCI}^2) \cdot P_{BCI} \cdot P_{BIC}^2 \cdot P_{BCB} \cdot P_{BIC} \cdot P_{BCB}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AC_2}}{\overline{BC_2}} \stackrel{C_2}{=} \frac{S_{AA_1B_1}}{S_{BA_1B_1}} \quad \frac{\overline{AC_1}}{\overline{BC_1}} \stackrel{C_1}{=} \frac{P_{ICA}}{P_{BCI}}$$

$$S_{AA_1B_1} \stackrel{B_1}{=} \frac{P_{IBA} \cdot S_{CAA_1}}{P_{ICBA}}$$

$$S_{BA_1B_1} \stackrel{B_1}{=} \frac{P_{CBI} \cdot S_{BAA_1}}{P_{ICBA}}$$

$$S_{CAA_1} \stackrel{A_1}{=} \frac{-P_{CAI} \cdot S_{BCA}}{P_{CIBA}}$$

$$S_{BAA_1} \stackrel{A_1}{=} \frac{-P_{BAI} \cdot S_{BCA}}{P_{CIBA}}$$

$$P_{CAI} \stackrel{A}{=} \frac{(16) \cdot P_{CIC} \cdot P_{CBI} \cdot (S_{BCI})^2}{(P_{BIC})^2 \cdot P_{BCB}}$$

$$P_{IBA} \stackrel{A}{=} \frac{-P_{CBI} \cdot P_{BIB} \cdot P_{BCI}}{P_{BIC} \cdot P_{BCB}}$$

$$P_{BAI} \stackrel{A}{=} \frac{(16) \cdot P_{BIB} \cdot P_{BCI} \cdot (S_{BCI})^2}{(P_{BIC})^2 \cdot P_{BCB}}$$

$$P_{ICA} \stackrel{A}{=} \frac{-P_{CIC} \cdot P_{CBI} \cdot P_{BCI}}{P_{BIC} \cdot P_{BCB}}$$

Example 6.166 (1.850, 14, 28) *Prove that the triangle formed by the points of contact of the sides of a given triangle with the excircles corresponding to these sides has the same area with the triangle formed by the points of contact of the sides of the triangle with the inscribed circle.*

Constructive description
((points B C I)
(incenter A I C B)
(inter I_a (t B B I) (l A I))
(foot X I B C) (foot Y I A C)
(foot Z I A B) (foot X_a I_a B C)
(inter I_b (t C C I) (l B I))
(inter I_c (t B B I) (l C I))
(foot Y_b I_b A C) (foot Z_c I_c B A)
($S_{XYZ} = S_{X_aY_bZ_c}$))

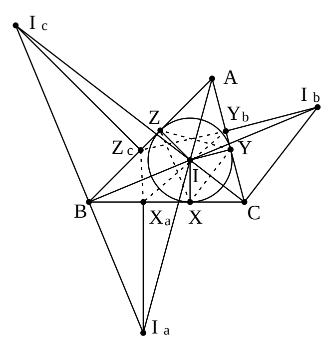


Figure 6-166

Example 6.167 (Gergonne Point) (0.050, 1, 7) *The lines joining the vertices of a triangle to the points of contact of the opposite sides with the inscribed circle are concurrent (The Gergonne point).*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(foot $X\ I\ B\ C$)
(foot $Y\ I\ A\ C$)
(foot $Z\ I\ A\ B$)
($\frac{\overline{BX}}{\overline{XC}}\frac{\overline{CY}}{\overline{YA}}\frac{\overline{AZ}}{\overline{ZB}} = 1$))

The eliminants

$$\frac{\overline{AZ}}{\overline{BZ}} \stackrel{Z}{=} \frac{P_{BAI}}{-P_{IBA}}$$
$$\frac{\overline{CY}}{\overline{AY}} \stackrel{Y}{=} \frac{P_{ICA}}{-P_{CAI}}$$
$$\frac{\overline{BX}}{\overline{CX}} \stackrel{X}{=} \frac{P_{CBI}}{-P_{BCI}}$$
$$P_{CAI} \stackrel{A}{=} \frac{(16)\cdot P_{CIC}\cdot P_{CBI}\cdot (S_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$
$$P_{IBA} \stackrel{A}{=} \frac{-P_{CBI}\cdot P_{BIB}\cdot P_{BCI}}{P_{BIC}\cdot P_{BCB}}$$
$$P_{ICA} \stackrel{A}{=} \frac{-P_{CIC}\cdot P_{CBI}\cdot P_{BCI}}{P_{BIC}\cdot P_{BCB}}$$
$$P_{BAI} \stackrel{A}{=} \frac{(16)\cdot P_{BIB}\cdot P_{BCI}\cdot (S_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$$

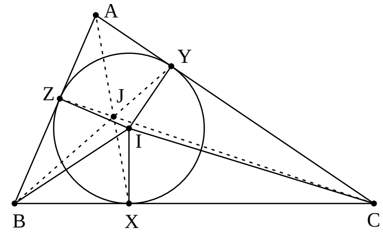


Figure 6-167

The machine proof

$$-\frac{\overline{AZ}}{\overline{BZ}}\cdot\frac{\overline{CY}}{\overline{AY}}\cdot\frac{\overline{BX}}{\overline{CX}} \stackrel{Z}{=} \frac{-P_{BAI}}{-P_{IBA}}\cdot\frac{\overline{CY}}{\overline{AY}}\cdot\frac{\overline{BX}}{\overline{CX}} \stackrel{Y}{=} \frac{P_{BAI}\cdot P_{ICA}}{P_{IBA}\cdot(-P_{CAI})}\cdot\frac{\overline{BX}}{\overline{CX}} \stackrel{X}{=} \frac{-P_{BAI}\cdot P_{ICA}\cdot P_{CBI}}{P_{IBA}\cdot P_{CAI}\cdot(-P_{BCI})}$$
$$\stackrel{A}{=} \frac{(16P_{BIB}\cdot P_{BCI}\cdot S^2_{BCI})\cdot(-P_{CIC}\cdot P_{CBI}\cdot P_{BCI})\cdot P_{CBI}\cdot P^2_{BIC}\cdot P_{BCB}\cdot P_{BIC}\cdot P_{BCB}}{(-P_{CBI}\cdot P_{BIB}\cdot P_{BCI})\cdot(16P_{CIC}\cdot P_{CBI}\cdot S^2_{BCI})\cdot P_{BCI}\cdot P_{BIC}\cdot P_{BCB}\cdot P^2_{BIC}\cdot P_{BCB}} \stackrel{simplify}{=} 1$$

Example 6.168 (Nagel Point) (0.650, 3, 31) *The lines joining the vertices of a triangle to the points of contact of the opposite sides with the excircles relative to those sides are concurrent (The Nagel point).*

Constructive description
((points $B\ C\ I$)
(incenter $A\ I\ C\ B$)
(inter I_a (t $B\ I\ B$) (l $A\ I$))
(inter I_b (l $B\ I$) (l $I_a\ C$))
(inter I_c (l $I_a\ B$) (l $C\ I$))
(foot $X_a\ I_a\ B\ C$) (foot $Y_b\ I_b\ A\ C$)
(foot $Z_c\ I_c\ A\ B$)
(inter J (l $A\ X_a$) (l $B\ Y_b$))
(inter K (l $C\ Z_c$) (l $B\ Y_b$)) ($\frac{\overline{BJ}}{\overline{Y_bJ}} = \frac{\overline{BK}}{\overline{Y_bK}}$))

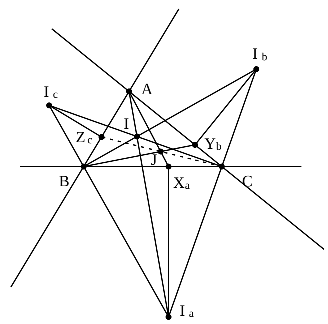


Figure 6-168

Example 6.169 (3.583, 12, 30) *Show that the line joining the incenter of the triangle ABC to the midpoint of the segment joining A to the Nagel point of ABC is bisected by the median issued from A.*

Constructive description
((points $B\ C\ I$) (incenter $A\ I\ B\ C$)
(inter I_a (l $A\ I$) (t $B\ B\ I$))
(inter I_b (l $C\ I_a$) (l $B\ I$))
(foot $X_a\ I_a\ B\ C$) (foot $Y_b\ I_b\ A\ C$)
(inter N (l $B\ Y_b$) (l $A\ X_a$))

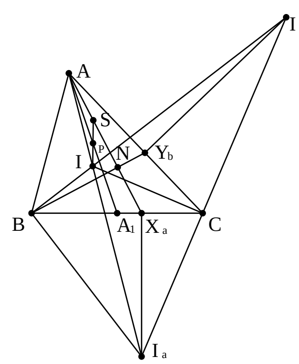


Figure 6-169

(midpoint A_1 B C) (midpoint S N A)
(inter P (l I S) (l A_1 A))
(midpoint P I S))

Example 6.170 (0.816, 12, 13) *The sides AB, AC intercept the segments DE, FG on the parallels to the side BC through the tritangent centers I and I_a. Show that: $2/BC = 1/DE + 1/FG$.*

Constructive description
((points B C I)
(incenter A I B C)
(inter I_a (t B B I) (l A I))
(inter D (p I B C) (l A B))
(inter E (l D I) (l A C))
(inter F (p I_a B C) (l A B))
(inter G (l F I_a) (l A C))
($\frac{\overline{BC}}{\overline{DE}} + \frac{\overline{BC}}{\overline{FG}} = 2$))

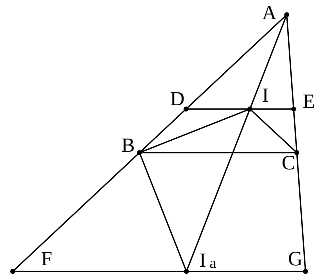


Figure 6-170

Example 6.171 (1.817, 27, 20) *Show that the perpendiculars to the internal bisectors of a triangle at the incenter meet the respective sides in three points lying on a line perpendicular to the line joining the incenter to the circumcenter of the triangle.*

Constructive description
((points B C I)
(incenter A I B C)
(circumcenter O A B C)
(inter X (t I I A) (l B C))
(inter Y (t I I B) (l A C))
(inter Z (t I I C) (l A B))
(perpendicular O I X Z))

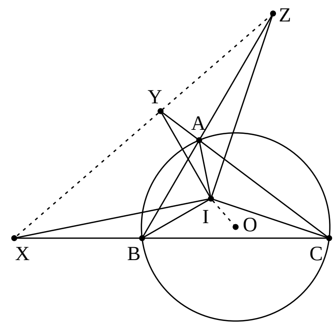


Figure 6-171

Example 6.172 (0.416, 4, 19) *The side BC of the triangle ABC touches the incircle (I) in X and the excircle (I_a) relative to BC in X_a. Show that the line AX_a passes through the diametric opposite X_1 of X on (I).*

Constructive description
((points B C I)
(incenter A I C B)
(inter I_a (t B I B) (l A I))
(foot X I B C)
(foot X_a I_a B C)
(inter X_1 (l I X) (l A X_a))
(midpoint I X X_1))

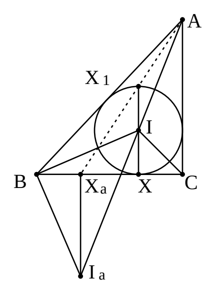


Figure 6-172

Example 6.173 (0.816, 6, 15) *With the notations of Example 6.172, show that if the line A_1I meets the altitude AD of ABC in P, then AP is equal to the inradius of ABC.*

Constructive description
((points B C I)
(incenter A I C B)
(foot X I B C)
(foot D A B C)
(midpoint A_1 B C)
(inter P (l I A_1) (l A D))
(eqdistance A P I X))

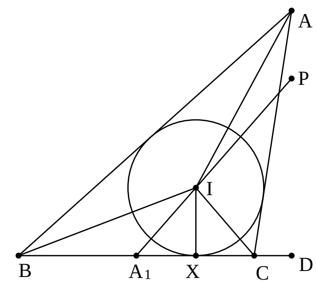


Figure 6-173

Example 6.174 (0.267, 10, 16) *With the notations of Example 6.172, if the parallels to AX_a through B, C meet the bisectors CI, BI in L, M show that the line LM is parallel to BC.*

Constructive description
((points B C I)
(incenter A I C B)
(inter I_a (t B I B) (l A I))
(foot X I B C)
(foot X_a I_a B C)
(inter L (p B A X_a) (l C I))
(inter M (p C A X_a) (l B I))
(parallel L M B C))

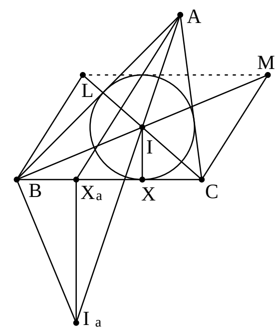


Figure 6-174

Example 6.175 (0.150, 2, 10) *Show that the trilinear polar (see Example 6.203) of the incenter of a triangle passes through the feet of the external bisectors, and this line is perpendicular to the line joining the incenter to the circumcenter, and this lines is perpendicular to the line joining the incenter to circumcenter.*

Constructive description
((points B C I)
(incenter A I B C)
(circumcenter O A B C)
(inter E (l B I) (l A C))
(inter F (l C I) (l B A))
(inter D (l A I) (l B C))
(inter X (l E F) (l B C))
(inter Z (l D E) (l A B))
(perpendicular X A A I))

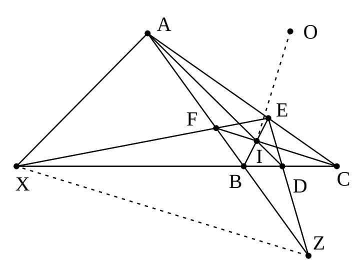


Figure 6-175

The machine proof

P_{IAX}

$\stackrel{n}{=} P_{CAI}\cdot S_{BEF}-P_{BAI}\cdot S_{CEF}$

$\stackrel{simplify}{=} P_{CAI}\cdot S_{BEF}-P_{BAI}\cdot S_{CEF}$

$\stackrel{n}{=} S_{BCAI}\cdot P_{CAI}\cdot S_{BAE}\cdot S_{BCI}-S_{BCAI}\cdot P_{BAI}\cdot S_{CIE}\cdot S_{BCA}$

$\stackrel{simplify}{=} P_{CAI}\cdot S_{BAE}\cdot S_{BCI}-P_{BAI}\cdot S_{CIE}\cdot S_{BCA}$

$\stackrel{n}{=} -S_{BCIA}\cdot P_{CAI}\cdot S_{BIA}\cdot S_{BCA}\cdot S_{BCI}-S_{BCIA}\cdot P_{BAI}\cdot S_{CIA}\cdot S_{BCA}\cdot S_{BCI}$

$\stackrel{simplify}{=} -(P_{CAI}\cdot S_{BIA}+P_{BAI}\cdot S_{CIA})\cdot S_{BCA}\cdot S_{BCI}$

$\stackrel{n}{=} -(0)\cdot(-2P_{CBI}\cdot P_{BCI}\cdot S_{BCI})\cdot S_{BCI}$

$\stackrel{simplify}{=} 0$

The eliminants

$P_{IAX}\stackrel{X}{=}\frac{P_{CAI}\cdot S_{BEF}-P_{BAI}\cdot S_{CEF}}{S_{BECF}}$

$S_{CEF}\stackrel{F}{=}\frac{S_{CIE}\cdot S_{BCA}}{-S_{BCAI}}$

$S_{BEF}\stackrel{F}{=}\frac{-S_{BAE}\cdot S_{BCI}}{S_{BCAI}}$

$S_{CIE}\stackrel{E}{=}\frac{S_{CIA}\cdot S_{BCI}}{S_{BCIA}}$

$S_{BAE}\stackrel{E}{=}\frac{-S_{BIA}\cdot S_{BCA}}{S_{BCIA}}$

$S_{BCA}\stackrel{A}{=}\frac{(-2)\cdot P_{CBI}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$

$S_{CIA}\stackrel{A}{=}\frac{P_{CIC}\cdot P_{CBI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$

$P_{BAI}\stackrel{A}{=}\frac{(16)\cdot P_{BIB}\cdot P_{BCI}\cdot(S_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$

$S_{BIA}\stackrel{A}{=}\frac{-P_{BIB}\cdot P_{BCI}\cdot S_{BCI}}{P_{BIC}\cdot P_{BCB}}$

$P_{CAI}\stackrel{A}{=}\frac{(16)\cdot P_{CIC}\cdot P_{CBI}\cdot(S_{BCI})^2}{(P_{BIC})^2\cdot P_{BCB}}$

The second result has also been proved by the program.

Example 6.176 (0.633, 3, 27) *Show that the mediators of the internal bisectors of the angles of a triangle meet the respective sides of the triangle in three collinear points.*

Constructive description
((points B C I)
(incenter A I B C)
(inter U (l A I) (l B C))
(inter V (l B I) (l A C))
(inter W (l C I) (l A B))
(inter P (b A U) (l B C))
(inter Q (b B V) (l A C))
(inter S (b C W) (l A B))
(inter S_1 (l A B) (l P Q))
($\frac{\overline{AS}}{\overline{BS}} = \frac{\overline{AS_1}}{\overline{BS_1}}$))

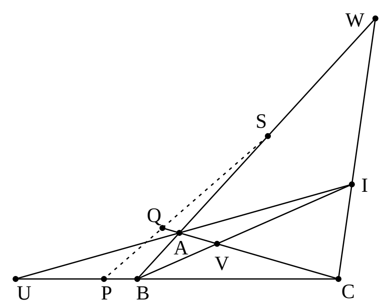


Figure 6-176

Example 6.177 (0.450, 6, 25) *Show that the lines joining the vertices of a triangle to the projections of the incenter upon the mediators of the respectively opposite sides meet in a point - the isotomic conjugate of the Gergonne point of the triangle.*

Constructive description
((points B C I) (incenter A I B C)
(foot X I B C) (midpoint A_1 B C)
(midpoint B_1 C A) (midpoint C_1 A B)
(inter D (t A_1 A_1 C) (p I B C))
(inter E (t B_1 B_2 C) (p I A C))
(inter F (t C_1 C_1 B) (p I A B))
(inter J (l A D) (l B E))
(inter Y (l A D) (l B C)) (midpoint A_1 X Y))

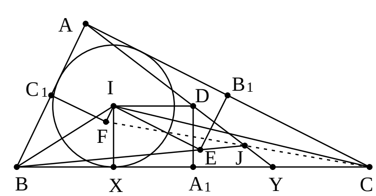


Figure 6-177

Example 6.178 (0.516, 3, 19) *Show that the line AI meets the sides XY, XZ in two points P, Q inverse with respect to the incircle (I) = XYZ, and the perpendiculars to AI at P, Q pass through the vertices B, C of the given triangle ABC.*

Constructive description
((points *B C I*)
(incenter *A I B C*)
(foot *X I B C*)
(foot *Y I A C*)
(foot *Z I A B*)
(inter *P* (l *I A*) (l *X Y*))
(inter *Q* (l *I A*) (l *X Z*))
(inversion *I X P Q*))

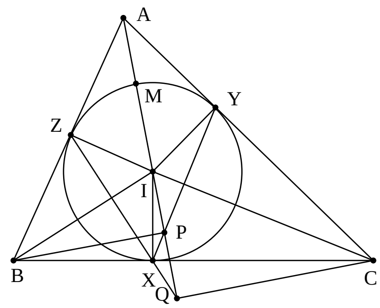


Figure 6-178

Definition *The symmetric of a median of a triangle with respect to the internal bisector issued from the same vertex is a symmedian of the triangle*

Example 6.179 (1.250, 6,18) *The three symmedians of a triangle are concurrent (The Lemoine Point or the symmedian point).*

Constructive description
((points *A B C*)
(midpoint M_1 *B C*)
(midpoint M_2 *A C*)
(midpoint M_3 *A B*)
(on B_1 (a *B C* M_2 *B A*))
(on C_1 (a *C A* M_3 *C B*))
(inter *I* (l *C* C_1) (l *B* B_1))
(eqangle *B A I* M_1 *A C*))

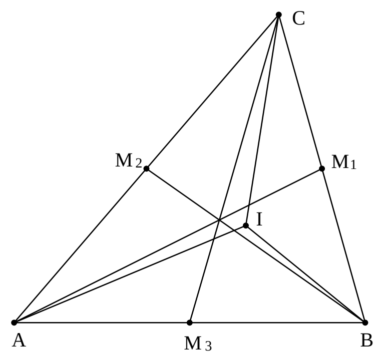


Figure 6-179

Example 6.180 (0.566, 5, 19) *The three symmetrics of the three lines joining a point and the three vertices of a triangle with respect to the internal bisectors issued from the same vertices are concurrent. (The isogonal conjugate point).*

Constructive description
((points *A B C M*)
(on A_1 (a *A C M A B*))
(on B_1 (a *B A M B C*))
(inter *N* (l *A* A_1) (l *B* B_1))
(eqangle *A C M N C B*))

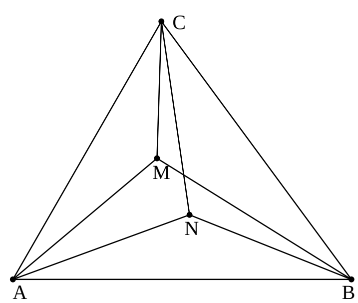


Figure 6-180

6.3.7 Intercept Triangles

Definition. *Let L, M, N be three points on the sides BC, CA, AB of triangle ABC. Then triangle LMN and the triangle determined by lines AL, BM, and CN are called the intercept triangles of triangle ABC for points L, M, N.*

Example 6.181 (0.033, 2, 5) *Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that $BA_1/BC = CB_1/CA = AC_1/AB = 1/3$. Show that the area of the triangle determined by lines AA_1, BB_1 and CC_1 is one seventh of the area of triangle ABC.*

We only need to show $\frac{S_{ABA_2}}{S_{ABC}} = 2/7$.

Constructive description: ((points $A\ B\ C$) (lratio $A_1\ B\ C\ 1/3$) (lratio $B_1\ C\ A\ 1/3$) (inter A_2 (l $A\ A_1$) (l $B\ B_1$)) ($2S_{ABC} = 7S_{ABA_2}$)).

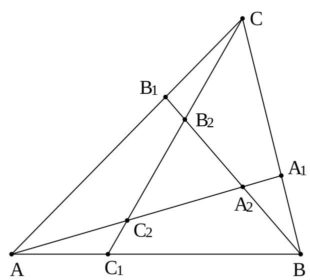


Figure 6-181

The machine proof

$$\frac{(2)\cdot S_{ABC}}{(7)\cdot S_{ABA_2}}$$
$$\stackrel{A_2}{=} \frac{(2)\cdot S_{ABC}\cdot S_{ABA_1B_1}}{(7)\cdot S_{ABB_1}\cdot S_{ABA_1}}$$
$$\stackrel{B_1}{=} \frac{(2)\cdot S_{ABC}\cdot(-\frac{2}{3}S_{ACA_1}+S_{ABA_1})}{(7)\cdot(\frac{2}{3}S_{ABC})\cdot S_{ABA_1}}$$
$$\stackrel{simplify}{=} \frac{-(2S_{ACA_1}-3S_{ABA_1})}{(7)\cdot S_{ABA_1}}$$
$$\stackrel{A_1}{=} \frac{-(-\frac{7}{3}S_{ABC})}{(7)\cdot(\frac{1}{3}S_{ABC})} \stackrel{simplify}{=} 1$$

The eliminants

$$S_{ABA_2} \stackrel{A_2}{=} \frac{S_{ABB_1}\cdot S_{ABA_1}}{S_{ABA_1B_1}}$$
$$S_{ABB_1} \stackrel{B_1}{=} \frac{2}{3}(S_{ABC})$$
$$S_{ABA_1B_1} \stackrel{B_1}{=} -\frac{1}{3}(2S_{ACA_1}-3S_{ABA_1})$$
$$S_{ABA_1} \stackrel{A_1}{=} \frac{1}{3}(S_{ABC})$$
$$S_{ACA_1} \stackrel{A_1}{=} -\frac{2}{3}(S_{ABC})$$

Example 6.182 (0.017, 3, 5) *Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that $BA_1/BC = CB_1/CA = AC_1/AB = r$. The intercept triangle determined by lines AA_1, BB_1 and CC_1 is $A_2B_2C_2$ (Figure 6-181). Show that $\frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{(2r-1)^2}{r^2-r+1}$.*

Constructive description: ((points $A\ B\ C$) (lratio $A_1\ B\ C\ r$) (lratio $B_1\ C\ A\ r$) (inter A_2 (l $A\ A_1$) (l $B\ B_1$)) ($(r^2-r+1)\cdot S_{A_2AB} = (-r^2+r)\cdot S_{ABC}$)).

The machine proof

$$\frac{(r^2-r+1)\cdot S_{ABA_2}}{-(r-1)\cdot r\cdot S_{ABC}}$$

$$\stackrel{A_2}{=} \frac{(r^2-r+1)\cdot S_{ABB_1}\cdot S_{ABA_1}}{-(r-1)\cdot r\cdot S_{ABC}\cdot S_{ABA_1B_1}}$$

$$\stackrel{B_1}{=} \frac{-(r^2-r+1)\cdot(-S_{ABC}\cdot r+S_{ABC})\cdot S_{ABA_1}}{(r-1)\cdot r\cdot S_{ABC}\cdot(S_{ACA_1}\cdot r-S_{ACA_1}+S_{ABA_1})}$$

$$\stackrel{simplify}{=} \frac{(r^2-r+1)\cdot S_{ABA_1}}{r\cdot(S_{ACA_1}\cdot r-S_{ACA_1}+S_{ABA_1})}$$

$$\stackrel{A_1}{=} \frac{(r^2-r+1)\cdot S_{ABC}\cdot r}{r\cdot(S_{ABC}\cdot r^2-S_{ABC}\cdot r+S_{ABC})} \stackrel{simplify}{=} 1$$

The eliminants

$$S_{ABA_2} \stackrel{A_2}{=} \frac{S_{ABB_1}\cdot S_{ABA_1}}{S_{ABA_1B_1}}$$

$$S_{ABA_1B_1} \stackrel{B_1}{=} S_{ACA_1}\cdot r-S_{ACA_1}+S_{ABA_1}$$

$$S_{ABB_1} \stackrel{B_1}{=} -\left((r-1)\cdot S_{ABC}\right)$$

$$S_{ACA_1} \stackrel{A_1}{=} (r-1)\cdot S_{ABC}$$

$$S_{ABA_1} \stackrel{A_1}{=} S_{ABC}\cdot r$$

Example 6.183 (0.466, 5, 16) *Use the same notations as 6.182. If $BA_1/A_1C = r_1$, $CB_1/B_1A = r_2$, $AC_1/C_1B = r_3$ then $\frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{(r_3r_2r_1-1)^2}{(r_2r_1+r_1+1)(r_3r_1+r_3+1)(r_3r_2+r_2+1)}$.*

Constructive description: (Formula derivation)

((points A B C) (mratio A_1 B C r_1) (mratio B_1 C A r_2) (mratio C_1 A B r_3)
(inter A_2 (l A A_1) (l B B_1)) (inter B_2 (l B B_1) (l C C_1)) (inter C_2 (l C C_1) (l A A_1)) ($\frac{S_{A_2B_2C_2}}{S_{ABC}}$))

Example 6.184 (0.033, 2, 4) *Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that $BA_1/A_1C = CB_1/B_1A = AC_1/C_1B = k$. Furthermore, let A_2, B_2, C_2 be points on the sides B_1C_1, C_1A_1, A_1C_1 of a triangle $A_1B_1C_1$ such that $C_1A_2/A_2B_1 = A_1B_2/B_2C_1 = B_1C_2/C_2A_1 = k$. Show that triangles ABC and $A_2B_2C_2$ are homothetic.*

Constructive description

((points A B C)
(lratio C_1 A B Y_T)
(lratio A_1 B C Y_T)
(lratio B_1 C A Y_T)
(lratio A_2 C_1 B_1 Y_T)
(lratio B_2 A_1 C_1 Y_T)
(lratio C_2 B_1 A_1 Y_T)
(inter O (l A A_2) (l B B_2))
(inter D (l C C_2) (l A B))
(parallel A_2 B_2 A B))

The machine proof

$$\frac{S_{ABA_2}}{S_{ABB_2}}$$

$$\stackrel{B_2}{=} \frac{S_{ABA_2}}{-S_{ABA_1}\cdot Y_T+S_{ABA_1}}$$

$$\stackrel{simplify}{=} \frac{-S_{ABA_2}}{(Y_T-1)\cdot S_{ABA_1}}$$

$$\stackrel{A_2}{=} \frac{-S_{ABB_1}\cdot Y_T}{(Y_T-1)\cdot S_{ABA_1}}$$

$$\stackrel{B_1}{=} \frac{-(-S_{ABC}\cdot Y_T+S_{ABC})\cdot Y_T}{(Y_T-1)\cdot S_{ABA_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ABC}\cdot Y_T}{S_{ABA_1}}$$

$$\stackrel{A_1}{=} \frac{S_{ABC}\cdot Y_T}{S_{ABC}\cdot Y_T}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{ABB_2} \stackrel{B_2}{=} -\left((Y_T-1)\cdot S_{ABA_1}\right)$$

$$S_{ABA_2} \stackrel{A_2}{=} S_{ABB_1}\cdot Y_T$$

$$S_{ABB_1} \stackrel{B_1}{=} -\left((Y_T-1)\cdot S_{ABC}\right)$$

$$S_{ABA_1} \stackrel{A_1}{=} S_{ABC}\cdot Y_T$$

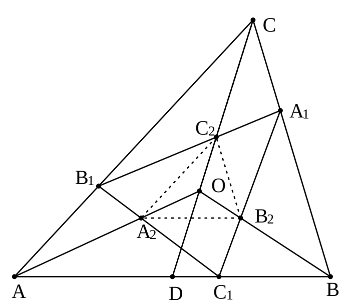


Figure 6-184

Example 6.185 (0.083, 4, 11) *Let M, N, and P be three points on the sides AB, BC and AC of a triangle ABC such that $AM/MB = BN/NC = CP/PA$. Show that the point of intersection of the medians of $\triangle MNP$ coincides with the point of intersection of the medians of $\triangle ABC$.*

Constructive description
((points $A\ B\ C$)
(lratio $M\ A\ B\ Y_T$)
(lratio $N\ B\ C\ Y_T$)
(lratio $P\ C\ A\ Y_T$)
(centroid $G\ A\ B\ C$)
(inter L (l $N\ P$) (l $M\ G$))
($\frac{\overline{PL}}{\overline{NL}}=-1$))

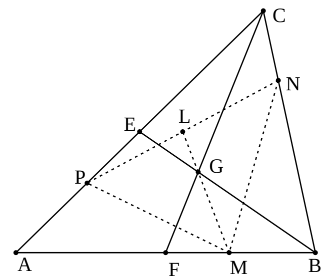


Figure 6-185

The machine proof

$$-\left(\frac{\overline{PL}}{\overline{NL}}\right)$$

$$\overset{L}{=}\frac{-S_{MPG}}{-(-S_{MNG})}$$

$$\overset{G}{=}\frac{-(S_{CMP}+S_{BMP}+S_{AMP})\cdot(3)}{(S_{CMN}+S_{BMN}+S_{AMN})\cdot(3)}$$

$$\overset{P}{=}\frac{-(S_{BCM}\cdot Y_T-S_{BCM}+2S_{ACM}\cdot Y_T-S_{ACM})}{S_{CMN}+S_{BMN}+S_{AMN}}$$

$$\overset{N}{=}\frac{-(S_{BCM}\cdot Y_T-S_{BCM}+2S_{ACM}\cdot Y_T-S_{ACM})}{-2S_{BCM}\cdot Y_T+S_{BCM}-S_{ACM}\cdot Y_T}$$

$$\overset{M}{=}\frac{-3S_{ABC}\cdot Y_T^2+3S_{ABC}\cdot Y_T-S_{ABC}}{-3S_{ABC}\cdot Y_T^2+3S_{ABC}\cdot Y_T-S_{ABC}}$$

$$\overset{simplify}{=}1$$

The eliminants

$$\frac{\overline{PL}}{\overline{NL}}\overset{L}{=}\frac{S_{MPG}}{S_{MNG}}$$

$$S_{MNG}\overset{G}{=}\frac{1}{3}\left(S_{CMN}+S_{BMN}+S_{AMN}\right)$$

$$S_{MPG}\overset{G}{=}\frac{1}{3}\left(S_{CMP}+S_{BMP}+S_{AMP}\right)$$

$$S_{AMP}\overset{P}{=}(Y_T-1)\cdot S_{ACM}$$

$$S_{BMP}\overset{P}{=}(Y_T-1)\cdot S_{BCM}$$

$$S_{CMP}\overset{P}{=}S_{ACM}\cdot Y_T$$

$$S_{AMN}\overset{N}{=}-\left(S_{ACM}\cdot Y_T\right)$$

$$S_{BMN}\overset{N}{=}-\left(S_{BCM}\cdot Y_T\right)$$

$$S_{CMN}\overset{N}{=}-\left((Y_T-1)\cdot S_{BCM}\right)$$

$$S_{ACM}\overset{M}{=}-\left(S_{ABC}\cdot Y_T\right)$$

$$S_{BCM}\overset{M}{=}-\left((Y_T-1)\cdot S_{ABC}\right)$$

Example 6.186 (0.266, 3, 13) *Let M, N, and P be the same as in Example 6.185. Show that the point of intersection of the medians of the triangle formed by lines AN, BP and CM coincides with the point of intersection of the medians of $\triangle ABC$.*

Constructive description
((points $A\ B\ C$)
(lratio $M\ A\ B\ Y_T$) (lratio $N\ B\ C\ Y_T$)
(lratio $P\ C\ A\ Y_T$) (centroid $G\ A\ B\ C$)
(inter X (l $B\ P$) (l $A\ N$))
(inter Y (l $B\ P$) (l $C\ M$))
(inter Z (l $A\ N$) (l $C\ M$))
(inter L (l $Y\ Z$) (l $X\ G$))
($\frac{\overline{YL}}{\overline{ZL}}=-1$))

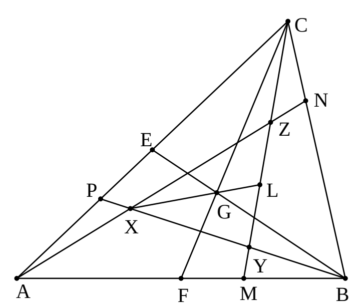


Figure 6-186

The machine proof

$$-\left(\frac{\overline{YL}}{\overline{ZL}}\right)$$

$$\stackrel{L}{=} \frac{-S_{GXY}}{-(-S_{GXZ})}$$

$$\stackrel{Z}{=} \frac{-S_{GXY}\cdot S_{ACNM}}{S_{CMX}\cdot S_{ANG}}$$

$$\stackrel{Y}{=} \frac{-S_{CMX}\cdot S_{BPG}\cdot S_{ACNM}}{S_{CMX}\cdot S_{ANG}\cdot S_{BCPM}}$$

$$\stackrel{simplify}{=} \frac{-S_{BPG}\cdot S_{ACNM}}{S_{ANG}\cdot S_{BCPM}}$$

$$\stackrel{G}{=} \frac{-(-S_{BCP}+S_{ABP})\cdot S_{ACNM}\cdot(3)}{(-S_{ACN}-S_{ABN})\cdot S_{BCPM}\cdot(3)}$$

$$\stackrel{P}{=} \frac{-(2S_{ABC}\cdot Y_T-S_{ABC})\cdot S_{ACNM}}{(S_{ACN}+S_{ABN})\cdot(S_{BCM}-S_{ACM}\cdot Y_T)}$$

$$\stackrel{simplify}{=} \frac{-(2Y_T-1)\cdot S_{ABC}\cdot S_{ACNM}}{(S_{ACN}+S_{ABN})\cdot(S_{BCM}-S_{ACM}\cdot Y_T)}$$

$$\stackrel{N}{=} \frac{-(2Y_T-1)\cdot S_{ABC}\cdot(S_{BCM}\cdot Y_T-S_{BCM}+S_{ACM})}{(2S_{ABC}\cdot Y_T-S_{ABC})\cdot(S_{BCM}-S_{ACM}\cdot Y_T)}$$

$$\stackrel{simplify}{=} \frac{-(S_{BCM}\cdot Y_T-S_{BCM}+S_{ACM})}{S_{BCM}-S_{ACM}\cdot Y_T}$$

$$\stackrel{M}{=} \frac{-(-S_{ABC}\cdot Y_T^2+S_{ABC}\cdot Y_T-S_{ABC})}{S_{ABC}\cdot Y_T^2-S_{ABC}\cdot Y_T+S_{ABC}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{YL}}{\overline{ZL}} \stackrel{L}{=} \frac{S_{GXY}}{S_{GXZ}}$$

$$S_{GXZ} \stackrel{Z}{=} \frac{S_{CMX}\cdot S_{ANG}}{S_{ACNM}}$$

$$S_{GXY} \stackrel{Y}{=} \frac{S_{CMX}\cdot S_{BPG}}{S_{BCPM}}$$

$$S_{ANG} \stackrel{G}{=} -\frac{1}{3}(S_{ACN}+S_{ABN})$$

$$S_{BPG} \stackrel{G}{=} -\frac{1}{3}(S_{BCP}-S_{ABP})$$

$$S_{BCPM} \stackrel{P}{=} S_{BCM}-S_{ACM}\cdot Y_T$$

$$S_{ABP} \stackrel{P}{=} -((Y_T-1)\cdot S_{ABC})$$

$$S_{BCP} \stackrel{P}{=} S_{ABC}\cdot Y_T$$

$$S_{ABN} \stackrel{N}{=} S_{ABC}\cdot Y_T$$

$$S_{ACN} \stackrel{N}{=} (Y_T-1)\cdot S_{ABC}$$

$$S_{ACNM} \stackrel{N}{=} S_{BCM}\cdot Y_T-S_{BCM}+S_{ACM}$$

$$S_{ACM} \stackrel{M}{=} -(S_{ABC}\cdot Y_T)$$

$$S_{BCM} \stackrel{M}{=} -((Y_T-1)\cdot S_{ABC})$$

Example 6.187 (0.533, 2, 32) *Through each of the vertices of a triangle ABC we draw two lines dividing the opposite side into three equal parts. These six lines determine a hexagon. Prove that the diagonals joining opposite vertices of this hexagon meet in a point.*

Constructive description

((points A B C) (lratio A_1 B C $1/3$)
(lratio A_2 B C $2/3$) (lratio B_1 C A $1/3$)
(lratio B_2 C A $2/3$) (lratio C_1 A B $1/3$)
(lratio C_2 A B $2/3$) (inter X (l A A_1) (l B B_2))
(inter Y (l C C_1) (l A A_2)) (inter Z (l B B_1) (l C C_2))
(inter U (l A A_2) (l B B_1)) (inter V (l C C_2) (l A A_1))
(inter W (l B B_2) (l C C_1)) (inter O (l Y V) (l U X))
(inter Z_O (l W Z) (l U X)) ($\frac{\overline{UO}}{\overline{XO}} = \frac{\overline{UZ_O}}{\overline{XZ_O}}$))

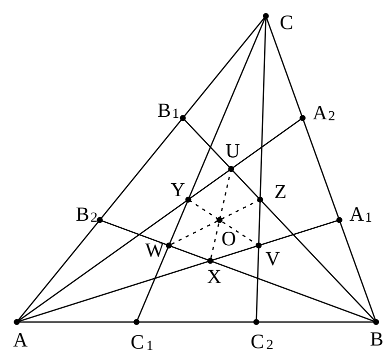


Figure 6-187

Example 6.188 (0.216, 4, 11) *Let M, N, P be points on the sides AB, BC and AC of a triangle ABC. Show that if M_1, N_1 and P_1 are points on sides AC, BA, and BC of a triangle ABC such that $MM_1 \parallel BC$, $NN_1 \parallel CA$ and $PP_1 \parallel AB$, then triangles MNP and $M_1N_1P_1$ have equal areas.*

Constructive description

((points $A\ B\ C$)
(on M (l $A\ B$)) (on N (l $B\ C$))
(on P (l $A\ C$))
(inter M_1 (l $A\ C$) (p $M\ B\ C$))
(inter N_1 (l $A\ B$) (p $N\ A\ C$))
(inter P_1 (l $B\ C$) (p $P\ A\ B$))
($S_{MNP} = S_{M_1N_1P_1}$))

The eliminants

$$S_{M_1N_1P_1} \stackrel{P_1}{=} \frac{S_{CM_1N_1} \cdot S_{ABP} + S_{BM_1N_1} \cdot S_{BCP}}{S_{ABC}}$$
$$S_{BM_1N_1} \stackrel{N_1}{=} \frac{S_{ABM_1} \cdot S_{ABN}}{S_{ABC}} \quad S_{CM_1N_1} \stackrel{N_1}{=} -(S_{CNM_1})$$
$$S_{ABM_1} \stackrel{M_1}{=} -S_{ACM} \quad S_{CNM_1} \stackrel{M_1}{=} -(S_{CMN})$$
$$S_{BCP} \stackrel{P}{=} -((\frac{\overline{AP}}{\overline{AC}} - 1) \cdot S_{ABC})$$
$$S_{ABP} \stackrel{P}{=} S_{ABC} \cdot \frac{\overline{AP}}{\overline{AC}}$$
$$S_{MNP} \stackrel{P}{=} S_{CMN} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{AMN} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{AMN}$$
$$S_{ABN} \stackrel{N}{=} S_{ABC} \cdot \frac{\overline{BN}}{\overline{BC}}$$
$$S_{AMN} \stackrel{N}{=} -(S_{ACM} \cdot \frac{\overline{BN}}{\overline{BC}})$$
$$S_{CMN} \stackrel{N}{=} -((\frac{\overline{BN}}{\overline{BC}} - 1) \cdot S_{BCM})$$

The machine proof

$$\frac{S_{MNP}}{S_{M_1N_1P_1}}$$
$$\stackrel{P_1}{=} \frac{S_{MNP} \cdot (-S_{ABC})}{-S_{CM_1N_1} \cdot S_{ABP} - S_{BM_1N_1} \cdot S_{BCP}}$$
$$\stackrel{N_1}{=} \frac{S_{MNP} \cdot (S_{ABC})^2}{-S_{CNM_1} \cdot S_{ABP} \cdot S_{ABC} + S_{BCP} \cdot S_{ABM_1} \cdot S_{ABN}}$$
$$\stackrel{M_1}{=} \frac{S_{MNP} \cdot (S_{ABC})^3}{-(-S_{CMN} \cdot S_{ABP} \cdot S_{ABC}^2 + S_{BCP} \cdot S_{ACM} \cdot S_{ABN} \cdot S_{ABC})}$$
$$\stackrel{simplify}{=} \frac{S_{MNP} \cdot (S_{ABC})^2}{S_{CMN} \cdot S_{ABP} \cdot S_{ABC} - S_{BCP} \cdot S_{ACM} \cdot S_{ABN}}$$
$$\stackrel{P}{=} \frac{(S_{CMN} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{AMN} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{AMN}) \cdot (S_{ABC})^2}{S_{CMN} \cdot S_{ABC}^2 \cdot \frac{\overline{AP}}{\overline{AC}} + S_{ACM} \cdot S_{ABN} \cdot S_{ABC} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{ACM} \cdot S_{ABN} \cdot S_{ABC}}$$
$$\stackrel{simplify}{=} \frac{(S_{CMN} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{AMN} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{AMN}) \cdot S_{ABC}}{S_{CMN} \cdot S_{ABC} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{ACM} \cdot S_{ABN} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{ACM} \cdot S_{ABN}}$$
$$\stackrel{N}{=} \frac{(-S_{BCM} \cdot \frac{\overline{BN}}{\overline{BC}} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{BCM} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{ACM} \cdot \frac{\overline{BN}}{\overline{BC}} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{ACM} \cdot \frac{\overline{BN}}{\overline{BC}}) \cdot S_{ABC}}{-S_{BCM} \cdot S_{ABC} \cdot \frac{\overline{BN}}{\overline{BC}} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{BCM} \cdot S_{ABC} \cdot \frac{\overline{AP}}{\overline{AC}} + S_{ACM} \cdot S_{ABC} \cdot \frac{\overline{BN}}{\overline{BC}} \cdot \frac{\overline{AP}}{\overline{AC}} - S_{ACM} \cdot S_{ABC} \cdot \frac{\overline{BN}}{\overline{BC}}} \stackrel{simplify}{=} 1$$

Example 6.189 (0.050, 2, 3) *Three parallel lines drawn through the vertices of a triangle ABC meet the respectively opposite sides in the points X, Y, Z. Show that area XYZ / area ABC = 2 / 1.*

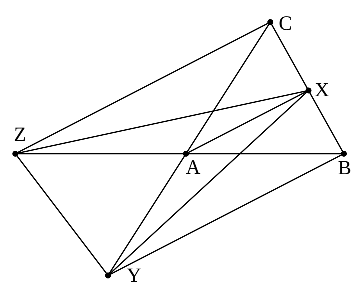


Figure 6-189

Constructive description

((points $A\ B\ C$)
(on X (l $B\ C$))
(inter Y (l $A\ C$) (p $B\ A\ X$))
(inter Z (l $A\ B$) (p $C\ A\ X$))
($2S_{BAC} = S_{XYZ}$))

The machine proof: $\frac{(-2) \cdot S_{ABC}}{S_{XYZ}} \stackrel{Z}{=} \frac{(-2) \cdot S_{ABC} \cdot S_{ABX}}{S_{BXY} \cdot S_{ACX} + S_{AXY} \cdot S_{ABC}} \stackrel{Y}{=} \frac{(-2) \cdot S_{ABC} \cdot S_{ABX} \cdot S_{ACX}}{-2S_{ACX} \cdot S_{ABX} \cdot S_{ABC}} \stackrel{simplify}{=} 1.$

The eliminants: $S_{XYZ} \stackrel{Z}{=} \frac{S_{BXY} \cdot S_{ACX} + S_{AXY} \cdot S_{ABC}}{S_{ABX}}$. $S_{AXY} \stackrel{Y}{=} -(S_{ABX})$. $S_{BXY} \stackrel{Y}{=} \frac{-S_{ABX} \cdot S_{ABC}}{S_{ACX}}$.

Example 6.190 (0.066, 1, 6) *Two doubly perspective triangles are in fact triply perspective.*

Constructive description

((points A B C O O_1)
(inter A_1 (l C O_1) (l A O))
(inter B_1 (l A O_1) (l B O))
(inter C_1 (l B O_1) (l C O))
(inter O_2 (l B A_1) (l A C_1))
(inter Z_{O_2} (l B_1 C) (l B A_1))
($\frac{\overline{BO_2}}{\overline{A_1O_2}} = \frac{\overline{BZ_{O_2}}}{\overline{A_1Z_{O_2}}}$))

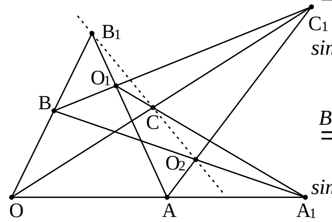


Figure 6-190

The machine proof

$$(\frac{\overline{BO_2}}{\overline{A_1O_2}})/(\frac{\overline{BZ_{O_2}}}{\overline{A_1Z_{O_2}}})$$

$$\overset{Z_{O_2}}{=} \frac{S_{CA_1B_1}}{-S_{BCB_1}} \cdot \frac{\overline{BO_2}}{\overline{A_1O_2}}$$

$$\overset{O_2}{=} \frac{(-S_{ABC_1})\cdot S_{CA_1B_1}}{-S_{BCB_1}\cdot(-S_{AA_1C_1})}$$

$$\overset{C_1}{=} \frac{-S_{BCO}\cdot S_{ABO_1}\cdot S_{CA_1B_1}\cdot(-S_{BCO_1O})}{S_{BCB_1}\cdot(-S_{BOO_1}\cdot S_{ACA_1})\cdot S_{BCO_1O}}$$

$$\overset{simplify}{=} \frac{-S_{BCO}\cdot S_{ABO_1}\cdot S_{CA_1B_1}}{S_{BCB_1}\cdot S_{BOO_1}\cdot S_{ACA_1}}$$

$$\overset{B_1}{=} \frac{-S_{BCO}\cdot S_{ABO_1}\cdot(-S_{BOO_1}\cdot S_{ACA_1})\cdot(-S_{ABO_1O})}{(-S_{BCO}\cdot S_{ABO_1})\cdot S_{BOO_1}\cdot S_{ACA_1}\cdot S_{ABO_1O}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BZ_{O_2}}}{\overline{A_1Z_{O_2}}} \overset{Z_{O_2}}{=} \frac{-S_{BCB_1}}{S_{CA_1B_1}}$$

$$\frac{\overline{BO_2}}{\overline{A_1O_2}} \overset{O_2}{=} \frac{S_{ABC_1}}{S_{AA_1C_1}}$$

$$S_{AA_1C_1} \overset{C_1}{=} \frac{S_{BOO_1}\cdot S_{ACA_1}}{S_{BCO_1O}}$$

$$S_{ABC_1} \overset{C_1}{=} \frac{S_{BCO}\cdot S_{ABO_1}}{S_{BCO_1O}}$$

$$S_{BCB_1} \overset{B_1}{=} \frac{S_{BCO}\cdot S_{ABO_1}}{S_{ABO_1O}}$$

$$S_{CA_1B_1} \overset{B_1}{=} \frac{-S_{BOO_1}\cdot S_{ACA_1}}{S_{ABO_1O}}$$

6.3.8 Equilateral Triangles

Example 6.191 (The Napoleon triangle) (0.433, 8, 22) *If equilateral triangles are erected externally (or internally) on the sides of any triangle, their centers form an equilateral triangle.*

The machine proof

$$\frac{P_{O_2O_1O_2}}{P_{O_2O_3O_2}}$$

$$\overset{O_3}{=} \frac{P_{O_2O_1O_2}}{P_{O_2GO_2}+\frac{1}{9}P_{BGB}\cdot r^2-\frac{8}{3}S_{BO_2G}\cdot r}$$

$$\overset{G}{=} \frac{(9)\cdot P_{O_2O_1O_2}}{\frac{9}{2}P_{BO_2B}+\frac{9}{2}P_{AO_2A}+\frac{1}{4}P_{ABA}\cdot r^2-\frac{9}{4}P_{ABA}-12S_{ABO_2}\cdot r}$$

$$\overset{O_1}{=} \frac{(36)\cdot(P_{O_2FO_2}+\frac{1}{9}P_{CFC}\cdot r^2-\frac{8}{3}S_{CO_2F}\cdot r)}{18P_{BO_2B}+18P_{AO_2A}+P_{ABA}\cdot r^2-9P_{ABA}-48S_{ABO_2}\cdot r}$$

$$\overset{F}{=} \frac{(4)\cdot(\frac{9}{2}P_{CO_2C}+\frac{9}{2}P_{BO_2B}+\frac{1}{4}P_{BCB}\cdot r^2-\frac{9}{4}P_{BCB}-12S_{BCO_2}\cdot r)}{18P_{BO_2B}+18P_{AO_2A}+P_{ABA}\cdot r^2-9P_{ABA}-48S_{ABO_2}\cdot r}$$

$$\overset{O_2}{=} \frac{4P_{CABE}\cdot r^2+18P_{CEC}+18P_{BEB}+P_{BCB}\cdot r^2-9P_{BCB}+4P_{AEA}\cdot r^2-48S_{BCE}\cdot r-48S_{ABE}\cdot r}{18P_{BEB}+4P_{BAE}\cdot r^2+4P_{AEA}\cdot r^2+18P_{AEA}+P_{ABA}\cdot r^2-9P_{ABA}-96S_{ABE}\cdot r}$$

$$\overset{E}{=} \frac{3P_{BCB}\cdot r^2+P_{ACA}\cdot r^2-2P_{ABC}\cdot r^2+9P_{ABA}-48S_{ABC}\cdot r}{9P_{BCB}+2P_{BAC}\cdot r^2+P_{ACA}\cdot r^2+P_{ABA}\cdot r^2-48S_{ABC}\cdot r}$$

$$\overset{cons}{=} \frac{9P_{BCB}+3P_{ACA}-6P_{ABC}+9P_{ABA}-48S_{ABC}\cdot r}{9P_{BCB}+6P_{BAC}+3P_{ACA}+3P_{ABA}-48S_{ABC}\cdot r}$$

$$\overset{py}{=} \frac{(4P_{BCB}+4P_{ACA}+4P_{ABA}-32S_{ABC}\cdot r)\cdot(2)}{(4P_{BCB}+4P_{ACA}+4P_{ABA}-32S_{ABC}\cdot r)\cdot(2)} \overset{simplify}{=} 1$$

Constructive description
((points A B C) (constant r^2-3),
(midpoint E A C) (tratio O_2 E A $\frac{1}{3}r$)
(midpoint F B C) (tratio O_1 F C $\frac{1}{3}r$)
(midpoint G A B) (tratio O_3 G B $\frac{1}{3}r$)
(eqdistance O_1 O_2 O_2 O_3))

The eliminants

$P_{O_2O_3O_2} \overset{O_3}{=} \frac{1}{9}\left(9P_{O_2GO_2}+P_{BGB}\cdot r^2-24S_{BO_2G}\cdot r\right)$
$S_{BO_2G} \overset{G}{=} \frac{1}{2}\left(S_{ABO_2}\right),\ P_{BGB} \overset{G}{=} \frac{1}{4}\left(P_{ABA}\right)$
$P_{O_2GO_2} \overset{G}{=} \frac{1}{4}\left(2P_{BO_2B}+2P_{AO_2A}-P_{ABA}\right)$
$P_{O_2O_1O_2} \overset{O_1}{=} \frac{1}{9}\left(9P_{O_2FO_2}+P_{CFC}\cdot r^2-24S_{CO_2F}\cdot r\right)$
$S_{CO_2F} \overset{F}{=} \frac{1}{2}\left(S_{BCO_2}\right),\ P_{CFC} \overset{F}{=} \frac{1}{4}\left(P_{BCB}\right)$
$P_{O_2FO_2} \overset{F}{=} \frac{1}{4}\left(2P_{CO_2C}+2P_{BO_2B}-P_{BCB}\right)$
$S_{ABO_2} \overset{O_2}{=} -\frac{1}{12}\left(P_{BAE}\cdot r-12S_{ABE}\right)$
$P_{AO_2A} \overset{O_2}{=} \frac{1}{9}\left((r^2+9)\cdot P_{AEA}\right)$
$S_{BCO_2} \overset{O_2}{=} -\frac{1}{12}\left(P_{CABE}\cdot r-12S_{BCE}\right)$
$P_{BO_2B} \overset{O_2}{=} \frac{1}{9}\left(9P_{BEB}+P_{AEA}\cdot r^2-24S_{ABE}\cdot r\right)$
$P_{CO_2C} \overset{O_2}{=} \frac{1}{9}\left(9P_{CEC}+P_{AEA}\cdot r^2\right)$
$P_{BAE} \overset{E}{=} \frac{1}{2}\left(P_{BAC}\right),\ S_{ABE} \overset{E}{=} \frac{1}{2}\left(S_{ABC}\right),$
$S_{BCE} \overset{E}{=} \frac{1}{2}\left(S_{ABC}\right),\ P_{AEA} \overset{E}{=} \frac{1}{4}\left(P_{ACA}\right)$
$P_{BEB} \overset{E}{=} \frac{1}{4}\left(2P_{BCB}-P_{ACA}+2P_{ABA}\right)$
$P_{CEC} \overset{E}{=} \frac{1}{4}\left(P_{ACA}\right),\ P_{CABE} \overset{E}{=} \frac{1}{2}\left(P_{BCB}-P_{ABC}\right)$
$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$
$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$

Example 6.192 (0.883, 15, 31) *Continuing from the above example, the lines AO_1, BO_2, CO_3 are concurrent.*

Constructive description
((constant r^2-3)
(points A B C)
(midpoint E A C) (tratio O_2 E A $\frac{1}{3}r$)
(midpoint F B C) (tratio O_1 F C $\frac{1}{3}r$)
(midpoint G A B) (tratio O_3 G B $\frac{1}{3}r$)
(inter N (l O_2 B) (l O_1 A))
(inter M (l O_2 B) (l O_3 C))
($\frac{\overline{BN}}{\overline{O_2N}} = \frac{\overline{BM}}{\overline{O_2M}}$))

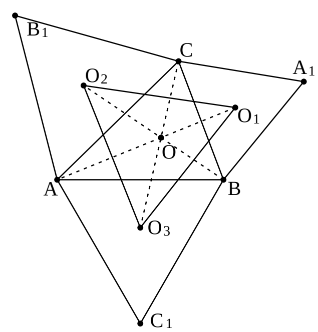


Figure 6.192

Example 6.193 (0.900, 18, 31) *In Example 6.192, if the three equilateral triangles become similar isosceles triangles, the conclusion is still true.*

Constructive description
((points A B C) (midpoint E A C) (tratio O_2 E A r) (midpoint F B C)
(tratio O_1 F C r) (midpoint G A B) (tratio O_3 G B r) (inter N (l O_2 B) (l O_1 A))
(inter M (l O_2 B) (l O_3 C)) ($\frac{\overline{BN}}{\overline{O_2N}} = \frac{\overline{BM}}{\overline{O_2M}}$))

Example 6.194 (0.367 26 54) *Let the circumcenters of the equilateral triangles erected externally (or internally) on the sides of any triangle be O_1, O_2 and O_3 (X_1, X_2, and X_3). Show that $S_{O_1O_2O_3} + S_{X_1X_2X_3} = S_{ABC}$.*

Constructive description
((points $A\ B\ C$) (constant r^2-3) (midpoint $E\ A\ C$) (tratio $O_2\ E\ A\ \frac{1}{3}r$)
(lratio $X_2\ E\ O_2\ -1$) (midpoint $F\ B\ C$) (tratio $O_1\ F\ C\ \frac{1}{3}r$) (lratio $X_1\ F\ O_1\ -1$)
(midpoint $G\ A\ B$) (tratio $O_3\ G\ B\ \frac{1}{3}r$) (lratio $X_3\ G\ O_3\ -1$) ($S_{O_1O_2O_3}+S_{X_1X_2X_3} = S_{ABC}$))

Example 6.195 (0.050, 5, 8) *Let equilaterals BCD, ABF, and ACE are erected externally on the sides of triangle ABC. Show that $AD = CF = BE$.*

Constructive description
((points $A\ B\ C$)
(constant r^2-3)
(midpoint $M\ A\ C$)
(tratio $E\ M\ A\ r$)
(midpoint $N\ B\ C$)
(tratio $D\ N\ C\ r$)
(eqdistance $B\ E\ A\ D$))

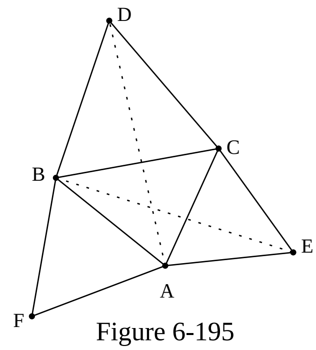


Figure 6-195

The machine proof

$$\frac{P_{BEB}}{P_{ADA}}$$

$$\stackrel{D}{=} \frac{P_{BEB}}{P_{CNC}\cdot r^2+P_{ANA}+8S_{ACN}\cdot r}$$

$$\stackrel{N}{=} \frac{P_{BEB}}{\frac{1}{4}P_{BCB}\cdot r^2-\frac{1}{4}P_{BCB}+\frac{1}{2}P_{ACA}+\frac{1}{2}P_{ABA}-4S_{ABC}\cdot r}$$

$$\stackrel{E}{=} \frac{(4)\cdot(P_{BMB}+P_{AMA}\cdot r^2-8S_{ABM}\cdot r)}{P_{BCB}\cdot r^2-P_{BCB}+2P_{ACA}+2P_{ABA}-16S_{ABC}\cdot r}$$

$$\stackrel{M}{=} \frac{(4)\cdot(\frac{1}{2}P_{BCB}+\frac{1}{4}P_{ACA}\cdot r^2-\frac{1}{4}P_{ACA}+\frac{1}{2}P_{ABA}-4S_{ABC}\cdot r)}{P_{BCB}\cdot r^2-P_{BCB}+2P_{ACA}+2P_{ABA}-16S_{ABC}\cdot r}$$

$$\stackrel{cons}{=} \frac{2P_{BCB}+2P_{ACA}+2P_{ABA}-16S_{ABC}\cdot r}{2P_{BCB}+2P_{ACA}+2P_{ABA}-16S_{ABC}\cdot r} \stackrel{simplify}{=} 1$$

The eliminants

$$P_{ADA}\stackrel{D}{=}P_{CNC}\cdot r^2+P_{ANA}+8S_{ACN}\cdot r$$
$$S_{ACN}\stackrel{N}{=}-\frac{1}{2}(S_{ABC})$$
$$P_{ANA}\stackrel{N}{=}-\frac{1}{4}(P_{BCB}-2P_{ACA}-2P_{ABA})$$
$$P_{CNC}\stackrel{N}{=}\frac{1}{4}(P_{BCB})$$
$$P_{BEB}\stackrel{E}{=}P_{BMB}+P_{AMA}\cdot r^2-8S_{ABM}\cdot r \quad S_{ABM}\stackrel{M}{=}\frac{1}{2}(S_{ABC})$$
$$P_{AMA}\stackrel{M}{=}\frac{1}{4}(P_{ACA})$$
$$P_{BMB}\stackrel{M}{=}\frac{1}{4}(2P_{BCB}-P_{ACA}+2P_{ABA})$$

6.3.9 Pedal Triangles

Definition. *From a point P three perpendicular lines are drawn to the three sides of a triangle. The triangle whose vertices are the three feet of the three perpendicular lines is called the pedal triangle of point P with respect to the given triangle.*

Example 6.196 (0.783, 6, 23) *The orthogonal projections from point D to BC, AC, and AB are E, F, and G respectively. Let O be the circumcenter of triangle ABC. Show that $\overline{DO}^2 = \overline{AO}^2(1-\frac{4S_{EFG}}{S_{ABC}})$.*

Constructive description

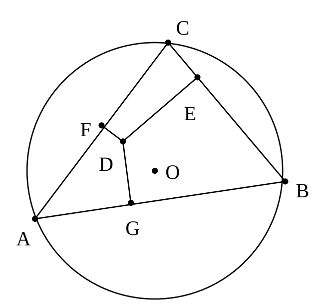


Figure 6-196

((points $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ C$)
(foot $E\ D\ B\ C$)
(foot $F\ D\ A\ C$)
(foot $G\ D\ A\ B$)
$(-\overline{DO}^2 \cdot S_{ABC} = -\overline{OA}^2 \cdot S_{ABC} + 4\overline{OA}^2 \cdot S_{EFG})$)

Example 6.197 (0.083, 2, 5) *Let K be the area of the pedal triangle of the orthocenter of triangle ABC with respect to triangle ABC. Show that* $\frac{K}{S_{ABC}} = \frac{P_{ABC}P_{ACB}P_{BAC}}{4\overline{AB}^2\overline{BC}^2\overline{AC}^2}$.

Constructive description
((points $A\ B\ C$)
(foot $E\ A\ B\ C$)
(foot $F\ B\ A\ C$)
(foot $G\ C\ A\ B$)
$(P_{ABC} \cdot P_{BAC} \cdot P_{ACB} \cdot S_{ABC} = 4\overline{AB}^2 \cdot \overline{BC}^2 \cdot \overline{AC}^2 \cdot S_{EFG})$)

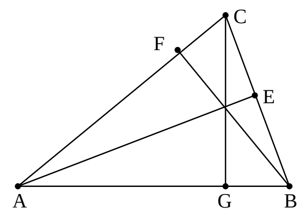


Figure 6-197

The machine proof

$$\frac{P_{BAC} \cdot P_{ACB} \cdot P_{ABC} \cdot S_{ABC}}{(\frac{1}{2}) \cdot P_{BCB} \cdot P_{ACA} \cdot P_{ABA} \cdot S_{EFG}}$$

$$\stackrel{G}{=} \frac{P_{BAC} \cdot P_{ACB} \cdot P_{ABC} \cdot S_{ABC} \cdot P_{ABA}}{(\frac{1}{2}) \cdot P_{BCB} \cdot P_{ACA} \cdot P_{ABA} \cdot (P_{BAC} \cdot S_{BEF} + P_{ABC} \cdot S_{AEF})}$$

$$\stackrel{simplify}{=} \frac{P_{BAC} \cdot P_{ACB} \cdot P_{ABC} \cdot S_{ABC}}{(\frac{1}{2}) \cdot P_{BCB} \cdot P_{ACA} \cdot (P_{BAC} \cdot S_{BEF} + P_{ABC} \cdot S_{AEF})}$$

$$\stackrel{F}{=} \frac{P_{BAC} \cdot P_{ACB} \cdot P_{ABC} \cdot S_{ABC} \cdot (P_{ACA})^2}{(\frac{1}{2}) \cdot P_{BCB} \cdot P_{ACA} \cdot (P_{BAC} \cdot P_{ACB} \cdot P_{ACA} \cdot S_{ABE} - P_{BAC} \cdot P_{ACA} \cdot P_{ABC} \cdot S_{ACE})}$$

$$\stackrel{simplify}{=} \frac{P_{ACB} \cdot P_{ABC} \cdot S_{ABC}}{(\frac{1}{2}) \cdot P_{BCB} \cdot (P_{ACB} \cdot S_{ABE} - P_{ABC} \cdot S_{ACE})}$$

$$\stackrel{E}{=} \frac{P_{ACB} \cdot P_{ABC} \cdot S_{ABC} \cdot (P_{BCB})^2}{(\frac{1}{2}) \cdot P_{BCB} \cdot (2P_{BCB} \cdot P_{ACB} \cdot P_{ABC} \cdot S_{ABC})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{EFG} \stackrel{G}{=} \frac{P_{BAC} \cdot S_{BEF} + P_{ABC} \cdot S_{AEF}}{P_{ABA}}$$

$$S_{AEF} \stackrel{F}{=} \frac{-P_{BAC} \cdot S_{ACE}}{P_{ACA}}$$

$$S_{BEF} \stackrel{F}{=} \frac{P_{ACB} \cdot S_{ABE}}{P_{ACA}}$$

$$S_{ACE} \stackrel{E}{=} \frac{-P_{ACB} \cdot S_{ABC}}{P_{BCB}}$$

$$S_{ABE} \stackrel{E}{=} \frac{P_{ABC} \cdot S_{ABC}}{P_{BCB}}$$

Example 6.198 (0.233, 14, 16) *Let K be the area of the pedal triangle of the centroid of triangle ABC with respect to triangle ABC. Show that* $\frac{K}{S_{ABC}} = \frac{\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2}{36R^2}$.

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(centroid $G\ A\ B\ C$)
(foot $E\ G\ B\ C$)
(foot $F\ G\ A\ C$)
(foot $G\ G\ A\ B$)
$(\overline{AB}^2 \cdot S_{ABC} + \overline{BC}^2 \cdot S_{ABC} + \overline{AC}^2 \cdot S_{ABC} = 36\overline{OA}^2 \cdot S_{EFG})$)

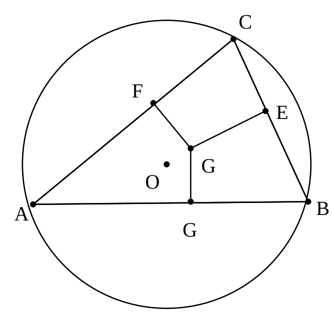


Figure 6-198

Example 6.199 (0.083, 2, 11) *Let K be the area of the pedal triangle of the circumcenter of triangle ABC with respect to triangle ABC. Show that* $S_{ABC} = 4K$.

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $E\ O\ B\ C$)
(foot $F\ O\ A\ C$)
(foot $G\ O\ A\ B$)
($S_{ABC} = 4S_{EFG}$))

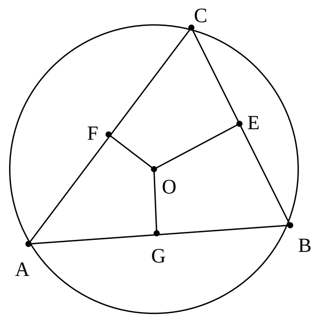


Figure 6-199

The eliminants

$S_{EFG} \overset{G}{=} \frac{P_{BAO} \cdot S_{BEF} + P_{ABO} \cdot S_{AEF}}{P_{ABA}}$

$S_{AEF} \overset{F}{=} \frac{-P_{CAO} \cdot S_{ACE}}{P_{ACA}}$

$S_{BEF} \overset{F}{=} \frac{P_{ACO} \cdot S_{ABE}}{P_{ACA}}$

$S_{ABE} \overset{E}{=} \frac{P_{CBO} \cdot S_{ABC}}{P_{BCB}}$

$S_{ACE} \overset{E}{=} \frac{-P_{BCO} \cdot S_{ABC}}{P_{BCB}}$

$P_{ABO} \overset{O}{=} \frac{1}{2}(P_{ABA})$

$P_{BCO} \overset{O}{=} \frac{1}{2}(P_{BCB})$

$P_{CAO} \overset{O}{=} \frac{1}{2}(P_{ACA})$

$P_{ACO} \overset{O}{=} \frac{1}{2}(P_{ACA})$

$P_{BAO} \overset{O}{=} \frac{1}{2}(P_{ABA})$

$P_{CBO} \overset{O}{=} \frac{1}{2}(P_{BCB})$

The machine proof

$\frac{S_{ABC}}{(4) \cdot S_{EFG}}$

$\overset{G}{=} \frac{S_{ABC} \cdot P_{ABA}}{(4) \cdot (P_{BAO} \cdot S_{BEF} + P_{ABO} \cdot S_{AEF})}$

$\overset{F}{=} \frac{S_{ABC} \cdot P_{ABA} \cdot (P_{ACA})^2}{(4) \cdot (-P_{CAO} \cdot P_{ACA} \cdot P_{ABO} \cdot S_{ACE} + P_{BAO} \cdot P_{ACO} \cdot P_{ACA} \cdot S_{ABE})}$

$\overset{simplify}{=} \frac{S_{ABC} \cdot P_{ABA} \cdot P_{ACA}}{(-4) \cdot (P_{CAO} \cdot P_{ABO} \cdot S_{ACE} - P_{BAO} \cdot P_{ACO} \cdot S_{ABE})}$

$\overset{E}{=} \frac{S_{ABC} \cdot P_{ABA} \cdot P_{ACA} \cdot (P_{BCB})^2}{(-4) \cdot (-P_{CBO} \cdot P_{BCB} \cdot P_{BAO} \cdot P_{ACO} \cdot S_{ABC} - P_{CAO} \cdot P_{BCO} \cdot P_{BCB} \cdot P_{ABO} \cdot S_{ABC})}$

$\overset{simplify}{=} \frac{P_{ABA} \cdot P_{ACA} \cdot P_{BCB}}{(4) \cdot (P_{CBO} \cdot P_{BAO} \cdot P_{ACO} + P_{CAO} \cdot P_{BCO} \cdot P_{ABO})}$

$\overset{O}{=} \frac{P_{ABA} \cdot P_{ACA} \cdot P_{BCB} \cdot ((2))^6}{(4) \cdot (16 P_{BCB} \cdot P_{ACA} \cdot P_{ABA})} \overset{simplify}{=} 1$

Example 6.200 (0.300, 2, 14) *Let K be the area of the pedal triangle of the incenter of triangle ABC with respect to triangle ABC. Show that* $\frac{K}{S_{ABC}} = \frac{r}{2R}$.

Constructive description
((points $A\ B\ I$)
(incenter $C\ I\ A\ B$)
(circumcenter $O\ A\ B\ C$)
(foot $E\ I\ B\ C$)
(foot $F\ I\ A\ C$)
(foot $G\ I\ A\ B$)
($\overline{IE}^2 \cdot S_{ABC}^2 = 4\overline{OA}^2 \cdot S_{EFG}^2$))

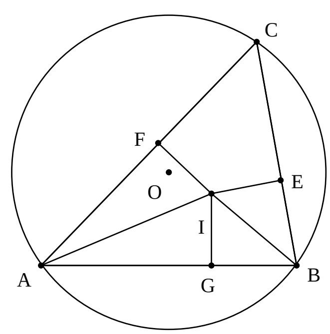


Figure 6-200

6.3.10 Miscellaneous

Example 6.201 (0.016, 1, 4) *AD, AA_1 are the altitude and the median of the triangle ABC; the parallels through A_1 to AB, AC meet AD in P, Q; show that (ADPQ) = -1.*

Constructive description
((points A B C)
(foot D A B C)
(midpoint A_1 B C)
(inter P (l A D) (p A_1 A B))
(inter Q (l A D) (p A_1 A C))
(harmonic A D P Q))

The machine proof

$$\left(-\frac{\overline{AP}}{\overline{DP}}\right)/\left(\frac{\overline{AQ}}{\overline{DQ}}\right)$$

$$\overset{Q}{=} \frac{-S_{ADA_1}}{-S_{ACA_1}} \cdot -\frac{\overline{AP}}{\overline{DP}}$$

$$\overset{P}{=} \frac{-(-S_{ABA_1})\cdot S_{ADA_1}}{S_{ACA_1}\cdot(-S_{ADA_1})}$$

$$\overset{simplify}{=} \frac{-S_{ABA_1}}{S_{ACA_1}}$$

$$\overset{A_1}{=} \frac{-(\frac{1}{2}S_{ABC})}{-\frac{1}{2}S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AQ}}{\overline{DQ}} \overset{Q}{=} \frac{S_{ACA_1}}{S_{ADA_1}} \quad \frac{\overline{AP}}{\overline{DP}} \overset{P}{=} \frac{S_{ABA_1}}{S_{ADA_1}}$$

$$S_{ACA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ABC})$$

$$S_{ABA_1} \overset{A_1}{=} \frac{1}{2}(S_{ABC})$$

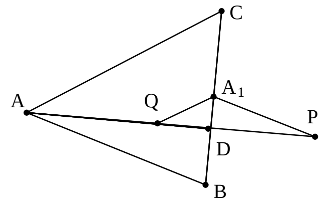


Figure 6-201

Example 6.202 (0.067, 3, 13) *If A_1, B_1, C_1 are the midpoint of the sides of the triangle ABC, prove that A_1A is the harmonic conjugate of A_1C with respect to A_1B_1, A_1C_1.*

Constructive description
((points A B C)
(midpoint A_1 C B)
(midpoint B_1 A C)
(midpoint C_1 B A)
(centroid G A B C)
(inter D (l C_1 C) (l A_1 B_1))
(harmonic C_1 D G C))

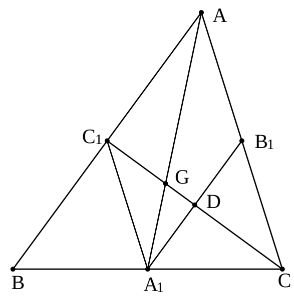


Figure 6-202

The machine proof

$$\left(\frac{\overline{C_1G}}{\overline{GD}}\right)/\left(\frac{\overline{CC_1}}{\overline{CD}}\right)$$

$$\overset{D}{=} \frac{(-S_{A_1C_1B_1G})\cdot S_{CA_1B_1}}{S_{CA_1C_1B_1}\cdot S_{A_1B_1G}}$$

$$\overset{G}{=} \frac{-(-3S_{A_1B_1C_1}+S_{CA_1B_1}+S_{BA_1B_1}+S_{AA_1B_1})\cdot S_{CA_1B_1}\cdot(3)}{S_{CA_1C_1B_1}\cdot(S_{CA_1B_1}+S_{BA_1B_1}+S_{AA_1B_1})\cdot(3)}$$

$$\overset{C_1}{=} \frac{(-S_{CA_1B_1}+\frac{1}{2}S_{BA_1B_1}+\frac{1}{2}S_{AA_1B_1})\cdot S_{CA_1B_1}}{(S_{CA_1B_1}-\frac{1}{2}S_{BA_1B_1}-\frac{1}{2}S_{AA_1B_1})\cdot(S_{CA_1B_1}+S_{BA_1B_1}+S_{AA_1B_1})}$$

$$\overset{simplify}{=} \frac{-S_{CA_1B_1}}{S_{CA_1B_1}+S_{BA_1B_1}+S_{AA_1B_1}}$$

$$\overset{B_1}{=} \frac{-(\frac{1}{2}S_{ACA_1})}{\frac{1}{2}S_{ABA_1}}$$

$$\overset{A_1}{=} \frac{-(-\frac{1}{2}S_{ABC})}{\frac{1}{2}S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{CC_1}}{\overline{CD}} \overset{D}{=} \frac{S_{CA_1C_1B_1}}{S_{CA_1B_1}}$$

$$\frac{\overline{C_1G}}{\overline{GD}} \overset{D}{=} \frac{-S_{A_1C_1B_1G}}{S_{A_1B_1G}}$$

$$S_{A_1B_1G} \overset{G}{=} \frac{1}{3}(S_{CA_1B_1}+S_{BA_1B_1}+S_{AA_1B_1})$$

$$S_{A_1C_1B_1G} \overset{G}{=} -\frac{1}{3}(3S_{A_1B_1C_1}-S_{CA_1B_1}-S_{BA_1B_1}-S_{AA_1B_1})$$

$$S_{CA_1C_1B_1} \overset{C_1}{=} \frac{1}{2}(2S_{CA_1B_1}-S_{BA_1B_1}-S_{AA_1B_1})$$

$$S_{A_1B_1C_1} \overset{C_1}{=} \frac{1}{2}(S_{BA_1B_1}+S_{AA_1B_1})$$

$$S_{AA_1B_1} \overset{B_1}{=} -\frac{1}{2}(S_{ACA_1})$$

$$S_{BA_1B_1} \overset{B_1}{=} \frac{1}{2}(S_{ABA_1})$$

$$S_{CA_1B_1} \overset{B_1}{=} \frac{1}{2}(S_{ACA_1})$$

$$S_{ABA_1} \overset{A_1}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ACA_1} \overset{A_1}{=} -\frac{1}{2}(S_{ABC})$$

Example 6.203 (0.250, 1, 12) *Let P be a point in the plane of the triangle ABC. Let $A_1 = BC\cap AP, B_1 = AC\cap BP, C_1 = AB\cap CP, A_2 = BC\cap B_1C_1, B_2 = AC\cap A_1C_1, C_2 = AB\cap A_1B_1$. Show that A_2, B_2, C_2 are collinear, and is called the trilinear polar of P for triangle ABC.*

Constructive description
((points A B C P)
(inter A_1 (l B C) (l A P))
(inter B_1 (l A C) (l B P))
(inter C_1 (l A B) (l C P))
(inter A_2 (l B C) (l B_1 C_1))
(inter B_2 (l A C) (l A_1 C_1))
(inter C_2 (l A B) (l A_1 B_1))
(inter Z_{C_2} (l B_2 A_2) (l A_1 B_1))
($\frac{\overline{A_1C_2}}{\overline{B_1C_2}} = \frac{\overline{A_1Z_{C_2}}}{\overline{B_1Z_{C_2}}}$))

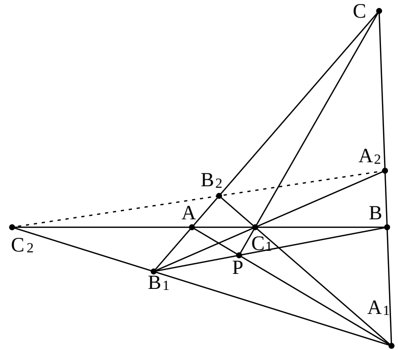


Figure 6-203

The machine proof

$$(\frac{\overline{A_1C_2}}{\overline{B_1C_2}})/(\frac{\overline{A_1Z_{C_2}}}{\overline{B_1Z_{C_2}}})$$

$$\stackrel{Z_{C_2}}{=} \frac{-S_{B_1A_2B_2}}{-S_{A_1A_2B_2}}\cdot\frac{\overline{A_1C_2}}{\overline{B_1C_2}}$$

$$\stackrel{C_2}{=} \frac{S_{ABA_1}\cdot S_{B_1A_2B_2}}{S_{A_1A_2B_2}\cdot S_{ABB_1}}$$

$$\stackrel{B_2}{=} \frac{S_{ABA_1}\cdot S_{A_1B_1C_1}\cdot S_{ACA_2}\cdot(-S_{AA_1CC_1})}{(-S_{A_1C_1A_2}\cdot S_{ACA_1})\cdot S_{ABB_1}\cdot S_{AA_1CC_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ABA_1}\cdot S_{A_1B_1C_1}\cdot S_{ACA_2}}{S_{A_1C_1A_2}\cdot S_{ACA_1}\cdot S_{ABB_1}}$$

$$\stackrel{A_2}{=} \frac{S_{ABA_1}\cdot S_{A_1B_1C_1}\cdot S_{CB_1C_1}\cdot S_{ABC}\cdot S_{BB_1CC_1}}{(-S_{A_1B_1C_1}\cdot S_{BCC_1})\cdot S_{ACA_1}\cdot S_{ABB_1}\cdot S_{BB_1CC_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ABA_1}\cdot S_{CB_1C_1}\cdot S_{ABC}}{-S_{BCC_1}\cdot S_{ACA_1}\cdot S_{ABB_1}}$$

$$\stackrel{C_1}{=} \frac{S_{ABA_1}\cdot(-S_{CPB_1}\cdot S_{ABC})\cdot S_{ABC}\cdot S_{ACBP}}{-(-S_{BCP}\cdot S_{ABC})\cdot S_{ACA_1}\cdot S_{ABB_1}\cdot(-S_{ACBP})}$$

$$\stackrel{simplify}{=} \frac{S_{ABA_1}\cdot S_{CPB_1}\cdot S_{ABC}}{S_{BCP}\cdot S_{ACA_1}\cdot S_{ABB_1}}$$

$$\stackrel{B_1}{=} \frac{S_{ABA_1}\cdot S_{BCP}\cdot S_{ACP}\cdot S_{ABC}\cdot S_{ABCP}}{S_{BCP}\cdot S_{ACA_1}\cdot S_{ABP}\cdot S_{ABC}\cdot S_{ABCP}}$$

$$\stackrel{simplify}{=} \frac{S_{ABA_1}\cdot S_{ACP}}{S_{ACA_1}\cdot S_{ABP}}$$

$$\stackrel{A_1}{=} \frac{(-S_{ABP}\cdot S_{ABC})\cdot S_{ACP}\cdot(-S_{ABPC})}{(-S_{ACP}\cdot S_{ABC})\cdot S_{ABP}\cdot(-S_{ABPC})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{A_1Z_{C_2}}}{\overline{B_1Z_{C_2}}} \stackrel{Z_{C_2}}{=} \frac{S_{A_1A_2B_2}}{S_{B_1A_2B_2}}$$

$$\frac{\overline{A_1C_2}}{\overline{B_1C_2}} \stackrel{C_2}{=} \frac{S_{ABA_1}}{S_{ABB_1}}$$

$$S_{A_1A_2B_2} \stackrel{B_2}{=} \frac{S_{A_1C_1A_2}\cdot S_{ACA_1}}{S_{AA_1CC_1}}$$

$$S_{B_1A_2B_2} \stackrel{B_2}{=} \frac{S_{A_1B_1C_1}\cdot S_{ACA_2}}{S_{AA_1CC_1}}$$

$$S_{A_1C_1A_2} \stackrel{A_2}{=} \frac{-S_{A_1B_1C_1}\cdot S_{BCC_1}}{S_{BB_1CC_1}}$$

$$S_{ACA_2} \stackrel{A_2}{=} \frac{S_{CB_1C_1}\cdot S_{ABC}}{S_{BB_1CC_1}}$$

$$S_{BCC_1} \stackrel{C_1}{=} \frac{-S_{BCP}\cdot S_{ABC}}{S_{ACBP}}$$

$$S_{CB_1C_1} \stackrel{C_1}{=} \frac{S_{CPB_1}\cdot S_{ABC}}{S_{ACBP}}$$

$$S_{ABB_1} \stackrel{B_1}{=} \frac{S_{ABP}\cdot S_{ABC}}{S_{ABCP}}$$

$$S_{CPB_1} \stackrel{B_1}{=} \frac{S_{BCP}\cdot S_{ACP}}{S_{ABCP}}$$

$$S_{ACA_1} \stackrel{A_1}{=} \frac{S_{ACP}\cdot S_{ABC}}{S_{ABPC}}$$

$$S_{ABA_1} \stackrel{A_1}{=} \frac{S_{ABP}\cdot S_{ABC}}{S_{ABPC}}$$

Example 6.204 (0.083, 1, 6) *Let P be a point in the plane of the triangle ABC. Let $A_1 = BC\cap AP, B_1 = AC\cap BP, C_1 = AB\cap CP, A_2 = BC\cap B_1C_1$. Show that A_1, A_2, B, C form a harmonic sequence.*

Constructive description

((points $A\ B\ C\ P$)
(inter A_1 (l $B\ C$) (l $A\ P$))
(inter B_1 (l $A\ C$) (l $B\ P$))
(inter C_1 (l $A\ B$) (l $C\ P$))
(inter A_2 (l $B\ C$) (l $B_1\ C_1$))
(harmonic $B\ C\ A_1\ A_2$))

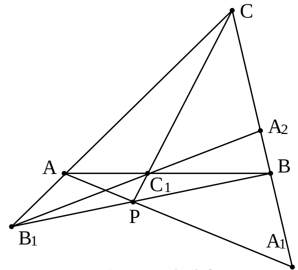


Figure 6-204

The machine proof

$$(-\frac{\overline{BA_1}}{\overline{CA_1}})/(\frac{\overline{BA_2}}{\overline{CA_2}})$$

$$\stackrel{A_2}{=} \frac{S_{CB_1C_1}}{S_{BB_1C_1}} \cdot -\frac{\overline{BA_1}}{\overline{CA_1}}$$

$$\stackrel{C_1}{=} \frac{-(-S_{CPB_1})\cdot S_{ABC}\cdot S_{ACBP}}{(-S_{BCP}\cdot S_{ABB_1})\cdot(-S_{ACBP})} \cdot \frac{\overline{BA_1}}{\overline{CA_1}}$$

$$\stackrel{simplify}{=} \frac{S_{CPB_1}\cdot S_{ABC}}{S_{BCP}\cdot S_{ABB_1}} \cdot \frac{\overline{BA_1}}{\overline{CA_1}}$$

$$\stackrel{B_1}{=} \frac{S_{BCP}\cdot S_{ACP}\cdot S_{ABC}\cdot S_{ABCP}}{S_{BCP}\cdot S_{ABP}\cdot S_{ABC}\cdot S_{ABCP}} \cdot \frac{\overline{BA_1}}{\overline{CA_1}}$$

$$\stackrel{simplify}{=} \frac{S_{ACP}}{S_{ABP}} \cdot \frac{\overline{BA_1}}{\overline{CA_1}}$$

$$\stackrel{A_1}{=} \frac{(-S_{ABP})\cdot S_{ACP}}{S_{ABP}\cdot(-S_{ACP})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BA_2}}{\overline{CA_2}} \stackrel{A_2}{=} \frac{S_{BB_1C_1}}{S_{CB_1C_1}}$$

$$S_{BB_1C_1} \stackrel{C_1}{=} \frac{-S_{BCP}\cdot S_{ABB_1}}{S_{ACBP}}$$

$$S_{CB_1C_1} \stackrel{C_1}{=} \frac{S_{CPB_1}\cdot S_{ABC}}{S_{ACBP}}$$

$$S_{ABB_1} \stackrel{B_1}{=} \frac{S_{ABP}\cdot S_{ABC}}{S_{ABCP}}$$

$$S_{CPB_1} \stackrel{B_1}{=} \frac{S_{BCP}\cdot S_{ACP}}{S_{ABCP}}$$

$$\frac{\overline{BA_1}}{\overline{CA_1}} \stackrel{A_1}{=} \frac{S_{ABP}}{S_{ACP}}$$

Example 6.205 (0.033, 1, 6) *With the usual notations for the triangle ABC, if EF meets BC in M, show that $(BCDM) = -1$.*

Constructive description

((points $A\ B\ C$)
(foot $D\ A\ B\ C$)
(foot $E\ B\ A\ C$)
(foot $F\ C\ A\ B$)
(inter H (l $A\ D$) (l $C\ F$))
(inter K (l $D\ F$) (l $B\ E$))
(inter M (l $B\ C$) (l $E\ F$))
(harmonic $B\ C\ D\ M$))

The machine proof

$$(-\frac{\overline{BD}}{\overline{CD}})/(\frac{\overline{BM}}{\overline{CM}})$$

$$\stackrel{M}{=} \frac{S_{CEF}}{S_{BEF}} \cdot -\frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{F}{=} \frac{-P_{BAC}\cdot S_{BCE}\cdot P_{ABA}}{P_{ABC}\cdot S_{ABE}\cdot P_{ABA}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{simplify}{=} \frac{-P_{BAC}\cdot S_{BCE}}{P_{ABC}\cdot S_{ABE}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{E}{=} \frac{-P_{BAC}\cdot P_{ACB}\cdot S_{ABC}\cdot P_{ACA}}{P_{ABC}\cdot P_{BAC}\cdot S_{ABC}\cdot P_{ACA}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{simplify}{=} \frac{-P_{ACB}}{P_{ABC}} \cdot \frac{\overline{BD}}{\overline{CD}}$$

$$\stackrel{D}{=} \frac{-P_{ABC}\cdot P_{ACB}}{P_{ABC}\cdot(-P_{ACB})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BM}}{\overline{CM}} \stackrel{M}{=} \frac{S_{BEF}}{S_{CEF}}, \quad S_{BEF} \stackrel{F}{=} \frac{P_{ABC}\cdot S_{ABE}}{P_{ABA}}$$

$$S_{CEF} \stackrel{F}{=} \frac{P_{BAC}\cdot S_{BCE}}{P_{ABA}}$$

$$S_{ABE} \stackrel{E}{=} \frac{P_{BAC}\cdot S_{ABC}}{P_{ACA}}$$

$$S_{BCE} \stackrel{E}{=} \frac{P_{ACB}\cdot S_{ABC}}{P_{ACA}}, \quad \frac{\overline{BD}}{\overline{CD}} \stackrel{D}{=} \frac{P_{ABC}}{-P_{ACB}}$$

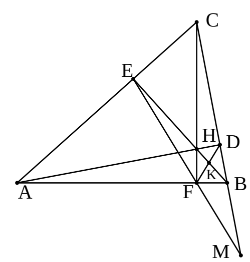


Figure 6-205

Example 6.206 (0.467, 13, 34) *The circle having for diameter the median AA_1 of the triangle ABC meets the circumcircle in L: show that $A(LDBC) = -1$, where AD is the altitude.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(midpoint A_1 B C)
(midpoint M A_1 A)
(inter L (cir M A) (cir O A))
(inter K (l B C) (l A L))
(harmonic K D B C))

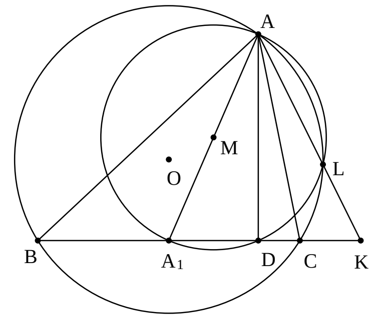


Figure 6-206

Example 6.207 (0.883, 10, 38) *If L, M, N are the traces of the lines AP, BP, CP on the sides BC, CA, AB of the triangle ABC, and L_1, M_1, N_1 the traces, on the same sides, of the trilinear polar of P for ABC, show that the midpoints of the segments LL_1, MM_1, NN_1 are collinear.*

Constructive description
((points A B C P) (inter L (l A P) (l B C))
(inter M (l B P) (l A C))
(inter N (l C P) (l A B))
(inter L_1 (l B C) (l M N))
(inter M_1 (l A C) (l L N))
(inter N_1 (l A B) (l L M))
(midpoint L_2 L L_1) (midpoint M_2 M M_1)
(midpoint N_2 N N_1) (collinear L_2 M_2 N_2))

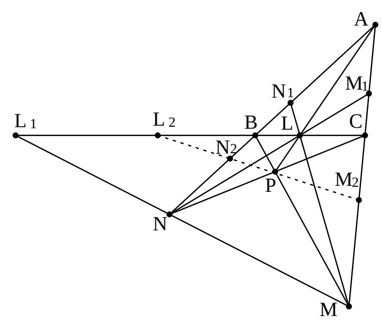


Figure 6-207

Example 6.208 (0.550, 2, 25) *Let L, M, N be the feet of the cevians AP, BP, CP of the triangle ABC and let P_1 be a point on the trilinear polar of P for ABC. If the lines AP_1 BP_1, CP_1 meet MN, NL, LM in X, Y, Z, show that the triangle XYZ is circumscribed about the triangle ABC.*

Constructive description
((points A B C P) (inter L (l A P) (l B C))
(inter M (l B P) (l A C)) (inter N (l C P) (l A B))
(inter L_1 (l B C) (l M N)) (inter N_1 (l A B) (l L M))
(on P_1 (l N_1 L_1)) (inter Y (l N L) (l B P_1))
(inter Z (l L M) (l C P_1)) (inter Z_Z (l Y A) (l N_1 L))
($\frac{\overline{N_1Z}}{\overline{LZ}} = \frac{\overline{N_1Z_Z}}{\overline{LZ_Z}}$))

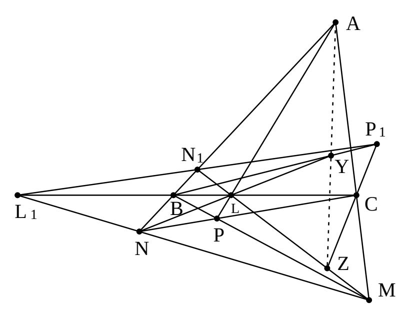


Figure 6-208

Example 6.209 (0.466, 4, 26) *If A_1 is the point of intersection of the side BC of the triangle ABC with the trilinear polar p of a point P on the circumcircle of ABC, show that the circle APA_1 passes through the midpoint of BC.*

Constructive description
((circle $A\ B\ C\ P$)
(circumcenter $O\ A\ B\ C$)
(inter M (l $B\ P$) (l $A\ C$))
(inter N (l $C\ P$) (l $A\ B$))
(inter A_1 (l $B\ C$) (l $M\ N$))
(midpoint $G\ B\ C$)
(cocircle $A\ P\ A_1\ G$))

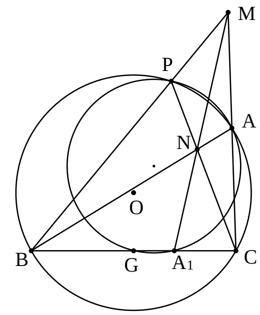


Figure 6-209

Example 6.210 (0.050, 1, 8) *Let ABC be a triangle with $\angle B = 2\angle C$, D the foot of the altitude on CB and M the midpoint of B and C. Show that $AB = 2DM$.*

Constructive description
((points $B\ C$)
(midpoint $M\ B\ C$)
(tratio $X\ M\ B\ H$)
(foot $Y\ C\ B\ X$)
(lratio $Z\ Y\ C\ -1$)
(inter A (l $B\ Z$) (l $C\ X$))
(inter D (l $B\ C$) (p $A\ M\ X$))
($\frac{\overline{BA}}{\overline{BZ}} = 2\frac{\overline{MD}}{\overline{CB}}$))

The eliminants

$\frac{\overline{MD}}{\overline{BC}} \overset{D}{=} \frac{-S_{MXA}}{S_{BCX}}$

$S_{MXA} \overset{A}{=} \frac{S_{CMX}\cdot S_{BXZ}}{-S_{BCZX}}$

$\frac{\overline{BA}}{\overline{BZ}} \overset{A}{=} \frac{S_{BCX}}{S_{BCZX}}$

$S_{BXZ} \overset{Z}{=} S_{BCX}$

$S_{CMX} \overset{X}{=} -\frac{1}{4}(P_{BMC}\cdot H)$

$S_{BCX} \overset{X}{=} -\frac{1}{4}(P_{CBM}\cdot H)$

$P_{BMC} \overset{M}{=} -\frac{1}{4}(P_{BCB})$ $P_{CBM} \overset{M}{=} \frac{1}{2}(P_{BCB})$

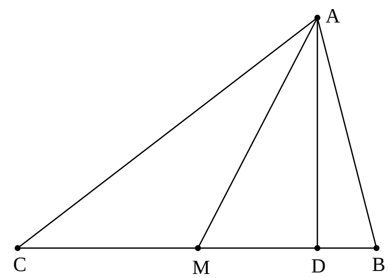


Figure 6-210

The machine proof: $\frac{\frac{\overline{BA}}{\overline{BZ}}}{(-2)\cdot\frac{\overline{MD}}{\overline{BC}}} \overset{D}{=} \frac{S_{BCX}}{(-2)\cdot(-S_{MXA})}\cdot\frac{\overline{BA}}{\overline{BZ}} \overset{A}{=} \frac{(S_{BCX})^2\cdot(-S_{BCZX})}{(2)\cdot S_{CMX}\cdot S_{BXZ}\cdot S_{BCZX}}$

$\overset{simplify}{=} \frac{-(S_{BCX})^2}{(2)\cdot S_{CMX}\cdot S_{BXZ}} \overset{Z}{=} \frac{-(S_{BCX})^2}{(2)\cdot S_{CMX}\cdot S_{BCX}} \overset{simplify}{=} \frac{-S_{BCX}}{(2)\cdot S_{CMX}} \overset{X}{=} \frac{-(-\frac{1}{4}P_{CBM}\cdot H)}{(2)\cdot(-\frac{1}{4}P_{BMC}\cdot H)}$

$\overset{simplify}{=} \frac{-P_{CBM}}{(2)\cdot P_{BMC}} \overset{M}{=} \frac{-(\frac{1}{2}P_{BCB})}{(2)\cdot(-\frac{1}{4}P_{BCB})} \overset{simplify}{=} 1.$

Example 6.211 (1.117, 33, 38) *The three adjoint circles of the direct group have a point in common.*

Constructive description
((points $A\ B\ C$)
(inter A_1 (t $C\ C\ A$) (b $B\ C$))
(inter B_1 (t $A\ A\ B$) (b $A\ C$))
(inter C_1 (t $B\ B\ C$) (b $A\ B$))
(inter N (cir $C_1\ B$) (cir $A_1\ B$))
(perp-biesct $B_1\ C\ N$))

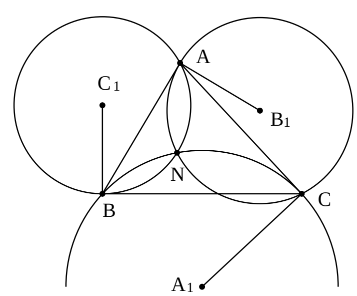


Figure 6-211

Example 6.212 (2.383, 68, 47) *The three adjoint circles of the indirect group have a point in common.*

Constructive description
((points A B C)
(inter A_1 (t B B A) (b C B))
(inter B_1 (t C C B) (b A C))
(inter C_1 (t A A C) (b B A))
(inter M (cir C_1 B) (cir A_1 B))
(perp-biesct B_1 C M))

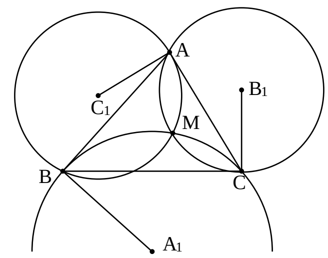


Figure 6-212

Definition. *The points N, M in Examples 6.211 and 6.212 are called the Brocard points of the triangle.*

Example 6.213 (1.000, 12, 50) *The Brocard points are a pair of isogonal points of the triangle.*

Constructive description
((points A B C)
(inter A_1 (t C C A) (b B C))
(inter C_1 (t B B C) (b A B))
(inter A_2 (t B B A) (b C B))
(inter B_2 (t C C B) (b A C))
(inter M (cir C_1 B) (cir A_1 B))
(inter N (cir B_2 C) (cir A_2 C))
(eqangle A B M N B C))

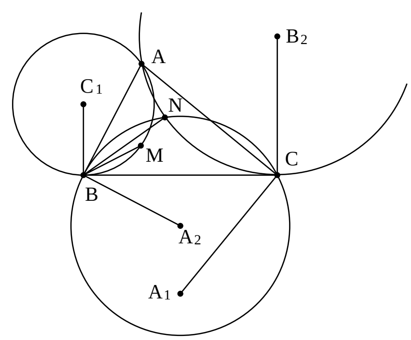


Figure 6-213

Example 6.214 (1.600, 27, 52) *The two Brocard points of a triangle are equidistant from the circumcenter of the triangle.*

Constructive description
((points A B C)
(circumcenter O A B C)
(inter A_1 (t C C A) (b B C))
(inter C_1 (t B B C) (b A B))
(inter A_2 (t B B A) (b C B))
(inter B_2 (t C C B) (b A C))
(inter M (cir C_1 B) (cir A_1 B))
(inter N (cir B_2 C) (cir A_2 C))
(perp-biesct O N M))

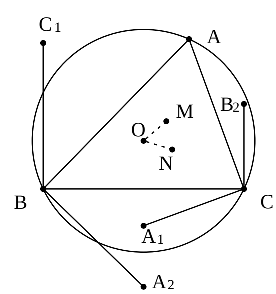


Figure 6-214

Example 6.215 (0.683, 12, 34) *For the Brocard point in Example 6.211 we have $\angle NAB = \angle NBC = \angle NCA$. Similar for the Brocard point in Example 6.212.*

Constructive description
((points A B C)
(inter A_1 (t C C A) (b B C))
(inter C_1 (t B B C) (b A B))
(inter N (cir C_1 B) (cir A_1 B))
(eqangle N A B N B C))

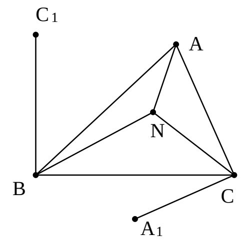


Figure 6-215

Example 6.216 (0.016, 1, 6) *Let D and E be two points on two sides AC and BC of triangle ABC such that $AD = BE$, $F = DE \cap AB$. Show that $FD \cdot AC = EF \cdot BC$.*

Constructive description
((points A B C)
(on D (l A C))
(inter E (l C B) (cir B $\overline{AD}^2$))
(inter F (l E D) (l A B))
(eq-product F D A C E F B C))

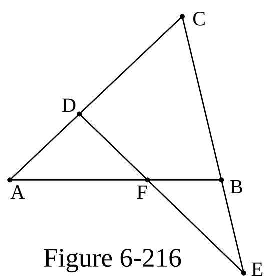


Figure 6-216

The machine proof

$$\frac{P_{DFD} \cdot P_{ACA}}{P_{EFE} \cdot P_{BCB}}$$

$$\stackrel{F}{=} \frac{P_{DED} \cdot S_{ABD}^2 \cdot P_{ACA} \cdot S_{ADBE}^2}{P_{DED} \cdot S_{ABE}^2 \cdot P_{BCB} \cdot S_{ADBE}^2}$$

$$\stackrel{simplify}{=} \frac{(S_{ABD})^2 \cdot P_{ACA}}{(S_{ABE})^2 \cdot P_{BCB}}$$

$$\stackrel{E}{=} \frac{(S_{ABD})^2 \cdot P_{ACA} \cdot P_{BCB}}{P_{ADA} \cdot S_{ABC}^2 \cdot P_{BCB}}$$

$$\stackrel{simplify}{=} \frac{(S_{ABD})^2 \cdot P_{ACA}}{P_{ADA} \cdot (S_{ABC})^2}$$

$$\stackrel{D}{=} \frac{(S_{ABC} \cdot \frac{\overline{AD}}{\overline{AC}})^2 \cdot P_{ACA}}{P_{ACA} \cdot \frac{\overline{AD}}{\overline{AC}}^2 \cdot (S_{ABC})^2} \stackrel{simplify}{=} 1$$

The eliminants

$$P_{EFE} \stackrel{F}{=} \frac{P_{DED} \cdot (S_{ABE})^2}{(S_{ADBE})^2}$$

$$P_{DFD} \stackrel{F}{=} \frac{P_{DED} \cdot (S_{ABD})^2}{(S_{ADBE})^2}$$

$$S_{ABE} \stackrel{E}{=} -\left(\frac{\overline{BE}}{\overline{BC}} \cdot S_{ABC}\right)$$

$$\frac{\overline{BE}}{\overline{BC}}^2 = \frac{P_{ADA}}{P_{BCB}}$$

$$P_{ADA} \stackrel{D}{=} P_{ACA} \cdot (\frac{\overline{AD}}{\overline{AC}})^2$$

$$S_{ABD} \stackrel{D}{=} S_{ABC} \cdot \frac{\overline{AD}}{\overline{AC}}$$

6.4 Quadrilaterals

6.4.1 General Quadrilaterals

Example 6.217 (0.050, 2, 4) *The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram.*

Constructive description
((points $A\ B\ C\ D$)
(midpoint $E\ A\ B$) (midpoint $F\ B\ C$)
(midpoint $G\ C\ D$) (midpoint $H\ D\ A$)
($\frac{\overline{HE}}{\overline{GF}} = 1$))

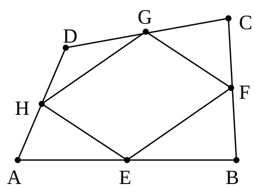


Figure 6-217

The machine proof

$$\frac{\overline{EH}}{\overline{FG}} \overset{H}{=} \frac{-S_{ADE}}{S_{AFDG}} \overset{G}{=} \frac{-S_{ADE}}{-S_{ADF}-\frac{1}{2}S_{ACD}} \overset{F}{=} \frac{(2)\cdot S_{ADE}}{-S_{ABD}} \overset{E}{=} \frac{(-2)\cdot(-\frac{1}{2}S_{ABD})}{S_{ABD}} \overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{EH}}{\overline{FG}} \overset{H}{=} \frac{-S_{ADE}}{S_{AFDG}}$
$S_{AFDG} \overset{G}{=} -\frac{1}{2}(2S_{ADF}+S_{ACD})$
$S_{ADF} \overset{F}{=} -\frac{1}{2}(S_{ACD}+S_{ABD})$
$S_{ADE} \overset{E}{=} -\frac{1}{2}(S_{ABD})$

Example 6.218 (0.083, 5, 13) *The area of the parallelogram whose vertices are the midpoints of the sides of a quadrilateral is equal to half the area of the given quadrilateral (Figure 6-217).*

Constructive description
((points $A\ B\ C\ D$)
(midpoint $E\ A\ B$)
(midpoint $F\ B\ C$)
(midpoint $G\ C\ D$)
(midpoint $H\ D\ A$)
($2S_{EFGH} = S_{ABCD}$))

The machine proof

$$\frac{(2)\cdot S_{EFGH}}{S_{ABCD}} \overset{H}{=} \frac{(2)\cdot(S_{EFG}+\frac{1}{2}S_{DEG}+\frac{1}{2}S_{AEG})}{S_{ABCD}} \overset{G}{=} \frac{S_{DEF}+S_{CEF}+\frac{1}{2}S_{CDE}-\frac{1}{2}S_{ADE}-\frac{1}{2}S_{ACE}}{S_{ABCD}} \overset{F}{=} \frac{2S_{CDE}+S_{BDE}+S_{BCE}-S_{ADE}-S_{ACE}}{(2)\cdot S_{ABCD}} \overset{E}{=} \frac{S_{BCD}+S_{ACD}+S_{ABD}+S_{ABC}}{(2)\cdot S_{ABCD}} \overset{area-co}{=} \frac{2S_{ACD}+2S_{ABC}}{(2)\cdot(S_{ACD}+S_{ABC})} \overset{simplify}{=} 1$$

The eliminants

$S_{EFGH} \overset{H}{=} \frac{1}{2}(2S_{EFG}+S_{DEG}+S_{AEG})$
$S_{AEG} \overset{G}{=} -\frac{1}{2}(S_{ADE}+S_{ACE})$
$S_{DEG} \overset{G}{=} \frac{1}{2}(S_{CDE})$
$S_{EFG} \overset{G}{=} \frac{1}{2}(S_{DEF}+S_{CEF})$
$S_{CEF} \overset{F}{=} \frac{1}{2}(S_{BCE})$
$S_{DEF} \overset{F}{=} \frac{1}{2}(S_{CDE}+S_{BDE})$
$S_{ACE} \overset{E}{=} -\frac{1}{2}(S_{ABC})$
$S_{ADE} \overset{E}{=} -\frac{1}{2}(S_{ABD})$
$S_{BCE} \overset{E}{=} \frac{1}{2}(S_{ABC})$
$S_{BDE} \overset{E}{=} \frac{1}{2}(S_{ABD})$
$S_{CDE} \overset{E}{=} \frac{1}{2}(S_{BCD}+S_{ACD})$
$S_{ABCD} = S_{ACD}+S_{ABC}$
$S_{BCD} = S_{ACD}-S_{ABD}+S_{ABC}$

Example 6.219 (0.067, 3, 13) *The lines joining the midpoints of the two pairs of opposite sides of a quadrilateral and the line joining the midpoints of the diagonals are concurrent and are bisected by their common point.*

Constructive description
((points $A\ B\ C\ D$)
(midpoint $E\ A\ B$)
(midpoint $G\ C\ D$)
(midpoint $N\ A\ C$)
(midpoint $M\ B\ D$)
(inter O (l $E\ G$) (l $N\ M$))
(midpoint $O\ N\ M$))

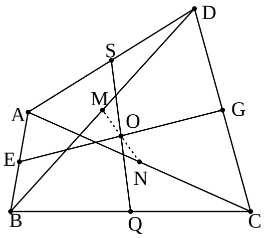


Figure 6-219

The machine proof

$-\frac{\overline{NO}}{\overline{MO}}$

$\stackrel{O}{=} \frac{-S_{EGN}}{S_{EGM}}$

$\stackrel{M}{=} \frac{-S_{EGN}}{\frac{1}{2}S_{DEG}+\frac{1}{2}S_{BEG}}$

$\stackrel{N}{=} \frac{(-2)\cdot(\frac{1}{2}S_{CEG}+\frac{1}{2}S_{AEG})}{S_{DEG}+S_{BEG}}$

$\stackrel{G}{=} \frac{-(-\frac{1}{2}S_{CDE}-\frac{1}{2}S_{ADE}-\frac{1}{2}S_{ACE})}{\frac{1}{2}S_{CDE}-\frac{1}{2}S_{BDE}-\frac{1}{2}S_{BCE}}$

$\stackrel{E}{=} \frac{\frac{1}{2}S_{BCD}+\frac{1}{2}S_{ACD}-\frac{1}{2}S_{ABD}-\frac{1}{2}S_{ABC}}{\frac{1}{2}S_{BCD}+\frac{1}{2}S_{ACD}-\frac{1}{2}S_{ABD}-\frac{1}{2}S_{ABC}}$

$\stackrel{simplify}{=} 1$

The eliminants

$\frac{\overline{NO}}{\overline{MO}} \stackrel{O}{=} \frac{S_{EGN}}{S_{EGM}}$

$S_{EGM} \stackrel{M}{=} \frac{1}{2}(S_{DEG}+S_{BEG})$

$S_{EGN} \stackrel{N}{=} \frac{1}{2}(S_{CEG}+S_{AEG})$

$S_{BEG} \stackrel{G}{=} -\frac{1}{2}(S_{BDE}+S_{BCE})$

$S_{DEG} \stackrel{G}{=} \frac{1}{2}(S_{CDE})$

$S_{AEG} \stackrel{G}{=} -\frac{1}{2}(S_{ADE}+S_{ACE})$

$S_{CEG} \stackrel{G}{=} -\frac{1}{2}(S_{CDE})$

$S_{BCE} \stackrel{E}{=} \frac{1}{2}(S_{ABC})$

$S_{BDE} \stackrel{E}{=} \frac{1}{2}(S_{ABD})$

$S_{ACE} \stackrel{E}{=} -\frac{1}{2}(S_{ABC})$

$S_{ADE} \stackrel{E}{=} -\frac{1}{2}(S_{ABD})$

$S_{CDE} \stackrel{E}{=} \frac{1}{2}(S_{BCD}+S_{ACD})$

Definition. *The intersection of the lines joining the midpoints of two pairs of opposite sides of a quadrilateral is called the centroid of the quadrilateral.*

Example 6.220 (0.050, 2, 6) *The four lines obtained by joining each vertex of a quadrilateral to the centroid of the triangle determined by the remaining three vertices are concurrent at the centroid of the given quadrilateral.*

Constructive description
((points $A\ B\ C\ D$)
(centroid $A_1\ B\ C\ D$)
(midpoint $N\ A\ C$) (midpoint $M\ B\ D$)
(inter J (l $A\ A_1$) (l $N\ M$))
(midpoint $J\ N\ M$))

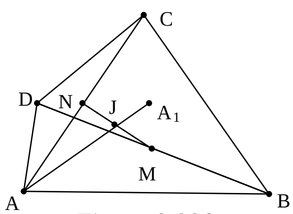


Figure 6-220

The machine proof

$-\frac{\overline{NJ}}{\overline{MJ}}$

$\stackrel{J}{=} \frac{-S_{AA_1N}}{S_{AA_1M}}$

$\stackrel{M}{=} \frac{-S_{AA_1N}}{-\frac{1}{2}S_{ADA_1}-\frac{1}{2}S_{ABA_1}}$

$\stackrel{N}{=} \frac{(2)\cdot(-\frac{1}{2}S_{ACA_1})}{S_{ADA_1}+S_{ABA_1}}$

$\stackrel{A_1}{=} \frac{-(S_{ACD}-S_{ABC})\cdot(3)^2}{(-3S_{ACD}+3S_{ABC})\cdot(3)}$

$\stackrel{simplify}{=} 1$

The eliminants

$\frac{\overline{NJ}}{\overline{MJ}} \stackrel{J}{=} \frac{S_{AA_1N}}{S_{AA_1M}}$

$S_{AA_1M} \stackrel{M}{=} -\frac{1}{2}\left(S_{ADA_1}+S_{ABA_1}\right)$

$S_{AA_1N} \stackrel{N}{=} -\frac{1}{2}\left(S_{ACA_1}\right)$

$S_{ABA_1} \stackrel{A_1}{=} \frac{1}{3}\left(S_{ABD}+S_{ABC}\right)$

$S_{ADA_1} \stackrel{A_1}{=} -\frac{1}{3}\left(S_{ACD}+S_{ABD}\right)$

$S_{ACA_1} \stackrel{A_1}{=} \frac{1}{3}\left(S_{ACD}-S_{ABC}\right)$

Example 6.221 (0.083, 4, 3) *The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the segment joining the midpoints of the diagonals.*

Constructive description

((points A B C D)
(midpoint E A C)
(midpoint F B D)
($\overline{AB}^2+\overline{CB}^2+\overline{CD}^2+\overline{DA}^2$ = $\overline{AC}^2+\overline{BD}^2+4\overline{EF}^2$))

The machine proof

$$\frac{P_{CDC}+P_{BCB}+P_{ADA}+P_{ABA}}{4P_{EFE}+P_{BDB}+P_{ACA}}$$
$$\stackrel{F}{=} \frac{P_{CDC}+P_{BCB}+P_{ADA}+P_{ABA}}{2P_{DED}+2P_{BEB}+P_{ACA}}$$
$$\stackrel{E}{=} \frac{P_{CDC}+P_{BCB}+P_{ADA}+P_{ABA}}{P_{CDC}+P_{BCB}+P_{ADA}+P_{ABA}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{EFE} \stackrel{F}{=} \tfrac{1}{4}(2P_{DED}+2P_{BEB}-P_{BDB})$$
$$P_{BEB} \stackrel{E}{=} \tfrac{1}{4}(2P_{BCB}-P_{ACA}+2P_{ABA})$$
$$P_{DED} \stackrel{E}{=} \tfrac{1}{4}(2P_{CDC}+2P_{ADA}-P_{ACA})$$

Example 6.222 (0.033, 7, 6) *The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the two lines joining the midpoints of the two pairs of opposite sides of the quadrilateral.*

Constructive description

((points A B C D)
(midpoint P A B)
(midpoint Q B C)
(midpoint S D A)
(midpoint R C D)
($\overline{AC}^2+\overline{BD}^2$ = $2\overline{QS}^2+2\overline{PR}^2$))

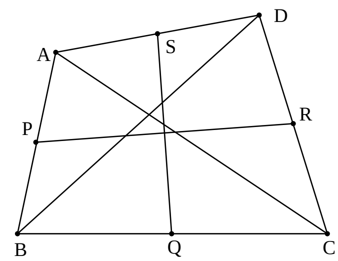


Figure 6-222

The machine proof

$$\frac{(\frac{1}{2})\cdot(P_{BDB}+P_{ACA})}{P_{QSQ}+P_{PRP}}$$
$$\stackrel{R}{=} \frac{(\frac{1}{2})\cdot(P_{BDB}+P_{ACA})}{P_{QSQ}+\frac{1}{2}P_{DPD}+\frac{1}{2}P_{CPC}-\frac{1}{4}P_{CDC}}$$
$$\stackrel{S}{=} \frac{(2)\cdot(P_{BDB}+P_{ACA})}{2P_{DQD}+2P_{DPD}+2P_{CPC}-P_{CDC}+2P_{AQA}-P_{ADA}}$$
$$\stackrel{Q}{=} \frac{(2)\cdot(P_{BDB}+P_{ACA})}{2P_{DPD}+2P_{CPC}+P_{BDB}-P_{BCB}-P_{ADA}+P_{ACA}+P_{ABA}}$$
$$\stackrel{P}{=} \frac{(2)\cdot(P_{BDB}+P_{ACA})}{2P_{BDB}+2P_{ACA}} \stackrel{simplify}{=} 1$$

The eliminants

$$P_{PRP} \stackrel{R}{=} \tfrac{1}{4}(2P_{DPD}+2P_{CPC}-P_{CDC})$$
$$P_{QSQ} \stackrel{S}{=} \tfrac{1}{4}(2P_{DQD}+2P_{AQA}-P_{ADA})$$
$$P_{AQA} \stackrel{Q}{=} -\tfrac{1}{4}(P_{BCB}-2P_{ACA}-2P_{ABA})$$
$$P_{DQD} \stackrel{Q}{=} \tfrac{1}{4}(2P_{CDC}+2P_{BDB}-P_{BCB})$$
$$P_{CPC} \stackrel{P}{=} \tfrac{1}{4}(2P_{BCB}+2P_{ACA}-P_{ABA})$$
$$P_{DPD} \stackrel{P}{=} \tfrac{1}{4}(2P_{BDB}+2P_{ADA}-P_{ABA})$$

Example 6.223 (0.050, 4, 10) *If a quadrilateral $ABCD$ has its opposite sides AD and BC (extended) meeting at W, while X and Y are the midpoints of the diagonals AC and BD, then $4S_{WXY} = S_{ABCD}$.*

Constructive description

((points B C D A)
(midpoint X C A)
(midpoint Y B D)
(inter W (l B C) (l D A))
(S_{BCDA} = $4S_{XYW}$))

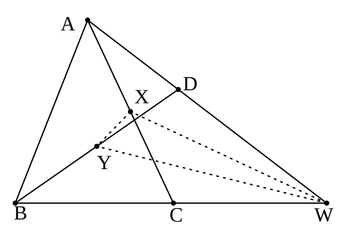


Figure 6-223

The machine proof

$$\frac{S_{BCDA}}{(4)\cdot S_{XYW}}$$

$$\stackrel{W}{=} \frac{S_{BCDA}\cdot S_{BDCA}}{(4)\cdot(S_{CXY}\cdot S_{BDA}-S_{CDA}\cdot S_{BXY})}$$

$$\stackrel{Y}{=} \frac{S_{BCDA}\cdot S_{BDCA}}{(4)\cdot(-\frac{1}{2}S_{CDX}\cdot S_{BDA}+\frac{1}{2}S_{CDA}\cdot S_{BDX}+\frac{1}{2}S_{BDA}\cdot S_{BCX})}$$

$$\stackrel{X}{=} \frac{-S_{BCDA}\cdot S_{BDCA}}{(2)\cdot(\frac{1}{2}S_{CDA}\cdot S_{BCD}-\frac{1}{2}S_{BDA}\cdot S_{BCA})}$$

$$\stackrel{area-co}{=} \frac{-(S_{BDA}+S_{BCD})\cdot(S_{BCA}-S_{BCD})}{-S_{BDA}\cdot S_{BCA}+S_{BDA}\cdot S_{BCD}-S_{BCA}\cdot S_{BCD}+S_{BCD}^2}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{XYW} \stackrel{W}{=} \frac{S_{CXY}\cdot S_{BDA}-S_{CDA}\cdot S_{BXY}}{S_{BDCA}}$$

$$S_{BXY} \stackrel{Y}{=} -\frac{1}{2}(S_{BDX})$$

$$S_{CXY} \stackrel{Y}{=} -\frac{1}{2}(S_{CDX}-S_{BCX})$$

$$S_{BCX} \stackrel{X}{=} \frac{1}{2}(S_{BCA})$$

$$S_{BDX} \stackrel{X}{=} \frac{1}{2}(S_{BDA}-S_{BCD})$$

$$S_{CDX} \stackrel{X}{=} \frac{1}{2}(S_{CDA})$$

$$S_{CDA}=S_{BDA}-S_{BCA}+S_{BCD}$$

$$S_{BDCA}=S_{BCA}-S_{BCD}$$

$$S_{BCDA}=S_{BDA}+S_{BCD}$$

Example 6.224 (0.216, 6, 12) *Let P be a point on the line joining the midpoints of the diagonals of the quadrilateral $ABCD$. Show that $S_{PAB}+S_{PCD}=S_{PDA}+S_{PBC}$.*

Constructive description
((points A B C D)
(midpoint N A C)
(midpoint M B D)
(lratio P N M r)
$(S_{PAB}+S_{PCD} = S_{PDA}+S_{PBC})$)

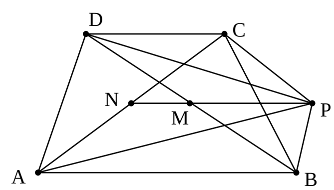


Figure 6-224

The machine proof

$$\frac{S_{CDP}+S_{ABP}}{S_{BCP}-S_{ADP}}$$

$$\stackrel{P}{=} \frac{S_{CDM}\cdot r-S_{CDN}\cdot r+S_{CDN}+S_{ABM}\cdot r-S_{ABN}\cdot r+S_{ABN}}{S_{BCM}\cdot r-S_{BCN}\cdot r+S_{BCN}-S_{ADM}\cdot r+S_{ADN}\cdot r-S_{ADN}}$$

$$\stackrel{M}{=} \frac{-S_{CDN}\cdot r+S_{CDN}+\frac{1}{2}S_{BCD}\cdot r-S_{ABN}\cdot r+S_{ABN}+\frac{1}{2}S_{ABD}\cdot r}{-S_{BCN}\cdot r+S_{BCN}+\frac{1}{2}S_{BCD}\cdot r+S_{ADN}\cdot r-S_{ADN}+\frac{1}{2}S_{ABD}\cdot r}$$

$$\stackrel{N}{=} \frac{-S_{BCD}\cdot r+S_{ACD}\cdot r-S_{ACD}-S_{ABD}\cdot r+S_{ABC}\cdot r-S_{ABC}}{-S_{BCD}\cdot r+S_{ACD}\cdot r-S_{ACD}-S_{ABD}\cdot r+S_{ABC}\cdot r-S_{ABC}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{ADP} \stackrel{P}{=} S_{ADM}\cdot r-S_{ADN}\cdot r+S_{ADN}$$

$$S_{BCP} \stackrel{P}{=} S_{BCM}\cdot r-S_{BCN}\cdot r+S_{BCN}$$

$$S_{ABP} \stackrel{P}{=} S_{ABM}\cdot r-S_{ABN}\cdot r+S_{ABN}$$

$$S_{CDP} \stackrel{P}{=} S_{CDM}\cdot r-S_{CDN}\cdot r+S_{CDN}$$

$$S_{ADM} \stackrel{M}{=} -\frac{1}{2}(S_{ABD}),\ S_{BCM} \stackrel{M}{=} \frac{1}{2}(S_{BCD})$$

$$S_{ABM} \stackrel{M}{=} \frac{1}{2}(S_{ABD}),\ S_{CDM} \stackrel{M}{=} \frac{1}{2}(S_{BCD})$$

$$S_{ADN} \stackrel{N}{=} -\frac{1}{2}(S_{ACD}),\ S_{BCN} \stackrel{N}{=} \frac{1}{2}(S_{ABC})$$

$$S_{ABN} \stackrel{N}{=} \frac{1}{2}(S_{ABC}),\ S_{CDN} \stackrel{N}{=} \frac{1}{2}(S_{ACD})$$

Example 6.225 (0.083, 2, 10) *Let $ABCD$ be a quadrilateral, F and E be the midpoints of AD and BC respectively. $M = BA\cap EF$, $N = CD\cap EF$. Show that $\frac{\overline{AM}}{\overline{BM}} = \frac{\overline{DN}}{\overline{CN}}$*

Constructive description

((points $A\ B\ C\ D$)
(midpoint $E\ B\ C$)
(midpoint $F\ A\ D$)
(inter M (l $A\ B$) (l $E\ F$))
(inter N (l $C\ D$) (l $E\ F$))
($\frac{\overline{AM}}{\overline{BM}} = \frac{\overline{DN}}{\overline{CN}}$))

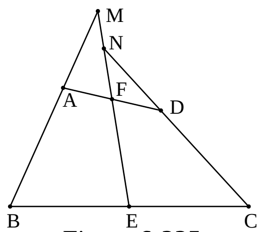


Figure 6-225

The machine proof

$$(\frac{\overline{AM}}{\overline{BM}})/(\frac{\overline{DN}}{\overline{CN}})$$

$$\overset{N}{=} \frac{S_{CEF}}{S_{DEF}} \cdot \frac{\overline{AM}}{\overline{BM}}$$

$$\overset{M}{=} \frac{S_{AEF} \cdot S_{CEF}}{S_{DEF} \cdot S_{BEF}}$$

$$\overset{F}{=} \frac{(-\frac{1}{2}S_{ADE})\cdot(-\frac{1}{2}S_{CDE}+\frac{1}{2}S_{ACE})}{(\frac{1}{2}S_{ADE})\cdot(-\frac{1}{2}S_{BDE}+\frac{1}{2}S_{ABE})}$$

$$\overset{simplify}{=} \frac{-(S_{CDE}-S_{ACE})}{S_{BDE}-S_{ABE}}$$

$$\overset{E}{=} \frac{-(\frac{1}{2}S_{BCD}+\frac{1}{2}S_{ABC})}{-\frac{1}{2}S_{BCD}-\frac{1}{2}S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{DN}}{\overline{CN}} \overset{N}{=} \frac{S_{DEF}}{S_{CEF}}$

$\frac{\overline{AM}}{\overline{BM}} \overset{M}{=} \frac{S_{AEF}}{S_{BEF}}$

$S_{BEF} \overset{F}{=} -\frac{1}{2}(S_{BDE}-S_{ABE})$

$S_{DEF} \overset{F}{=} \frac{1}{2}(S_{ADE})$

$S_{CEF} \overset{F}{=} -\frac{1}{2}(S_{CDE}-S_{ACE})$

$S_{AEF} \overset{F}{=} -\frac{1}{2}(S_{ADE})$

$S_{ABE} \overset{E}{=} \frac{1}{2}(S_{ABC})$

$S_{BDE} \overset{E}{=} -\frac{1}{2}(S_{BCD})$

$S_{ACE} \overset{E}{=} -\frac{1}{2}(S_{ABC})$

$S_{CDE} \overset{E}{=} \frac{1}{2}(S_{BCD})$

Example 6.226 (1.333, 20, 37) *Let $ABCD$ be a quadrilateral with $AB = CD$, F and E be the midpoints of AD and BC respectively. $M = BA \cap EF$, $N = CD \cap EF$. Show that $\angle BME = \angle ENC$ (Figure 6-225).*

Constructive description

((points $A\ B\ C\ X$) (inter D (l $C\ X$) (cir $C\ \overline{AB}^2$)) (midpoint $F\ A\ D$) (midpoint $E\ B\ C$)
(inter M (l $A\ B$) (l $E\ F$)) (inter N (l $C\ D$) (l $E\ F$)) (eqangle $B\ M\ E\ E\ N\ C$))

Example 6.227 (0.300, 3, 18) *Let $ABCD$ be a quadrilateral such that $AB = CD$. F and E are the midpoints of AD and BC respectively. $M = BA \cap EF$, $N = CD \cap EF$. Show that $AM = DN$.*

Constructive description
((points $A\ B\ C\ X$)
(inter D (l $C\ X$) (cir $C\ \overline{AB}^2$))
(midpoint $F\ A\ D$)
(midpoint $E\ B\ C$)
(inter M (l $A\ B$) (l $E\ F$))
(inter N (l $C\ D$) (l $E\ F$))
(eqdistance $A\ M\ D\ N$))

The eliminants

$P_{DND} \overset{N}{=} \frac{P_{CDC} \cdot S_{DFE}^2}{S_{CFDE}^2}$, $P_{AMA} \overset{M}{=} \frac{P_{ABA} \cdot S_{AFE}^2}{S_{AFBE}^2}$

$S_{AFBE} \overset{E}{=} -\frac{1}{2}(2S_{ABF} - S_{ABC})$

$S_{DFE} \overset{E}{=} \frac{1}{2}(S_{CDF} + S_{BDF})$

$S_{CFDE} \overset{E}{=} -\frac{1}{2}(2S_{CDF} - S_{BCD})$

$S_{AFE} \overset{E}{=} -\frac{1}{2}(S_{ACF} + S_{ABF})$

$S_{BDF} \overset{F}{=} \frac{1}{2}(S_{ABD})$, $S_{CDF} \overset{F}{=} \frac{1}{2}(S_{ACD})$

$S_{ABF} \overset{F}{=} \frac{1}{2}(S_{ABD})$, $S_{ACF} \overset{F}{=} \frac{1}{2}(S_{ACD})$

$S_{ABD} \overset{D}{=} -\left(\frac{\overline{XD}}{\overline{CX}} \cdot S_{ABC} - \frac{\overline{CD}}{\overline{CX}} \cdot S_{ABX}\right)$

$P_{CDC} \overset{D}{=} \frac{\overline{CD}}{\overline{CX}}^2 \cdot P_{CXC}$

$S_{ACD} \overset{D}{=} -\left(\frac{\overline{CD}}{\overline{CX}} \cdot S_{ACX}\right)$, $S_{BCD} \overset{D}{=} -\left(\frac{\overline{CD}}{\overline{CX}} \cdot S_{BCX}\right)$

$\frac{\overline{CD}}{\overline{CX}}^2 = \frac{P_{ABA}}{P_{CXC}}$, $\frac{\overline{CD}}{\overline{CX}}^2 = \frac{P_{ABA}}{P_{CXC}}$

$S_{ABC} = S_{ABC}$, $S_{BCX} = S_{ACX} - S_{ABX} + S_{ABC}$

The machine proof

$$\frac{P_{AMA}}{P_{DND}} \overset{N}{=} \frac{P_{AMA} \cdot S_{CFDE}^2}{P_{CDC} \cdot S_{DFE}^2} \overset{M}{=} \frac{P_{ABA} \cdot S_{AFE}^2 \cdot S_{CFDE}^2}{P_{CDC} \cdot S_{DFE}^2 \cdot S_{AFBE}^2}$$

$$\overset{E}{=} \frac{P_{ABA} \cdot (-\frac{1}{2}S_{ACF} - \frac{1}{2}S_{ABF})^2 \cdot (-S_{CDF} + \frac{1}{2}S_{BCD})^2}{P_{CDC} \cdot (\frac{1}{2}S_{CDF} + \frac{1}{2}S_{BDF})^2 \cdot (-S_{ABF} + \frac{1}{2}S_{ABC})^2}$$

$$\overset{F}{=} \frac{P_{ABA} \cdot (\frac{1}{2}S_{ACD} + \frac{1}{2}S_{ABD})^2 \cdot (-S_{BCD} + S_{ACD})^2}{P_{CDC} \cdot (\frac{1}{2}S_{ACD} + \frac{1}{2}S_{ABD})^2 \cdot (S_{ABD} - S_{ABC})^2}$$

$$\overset{simplify}{=} \frac{P_{ABA} \cdot (S_{BCD} - S_{ACD})^2}{P_{CDC} \cdot (S_{ABD} - S_{ABC})^2}$$

$$\overset{D}{=} \frac{(P_{ABA} \cdot S_{BCX}^2 - 2P_{ABA} \cdot S_{BCX} \cdot S_{ACX} + P_{ABA} \cdot S_{ACX}^2) \cdot P_{ABA} \cdot P_{CXC}^2}{(P_{CXC} \cdot P_{ABA}^2 \cdot S_{ABX}^2 - 2P_{CXC} \cdot P_{ABA}^2 \cdot S_{ABX} \cdot S_{ABC} + P_{CXC} \cdot P_{ABA}^2 \cdot S_{ABC}^2) \cdot P_{CXC}}$$

$$\overset{simplify}{=} \frac{(S_{BCX} - S_{ACX})^2}{(S_{ABX} - S_{ABC})^2} \overset{area-co}{=} \frac{(-S_{ABX} + S_{ABC})^2}{(S_{ABX} - S_{ABC})^2} \overset{simplify}{=} 1$$

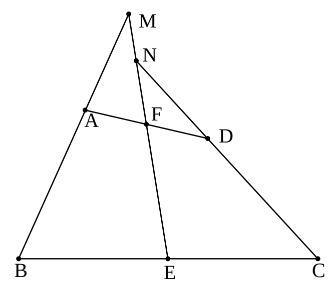


Figure 6-227

Example 6.228 (0.416, 12, 36) *Let ABCD be a quadrilateral such that AB = CD. P and Q are the midpoints of AD and BC; N and M are the midpoints of AC and BD. Show that* $PQ \perp NM$.

Constructive description
((points $A\ B\ C\ X$)
(inter D (l $C\ X$) (cir $C\ \overline{AB}^2$))
(midpoint $P\ A\ D$)
(midpoint $Q\ B\ C$)
(midpoint $N\ A\ C$)
(midpoint $M\ B\ D$)
(perpendicular $N\ M\ P\ Q$))

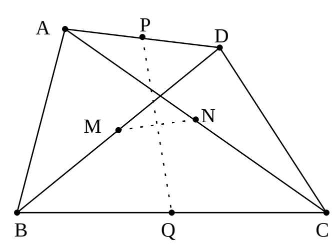


Figure 6-228

Example 6.229 (0.866, 10, 20) *The sides BA, CD of the quadrilateral ABCD meet in O, and the sides DA, CB meet O_1. Along OA, OC, O_1A, O_1C are measured off, respectively, OE, OF, O_1E_1, O_1F_1 equal to AB, DC, AD, BC. Prove that EF is parallel to E_1F_1.*

Constructive description
((points A B C D)
(inter O (l A B) (l C D))
(inter O_1 (l A D) (l B C))
(lratio E O B $-\frac{\overline{AB}}{\overline{BO}}$)
(lratio F O C $-\frac{\overline{DC}}{\overline{CO}}$)
(lratio E_1 O_1 A $-\frac{\overline{AD}}{\overline{AO_1}}$)
(lratio F_1 O_1 B $-\frac{\overline{BC}}{\overline{BO_1}}$)
($S_{EFE_1} = S_{EFF_1}$))

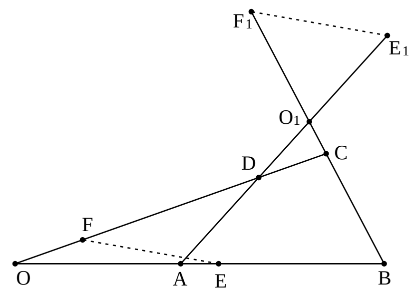


Figure 6-229

Example 6.230 (0.617, 13, 70) *ABCD is a quadrilateral, P, Q, R, S the midpoints of its sides taken in order, U, V, the midpoints of the diagonals, O any point; OP, OQ, OR, OS, OU, OV are divided in the same ratio in P_1, Q_1, R_1, S_1, U_1, V_1. Prove that P_1R_1, Q_1S_1, U_1V_1, are concurrent.*

Constructive description
((points A B C D O)
(midpoint P A B) (midpoint Q B C)
(midpoint R C D) (midpoint S D A)
(midpoint U A C) (midpoint V D B)
(lratio P_1 O P Y_T) (lratio Q_1 O Q Y_T)
(lratio R_1 O R Y_T) (lratio S_1 O S Y_T)
(lratio U_1 O U Y_T) (lratio V_1 O V Y_T)
(inter J (l P_1 R_1) (l Q_1 S_1)) (inter Z_1 (l P_1 R_1) (l U_1 V_1))
(inter Z_2 (l U_1 V_1) (l Q_1 S_1)) ($\frac{\overline{U_1Z_1}}{\overline{V_1Z_1}} = \frac{\overline{U_1Z_2}}{\overline{V_1Z_2}}$))

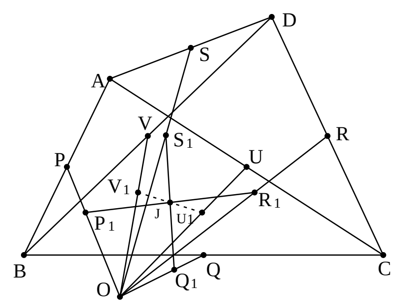


Figure 6-230

Example 6.231 (Theorem of Pratt-Wu) (5.133, 72, 123) *Given a quadrilateral ABDC, let HE, EF, FG, GH be the tangents of circles CAB, ABD, BDC, DCA at A, B, D and C, respectively. Then $HA \cdot EB \cdot FD \cdot GC = AE \cdot BF \cdot DG \cdot CH$.*

Constructive description
((points A B C D)
(midpoint L A B) (midpoint M B D)
(midpoint N C D) (midpoint P C A)
(inter X (t L L A) (t P P A))
(inter Y (t L L B) (t M M B))
(inter Z (t M M D) (t N N D))

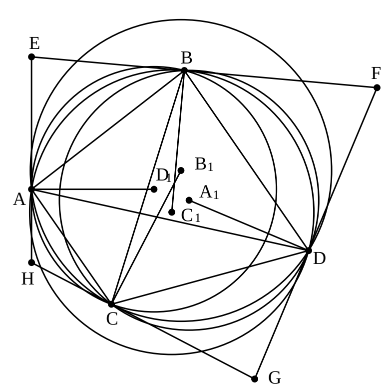


Figure 6-231

(inter W (t N N C) (t P P C))
(tratio O A X I) (inter H (t C C W) (l O A))
(tratio U D Z J) (inter F (t B B Y) (l U D))
(inter E (l B F) (l A O))
(inter G (l D U) (l C H)) ($\frac{\overline{HA}}{\overline{AE}} \cdot \frac{\overline{BE}}{\overline{FB}} = \frac{\overline{DG}}{\overline{FD}} \cdot \frac{\overline{HC}}{\overline{CG}}$))

Example 6.232 (0.00, 1, 3) *Let P, Q be the midpoints of the diagonals of a trapezoid $ABCD$. Then PQ is parallel to the two parallel sides of $ABCD$.*

Constructive description
((points A B C)
(on D (p C A B))
(midpoint Q A C)
(midpoint P B D)
(parallel P Q A B))

The machine proof

$$\frac{S_{ABP}}{S_{ABQ}}$$

$$\overset{P}{=} \frac{\frac{1}{2}S_{ABD}}{S_{ABQ}}$$

$$\overset{Q}{=} \frac{S_{ABD}}{(2)\cdot(\frac{1}{2}S_{ABC})}$$

$$\overset{D}{=} \frac{S_{ABC}}{S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{ABP} \overset{P}{=} \frac{1}{2}(S_{ABD})$$
$$S_{ABQ} \overset{Q}{=} \frac{1}{2}(S_{ABC})$$
$$S_{ABD} \overset{D}{=} S_{ABC}$$

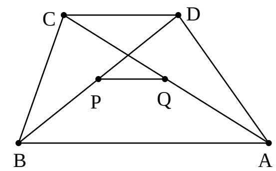


Figure 6-232

Example 6.233 (0.050, 2, 3) *Let P, Q be the midpoints of the diagonals of a trapezoid $ABCD$. Then PQ is half of the difference of the two parallel sides of $ABCD$.*

Constructive description
((points A B C)
(on D (p A B C))
(midpoint P B D)
(midpoint Q A C)
($\frac{\overline{BC}}{\overline{PQ}} - \frac{\overline{AD}}{\overline{PQ}} = 2$))

The machine proof

$$\frac{1}{2}\left(\frac{\overline{BC}}{\overline{PQ}} - \frac{\overline{AD}}{\overline{PQ}}\right)$$

$$\overset{Q}{=} \frac{S_{ACP}\cdot S_{ACD} - S_{ACP}\cdot S_{ABC}}{(2)\cdot S_{ACP}^2}$$

$$\overset{simplify}{=} \frac{S_{ACD} - S_{ABC}}{(2)\cdot S_{ACP}}$$

$$\overset{P}{=} \frac{S_{ACD} - S_{ABC}}{(2)\cdot(\frac{1}{2}S_{ACD} - \frac{1}{2}S_{ABC})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AD}}{\overline{PQ}} \overset{Q}{=} \frac{-S_{ACD}}{S_{ACP}}$$
$$\frac{\overline{BC}}{\overline{PQ}} \overset{Q}{=} \frac{-S_{ABC}}{S_{ACP}}$$
$$S_{ACP} \overset{P}{=} \frac{1}{2}(S_{ACD} - S_{ABC})$$

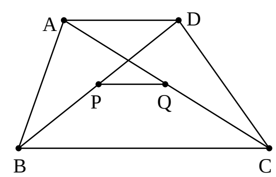


Figure 6-233

Example 6.234 (0.050, 3, 7) *Let $ABCD$ be a trapezoid and O be the intersection of its diagonals AC and BD. The line passing through O and parallel to AB meet AD and BC at F and E. Show that O is the midpoint of EF.*

Constructive description

((points A B C)
(on D (p C A B))
(inter O (l A C) (l B D))
(inter E (l B C) (p O A B))
(inter F (l A D) (p O A B))
(midpoint O E F))

The machine proof

$$-\frac{\overline{OE}}{\overline{OF}}$$
$$\stackrel{F}{=} \frac{S_{AODE}}{S_{ADO}}$$
$$\stackrel{E}{=} \frac{S_{BCO}\cdot S_{ABD}+S_{ADO}\cdot S_{ABC}+S_{ACD}\cdot S_{ABO}}{S_{ADO}\cdot(-S_{ABC})}$$
$$\stackrel{O}{=} \frac{-S_{ABCD}^2\cdot S_{BCD}\cdot S_{ABD}\cdot S_{ABC}\cdot S_{ABCD}}{(-S_{ACD}\cdot S_{ABD})\cdot S_{ABC}\cdot(S_{ABCD})^3}$$
$$\stackrel{simplify}{=} \frac{S_{BCD}}{S_{ACD}}$$
$$\stackrel{D}{=} \frac{-S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}}{-S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{OE}}{\overline{OF}} \stackrel{F}{=} \frac{-S_{AODE}}{S_{ADO}}$$
$$S_{AODE} \stackrel{E}{=} \frac{S_{BCO}\cdot S_{ABD}+S_{ADO}\cdot S_{ABC}+S_{ACD}\cdot S_{ABO}}{-S_{ABC}}$$
$$S_{ABO} \stackrel{O}{=} \frac{S_{ABD}\cdot S_{ABC}}{S_{ABCD}}$$
$$S_{ADO} \stackrel{O}{=} \frac{-S_{ACD}\cdot S_{ABD}}{S_{ABCD}}$$
$$S_{BCO} \stackrel{O}{=} \frac{S_{BCD}\cdot S_{ABC}}{S_{ABCD}}$$
$$S_{ACD} \stackrel{D}{=} -\left(S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}\right)$$
$$S_{BCD} \stackrel{D}{=} -\left(S_{ABC}\cdot\frac{\overline{CD}}{\overline{AB}}\right)$$

Other properties of the general quadrilateral can be found in Examples 2.62, 6.2, 6.3, and 6.17.

6.4.2 Complete Quadrilaterals

Definition. *By a complete quadrilateral, we mean the figure consisting of four points (any three of them are not collinear) and the six lines joining any two of them. As in Figure 6-235, A, B, C, D are the vertices of the complete quadrilateral. AB and CD, AC and BD, AD and BC are called opposite sides of the complete quadrilateral respectively.*

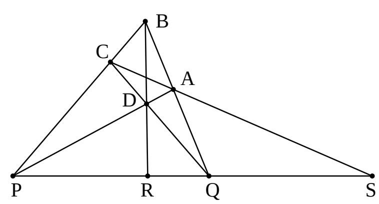


Figure 6-235

Example 6.235 (0.083, 1, 10) *The line joining the intersections of two pairs of opposite sides of a complete quadrilateral is divided harmonically by the remaining pair of opposite sides of the complete quadrilateral. In Figure 6-235, this means $(PQRS) = -1$.*

For a machine proof of this example, see Example 6.17 on page 273.

Example 6.235 can be used to define the concept of harmonic sequence: four collinear points P, Q, R, S are said to be a *harmonic sequence* if there exists a complete quadrilateral

$ABCD$ such that $P = BC \cap AD$, $Q = AB \cap CD$, $R = BD \cap PQ$, and $S = AC \cap PQ$. By the following example, the above definition is independent of the choices of $ABCD$.

Example 6.236 (0.466, 1, 14) *Let $ABCD$ and $EFGH$ be two complete quadrilaterals such that $P = AB \cap CD = EF \cap HG$, $Q = AD \cap BC = EH \cap FG$, and $S = AC \cap PQ = EG \cap PQ$ are collinear. Then BD, FH, PQ are concurrent.*

Constructive description
((points A B C D E)
(inter P (l A B) (l C D))
(inter Q (l B C) (l A D))
(inter S (l A C) (l P Q))
(on G (l E S))
(inter F (l P E) (l Q G))
(inter H (l E Q) (l P G))
(inter R (l B D) (l P Q))
(inter O (l F H) (l P Q))
($\frac{\overline{PR}}{\overline{QR}} = \frac{\overline{PO}}{\overline{QO}}$))

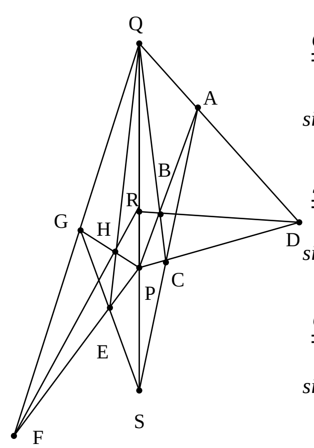


Figure 6-236

The machine proof

$$\left(\frac{\overline{PR}}{\overline{QR}}\right)/\left(\frac{\overline{PO}}{\overline{QO}}\right) \overset{O}{=} \frac{S_{QFH}}{S_{PFH}} \cdot \frac{\overline{PR}}{\overline{QR}}$$

$$\overset{R}{=} \frac{S_{BDP} \cdot S_{QFH}}{S_{PFH} \cdot S_{BDQ}}$$

$$\overset{H}{=} \frac{S_{BDP} \cdot S_{PQG} \cdot S_{EQF} \cdot (-S_{EPQG})}{S_{PGF} \cdot S_{EPQ} \cdot S_{BDQ} \cdot S_{EPQG}}$$

$$\overset{simplify}{=} \frac{-S_{BDP} \cdot S_{PQG} \cdot S_{EQF}}{S_{PGF} \cdot S_{EPQ} \cdot S_{BDQ}}$$

$$\overset{F}{=} \frac{-S_{BDP} \cdot S_{PQG} \cdot (-S_{EQG} \cdot S_{EPQ}) \cdot S_{EQPG}}{(-S_{PQG} \cdot S_{EPG}) \cdot S_{EPQ} \cdot S_{BDQ} \cdot S_{EQPG}}$$

$$\overset{simplify}{=} \frac{-S_{BDP} \cdot S_{EQG}}{S_{EPG} \cdot S_{BDQ}}$$

$$\overset{G}{=} \frac{-S_{BDP} \cdot S_{EQS} \cdot \frac{\overline{EG}}{\overline{ES}}}{S_{EPS} \cdot \frac{\overline{EG}}{\overline{ES}} \cdot S_{BDQ}}$$

$$\overset{simplify}{=} \frac{-S_{BDP} \cdot S_{EQS}}{S_{EPS} \cdot S_{BDQ}}$$

$$\overset{S}{=} \frac{-S_{BDP} \cdot S_{EPQ} \cdot S_{ACQ} \cdot (-S_{APCQ})}{S_{EPQ} \cdot S_{ACP} \cdot S_{BDQ} \cdot (-S_{APCQ})}$$

$$\overset{simplify}{=} \frac{-S_{BDP} \cdot S_{ACQ}}{S_{ACP} \cdot S_{BDQ}}$$

$$\overset{Q}{=} \frac{-S_{BDP} \cdot (-S_{ACD} \cdot S_{ABC}) \cdot (-S_{ABDC})}{S_{ACP} \cdot S_{BCD} \cdot S_{ABD} \cdot (-S_{ABDC})}$$

$$\overset{simplify}{=} \frac{S_{BDP} \cdot S_{ACD} \cdot S_{ABC}}{S_{ACP} \cdot S_{BCD} \cdot S_{ABD}}$$

$$\overset{P}{=} \frac{(-S_{BCD} \cdot S_{ABD}) \cdot S_{ACD} \cdot S_{ABC} \cdot S_{ACBD}}{(-S_{ACD} \cdot S_{ABC}) \cdot S_{BCD} \cdot S_{ABD} \cdot S_{ACBD}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{PO}}{\overline{QO}} \overset{O}{=} \frac{S_{PFH}}{S_{QFH}}$$

$$\frac{\overline{PR}}{\overline{QR}} \overset{R}{=} \frac{S_{BDP}}{S_{BDQ}}$$

$$S_{PFH} \overset{H}{=} \frac{S_{PGF} \cdot S_{EPQ}}{-S_{EPQG}}$$

$$S_{QFH} \overset{H}{=} \frac{S_{PQG} \cdot S_{EQF}}{S_{EPQG}}$$

$$S_{PGF} \overset{F}{=} \frac{-S_{PQG} \cdot S_{EPG}}{S_{EQPG}}$$

$$S_{EQF} \overset{F}{=} \frac{-S_{EQG} \cdot S_{EPQ}}{S_{EQPG}}$$

$$S_{EPG} \overset{G}{=} S_{EPS} \cdot \frac{\overline{EG}}{\overline{ES}}$$

$$S_{EQG} \overset{G}{=} S_{EQS} \cdot \frac{\overline{EG}}{\overline{ES}}$$

$$S_{EPS} \overset{S}{=} \frac{S_{EPQ} \cdot S_{ACP}}{-S_{APCQ}}$$

$$S_{EQS} \overset{S}{=} \frac{S_{EPQ} \cdot S_{ACQ}}{-S_{APCQ}}$$

$$S_{BDQ} \overset{Q}{=} \frac{S_{BCD} \cdot S_{ABD}}{-S_{ABDC}}$$

$$S_{ACQ} \overset{Q}{=} \frac{S_{ACD} \cdot S_{ABC}}{S_{ABDC}}$$

$$S_{ACP} \overset{P}{=} \frac{-S_{ACD} \cdot S_{ABC}}{S_{ACBD}}$$

$$S_{BDP} \overset{P}{=} \frac{-S_{BCD} \cdot S_{ABD}}{S_{ACBD}}$$

Example 6.237 (0.216, 2, 24) *If the intersections of five corresponding sides of two complete quadrilaterals are on the same line l. Then the remaining sides also meet in l.*

Constructive description
((points A B C D P)
(lratio U A D r_1) (lratio V B C r_2)

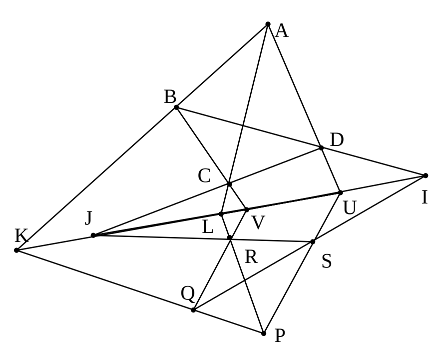


Figure 6-237

(inter I (l D B) (l U V))
(inter J (l C D) (l U V))
(inter K (l A B) (l U V))
(lratio S U P r_3)
(inter Q (l I S) (l K P))
(inter R (l V Q) (l S J))
(inter L (l P R) (l U V))
(inter N (l A C) (l U V)) $\left(\frac{\overline{UL}}{\overline{VL}} = \frac{\overline{UN}}{\overline{VN}}\right)$)

The machine proof

$$\left(\frac{\overline{UL}}{\overline{VL}}\right)/\left(\frac{\overline{UN}}{\overline{VN}}\right)$$

$$\overset{N}{=} \frac{S_{ACV}}{S_{ACU}} \cdot \frac{\overline{UL}}{\overline{VL}}$$

$$\overset{L}{=} \frac{(-S_{PUR}) \cdot S_{ACV}}{S_{ACU} \cdot (-S_{PVR})}$$

$$\overset{R}{=} \frac{S_{VSQ} \cdot S_{PUJ} \cdot S_{ACV} \cdot S_{VJQS}}{S_{ACU} \cdot S_{VJS} \cdot S_{PVQ} \cdot (-S_{VJQS})}$$

$$\overset{simplify}{=} \frac{S_{VSQ} \cdot S_{PUJ} \cdot S_{ACV}}{-S_{ACU} \cdot S_{VJS} \cdot S_{PVQ}}$$

$$\overset{Q}{=} \frac{S_{VIS} \cdot S_{PKS} \cdot S_{PUJ} \cdot S_{ACV} \cdot S_{PIKS}}{-S_{ACU} \cdot S_{VJS} \cdot S_{PIS} \cdot S_{PVK} \cdot (-S_{PIKS})}$$

$$\overset{simplify}{=} \frac{S_{VIS} \cdot S_{PKS} \cdot S_{PUJ} \cdot S_{ACV}}{S_{ACU} \cdot S_{VJS} \cdot S_{PIS} \cdot S_{PVK}}$$

$$\overset{S}{=} \frac{S_{PVI} \cdot r_3 \cdot (S_{PUK} \cdot r_3 - S_{PUK}) \cdot S_{PUJ} \cdot S_{ACV}}{S_{ACU} \cdot S_{PVJ} \cdot r_3 \cdot (S_{PUI} \cdot r_3 - S_{PUI}) \cdot S_{PVK}}$$

$$\overset{simplify}{=} \frac{S_{PVI} \cdot S_{PUK} \cdot S_{PUJ} \cdot S_{ACV}}{S_{ACU} \cdot S_{PVJ} \cdot S_{PUI} \cdot S_{PVK}}$$

$$\overset{K}{=} \frac{S_{PVI} \cdot S_{PUV} \cdot S_{ABU} \cdot S_{PUJ} \cdot S_{ACV} \cdot (-S_{AUBV})}{S_{ACU} \cdot S_{PVJ} \cdot S_{PUI} \cdot S_{PUV} \cdot S_{ABV} \cdot (-S_{AUBV})}$$

$$\overset{simplify}{=} \frac{S_{PVI} \cdot S_{ABU} \cdot S_{PUJ} \cdot S_{ACV}}{S_{ACU} \cdot S_{PVJ} \cdot S_{PUI} \cdot S_{ABV}}$$

$$\overset{J}{=} \frac{S_{PVI} \cdot S_{ABU} \cdot S_{PUV} \cdot S_{CDU} \cdot S_{ACV} \cdot (-S_{CUDV})}{S_{ACU} \cdot S_{PUV} \cdot S_{CDV} \cdot S_{PUI} \cdot S_{ABV} \cdot (-S_{CUDV})}$$

$$\overset{simplify}{=} \frac{S_{PVI} \cdot S_{ABU} \cdot S_{CDU} \cdot S_{ACV}}{S_{ACU} \cdot S_{CDV} \cdot S_{PUI} \cdot S_{ABV}}$$

$$\overset{I}{=} \frac{S_{PUV} \cdot S_{BDV} \cdot S_{ABU} \cdot S_{CDU} \cdot S_{ACV} \cdot (-S_{BUDV})}{S_{ACU} \cdot S_{CDV} \cdot S_{PUV} \cdot S_{BDU} \cdot S_{ABV} \cdot (-S_{BUDV})}$$

$$\overset{simplify}{=} \frac{S_{BDV} \cdot S_{ABU} \cdot S_{CDU} \cdot S_{ACV}}{S_{ACU} \cdot S_{CDV} \cdot S_{BDU} \cdot S_{ABV}}$$

$$\overset{V}{=} \frac{(-S_{BCD} \cdot r_2) \cdot S_{ABU} \cdot S_{CDU} \cdot (S_{ABC} \cdot r_2 - S_{ABC})}{S_{ACU} \cdot (-S_{BCD} \cdot r_2 + S_{BCD}) \cdot S_{BDU} \cdot S_{ABC} \cdot r_2}$$

$$\overset{simplify}{=} \frac{S_{ABU} \cdot S_{CDU}}{S_{ACU} \cdot S_{BDU}}$$

$$\overset{U}{=} \frac{S_{ABD} \cdot r_1 \cdot (-S_{ACD} \cdot r_1 + S_{ACD})}{S_{ACD} \cdot r_1 \cdot (-S_{ABD} \cdot r_1 + S_{ABD})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{UN}}{\overline{VN}} \overset{N}{=} \frac{S_{ACU}}{S_{ACV}}$$

$$\frac{\overline{UL}}{\overline{VL}} \overset{L}{=} \frac{S_{PUR}}{S_{PVR}}$$

$$S_{PVR} \overset{R}{=} \frac{S_{VJS} \cdot S_{PVQ}}{S_{VJQS}}$$

$$S_{PUR} \overset{R}{=} \frac{S_{VSQ} \cdot S_{PUJ}}{-S_{VJQS}}$$

$$S_{PVQ} \overset{Q}{=} \frac{S_{PIS} \cdot S_{PVK}}{S_{PIKS}}$$

$$S_{VSQ} \overset{Q}{=} \frac{S_{VIS} \cdot S_{PKS}}{-S_{PIKS}}$$

$$S_{PIS} \overset{S}{=} (r_3 - 1) \cdot S_{PUI}$$

$$S_{VJS} \overset{S}{=} S_{PVJ} \cdot r_3$$

$$S_{PKS} \overset{S}{=} (r_3 - 1) \cdot S_{PUK}$$

$$S_{VIS} \overset{S}{=} S_{PVI} \cdot r_3$$

$$S_{PVK} \overset{K}{=} \frac{S_{PUV} \cdot S_{ABV}}{-S_{AUBV}}$$

$$S_{PUK} \overset{K}{=} \frac{S_{PUV} \cdot S_{ABU}}{-S_{AUBV}}$$

$$S_{PVJ} \overset{J}{=} \frac{S_{PUV} \cdot S_{CDV}}{-S_{CUDV}}$$

$$S_{PUJ} \overset{J}{=} \frac{S_{PUV} \cdot S_{CDU}}{-S_{CUDV}}$$

$$S_{PUI} \overset{I}{=} \frac{S_{PUV} \cdot S_{BDU}}{-S_{BUDV}}$$

$$S_{PVI} \overset{I}{=} \frac{S_{PUV} \cdot S_{BDV}}{-S_{BUDV}}$$

$$S_{ABV} \overset{V}{=} S_{ABC} \cdot r_2$$

$$S_{CDV} \overset{V}{=} -\left((r_2 - 1) \cdot S_{BCD}\right)$$

$$S_{ACV} \overset{V}{=} (r_2 - 1) \cdot S_{ABC}$$

$$S_{BDV} \overset{V}{=} -\left(S_{BCD} \cdot r_2\right)$$

$$S_{BDU} \overset{U}{=} -\left((r_1 - 1) \cdot S_{ABD}\right)$$

$$S_{ACU} \overset{U}{=} S_{ACD} \cdot r_1$$

$$S_{CDU} \overset{U}{=} -\left((r_1 - 1) \cdot S_{ACD}\right)$$

$$S_{ABU} \overset{U}{=} S_{ABD} \cdot r_1$$

Definition. *The line joining the midpoints of a pair of opposite sides of a complete quadri-*

lateral bisects the segment between the intersections of the other two pairs of opposite sides of the quadrilateral. For a machine proof of this theorem see Example 2.36 on page 74. Let us call this line the Gauss line for the given complete quadrilateral.

Example 6.238 (Gauss Point Theorem) (0.933, 26, 56) *Given five points, we have five Gauss' lines. These five Gauss' lines are concurrent. Let us call this point of concurrency the Gauss point for the given five points.*

Constructive description
((points A_0 A_1 A_2 A_3 A_4)
(inter X (l A_3 A_4) (l A_1 A_0))
(inter Y (l A_2 A_3) (l A_0 A_4))
(inter Z (l A_4 A_3) (l A_1 A_2))
(midpoint M_1 A_1 A_3) (midpoint M_2 X A_2)
(midpoint M_3 A_0 A_2) (midpoint M_4 Y A_1)
(midpoint M_5 A_1 A_4) (midpoint M_6 A_0 Z)
(inter I (l M_3 M_4) (l M_1 M_2))
(inter Z_I (l M_6 M_5) (l M_1 M_2))
($\frac{\overline{M_1 I}}{\overline{M_2 I}} = \frac{\overline{M_1 Z_I}}{\overline{M_2 Z_I}}$))

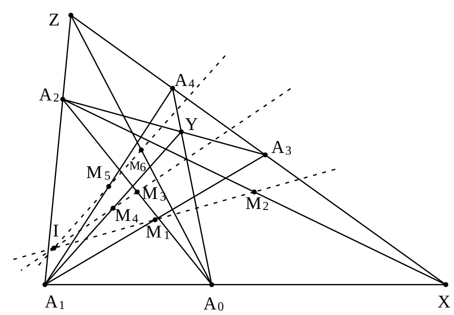


Figure 6-238

Example 6.239 (0.033, 3, 6) *The centroids of the four triangles determined by the vertices of a quadrilateral taken three at a time form a quadrilateral homothetic to the given quadrilateral in the ratio* $1/3$.

Constructive description
((points $A\ B\ C\ D$)
(midpoint $Q\ B\ C$)
(midpoint $P\ B\ A$)
(midpoint $R\ C\ D$)
(inter A_1 (l $D\ Q$) (l $B\ R$))
(inter D_1 (l $A\ Q$) (l $C\ P$))
(inter J (l $A\ A_1$) (l $D\ D_1$))
($\frac{\overline{AJ}}{\overline{A_1J}}$ = −3))

The machine proof

$$-\frac{1}{3}\left(\frac{\overline{AJ}}{\overline{A_1J}}\right)$$
$$\stackrel{J}{=} \frac{S_{ADD_1}}{(-3)\cdot(-S_{DA_1D_1})}$$
$$\stackrel{D_1}{=} \frac{S_{ADQ}\cdot S_{ACP}\cdot S_{ACQP}}{(3)\cdot S_{CQP}\cdot S_{ADA_1}\cdot S_{ACQP}}$$
$$\stackrel{simplify}{=} \frac{S_{ADQ}\cdot S_{ACP}}{(3)\cdot S_{CQP}\cdot S_{ADA_1}}$$
$$\stackrel{A_1}{=} \frac{S_{ADQ}\cdot S_{ACP}\cdot(-S_{BDRQ})}{(3)\cdot S_{CQP}\cdot(-S_{BDR}\cdot S_{ADQ})}$$
$$\stackrel{simplify}{=} \frac{S_{ACP}\cdot S_{BDRQ}}{(3)\cdot S_{CQP}\cdot S_{BDR}}$$
$$\stackrel{R}{=} \frac{S_{ACP}\cdot(-\frac{1}{2}S_{CDQ}+S_{BDQ})}{(3)\cdot S_{CQP}\cdot(-\frac{1}{2}S_{BCD})}$$
$$\stackrel{P}{=} \frac{(-\frac{1}{2}S_{ABC})\cdot(S_{CDQ}-2S_{BDQ})}{(3)\cdot(\frac{1}{2}S_{ACQ})\cdot S_{BCD}}$$
$$\stackrel{Q}{=} \frac{-S_{ABC}\cdot(\frac{3}{2}S_{BCD})}{(3)\cdot(-\frac{1}{2}S_{ABC})\cdot S_{BCD}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AJ}}{\overline{A_1J}} \stackrel{J}{=} \frac{S_{ADD_1}}{-S_{DA_1D_1}}$$
$$S_{DA_1D_1} \stackrel{D_1}{=} \frac{S_{CQP}\cdot S_{ADA_1}}{S_{ACQP}}$$
$$S_{ADD_1} \stackrel{D_1}{=} \frac{S_{ADQ}\cdot S_{ACP}}{S_{ACQP}}$$
$$S_{ADA_1} \stackrel{A_1}{=} \frac{S_{BDR}\cdot S_{ADQ}}{S_{BDRQ}}$$
$$S_{BDR} \stackrel{R}{=} -\frac{1}{2}(S_{BCD})$$
$$S_{BDRQ} \stackrel{R}{=} -\frac{1}{2}(S_{CDQ}-2S_{BDQ})$$
$$S_{CQP} \stackrel{P}{=} \frac{1}{2}(S_{ACQ})$$
$$S_{ACP} \stackrel{P}{=} -\frac{1}{2}(S_{ABC})$$
$$S_{ACQ} \stackrel{Q}{=} -\frac{1}{2}(S_{ABC})$$
$$S_{BDQ} \stackrel{Q}{=} -\frac{1}{2}(S_{BCD})$$
$$S_{CDQ} \stackrel{Q}{=} \frac{1}{2}(S_{BCD})$$

Example 6.240 (5.866, 51, 69) *The orthocenters of the four triangles formed by four lines taken three at a time are collinear.*

Constructive description
((points $A\ B\ C$)
(on D (l $A\ B$)) (on E (l $A\ C$))
(inter F (l $E\ D$) (l $C\ B$))
(foot $G_1\ C\ A\ B$) (foot $F_{H_1}\ A\ B\ C$)
(inter H_1 (l $C\ G_1$) (l $A\ F_{H_1}$))
(foot $G_2\ E\ A\ B$) (foot $F_{H_2}\ A\ E\ D$)
(inter H_2 (l $E\ G_2$) (l $A\ F_{H_2}$))
(foot $G_3\ F\ A\ B$) (foot $F_{H_3}\ B\ E\ D$)
(inter H_3 (l $F\ G_3$) (l $B\ F_{H_3}$))
(inter H_4 (l $F\ G_3$) (l $H_1\ H_2$)) ($\frac{\overline{FH_3}}{\overline{G_3H_3}} = \frac{\overline{FH_4}}{\overline{G_3H_4}}$))

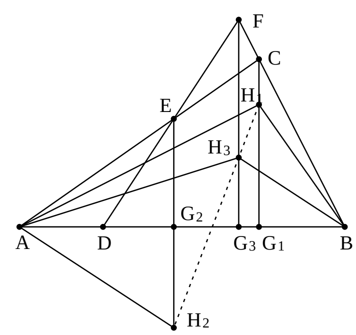


Figure 6-240

6.4.3 Parallelograms

The parallelogram has a special position in the area method. The reason is that when eliminating points from length ratios, we need to add auxiliary parallelograms. Thus some properties (2.11, 2.12, 3.6, 3.7, 3.11) of parallelograms are often used in the proofs produced by the area method.

Example 6.241 (0.016, 1, 3) *Let O be the intersection of the two diagonals AC and BD of a parallelogram ABCD. Show that O is the midpoint of AC.*

Constructive description
((points A B C) (pratio D A B C 1) (inter O (l B D) (l A C)) (midpoint O A C))

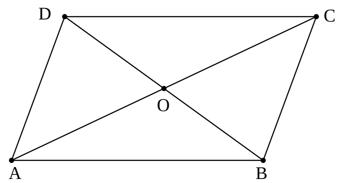


Figure 6-241

The machine proof

$-\frac{\overline{AO}}{\overline{CO}}$

$\overset{O}{=} \frac{-S_{ABD}}{-S_{BCD}}$

$\overset{D}{=} \frac{S_{ABC}}{S_{ABC}}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{AO}}{\overline{CO}} \overset{O}{=} \frac{S_{ABD}}{-S_{BCD}}$

$S_{BCD} \overset{D}{=} S_{ABC}$

$S_{ABD} \overset{D}{=} S_{ABC}$

Example 6.242 (0.066, 3, 10) *Let l be a line passing through the vertex of M of a parallelogram $MNPQ$ and intersecting the lines NP, PQ, NQ in points R, S, T. Show that* $\frac{\overline{MT}}{\overline{MR}} = \frac{\overline{ST}}{\overline{SM}}$ *(or $1/MR + 1/MS = 1/MT$).*

Constructive description
((points M N P)
(pratio Q M N P 1) (lratio R N P r)
(inter S (l M R) (l P Q))
(inter T (l M R) (l N Q))
($\frac{\overline{MT}}{\overline{MR}} = \frac{\overline{ST}}{\overline{SM}}$))

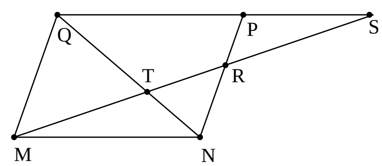


Figure 6-242

The machine proof

$\frac{\frac{\overline{MT}}{\overline{MR}}}{-\frac{\overline{ST}}{\overline{MS}}}$

$\overset{T}{=} \frac{S_{MNQ}\cdot S_{MNSQ}}{-S_{NQS}\cdot S_{MNRQ}}$

$\overset{S}{=} \frac{S_{MNQ}\cdot(-S_{MPRQ}\cdot S_{MNQ}+S_{NPQ}\cdot S_{MQR})\cdot(-S_{MPRQ})}{-(-S_{NPQ}\cdot S_{MQR})\cdot S_{MNRQ}\cdot(-S_{MPRQ})}$

$\overset{simplify}{=} \frac{-S_{MNQ}\cdot(S_{MPRQ}\cdot S_{MNQ}-S_{NPQ}\cdot S_{MQR})}{S_{NPQ}\cdot S_{MQR}\cdot S_{MNRQ}}$

$\overset{R}{=} \frac{-S_{MNQ}\cdot(S_{NPQ}\cdot S_{MPQ}\cdot r+S_{MPQ}\cdot S_{MNQ})}{S_{NPQ}\cdot(-S_{MPQ}\cdot r+S_{MNQ}\cdot r-S_{MNQ})\cdot(S_{NPQ}\cdot r+S_{MNQ})}$

$\overset{simplify}{=} \frac{S_{MNQ}\cdot S_{MPQ}}{S_{NPQ}\cdot(S_{MPQ}\cdot r-S_{MNQ}\cdot r+S_{MNQ})}$

$\overset{Q}{=} \frac{S_{MNP}^2}{S_{MNP}^2}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{ST}}{\overline{MS}} \overset{T}{=} \frac{S_{NQS}}{S_{MNSQ}}$

$\frac{\overline{MT}}{\overline{MR}} \overset{T}{=} \frac{S_{MNQ}}{S_{MNRQ}}$

$S_{NQS} \overset{S}{=} \frac{S_{NPQ}\cdot S_{MQR}}{S_{MPRQ}}$

$S_{MNSQ} \overset{S}{=} \frac{S_{MPRQ}\cdot S_{MNQ}-S_{NPQ}\cdot S_{MQR}}{S_{MPRQ}}$

$S_{MNRQ} \overset{R}{=} S_{NPQ}\cdot r+S_{MNQ}$

$S_{MQR} \overset{R}{=} -(S_{MPQ}\cdot r-S_{MNQ}\cdot r+S_{MNQ})$

$S_{MPRQ} \overset{R}{=} S_{NPQ}\cdot r-S_{NPQ}+S_{MPQ}$

$S_{NPQ} \overset{Q}{=} S_{MNP}$

$S_{MPQ} \overset{Q}{=} S_{MNP}$

$S_{MNQ} \overset{Q}{=} S_{MNP}$

Example 6.243 (0.066, 2, 9) *In the parallelogram ABCD, AE is drawn parallel to BD; show that $A(ECBD) = -1$.*

Constructive description
((points A B C)
(pratio D A B C 1)
(pratio E A B D 1)
(inter F (l B E) (l A D))
(inter G (l E B) (l A C))
(harmonic E G F B))

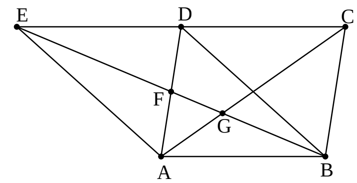


Figure 6-243

The machine proof

$$\left(\frac{\overline{EF}}{\overline{FG}}\right)/\left(\frac{\overline{BE}}{\overline{BG}}\right)$$

$$\overset{G}{=} \frac{(-S_{AECF})\cdot(-S_{ABC})}{(-S_{ABCE})\cdot S_{ACF}}$$

$$\overset{F}{=} \frac{-(-S_{ABDE}\cdot S_{ACE}+S_{ACD}\cdot S_{ABE})\cdot S_{ABC}\cdot S_{ABDE}}{S_{ABCE}\cdot S_{ACD}\cdot S_{ABE}\cdot S_{ABDE}}$$

$$\overset{simplify}{=} \frac{(S_{ABDE}\cdot S_{ACE}-S_{ACD}\cdot S_{ABE})\cdot S_{ABC}}{S_{ABCE}\cdot S_{ACD}\cdot S_{ABE}}$$

$$\overset{E}{=} \frac{(S_{ACD}\cdot S_{ABD}+2S_{ABD}\cdot S_{ABC})\cdot S_{ABC}}{(S_{ACD}+2S_{ABC})\cdot S_{ACD}\cdot S_{ABD}}$$

$$\overset{simplify}{=} \frac{S_{ABC}}{S_{ACD}}$$

$$\overset{D}{=} \frac{S_{ABC}}{S_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BE}}{\overline{BG}} \overset{G}{=} \frac{S_{ABCE}}{S_{ABC}}$$

$$\frac{\overline{EF}}{\overline{FG}} \overset{G}{=} \frac{-S_{AECF}}{S_{ACF}}$$

$$S_{ACF} \overset{F}{=} \frac{S_{ACD}\cdot S_{ABE}}{S_{ABDE}}$$

$$S_{AECF} \overset{F}{=} \frac{-(S_{ABDE}\cdot S_{ACE}-S_{ACD}\cdot S_{ABE})}{S_{ABDE}}$$

$$S_{ABCE} \overset{E}{=} S_{ACD}+2S_{ABC}$$

$$S_{ABE} \overset{E}{=} S_{ABD}$$

$$S_{ACE} \overset{E}{=} S_{ACD}+S_{ABC}$$

$$S_{ABDE} \overset{E}{=} (2S_{ABD})$$

$$S_{ACD} \overset{D}{=} S_{ABC}$$

Example 6.244 (0.116, 2, 12) *The diagonals of a parallelogram and those of its inscribed parallelogram are concurrent.*

Constructive description

((points $A\ B\ C\ E$)
(pratio $D\ C\ B\ A\ 1$)
(inter F (l $A\ E$) (p $D\ B\ E$))
(inter G (l $B\ E$) (p $C\ A\ E$))
(inter H (l $D\ F$) (l $C\ G$))
(inter O (l $A\ C$) (l $B\ D$))
(inter Z_2 (l $B\ D$) (l $E\ H$))
(inter Z_1 (l $A\ C$) (l $E\ H$))
($\frac{\overline{EZ_1}}{\overline{HZ_1}} \cdot \frac{\overline{HZ_2}}{\overline{EZ_2}} = 1$))

The machine proof

$$\frac{\overline{EZ_1}}{\overline{HZ_1}} \cdot \frac{\overline{HZ_2}}{\overline{EZ_2}}$$

$$\stackrel{Z_1}{=} \frac{S_{ACE}}{S_{ACH}} \cdot \frac{\overline{HZ_2}}{\overline{EZ_2}}$$

$$\stackrel{Z_2}{=} \frac{S_{ACE} \cdot S_{BDH}}{S_{ACH} \cdot (-S_{BED})}$$

$$\stackrel{H}{=} \frac{-S_{ACE} \cdot (-S_{CDG} \cdot S_{BDF}) \cdot S_{CDGF}}{S_{CDF} \cdot S_{ACG} \cdot S_{BED} \cdot (-S_{CDGF})}$$

$$\stackrel{simplify}{=} \frac{-S_{ACE} \cdot S_{CDG} \cdot S_{BDF}}{S_{CDF} \cdot S_{ACG} \cdot S_{BED}}$$

$$\stackrel{G}{=} \frac{-S_{ACE} \cdot S_{ACED} \cdot S_{BCE} \cdot S_{BDF} \cdot S_{ABE}}{S_{CDF} \cdot (-S_{BCE} \cdot S_{ACE}) \cdot S_{BED} \cdot S_{ABE}}$$

$$\stackrel{simplify}{=} \frac{S_{ACED} \cdot S_{BDF}}{S_{CDF} \cdot S_{BED}}$$

$$\stackrel{F}{=} \frac{S_{ACED} \cdot (-S_{BED} \cdot S_{AED}) \cdot (-S_{ABE})}{(-S_{BCED} \cdot S_{AED}) \cdot S_{BED} \cdot (-S_{ABE})}$$

$$\stackrel{simplify}{=} \frac{S_{ACED}}{S_{BCED}}$$

$$\stackrel{D}{=} \frac{S_{ABE}}{S_{ABE}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{EZ_1}}{\overline{HZ_1}} \stackrel{Z_1}{=} \frac{S_{ACE}}{S_{ACH}}$$

$$\frac{\overline{HZ_2}}{\overline{EZ_2}} \stackrel{Z_2}{=} \frac{S_{BDH}}{-S_{BED}}$$

$$S_{ACH} \stackrel{H}{=} \frac{S_{CDF} \cdot S_{ACG}}{S_{CDGF}}$$

$$S_{BDH} \stackrel{H}{=} \frac{S_{CDG} \cdot S_{BDF}}{S_{CDGF}}$$

$$S_{ACG} \stackrel{G}{=} \frac{-S_{BCE} \cdot S_{ACE}}{S_{ABE}}$$

$$S_{CDG} \stackrel{G}{=} \frac{S_{ACED} \cdot S_{BCE}}{S_{ABE}}$$

$$S_{CDF} \stackrel{F}{=} \frac{S_{BCED} \cdot S_{AED}}{S_{ABE}}$$

$$S_{BDF} \stackrel{F}{=} \frac{S_{BED} \cdot S_{AED}}{S_{ABE}}$$

$$S_{BCED} \stackrel{D}{=} S_{ABE}$$

$$S_{ACED} \stackrel{D}{=} S_{ABE}$$

Example 6.245 (0.066, 2, 8) *Let $ABCD$ be a parallelogram. Then the feet from A, B, C, D to the diagonals of the parallelogram form a parallelogram.*

Constructive description

((points $A\ B\ C$)
(pratio $D\ A\ B\ C\ 1$)
(foot $F\ A\ B\ D$)
(foot $E\ B\ A\ C$)
(foot $H\ C\ B\ D$)
(foot $G\ D\ A\ C$)
($\frac{\overline{EH}}{\overline{FG}} = 1$))

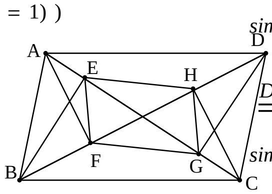


Figure 6-245

The machine proof

$$\frac{\overline{EH}}{\overline{FG}}$$

$$\stackrel{G}{=} \frac{-S_{ACH}}{S_{ACF}}$$

$$\stackrel{H}{=} \frac{-(P_{CBD} \cdot S_{ACD} - P_{BDC} \cdot S_{ABC})}{S_{ACF} \cdot P_{BDB}}$$

$$\stackrel{F}{=} \frac{-(P_{CBD} \cdot S_{ACD} - P_{BDC} \cdot S_{ABC}) \cdot P_{BDB}}{(-P_{ADB} \cdot S_{ABC} + P_{ABD} \cdot S_{ACD}) \cdot P_{BDB}}$$

$$\stackrel{simplify}{=} \frac{P_{CBD} \cdot S_{ACD} - P_{BDC} \cdot S_{ABC}}{P_{ADB} \cdot S_{ABC} - P_{ABD} \cdot S_{ACD}}$$

$$\stackrel{D}{=} \frac{P_{BCB} \cdot S_{ABC} - P_{ABA} \cdot S_{ABC}}{P_{BCB} \cdot S_{ABC} - P_{ABA} \cdot S_{ABC}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{EH}}{\overline{FG}} \stackrel{G}{=} \frac{-S_{ACH}}{S_{ACF}}$$

$$S_{ACH} \stackrel{H}{=} \frac{P_{CBD} \cdot S_{ACD} - P_{BDC} \cdot S_{ABC}}{P_{BDB}}$$

$$S_{ACF} \stackrel{F}{=} \frac{-(P_{ADB} \cdot S_{ABC} - P_{ABD} \cdot S_{ACD})}{P_{BDB}}$$

$$P_{ABD} \stackrel{D}{=} P_{ABC} + P_{ABA}$$

$$P_{ADB} \stackrel{D}{=} P_{BCB} + P_{ABC}$$

$$P_{BDC} \stackrel{D}{=} P_{ABC} + P_{ABA}$$

$$S_{ACD} \stackrel{D}{=} S_{ABC}$$

$$P_{CBD} \stackrel{D}{=} P_{BCB} + P_{ABC}$$

Example 6.246 (0.083, 10, 8) *If squares are erected externally (or internally) on the sides of any parallelogram, their centers form a square.*

Constructive description
((points A B C)
(pratio D A B C 1)
(constant i^2 -1)
(midpoint L A D) (pe-square S L D)
(midpoint N B C) (pe-square Q N B)
(midpoint T A B) (pe-square P T A)
($\overrightarrow{PS} - i \cdot \overrightarrow{PQ} = 0$))

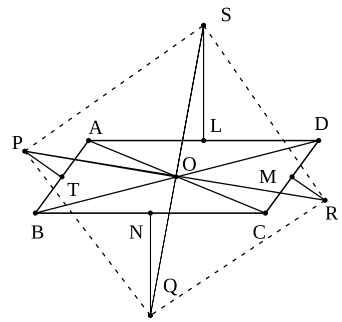


Figure 6-246

The machine proof

$$\overrightarrow{QP} \cdot i - \overrightarrow{SP}$$

$$\stackrel{n}{=} -\overrightarrow{T} \cdot i^2 + 2\overrightarrow{T} \cdot i - \overrightarrow{T} - \overrightarrow{Q} \cdot i + \overrightarrow{S} + \overrightarrow{A} \cdot i^2 - \overrightarrow{A} \cdot i$$

$$\stackrel{n}{=} -(\overrightarrow{Q} \cdot i - \overrightarrow{S} + \tfrac{1}{2}\overrightarrow{B} \cdot i^2 - \overrightarrow{B} \cdot i + \tfrac{1}{2}\overrightarrow{B} - \tfrac{1}{2}\overrightarrow{A} \cdot i^2 + \tfrac{1}{2}\overrightarrow{A})$$

$$\stackrel{n}{=} (-\tfrac{1}{2}) \cdot (-2\overrightarrow{N} \cdot i^2 + 2\overrightarrow{N} \cdot i - 2\overrightarrow{S} + 3\overrightarrow{B} \cdot i^2 - 2\overrightarrow{B} \cdot i + \overrightarrow{B} - \overrightarrow{A} \cdot i^2 + \overrightarrow{A})$$

$$\stackrel{n}{=} (\tfrac{1}{2}) \cdot (2\overrightarrow{S} + \overrightarrow{C} \cdot i^2 - \overrightarrow{C} \cdot i - 2\overrightarrow{B} \cdot i^2 + \overrightarrow{B} \cdot i - \overrightarrow{B} + \overrightarrow{A} \cdot i^2 - \overrightarrow{A})$$

$$\stackrel{n}{=} (\tfrac{1}{2}) \cdot (-2\overrightarrow{L} \cdot i + 2\overrightarrow{L} + 2\overrightarrow{D} \cdot i + \overrightarrow{C} \cdot i^2 - \overrightarrow{C} \cdot i - 2\overrightarrow{B} \cdot i^2 + \overrightarrow{B} \cdot i - \overrightarrow{B} + \overrightarrow{A} \cdot i^2 - \overrightarrow{A})$$

$$\stackrel{n}{=} -\overrightarrow{L} \cdot i + \overrightarrow{L} + \overrightarrow{D} \cdot i - \tfrac{1}{2}\overrightarrow{C} \cdot i - \tfrac{1}{2}\overrightarrow{C} + \tfrac{1}{2}\overrightarrow{B} \cdot i + \tfrac{1}{2}\overrightarrow{B} - \overrightarrow{A}$$

$$\stackrel{n}{=} (-\tfrac{1}{2}) \cdot (-\overrightarrow{D} \cdot i - \overrightarrow{D} + \overrightarrow{C} \cdot i + \overrightarrow{C} - \overrightarrow{B} \cdot i - \overrightarrow{B} + \overrightarrow{A} \cdot i + \overrightarrow{A})$$

$$\stackrel{simplify}{=} (\tfrac{1}{2}) \cdot (i+1) \cdot (\overrightarrow{D} - \overrightarrow{C} + \overrightarrow{B} - \overrightarrow{A}) \stackrel{simplify}{=} 0$$

The eliminants

$$\overrightarrow{SP} \stackrel{P}{=} -(\overrightarrow{T} \cdot i - \overrightarrow{T} + \overrightarrow{S} - \overrightarrow{A} \cdot i)$$

$$\overrightarrow{QP} \stackrel{P}{=} -(\overrightarrow{T} \cdot i - \overrightarrow{T} + \overrightarrow{Q} - \overrightarrow{A} \cdot i)$$

$$\overrightarrow{T} \stackrel{T}{=} \frac{1}{2}(\overrightarrow{B} + \overrightarrow{A})$$

$$\overrightarrow{Q} \stackrel{Q}{=} -(\overrightarrow{N} \cdot i - \overrightarrow{N} - \overrightarrow{B} \cdot i)$$

$$\overrightarrow{N} \stackrel{N}{=} \frac{1}{2}(\overrightarrow{C} + \overrightarrow{B})$$

$$\overrightarrow{S} \stackrel{S}{=} -(\overrightarrow{L} \cdot i - \overrightarrow{L} - \overrightarrow{D} \cdot i)$$

$$\overrightarrow{L} \stackrel{L}{=} \frac{1}{2}(\overrightarrow{D} + \overrightarrow{A})$$

$$\overrightarrow{D} \stackrel{D}{=} \overrightarrow{C} - \overrightarrow{B} + \overrightarrow{A}$$

Example 6.247 (0.250, 7, 8) *If squares are erected externally on two opposite sides and internally on the other two sides of any parallelogram, their centers form a parallelogram.*

Constructive description:
((points A B C)
(pratio D A B C 1)
(midpoint L A D)
(constant i^2 -1)
(pe-square S L D)
(midpoint N B C) (pe-square Q N B)
(midpoint T A B) (ne-square P T A)
(midpoint M C D) (ne-square R M C)
($\overrightarrow{PS} = \overrightarrow{QR}$))

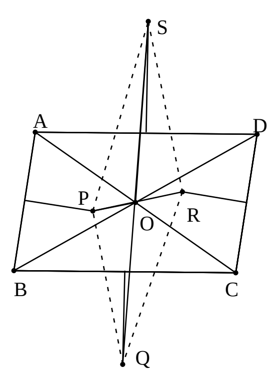


Figure 6-247

The machine proof

$$\frac{-\overrightarrow{SP}}{\overrightarrow{QR}} \overset{R}{=} \frac{-\overrightarrow{SP}}{\overrightarrow{M}\cdot i+\overrightarrow{M}-\overrightarrow{Q}-\overrightarrow{C}\cdot i}$$
$$\overset{M}{=} \frac{-\overrightarrow{SP}}{-\overrightarrow{Q}+\frac{1}{2}\overrightarrow{D}\cdot i+\frac{1}{2}\overrightarrow{D}-\frac{1}{2}\overrightarrow{C}\cdot i+\frac{1}{2}\overrightarrow{C}}$$
$$\overset{P}{=} \frac{(2)\cdot(\overrightarrow{T}\cdot i+\overrightarrow{T}-\overrightarrow{S}-\overrightarrow{A}\cdot i)}{2\overrightarrow{Q}-\overrightarrow{D}\cdot i-\overrightarrow{D}+\overrightarrow{C}\cdot i-\overrightarrow{C}}$$
$$\overset{T}{=} \frac{(2)\cdot(-\overrightarrow{S}+\frac{1}{2}\overrightarrow{B}\cdot i+\frac{1}{2}\overrightarrow{B}-\frac{1}{2}\overrightarrow{A}\cdot i+\frac{1}{2}\overrightarrow{A})}{2\overrightarrow{Q}-\overrightarrow{D}\cdot i-\overrightarrow{D}+\overrightarrow{C}\cdot i-\overrightarrow{C}}$$
$$\overset{Q}{=} \frac{-(2\overrightarrow{S}-\overrightarrow{B}\cdot i-\overrightarrow{B}+\overrightarrow{A}\cdot i-\overrightarrow{A})}{-2\overrightarrow{N}\cdot i+2\overrightarrow{N}-\overrightarrow{D}\cdot i-\overrightarrow{D}+\overrightarrow{C}\cdot i-\overrightarrow{C}+2\overrightarrow{B}\cdot i}$$
$$\overset{N}{=} \frac{2\overrightarrow{S}-\overrightarrow{B}\cdot i-\overrightarrow{B}+\overrightarrow{A}\cdot i-\overrightarrow{A}}{\overrightarrow{D}\cdot i+\overrightarrow{D}-\overrightarrow{B}\cdot i-\overrightarrow{B}}$$
$$\overset{simplify}{=} \frac{2\overrightarrow{S}-\overrightarrow{B}\cdot i-\overrightarrow{B}+\overrightarrow{A}\cdot i-\overrightarrow{A}}{(i+1)\cdot(\overrightarrow{D}-\overrightarrow{B})}$$
$$\overset{S}{=} \frac{-2\overrightarrow{L}\cdot i+2\overrightarrow{L}+2\overrightarrow{D}\cdot i-\overrightarrow{B}\cdot i-\overrightarrow{B}+\overrightarrow{A}\cdot i-\overrightarrow{A}}{(i+1)\cdot(\overrightarrow{D}-\overrightarrow{B})}$$
$$\overset{L}{=} \frac{-(-\overrightarrow{D}\cdot i-\overrightarrow{D}+\overrightarrow{B}\cdot i+\overrightarrow{B})}{(i+1)\cdot(\overrightarrow{D}-\overrightarrow{B})} \overset{simplify}{=} 1$$

The eliminants

$$\overrightarrow{QR} \overset{R}{=} \overrightarrow{M}\cdot i+\overrightarrow{M}-\overrightarrow{Q}-\overrightarrow{C}\cdot i$$
$$\overrightarrow{M} \overset{M}{=} \frac{1}{2}(\overrightarrow{D}+\overrightarrow{C})$$
$$\overrightarrow{SP} \overset{P}{=} \overrightarrow{T}\cdot i+\overrightarrow{T}-\overrightarrow{S}-\overrightarrow{A}\cdot i$$
$$\overrightarrow{T} \overset{T}{=} \frac{1}{2}(\overrightarrow{B}+\overrightarrow{A})$$
$$\overrightarrow{Q} \overset{Q}{=} -(\overrightarrow{N}\cdot i-\overrightarrow{N}-\overrightarrow{B}\cdot i)$$
$$\overrightarrow{N} \overset{N}{=} \frac{1}{2}(\overrightarrow{C}+\overrightarrow{B})$$
$$\overrightarrow{S} \overset{S}{=} -(\overrightarrow{L}\cdot i-\overrightarrow{L}-\overrightarrow{D}\cdot i)$$
$$\overrightarrow{L} \overset{L}{=} \frac{1}{2}(\overrightarrow{D}+\overrightarrow{A})$$

Example 6.248 (0.450, 5, 26) *The diagonals of the square in Example 6.246 pass through the center of the parallelogram (Figure 6-246).*

Constructive description ((points A B C) (pratio D A B C 1) (midpoint L A D) (constant i^2 -1) (pe-square S L D) (midpoint N B C) (pe-square Q N B) (midpoint O Q S) ($\overrightarrow{AO}-\overrightarrow{OC}=0$))

The machine proof.

$$\overrightarrow{CO}+\overrightarrow{AO}$$
$$\overset{n}{=} \tfrac{1}{2}\overrightarrow{CQ}+\tfrac{1}{2}\overrightarrow{CS}+\tfrac{1}{2}\overrightarrow{AQ}+\tfrac{1}{2}\overrightarrow{AS}$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(-2\overrightarrow{N}\cdot i+2\overrightarrow{N}+\overrightarrow{CS}-\overrightarrow{C}+2\overrightarrow{B}\cdot i+\overrightarrow{AS}-\overrightarrow{A})$$
$$\overset{n}{=} (-\tfrac{1}{2})\cdot(-\overrightarrow{CS}+\overrightarrow{C}\cdot i-\overrightarrow{B}\cdot i-\overrightarrow{B}-\overrightarrow{AS}+\overrightarrow{A})$$
$$\overset{n}{=} (\tfrac{1}{2})\cdot(-2\overrightarrow{L}\cdot i+2\overrightarrow{L}+2\overrightarrow{D}\cdot i-\overrightarrow{C}\cdot i-\overrightarrow{C}+\overrightarrow{B}\cdot i+\overrightarrow{B}-2\overrightarrow{A})$$
$$\overset{n}{=} (-\tfrac{1}{2})\cdot(-\overrightarrow{D}\cdot i-\overrightarrow{D}+\overrightarrow{C}\cdot i+\overrightarrow{C}-\overrightarrow{B}\cdot i-\overrightarrow{B}+\overrightarrow{A}\cdot i+\overrightarrow{A})$$
$$\overset{simplify}{=} (\tfrac{1}{2})\cdot(i+1)\cdot(\overrightarrow{D}-\overrightarrow{C}+\overrightarrow{B}-\overrightarrow{A})$$
$$\overset{simplify}{=} 0$$

The eliminants

$$\overrightarrow{AO} \overset{O}{=} \frac{1}{2}(\overrightarrow{AQ}+\overrightarrow{AS})$$
$$\overrightarrow{CO} \overset{O}{=} \frac{1}{2}(\overrightarrow{CQ}+\overrightarrow{CS})$$
$$\overrightarrow{AQ} \overset{Q}{=} -(\overrightarrow{N}\cdot i-\overrightarrow{N}-\overrightarrow{B}\cdot i+\overrightarrow{A})$$
$$\overrightarrow{CQ} \overset{Q}{=} -(\overrightarrow{N}\cdot i-\overrightarrow{N}+\overrightarrow{C}-\overrightarrow{B}\cdot i)$$
$$\overrightarrow{N} \overset{N}{=} \frac{1}{2}(\overrightarrow{C}+\overrightarrow{B})$$
$$\overrightarrow{AS} \overset{S}{=} -(\overrightarrow{L}\cdot i-\overrightarrow{L}-\overrightarrow{D}\cdot i+\overrightarrow{A})$$
$$\overrightarrow{CS} \overset{S}{=} -(\overrightarrow{L}\cdot i-\overrightarrow{L}-\overrightarrow{D}\cdot i+\overrightarrow{C})$$
$$\overrightarrow{L} \overset{L}{=} \frac{1}{2}(\overrightarrow{D}+\overrightarrow{A})$$
$$\overrightarrow{D} \overset{D}{=} \overrightarrow{C}-\overrightarrow{B}+\overrightarrow{A}$$

Example 6.249 (0.250, 4, 15) *If similar rectangles are erected externally on two opposite and internally on the other two sides of any parallelogram, their centers form a parallelogram.*

Constructive description
((points A B C)
(pratio D A B C 1)
(tratio A_1 A D r_1)
(pratio D_1 A_1 A D 1)
(inter S (l A D_1) (l D A_1))
(midpoint N B C)
(pratio Q N A_1 A 1/2)
(tratio A_2 A B r_2)
(pratio B_1 A_2 A B 1)
(inter P (l A B_1) (l B A_2))
(midpoint M C D)
(pratio R M A_2 A 1/2)
($\frac{\overline{PS}}{\overline{QR}}$ = 1))

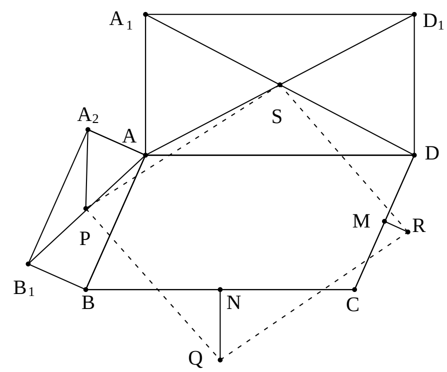


Figure 6-249

The machine proof.

$$-\frac{\overline{SP}}{\overline{QR}}$$

$$\stackrel{R}{=} \frac{-S_{ASA_2P}}{S_{AQA_2M}}$$

$$\stackrel{M}{=} \frac{-S_{ASA_2P}}{S_{AQA_2}-\frac{1}{2}S_{ADA_2}-\frac{1}{2}S_{ACA_2}}$$

$$\stackrel{P}{=} \frac{(-2)\cdot(-S_{ABB_1A_2}\cdot S_{ASA_2}-S_{AA_2B_1}\cdot S_{ABA_2})}{(2S_{AQA_2}-S_{ADA_2}-S_{ACA_2})\cdot(-S_{ABB_1A_2})}$$

$$\stackrel{B_1}{=} \frac{(-2)\cdot(2S_{ASA_2}\cdot S_{ABA_2}-S_{ABA_2}^2)}{(2S_{AQA_2}-S_{ADA_2}-S_{ACA_2})\cdot(2S_{ABA_2})}$$

$$\stackrel{simplify}{=} \frac{-(2S_{ASA_2}-S_{ABA_2})}{2S_{AQA_2}-S_{ADA_2}-S_{ACA_2}}$$

$$\stackrel{A_2}{=} \frac{-(\frac{1}{2}P_{BAS}\cdot r_2-\frac{1}{4}P_{ABA}\cdot r_2)}{\frac{1}{2}P_{BAQ}\cdot r_2-\frac{1}{4}P_{BAD}\cdot r_2-\frac{1}{4}P_{BAC}\cdot r_2}$$

$$\stackrel{simplify}{=} \frac{-(2P_{BAS}-P_{ABA})}{2P_{BAQ}-P_{BAD}-P_{BAC}}$$

$$\stackrel{Q}{=} \frac{-(2P_{BAS}-P_{ABA})}{2P_{BAN}-P_{BAA_1}-P_{BAD}-P_{BAC}}$$

$$\stackrel{N}{=} \frac{-(2P_{BAS}-P_{ABA})}{-P_{BAA_1}-P_{BAD}+P_{ABA}}$$

$$\stackrel{S}{=} \frac{-S_{ADD_1A_1}\cdot P_{ABA}+2P_{BAD_1}\cdot S_{ADA_1}}{(P_{BAA_1}+P_{BAD}-P_{ABA})\cdot S_{ADD_1A_1}}$$

$$\stackrel{D_1}{=} \frac{-(-2P_{BAA_1}\cdot S_{ADA_1}-2P_{BAD}\cdot S_{ADA_1}+2P_{ABA}\cdot S_{ADA_1})}{(P_{BAA_1}+P_{BAD}-P_{ABA})\cdot(2S_{ADA_1})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{SP}}{\overline{QR}} \stackrel{R}{=} \frac{S_{ASA_2P}}{S_{AQA_2M}}$$

$$S_{AQA_2M} \stackrel{M}{=} \frac{1}{2}(2S_{AQA_2}-S_{ADA_2}-S_{ACA_2})$$

$$S_{ASA_2P} \stackrel{P}{=} \frac{S_{ABB_1A_2}\cdot S_{ASA_2}+S_{AA_2B_1}\cdot S_{ABA_2}}{S_{ABB_1A_2}}$$

$$S_{AA_2B_1} \stackrel{B_1}{=} -(S_{ABA_2})$$

$$S_{ABB_1A_2} \stackrel{B_1}{=} 2(S_{ABA_2})$$

$$S_{ACA_2} \stackrel{A_2}{=} \frac{1}{4}(P_{BAC}\cdot r_2)$$

$$S_{ADA_2} \stackrel{A_2}{=} \frac{1}{4}(P_{BAD}\cdot r_2)$$

$$S_{AQA_2} \stackrel{A_2}{=} \frac{1}{4}(P_{BAQ}\cdot r_2)$$

$$S_{ABA_2} \stackrel{A_2}{=} \frac{1}{4}(P_{ABA}\cdot r_2)$$

$$S_{ASA_2} \stackrel{A_2}{=} \frac{1}{4}(P_{BAS}\cdot r_2)$$

$$P_{BAQ} \stackrel{Q}{=} \frac{1}{2}(2P_{BAN}-P_{BAA_1})$$

$$P_{BAN} \stackrel{N}{=} \frac{1}{2}(P_{BAC}+P_{ABA})$$

$$P_{BAS} \stackrel{S}{=} \frac{P_{BAD_1}\cdot S_{ADA_1}}{S_{ADD_1A_1}}$$

$$P_{BAD_1} \stackrel{D_1}{=} P_{BAA_1}+P_{BAD}$$

$$S_{ADD_1A_1} \stackrel{D_1}{=} 2(S_{ADA_1})$$

Example 6.250 (0.083, 3, 8) *Let A_1, B_1, C_1, D_1 be points on the sides CD, DA, AB, BC of a parallelogram $ABCD$ such that $CA_1/CD = DB_1/DA = AC_1/AB = BD_1/BC = 1/3$. Show that the area of the quadrilateral formed by the lines AA_1 BB_1, CC_1, DD_1 is one thirteenth* *of the area of parallelogram ABCD.*

Note that we only need to show $3S_{ABCD} = 13S_{ABA_2}$.

Constructive description
((points A B C)
(pratio D A B C 1)
(lratio A_1 C D 1/3)
(lratio B_1 D A 1/3)
(inter A_2 (l A A_1) (l B B_1))
($3S_{ABCD} = 13S_{ABA_2}$))

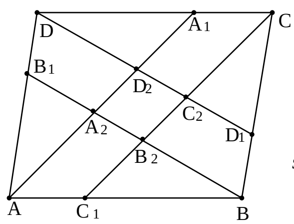


Figure 6-250

The machine proof

$$\frac{(3)\cdot S_{ABCD}}{(13)\cdot S_{ABA_2}}$$
$$\stackrel{A_2}{=} \frac{(3)\cdot S_{ABCD}\cdot S_{ABA_1B_1}}{(13)\cdot S_{ABB_1}\cdot S_{ABA_1}}$$
$$\stackrel{B_1}{=} \frac{(3)\cdot S_{ABCD}\cdot(-\frac{2}{3}S_{ADA_1}+S_{ABA_1})}{(13)\cdot(\frac{2}{3}S_{ABD})\cdot S_{ABA_1}}$$
$$\stackrel{A_1}{=} \frac{(-3)\cdot S_{ABCD}\cdot(-\frac{4}{3}S_{ACD}-S_{ABD}-2S_{ABC})}{(26)\cdot S_{ABD}\cdot(\frac{1}{3}S_{ABD}+\frac{2}{3}S_{ABC})}$$
$$\stackrel{D}{=} \frac{(3)\cdot(2S_{ABC})\cdot(13S_{ABC})}{(26)\cdot S_{ABC}\cdot(3S_{ABC})}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$S_{ABA_2} \stackrel{A_2}{=} \frac{S_{ABB_1}\cdot S_{ABA_1}}{S_{ABA_1B_1}}$$
$$S_{ABB_1} \stackrel{B_1}{=} \frac{2}{3}(S_{ABD})$$
$$S_{ABA_1B_1} \stackrel{B_1}{=} \frac{(2S_{ADA_1}-3S_{ABA_1})}{-3}$$
$$S_{ABA_1} \stackrel{A_1}{=} \frac{1}{3}(S_{ABD}+2S_{ABC})$$
$$S_{ADA_1} \stackrel{A_1}{=} -\frac{2}{3}(S_{ACD})$$
$$S_{ABD} \stackrel{D}{=} S_{ABC}$$
$$S_{ACD} \stackrel{D}{=} S_{ABC}$$
$$S_{ABCD} \stackrel{D}{=} 2(S_{ABC})$$

Example 6.251 (0.466, 7, 8) *Use the same notations as Example 6.250. If* $CA_1/CD = AC_1/AB = r_1, DB_1/DA = BD_1/BC = r_2$ *then* $\frac{S_{A_2B_2C_2D_2}}{S_{ABCD}} = \frac{r_1\cdot r_2}{r_2\cdot r_1-r_2-r_1+2}$.

We only need to show that $\frac{S_{A_2AB}}{S_{ABCD}} = \frac{1-r_2}{2\cdot(r_2\cdot r_1-r_2-r_1+2)}$.

Constructive description (Formula derivation)
((points A B C) (pratio D A B C 1) (lratio A_1 C D r_1)
(lratio B_1 D A r_2) (lratio C_1 A B r_1) (lratio D_1 B C r_2)
(inter A_2 (l B B_1) (l A A_1)) (inter B_2 (l C C_1) (l B B_1))
(inter C_2 (l D D_1) (l C C_1)) (inter D_2 (l A A_1) (l D D_1)) ($S_{A_2AB} = S_{ABCD}$))

Example 6.252 (0.066, 2, 11) *Let P and Q be two points on side BC and AD of a parallelogram such that PQ* ∥ *AB;* $M = AP \cap BQ$, $N = DP \cap QC$. *Show that* $MN = \frac{1}{2}AD$.

Constructive description
((points A B C)
(pratio D C B A 1)
(on P (l B C))
(inter Q (l A D) (p P A B))
(inter M (l A P) (l B Q))
(inter N (l D P) (l C Q))
($2S_{AMN} = -S_{ADN}$))

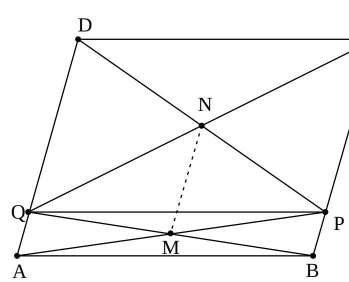


Figure 6-252

The machine proof

$$\frac{(-2)\cdot S_{AMN}}{S_{ADN}}$$
$$\overset{N}{=} \frac{(-2)\cdot(-S_{CPQ}\cdot S_{ADM})\cdot(-S_{CDQP})}{(-S_{CDQ}\cdot S_{ADP})\cdot(-S_{CDQP})}$$
$$\overset{simplify}{=} \frac{(-2)\cdot S_{CPQ}\cdot S_{ADM}}{S_{CDQ}\cdot S_{ADP}}$$
$$\overset{M}{=} \frac{(-2)\cdot S_{CPQ}\cdot S_{ADP}\cdot S_{ABQ}}{S_{CDQ}\cdot S_{ADP}\cdot S_{ABPQ}}$$
$$\overset{simplify}{=} \frac{(-2)\cdot S_{CPQ}\cdot S_{ABQ}}{S_{CDQ}\cdot S_{ABPQ}}$$
$$\overset{Q}{=} \frac{(-2)\cdot S_{ACP}\cdot S_{ABP}\cdot(-S_{ABD})}{S_{CDP}\cdot(S_{ADP}\cdot S_{ABP}-S_{ABP}\cdot S_{ABD})}$$
$$\overset{simplify}{=} \frac{(2)\cdot S_{ACP}\cdot S_{ABD}}{S_{CDP}\cdot(S_{ADP}-S_{ABD})}$$
$$\overset{P}{=} \frac{(2)\cdot(S_{ABC}\cdot\frac{\overline{BP}}{\overline{BC}}-S_{ABC})\cdot S_{ABD}}{(-S_{BCD}\cdot\frac{\overline{BP}}{\overline{BC}}+S_{BCD})\cdot(-S_{ACD}\cdot\frac{\overline{BP}}{\overline{BC}}+S_{ABD}\cdot\frac{\overline{BP}}{\overline{BC}}-2S_{ABD})}$$
$$\overset{simplify}{=} \frac{(2)\cdot S_{ABC}\cdot S_{ABD}}{S_{BCD}\cdot(S_{ACD}\cdot\frac{\overline{BP}}{\overline{BC}}-S_{ABD}\cdot\frac{\overline{BP}}{\overline{BC}}+2S_{ABD})}$$
$$\overset{D}{=} \frac{(2)\cdot S_{ABC}^2}{S_{ABC}\cdot(2S_{ABC})} \overset{simplify}{=} 1$$

The eliminants

$$S_{ADN}\overset{N}{=}\frac{S_{CDQ}\cdot S_{ADP}}{S_{CDQP}}$$
$$S_{AMN}\overset{N}{=}\frac{S_{CPQ}\cdot S_{ADM}}{S_{CDQP}}$$
$$S_{ADM}\overset{M}{=}\frac{S_{ADP}\cdot S_{ABQ}}{S_{ABPQ}}$$
$$S_{ABPQ}\overset{Q}{=}\frac{(S_{ADP}-S_{ABD})\cdot S_{ABP}}{-S_{ABD}}$$
$$S_{CDQ}\overset{Q}{=}S_{CDP}$$
$$S_{ABQ}\overset{Q}{=}S_{ABP}$$
$$S_{CPQ}\overset{Q}{=}S_{ACP}$$
$$S_{ADP}\overset{P}{=}-\left(S_{ACD}\cdot\frac{\overline{BP}}{\overline{BC}}-S_{ABD}\cdot\frac{\overline{BP}}{\overline{BC}}+S_{ABD}\right)$$
$$S_{CDP}\overset{P}{=}-\left((\frac{\overline{BP}}{\overline{BC}}-1)\cdot S_{BCD}\right)$$
$$S_{ACP}\overset{P}{=}(\frac{\overline{BP}}{\overline{BC}}-1)\cdot S_{ABC}$$
$$S_{ACD}\overset{D}{=}S_{ABC}$$
$$S_{BCD}\overset{D}{=}S_{ABC}$$
$$S_{ABD}\overset{D}{=}S_{ABC}$$

Example 6.253 (0.033, 2, 4) *Let ABCD be a rectangular, and EFGH a parallelogram inscribed in ABCD such that the sides of EFGH are parallel to the diagonals of ABCD. Show that the perimeter of EFGH is fixed.*

Constructive description
((points A B)
(tratio C B A r)
(pratio D A B C 1)
(on E (l A B))
(inter F (l B C) (p E A C))
(inter H (l A D) (p E B D))
($\frac{\overline{EF}}{\overline{AC}}+\frac{\overline{EH}}{\overline{BD}}$ = 1))

The machine proof

$$\frac{\overline{EH}}{\overline{BD}}+\frac{\overline{EF}}{\overline{AC}}$$
$$\overset{H}{=} \frac{-\frac{\overline{EF}}{\overline{AC}}\cdot S_{ABD}+S_{ADE}}{-S_{ABD}}$$
$$\overset{F}{=} \frac{S_{BCE}\cdot S_{ABD}-S_{ADE}\cdot S_{ABC}}{S_{ABD}\cdot S_{ABC}}$$
$$\overset{E}{=} \frac{S_{ABD}\cdot S_{ABC}}{S_{ABD}\cdot S_{ABC}}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{EH}}{\overline{BD}}\overset{H}{=}\frac{S_{ADE}}{-S_{ABD}} \quad \frac{\overline{EF}}{\overline{AC}}\overset{F}{=}\frac{S_{BCE}}{S_{ABC}}$$
$$S_{ADE}\overset{E}{=}-\left(S_{ABD}\cdot\frac{\overline{AE}}{\overline{AB}}\right)$$
$$S_{BCE}\overset{E}{=}-\left((\frac{\overline{AE}}{\overline{AB}}-1)\cdot S_{ABC}\right)$$

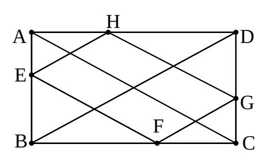


Figure 6-253

Example 6.254 (0.116, 3, 15) *Three parallel lines are cut by three parallel transversals in the points A, B, C; A_1, B_1, C_1; A_2, B_2, C_2. Show that B_2C, C_1A_2, AB_1 are concurrent.*

Constructive description ((points A B A_1) (on C (l A B)) (pratio B_1 B A A_1 1) (inter C_1 (l A_1 B_1) (p C A A_1)) (on A_2 (l A A_1)) (inter B_2 (l B B_1) (p A_2 A B)) (inter C_2 (l C C_1) (l A_2 B_2)) (inter I (l C B_2) (l C_1 A_2)) (inter Z_2 (l C_1 A_2) (l A B_1))

(inter Z_1 (l C B_2) (l A B_1)) ($\frac{\overline{AZ_1}}{\overline{B_1Z_1}}\cdot\frac{\overline{B_1Z_2}}{\overline{AZ_2}} = 1$))

The machine proof

$$\frac{\overline{AZ_1}}{\overline{B_1Z_1}}\cdot\frac{\overline{B_1Z_2}}{\overline{AZ_2}}$$

$$\overset{Z_1}{=} \frac{S_{ACB_2}}{-S_{CB_1B_2}}\cdot\frac{\overline{B_1Z_2}}{\overline{AZ_2}}$$

$$\overset{Z_2}{=} \frac{-S_{ACB_2}\cdot S_{B_1C_1A_2}}{S_{CB_1B_2}\cdot S_{AC_1A_2}}$$

$$\overset{B_2}{=} \frac{-S_{ACA_2}\cdot S_{B_1C_1A_2}\cdot(-S_{ABB_1})}{S_{AB_1BA_2}\cdot S_{BCB_1}\cdot S_{AC_1A_2}}$$

$$\overset{A_2}{=} \frac{(-S_{AA_1C}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})\cdot(-S_{AB_1C_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}}+S_{AB_1C_1})\cdot S_{ABB_1}}{(-S_{ABB_1}+S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})\cdot S_{BCB_1}\cdot(-S_{AA_1C_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})}$$

$$\overset{simplify}{=} \frac{S_{AA_1C}\cdot(\frac{\overline{AA_2}}{\overline{AA_1}}-1)\cdot S_{AB_1C_1}\cdot S_{ABB_1}}{(S_{ABB_1}-S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})\cdot S_{BCB_1}\cdot S_{AA_1C_1}}$$

$$\overset{C_1}{=} \frac{S_{AA_1C}\cdot(\frac{\overline{AA_2}}{\overline{AA_1}}-1)\cdot S_{ACA_1B_1}\cdot S_{AA_1B_1}\cdot S_{ABB_1}}{(S_{ABB_1}-S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})\cdot S_{BCB_1}\cdot S_{AA_1C}\cdot(-S_{AA_1B_1})}$$

$$\overset{simplify}{=} \frac{-(\frac{\overline{AA_2}}{\overline{AA_1}}-1)\cdot S_{ACA_1B_1}\cdot S_{ABB_1}}{(S_{ABB_1}-S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}})\cdot S_{BCB_1}}$$

$$\overset{B_1}{=} \frac{-(\frac{\overline{AA_2}}{\overline{AA_1}}-1)\cdot(-S_{AA_1C}-S_{ABA_1})\cdot S_{ABA_1}}{(-S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}}+S_{ABA_1})\cdot(-S_{BA_1C})}$$

$$\overset{simplify}{=} \frac{S_{AA_1C}+S_{ABA_1}}{S_{BA_1C}}$$

$$\overset{C}{=} \frac{-S_{ABA_1}\cdot\frac{\overline{AC}}{\overline{AB}}+S_{ABA_1}}{-S_{ABA_1}\cdot\frac{\overline{AC}}{\overline{AB}}+S_{ABA_1}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AZ_1}}{\overline{B_1Z_1}} \overset{Z_1}{=} \frac{S_{ACB_2}}{-S_{CB_1B_2}}$$

$$\frac{\overline{B_1Z_2}}{\overline{AZ_2}} \overset{Z_2}{=} \frac{S_{B_1C_1A_2}}{S_{AC_1A_2}}$$

$$S_{CB_1B_2} \overset{B_2}{=} \frac{S_{AB_1BA_2}\cdot S_{BCB_1}}{-S_{ABB_1}}$$

$$S_{ACB_2} \overset{B_2}{=} S_{ACA_2}$$

$$S_{AC_1A_2} \overset{A_2}{=} -\left(S_{AA_1C_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}}\right)$$

$$S_{AB_1BA_2} \overset{A_2}{=} -\left(S_{ABB_1}-S_{ABA_1}\cdot\frac{\overline{AA_2}}{\overline{AA_1}}\right)$$

$$S_{B_1C_1A_2} \overset{A_2}{=} -\left((\frac{\overline{AA_2}}{\overline{AA_1}}-1)\cdot S_{AB_1C_1}\right)$$

$$S_{ACA_2} \overset{A_2}{=} -\left(S_{AA_1C}\cdot\frac{\overline{AA_2}}{\overline{AA_1}}\right)$$

$$S_{AA_1C_1} \overset{C_1}{=} S_{AA_1C}$$

$$S_{AB_1C_1} \overset{C_1}{=} -\left(S_{ACA_1B_1}\right)$$

$$S_{BCB_1} \overset{B_1}{=} -\left(S_{BA_1C}\right)$$

$$S_{ABB_1} \overset{B_1}{=} S_{ABA_1}$$

$$S_{ACA_1B_1} \overset{B_1}{=} -\left(S_{AA_1C}+S_{ABA_1}\right)$$

$$S_{BA_1C} \overset{C}{=} -\left((\frac{\overline{AC}}{\overline{AB}}-1)\cdot S_{ABA_1}\right)$$

$$S_{AA_1C} \overset{C}{=} -\left(S_{ABA_1}\cdot\frac{\overline{AC}}{\overline{AB}}\right)$$

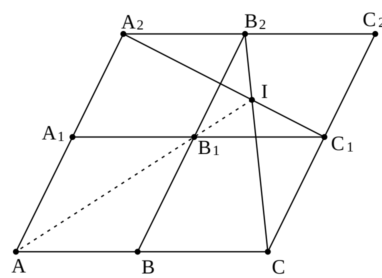


Figure 6-254

Example 6.255 (0.233, 5, 11) *A line passing through the intersection O of the diagonals of parallelogram $ABCD$ meets the four sides at E, F, G, H. Show that $EF = GH$.*

Constructive description

((points A B C) (pratio D C B A 1)
(inter O (l A C) (l B D))
(on G (l A B))
(inter F (l C D) (l O G))
(inter E (l A D) (l O G))
(inter H (l B C) (l O G))
($\frac{\overline{FE}}{\overline{GH}} = -1$))

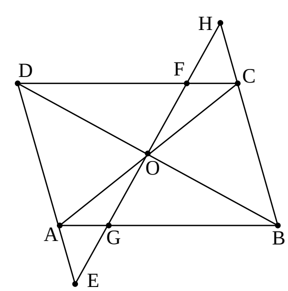


Figure 6-255

Example 6.256 (0.050, 5, 5) *Let $ABCD$ be a parallelogram and P, Q, R, S be points on AB, BC, CD, DA such that $AP = CR$ and $BQ = DS$. Show that $PQRS$ is also a parallelogram*

Constructive description: ((points $A\ B\ C$) (pratio $D\ A\ B\ C\ 1$) (lratio $S\ D\ A\ r_2$) (lratio $P\ A\ B\ r_1$) (lratio $R\ C\ D\ r_1$) (lratio $Q\ B\ C\ r_2$) ($\frac{\overline{QR}}{\overline{PS}} = 1$))

The machine proof

$$\frac{\overline{RQ}}{\overline{SP}}$$

$$\overset{Q}{=} \frac{S_{BCR}}{-S_{BSCP}}$$

$$\overset{R}{=} \frac{-S_{BCD}\cdot r_1}{S_{BSCP}}$$

$$\overset{P}{=} \frac{-S_{BCD}\cdot r_1}{-S_{BCS} - S_{ABC}\cdot r_1 + S_{ABC}}$$

$$\overset{S}{=} \frac{S_{BCD}\cdot r_1}{-S_{BCD}\cdot r_2 + S_{BCD} + S_{ABC}\cdot r_2 + S_{ABC}\cdot r_1 - S_{ABC}}$$

$$\overset{D}{=} \frac{-S_{ABC}\cdot r_1}{-S_{ABC}\cdot r_1} \overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{RQ}}{\overline{SP}} \overset{Q}{=} \frac{S_{BCR}}{-S_{BSCP}}$ $\quad S_{BCR} \overset{R}{=} S_{BCD}\cdot r_1$

$S_{BSCP} \overset{P}{=} -\left(S_{BCS} + S_{ABC}\cdot r_1 - S_{ABC}\right)$

$S_{BCS} \overset{S}{=} -\left(S_{BCD}\cdot r_2 - S_{BCD} - S_{ABC}\cdot r_2\right)$

$S_{BCD} \overset{D}{=} S_{ABC}$

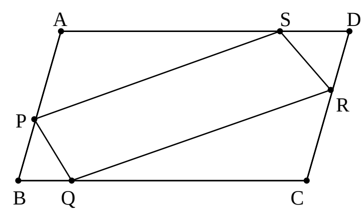


Figure 6-256

Example 6.257 (0.400, 20, 16) *Let $ABCD$ be a parallelogram and P, Q, R, S are points in AB, BC, CD, DA such that $r_1 = \frac{\overline{AP}}{\overline{AB}} = \frac{\overline{CR}}{\overline{CD}}$ and $r_2 = \frac{\overline{BQ}}{\overline{BC}} = \frac{\overline{DS}}{\overline{DA}}$. Show that $\frac{S_{PQRS}}{S_{ABCD}} = 2r_2r_1 - r_2 - r_1 + 1$. (Figure 6-256)*

Constructive description

((points $A\ B\ C$) (pratio $D\ A\ B\ C\ 1$) (lratio $S\ D\ A\ r_2$) (lratio $P\ A\ B\ r_1$) (lratio $R\ C\ D\ r_1$) (lratio $Q\ B\ C\ r_2$) ($S_{PQRS} = (2r_2\cdot r_1 - r_2 - r_1 + 1)\cdot S_{ABCD}$))

Example 6.258 (0.266, 6, 24) *Let C and F be any points on the respective sides AE and BD of a parallelogram $AEBD$. Let M and N denote the points of intersection of CD and FA and of EF and BC. Let the line MN meet DA at P and EB at Q. Then $AP = QB$.*

Constructive description

((points $A\ E\ B$)
(pratio $D\ B\ E\ A\ 1$)
(on C (l $A\ E$))
(on F (l $B\ D$))
(inter M (l $D\ C$) (l $A\ F$))
(inter N (l $E\ F$) (l $B\ C$))
(inter P (l $M\ N$) (l $A\ D$))
(inter Q (l $M\ N$) (l $E\ B$))
($\frac{\overline{AP}}{\overline{DP}} = \frac{\overline{BQ}}{\overline{EQ}}$))

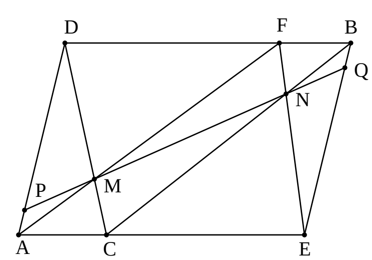


Figure 6-258

Example 6.259 (0.400, 10, 22) *Let ABC be any triangle and ABDE, ACFG any parallelograms described on AB and AC. Let DE and FG meet in H and draw BL and CM equal and parallel to HA. Then area(BCML) = area(ABDE) + area(ACFG).*

Constructive description
((points $A\ B\ C\ D\ F$)
(pratio $E\ A\ B\ D\ 1$)
(pratio $G\ A\ C\ F\ 1$)
(inter H (l $D\ E$) (l $F\ G$))
(inter R (l $B\ C$) (l $H\ A$))
(pratio $L\ B\ H\ A\ 1$)
($S_{ACF} + S_{BAD} = S_{CBL}$))

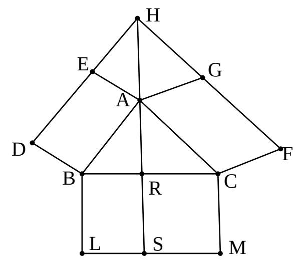


Figure 6-259

Example 6.260 (0.467, 46, 18) *The circle through the vertices A, B, C of a parallelogram ABCD meets DA, DC in the points A_1, C_1. Prove that $A_1D/A_1C_1 = A_1C/A_1B$.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(pratio $D\ A\ B\ C\ 1$)
(inter A_1 (l $A\ D$) (cir $O\ A$))
(inter C_1 (l $C\ D$) (cir $O\ C$))
(eq-product $A_1\ D\ A_1\ B\ A_1\ C_1\ A_1\ C$))

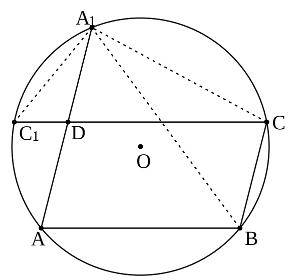


Figure 6-260

Example 6.261 (0.383, 8, 16) *Given the parallelogram $MDOM_1$, the vertex O is joined to the midpoint C of MM_1. If the internal and external bisectors of the angle COD meet MD in A and B, show that $MD^2 = MA \cdot MB$.*

Constructive description
((points $I\ O\ C$)
(incenter $D\ I\ O\ C$)
(midpoint $E\ O\ D$)
(pratio $M\ D\ E\ C\ 1$)
(pratio $M_1\ O\ E\ C\ 1$)
(inter A (l $O\ I$) (l $D\ M$))
(inter B (l $D\ M$) (t $O\ O\ I$))
($\frac{MD}{MA} = \frac{MB}{MD}$))

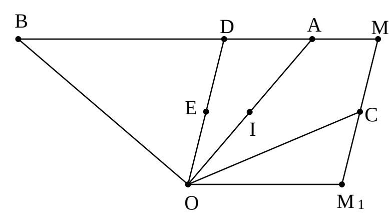


Figure 6-261

Example 6.262 (0.633, 46, 34) *ABCD is a parallelogram with center O. E is the midpoint of OA. F is the midpoint of OB. G is the midpoint of OC. H is the midpoint of OD. $P = DE \cap CF$; $Q = AF \cap DG$; $R = AH \cap BG$; $S = BE \cap CH$. Prove that PQRS is a parallelogram.*

Constructive description
((points $A\ B\ C\ O$)
(pratio $D\ A\ B\ C\ 1$)
(midpoint $E\ O\ A$)
(midpoint $F\ O\ B$)
(midpoint $G\ O\ C$)
(midpoint $H\ O\ D$)
(inter P (l $D\ E$) (l $C\ F$))
(inter Q (l $A\ F$) (l $D\ G$))
(inter R (l $A\ H$) (l $B\ G$))
(inter S (l $B\ E$) (l $C\ H$))
($\frac{\overline{PQ}}{\overline{SR}} = 1$))

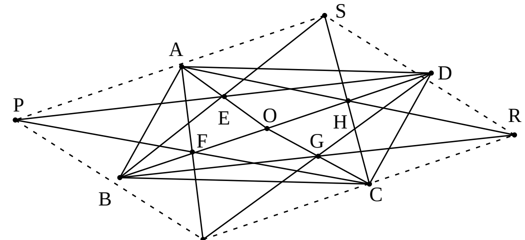


Figure 262

Example 6.263 (0.200, 5, 11) *ABCD is a parallelogram. $O = AC \cap BD$. Two perpendicular lines passing through O cut the sides AB, BC, CD, DA in points E, H, F, G. Prove that EHCG is a rhombus.*

Constructive description
((points $A\ B\ C$)
(pratio $D\ A\ B\ C\ 1$)
(inter O (l $A\ C$) (l $B\ D$))
(on E (l $A\ B$))
(inter F (l $C\ D$) (l $E\ O$))
(inter G (l $A\ D$) (t $O\ O\ E$))
(inter H (l $B\ C$) (l $O\ G$))
($\frac{\overline{EH}}{\overline{GF}} = 1$))

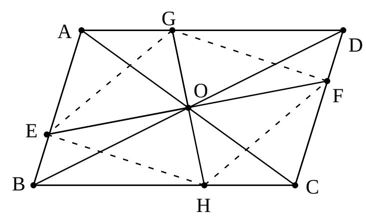


Figure 6-263

6.4.4 Squares

Example 6.264 (0.016, 1, 2) *On the two sides AB and AC of triangle ABC, two squares ABEF and ACGH are drawn externally. Show that $S_{ABC} = S_{AHF}$.*

Constructive description
((points $A\ B\ C$) (tratio $F\ A\ B\ 1$) (tratio $H\ A\ C\ -1$) ($S_{ABC} = S_{AHF}$))

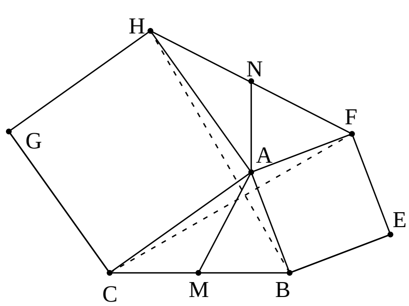


Figure 6-264

The machine proof

$$\frac{S_{ABC}}{-S_{AFH}} \stackrel{H}{=} \frac{S_{ABC}}{-(-\frac{1}{4}P_{CAF})} \stackrel{F}{=} \frac{(4)\cdot S_{ABC}}{4S_{ABC}} \stackrel{simplify}{=} 1$$

The eliminants

$$S_{AFH} \stackrel{H}{=} -\frac{1}{4}(P_{CAF})$$

$$P_{CAF} \stackrel{F}{=} 4(S_{ABC})$$

Example 6.265 (0.050, 3, 5) *On the two sides AB and AC of triangle ABC, two squares $ABEF$ and $ACGH$ are drawn externally. M is the midpoint of BC. Show that $FH = 2AM$ (Figure 6-264).*

Constructive description

((points A B C)
(midpoint M B C)
(tratio F A B 1)
(tratio H A C −1)
($P_{HFH} = 4P_{MAM}$))

The machine proof

$$\frac{P_{FHF}}{(4)\cdot P_{AMA}} \overset{H}{=} \frac{P_{AFA}+P_{ACA}+8S_{ACF}}{(4)\cdot P_{AMA}} \overset{F}{=} \frac{2P_{BAC}+P_{ACA}+P_{ABA}}{(4)\cdot P_{AMA}} \overset{M}{=} \frac{2P_{BAC}+P_{ACA}+P_{ABA}}{(4)\cdot(-\frac{1}{4}P_{BCB}+\frac{1}{2}P_{ACA}+\frac{1}{2}P_{ABA})} \overset{py}{=} \frac{-(-2P_{BCB}+4P_{ACA}+4P_{ABA})}{(P_{BCB}-2P_{ACA}-2P_{ABA})\cdot(2)} \overset{simplify}{=} 1$$

The eliminants

$P_{FHF} \overset{H}{=} P_{AFA}+P_{ACA}+8S_{ACF}$
$S_{ACF} \overset{F}{=} \frac{1}{4}(P_{BAC})$
$P_{AFA} \overset{F}{=} P_{ABA}$
$P_{AMA} \overset{M}{=} -\frac{1}{4}(P_{BCB}-2P_{ACA}-2P_{ABA})$
$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$

Example 6.266 (0.016, 3, 3) *On the two sides AB and AC of triangle ABC, two squares $ABEF$ and $ACGH$ are drawn externally. Passing through A a perpendicular to BC is drawn which cuts FH in N. Show that N is the midpoint of FH. (Figure 6-264)*

Constructive description

((points A B C)
(tratio F A B 1)
(tratio H A C −1)
(midpoint N F H)
(perpendicular N A B C))

The machine proof

$$\frac{P_{CBN}}{P_{ABC}} \overset{N}{=} \frac{\frac{1}{2}P_{CBH}+\frac{1}{2}P_{CBF}}{P_{ABC}} \overset{H}{=} \frac{P_{CBF}+P_{ABC}-4S_{ABC}}{(2)\cdot P_{ABC}} \overset{F}{=} \frac{2P_{ABC}}{(2)\cdot P_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$P_{CBN} \overset{N}{=} \frac{1}{2}(P_{CBH}+P_{CBF})$
$P_{CBH} \overset{H}{=} P_{ABC}-4S_{ABC}$
$P_{CBF} \overset{F}{=} P_{ABC}+4S_{ABC}$

Example 6.267 (0.033, 3, 2) *On the two sides AB and AC of triangle ABC, two squares $ABEF$ and $ACGH$ are drawn externally. Show that $FC = BH$. (Figure 6-264)*

Constructive description

((points A B C)
(tratio F A B 1)
(tratio H A C −1)
(eqdistance F C B H))

The machine proof

$$\frac{P_{CFC}}{P_{BHB}} \overset{H}{=} \frac{P_{CFC}}{P_{ACA}+P_{ABA}-8S_{ABC}} \overset{F}{=} \frac{P_{ACA}+P_{ABA}-8S_{ABC}}{P_{ACA}+P_{ABA}-8S_{ABC}} \overset{simplify}{=} 1$$

The eliminants

$P_{BHB} \overset{H}{=} P_{ACA}+P_{ABA}-8S_{ABC}$
$P_{CFC} \overset{F}{=} P_{ACA}+P_{ABA}-8S_{ABC}$

Example 6.268 (0.066, 4, 6) *On the two sides AB and AC of triangle ABC, two squares ABEF and ACGH are drawn externally. Show that $FC \perp BH$. (Figure 6-264)*

Constructive description

((points A B C)
(tratio F A B 1)
(tratio H A C -1)
(perpendicular F C B H))

The machine proof

$$\frac{P_{FBH}}{P_{CBH}}$$

$$\stackrel{H}{=} \frac{-4S_{ABCF}+P_{ABF}}{P_{ABC}-4S_{ABC}}$$

$$\stackrel{F}{=} \frac{-(P_{BAC}-P_{ABA}+4S_{ABC})}{P_{ABC}-4S_{ABC}}$$

$$\stackrel{py}{=} \frac{-(-P_{BCB}+P_{ACA}-P_{ABA}+8S_{ABC})\cdot(2)}{(P_{BCB}-P_{ACA}+P_{ABA}-8S_{ABC})\cdot(2)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{CBH} \stackrel{H}{=} P_{ABC}-4S_{ABC}$$

$$P_{FBH} \stackrel{H}{=} -\left(4S_{ABCF}-P_{ABF}\right)$$

$$P_{ABF} \stackrel{F}{=} P_{ABA}$$

$$S_{ABCF} \stackrel{F}{=} \frac{1}{4}\left(P_{BAC}+4S_{ABC}\right)$$

$$P_{ABC} = \frac{(P_{BCB}-P_{ACA}+P_{ABA})}{2}$$

$$P_{BAC} = \frac{(P_{BCB}-P_{ACA}-P_{ABA})}{-2}$$

Example 6.269 (0.033, 2, 8) *On the two sides AB and AC of triangle ABC, two squares ABDM and ACEN are drawn externally. F and G are the feet of the perpendiculars drawn from points D and E to BC. Show that $S_{ABC} = S_{DFB} + S_{CGE}$.*

Constructive description

((points A B C)
(tratio D B A 1)
(foot F D B C)
(tratio E C A -1)
(foot G E B C)
($S_{ABC} = S_{BDF} + S_{CGE}$))

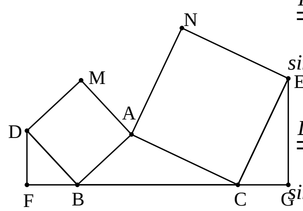


Figure 6-269

The machine proof

$$\frac{S_{ABC}}{-(S_{CEG}-S_{BDF})}$$

$$\stackrel{G}{=} \frac{S_{ABC}\cdot P_{BCB}}{-(P_{BCE}\cdot S_{BCE}-P_{BCB}\cdot S_{BDF})}$$

$$\stackrel{E}{=} \frac{S_{ABC}\cdot P_{BCB}}{-(-P_{BCB}\cdot S_{BDF}-P_{ACB}\cdot S_{ABC})}$$

$$\stackrel{F}{=} \frac{S_{ABC}\cdot (P_{BCB})^2}{-P_{CBD}\cdot P_{BCB}\cdot S_{BCD}+P_{BCB}\cdot P_{ACB}\cdot S_{ABC}}$$

$$\stackrel{simplify}{=} \frac{S_{ABC}\cdot P_{BCB}}{-(P_{CBD}\cdot S_{BCD}-P_{ACB}\cdot S_{ABC})}$$

$$\stackrel{D}{=} \frac{S_{ABC}\cdot P_{BCB}}{-(-P_{ACB}\cdot S_{ABC}-P_{ABC}\cdot S_{ABC})}$$

$$\stackrel{simplify}{=} \frac{P_{BCB}}{P_{ACB}+P_{ABC}}$$

$$\stackrel{py}{=} \frac{P_{BCB}\cdot((2))^2}{4P_{BCB}} \stackrel{simplify}{=} 1$$

The eliminants

$$S_{CEG} \stackrel{G}{=} \frac{P_{BCE}\cdot S_{BCE}}{P_{BCB}}$$

$$S_{BCE} \stackrel{E}{=} \frac{1}{4}\left(P_{ACB}\right)$$

$$P_{BCE} \stackrel{E}{=} -4\left(S_{ABC}\right)$$

$$S_{BDF} \stackrel{F}{=} \frac{-P_{CBD}\cdot S_{BCD}}{P_{BCB}}$$

$$S_{BCD} \stackrel{D}{=} \frac{1}{4}\left(P_{ABC}\right)$$

$$P_{CBD} \stackrel{D}{=} -4\left(S_{ABC}\right)$$

$$P_{ABC} = \frac{(P_{BCB}-P_{ACA}+P_{ABA})}{2}$$

$$P_{ACB} = \frac{(P_{BCB}+P_{ACA}-P_{ABA})}{2}$$

Example 6.270 (0.050, 6, 8) *On the two sides AB and AC of triangle ABC, two squares ABEF and ACGH are drawn externally. P is the midpoint of EG. Show that $BP = CP$.*

Constructive description

((points A B C)
(tratio E B A 1)

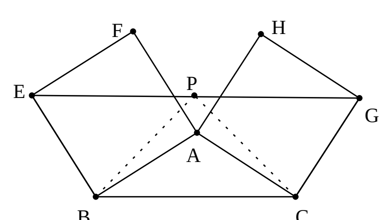


Figure 6-270

(tratio $G\ C\ A\ -1$)
(midpoint $P\ E\ G$)
(eqdistance $B\ P\ C\ P$))

The machine proof

$$\frac{P_{BPB}}{P_{CPC}}$$
$$\stackrel{P}{=} \frac{-\frac{1}{4}P_{EGE}+\frac{1}{2}P_{BGB}+\frac{1}{2}P_{BEB}}{-\frac{1}{4}P_{EGE}+\frac{1}{2}P_{CGC}+\frac{1}{2}P_{CEC}}$$
$$\stackrel{G}{=} \frac{P_{CEC}-2P_{BEB}-2P_{BCB}-P_{ACA}-8S_{ACE}-16S_{ABC}}{-P_{CEC}-P_{ACA}-8S_{ACE}}$$
$$\stackrel{E}{=} \frac{-(-P_{BCB}+2P_{BAC}-P_{ACA}-P_{ABA})}{P_{BCB}-2P_{BAC}+P_{ACA}+P_{ABA}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$P_{CPC} \stackrel{P}{=} -\frac{1}{4}(P_{EGE}-2P_{CGC}-2P_{CEC})$
$P_{BPB} \stackrel{P}{=} -\frac{1}{4}(P_{EGE}-2P_{BGB}-2P_{BEB})$
$P_{CGC} \stackrel{G}{=} P_{ACA}$
$P_{BGB} \stackrel{G}{=} P_{BCB}+P_{ACA}+8S_{ABC}$
$P_{EGE} \stackrel{G}{=} P_{CEC}+P_{ACA}-8S_{ACE}$
$S_{ACE} \stackrel{E}{=} -\frac{1}{4}(P_{BAC}+4S_{ABC})$
$P_{BEB} \stackrel{E}{=} P_{ABA}$
$P_{CEC} \stackrel{E}{=} P_{BCB}+P_{ABA}+8S_{ABC}$

Example 6.271 (0.033, 3, 6) *Let $ABCD$ be a square and G a point on CD. A square $GCEF$ is erected externally. Show that $DE \perp BG$ and $DE = BG$.*

Constructive description
((points $D\ C$)
(tratio $B\ C\ D\ 1$)
(lratio $G\ C\ D\ r$)
(lratio $E\ C\ B\ -r$)
(perpendicular $D\ E\ B\ G$))

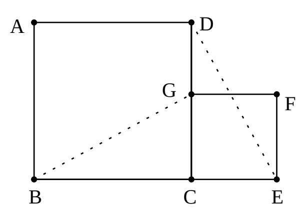


Figure 6-271

The machine proof

$$\frac{P_{DBG}}{P_{GBE}}$$
$$\stackrel{E}{=} \frac{P_{DBG}}{P_{CBG}\cdot r+P_{CBG}}$$
$$\stackrel{simplify}{=} \frac{P_{DBG}}{(r+1)\cdot P_{CBG}}$$
$$\stackrel{G}{=} \frac{-P_{DBC}\cdot r+P_{DBC}+P_{DBD}\cdot r}{(r+1)\cdot P_{CBC}}$$
$$\stackrel{B}{=} \frac{-(-P_{DCD}\cdot r-P_{DCD})}{(r+1)\cdot P_{DCD}}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$P_{GBE} \stackrel{E}{=} (r+1)\cdot P_{CBG}$
$P_{CBG} \stackrel{G}{=} P_{CBC}$
$P_{DBG} \stackrel{G}{=} -(P_{DBC}\cdot r-P_{DBC}-P_{DBD}\cdot r)$
$P_{CBC} \stackrel{B}{=} P_{DCD}$
$P_{DBD} \stackrel{B}{=} 2(P_{DCD})$
$P_{DBC} \stackrel{B}{=} P_{DCD}$

Example 6.272 (0.183, 6, 17) *On the two sides AB and AC of triangle ABC, two squares $ABDE$ and $ACFG$ are drawn externally. Let P and Q be the centers of squares $ABDE$ and $ACFG$, M and L the midpoints of BC and EG. Show that $PMQL$ is a square.*

Constructive description
((points $A\ B\ C$)
(midpoint $M\ B\ C$)
(tratio $E\ A\ B\ 1$)
(midpoint $P\ B\ E$)
(tratio $G\ A\ C\ -1$)
(midpoint $Q\ C\ G$)
(midpoint $L\ E\ G$)

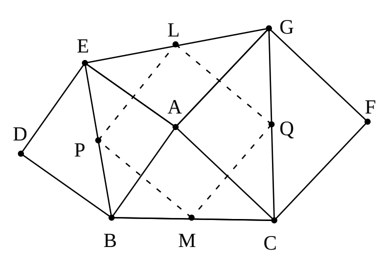


Figure 6-272

$(\frac{\overline{PM}}{\overline{LQ}} = 1))$

The machine proof

$$\frac{\overline{MP}}{\overline{QL}}$$

$$\stackrel{L}{=} \frac{S_{MEPG}}{S_{EGQ}}$$

$$\stackrel{Q}{=} \frac{S_{MEPG}}{\frac{1}{2}S_{CEG}}$$

$$\stackrel{G}{=} \frac{(2)\cdot(-\frac{1}{4}P_{CMAP}+S_{MEP}+S_{AMP})}{\frac{1}{4}P_{ACE}+S_{ACE}}$$

$$\stackrel{P}{=} \frac{(-2)\cdot(\frac{1}{2}P_{CAE}-P_{CAM}+\frac{1}{2}P_{BAC}-2S_{BME}-2S_{AME}+2S_{ABM})}{P_{ACE}+4S_{ACE}}$$

$$\stackrel{E}{=} \frac{-(-2P_{CAM}-P_{BAM}+P_{BAC}+P_{ABM}+4S_{ABC})}{P_{BAC}+P_{ACA}-4S_{ABC}}$$

$$\stackrel{M}{=} \frac{\frac{1}{2}P_{BAC}+P_{ACA}-\frac{1}{2}P_{ABC}+\frac{1}{2}P_{ABA}-4S_{ABC}}{P_{BAC}+P_{ACA}-4S_{ABC}}$$

$$\stackrel{py}{=} \frac{(-4P_{BCB}+12P_{ACA}+4P_{ABA}-32S_{ABC})\cdot(2)}{((2))^3\cdot(-P_{BCB}+3P_{ACA}+P_{ABA}-8S_{ABC})}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{MP}}{\overline{QL}} \stackrel{L}{=} \frac{S_{MEPG}}{S_{EGQ}}$$

$$S_{EGQ} \stackrel{Q}{=} \frac{1}{2}\left(S_{CEG}\right)$$

$$S_{CEG} \stackrel{G}{=} \frac{1}{4}\left(P_{ACE}+4S_{ACE}\right)$$

$$S_{MEPG} \stackrel{G}{=} -\frac{1}{4}\left(P_{CMAP}-4S_{MEP}-4S_{AMP}\right)$$

$$S_{AMP} \stackrel{P}{=} \frac{1}{2}\left(S_{AME}-S_{ABM}\right)$$

$$S_{MEP} \stackrel{P}{=} \frac{1}{2}\left(S_{BME}\right)$$

$$P_{CMAP} \stackrel{P}{=} \frac{1}{2}\left(P_{CAE}-2P_{CAM}+P_{BAC}\right)$$

$$S_{ACE} \stackrel{E}{=} \frac{1}{4}\left(P_{BAC}\right)$$

$$P_{ACE} \stackrel{E}{=} P_{ACA}-4S_{ABC}$$

$$S_{AME} \stackrel{E}{=} \frac{1}{4}\left(P_{BAM}\right)$$

$$S_{BME} \stackrel{E}{=} -\frac{1}{4}\left(P_{ABM}-4S_{ABM}\right)$$

$$P_{CAE} \stackrel{E}{=} 4\left(S_{ABC}\right)$$

$$P_{ABM} \stackrel{M}{=} \frac{1}{2}\left(P_{ABC}\right)$$

$$P_{BAM} \stackrel{M}{=} \frac{1}{2}\left(P_{BAC}+P_{ABA}\right)$$

$$P_{CAM} \stackrel{M}{=} \frac{1}{2}\left(P_{BAC}+P_{ACA}\right)$$

$$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$$

$$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$$

Example 6.273 (0.016, 1, 3) *Let ABCD be a square and l be a line passing through B. From A and C perpendiculars are drawn to l and the feet are A_1 and C_1 respectively. Show that $AA_1 = BC_1$.*

Constructive description

((points B C X)
(tratio A B C 1)
(foot A_1 A B X)
(foot C_1 C B X)
(eqdistance A A_1 B C_1))

The machine proof

$$\frac{P_{AA_1A}}{P_{BC_1B}}$$

$$\stackrel{C_1}{=} \frac{P_{AA_1A}\cdot P_{BXB}}{P^2_{CBX}}$$

$$\stackrel{A_1}{=} \frac{(16S^2_{BXA})\cdot P_{BXB}}{(P_{CBX})^2\cdot P_{BXB}}$$

$$\stackrel{simplify}{=} \frac{(16)\cdot(S_{BXA})^2}{(P_{CBX})^2}$$

$$\stackrel{A}{=} \frac{(16)\cdot((\frac{1}{4}P_{CBX}))^2}{(P_{CBX})^2}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{BC_1B} \stackrel{C_1}{=} \frac{(P_{CBX})^2}{P_{BXB}}$$

$$P_{AA_1A} \stackrel{A_1}{=} \frac{(16)\cdot(S_{BXA})^2}{P_{BXB}}$$

$$S_{BXA} \stackrel{A}{=} \frac{1}{4}\left(P_{CBX}\right)$$

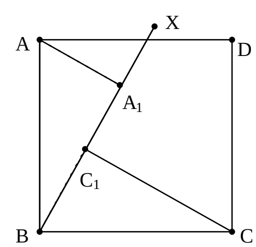


Figure 6-273

Example 6.274 [4] (0.067, 5, 13) *Starting with any triangle ABC, construct the exterior (or interior) squares BCDE, ACFG, and BAHK; then construct parallelograms FCDQ and EBKP. Show that PAQ is an isosceles.*

Constructive description
((points $A\ B\ C$)
(tratio $F\ C\ A\ 1$)
(tratio $D\ C\ B\ -1$)
(pratio $E\ D\ C\ B\ 1$)
(tratio $K\ B\ A\ -1$)
(pratio $Q\ F\ C\ D\ 1$)
(pratio $P\ K\ B\ E\ 1$)
(eqdistance $A\ Q\ A\ P$))

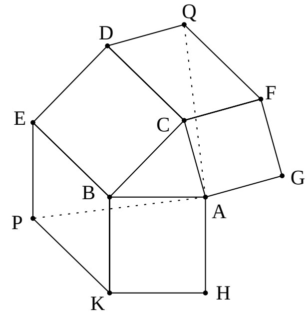


Figure 6-274

The machine proof

$$\frac{P_{AQA}}{P_{APA}}$$
$$\overset{P}{=} \frac{P_{AQA}}{2P_{EBK}+P_{AKA}+P_{AEA}-P_{ABA}}$$
$$\overset{Q}{=} \frac{2P_{FCD}+P_{ADA}+P_{AFA}-P_{ACA}}{2P_{EBK}+P_{AKA}+P_{AEA}-P_{ABA}}$$
$$\overset{K}{=} \frac{2P_{FCD}+P_{ADA}+P_{AFA}-P_{ACA}}{P_{AEA}+P_{ABA}+8S_{ABE}}$$
$$\overset{E}{=} \frac{2P_{FCD}+P_{ADA}+P_{AFA}-P_{ACA}}{P_{ADA}-P_{ACA}+2P_{ABA}+8S_{ABD}-8S_{ABC}}$$
$$\overset{D}{=} \frac{P_{BCB}+P_{AFA}+8S_{BCF}-8S_{ABC}}{P_{BCB}-2P_{ABC}+2P_{ABA}-8S_{ABC}}$$
$$\overset{F}{=} \frac{P_{BCB}-2P_{ACB}+2P_{ACA}-8S_{ABC}}{P_{BCB}-2P_{ABC}+2P_{ABA}-8S_{ABC}}$$
$$\overset{py}{=} \frac{(2P_{ACA}+2P_{ABA}-16S_{ABC})\cdot(2)}{(2P_{ACA}+2P_{ABA}-16S_{ABC})\cdot(2)}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$P_{APA}\overset{P}{=}2P_{EBK}+P_{AKA}+P_{AEA}-P_{ABA}$$
$$P_{AQA}\overset{Q}{=}2P_{FCD}+P_{ADA}+P_{AFA}-P_{ACA}$$
$$P_{AKA}\overset{K}{=}2(P_{ABA})$$
$$P_{EBK}\overset{K}{=}4(S_{ABE})$$
$$S_{ABE}\overset{E}{=}S_{ABD}-S_{ABC}$$
$$P_{AEA}\overset{E}{=}P_{ADA}-P_{ACA}+P_{ABA}$$
$$S_{ABD}\overset{D}{=}-\frac{1}{4}(P_{ABC}-4S_{ABC})$$
$$P_{ADA}\overset{D}{=}P_{BCB}+P_{ACA}-8S_{ABC}$$
$$P_{FCD}\overset{D}{=}4(S_{BCF})$$
$$S_{BCF}\overset{F}{=}-\frac{1}{4}(P_{ACB})$$
$$P_{AFA}\overset{F}{=}2(P_{ACA})$$
$$P_{ABC}=\frac{1}{2}(P_{BCB}-P_{ACA}+P_{ABA})$$
$$P_{ACB}=\frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$$

Example 6.275 (0.516, 9, 32) *On the four sides of a quadrilateral ABCD, four squares are drawn externally. Show that the two segments joining the two centers of the squares on two opposite sides are perpendicular and have the same length.*

Constructive description
((points $A\ B\ C\ D$)
(tratio $E\ A\ B\ 1$)
(midpoint $L\ B\ E$)
(tratio $G\ A\ D\ -1$)
(midpoint $P\ D\ G$)
(tratio $F\ C\ B\ -1$)
(midpoint $M\ B\ F$)
(tratio $H\ C\ D\ 1$)
(midpoint $N\ D\ H$)
(eqdistance $M\ P\ L\ N$))

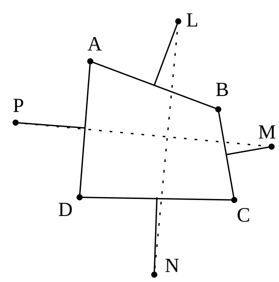


Figure 6-275

[4]This example is from Amer. Math. Mon. 75(1968), p.899.

Example 6.276 (0.067 2 14) *On the hypotenuse AB of right triangle ABC a square ABFE is erected. Let P be the intersection of the diagonals AF and BE of ABFE. Show that* $\angle ACP = \angle PCB$.

Constructive description
((points B C)
(tratio A C B r)
(tratio F B A -1)
(tratio E A B 1)
(inter P (l B E) (l A F))
(eqangle A C P P C B))

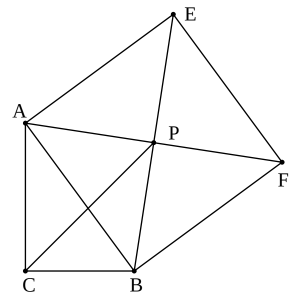


Figure 6-276

The machine proof

$$\frac{(-S_{CAP})\cdot P_{BCP}}{(-S_{BCP})\cdot P_{ACP}}$$

$$\stackrel{P}{=} \frac{S_{CAF}\cdot S_{BAE}\cdot(-P_{BCF}\cdot S_{BAE})\cdot(S_{BAEF})^2}{(-S_{BAF}\cdot S_{BCE})\cdot P_{ACE}\cdot S_{BAF}\cdot((-S_{BAEF}))^2}$$

$$\stackrel{simplify}{=} \frac{S_{CAF}\cdot(S_{BAE})^2\cdot P_{BCF}}{(S_{BAF})^2\cdot S_{BCE}\cdot P_{ACE}}$$

$$\stackrel{E}{=} \frac{S_{CAF}\cdot((-\frac{1}{4}P_{BAB}))^2\cdot P_{BCF}}{(S_{BAF})^2\cdot(-\frac{1}{4}P_{CBA}+S_{BCA})\cdot(P_{CAC}-4S_{BCA})}$$

$$\stackrel{F}{=} \frac{-(-\frac{1}{4}P_{BAC}+S_{BCA})\cdot(P_{BAB})^2\cdot(P_{BCB}-4S_{BCA})}{(4)\cdot((-\frac{1}{4}P_{BAB}))^2\cdot(P_{CBA}-4S_{BCA})\cdot(P_{CAC}-4S_{BCA})}$$

$$\stackrel{simplify}{=} \frac{(P_{BAC}-4S_{BCA})\cdot(P_{BCB}-4S_{BCA})}{(P_{CBA}-4S_{BCA})\cdot(P_{CAC}-4S_{BCA})}$$

$$\stackrel{A}{=} \frac{(P_{BCB}\cdot r^2+P_{BCB}\cdot r)\cdot(P_{BCB}\cdot r+P_{BCB})}{(P_{BCB}\cdot r+P_{BCB})\cdot(P_{BCB}\cdot r^2+P_{BCB}\cdot r)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{ACP} \stackrel{P}{=} \frac{P_{ACE}\cdot S_{BAF}}{S_{BAEF}}$$
$$S_{BCP} \stackrel{P}{=} \frac{S_{BAF}\cdot S_{BCE}}{S_{BAEF}}$$
$$P_{BCP} \stackrel{P}{=} \frac{P_{BCF}\cdot S_{BAE}}{S_{BAEF}}$$
$$S_{CAP} \stackrel{P}{=} \frac{S_{CAF}\cdot S_{BAE}}{S_{BAEF}}$$
$$P_{ACE} \stackrel{E}{=} P_{CAC}-4S_{BCA}$$
$$S_{BCE} \stackrel{E}{=} -\frac{1}{4}(P_{CBA}-4S_{BCA})$$
$$S_{BAE} \stackrel{E}{=} -\frac{1}{4}(P_{BAB})$$
$$S_{BAF} \stackrel{F}{=} -\frac{1}{4}(P_{BAB})$$
$$P_{BCF} \stackrel{F}{=} P_{BCB}-4S_{BCA}$$
$$S_{CAF} \stackrel{F}{=} -\frac{1}{4}(P_{BAC}-4S_{BCA})$$
$$P_{CAC} \stackrel{A}{=} P_{BCB}\cdot(r)^2$$
$$P_{CBA} \stackrel{A}{=} P_{BCB}$$
$$S_{BCA} \stackrel{A}{=} -\frac{1}{4}(P_{BCB}\cdot r)$$
$$P_{BAC} \stackrel{A}{=} P_{BCB}\cdot(r)^2$$

Example 6.277 (0.617, 65, 21) *ABCD is a square. G is a point on AD such that* $AG = AD$. *E, F are points on AB, BC such that* $EF \parallel AC$. $H = EG \cap FD$. *Show that* $AH = BC$.

Constructive description
((points B C)
(tratio A B C 1)
(pratio D A B C 1)
(lratio G A D -1)
(lratio F B C r)
(inter E (l A B) (p F A C))
(inter H (l G E) (l F D))
(eqdistance A H B C))

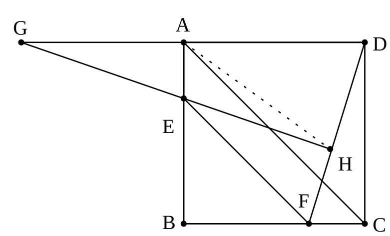


Figure 6-277

6.4.5 Cyclic Quadrilaterals

Definition *A quadrilateral whose vertices lie on the same circle is said to be cyclic.*

To prove theorems about cyclic quadrilaterals, we use the techniques developed in Section 3.6.

Example 6.278 (Ptolemy's Theorem) (0.050, 2, 12) *Let A, B, C, D be four cyclic points. Then $\widetilde{AB} \cdot \widetilde{CD} + \widetilde{AD} \cdot \widetilde{BC} = \widetilde{AC} \cdot \widetilde{BD}$.*

Constructive description
((circle $A\ B\ C\ D$) $(\widetilde{AB}\cdot\widetilde{CD}+\widetilde{AD}\cdot\widetilde{BC} = \widetilde{AC}\cdot\widetilde{BD})$)

The machine proof

$$\frac{\widetilde{CD}\cdot\widetilde{AB}+\widetilde{BC}\cdot\widetilde{AD}}{\widetilde{BD}\cdot\widetilde{AC}}$$

$$\overset{chord}{=} \frac{\sin(CD)\cdot\sin(AB)\cdot d^2+\sin(BC)\cdot\sin(AD)\cdot d^2}{\sin(BD)\cdot d\cdot\sin(AC)\cdot d}$$

$$\overset{simplify}{=} \frac{\sin(CD)\cdot\sin(AB)+\sin(BC)\cdot\sin(AD)}{\sin(BD)\cdot\sin(AC)}$$

$$\overset{co-cir}{=} \frac{S_D\cdot S_C\cdot C_B-S_C\cdot S_B\cdot C_D}{(S_D\cdot C_B-S_B\cdot C_D)\cdot S_C}$$

$$\overset{simplify}{=} 1$$

The eliminants

$\widetilde{AC}=\sin(AC)\cdot d$ $\widetilde{BD}=\sin(BD)\cdot d$

$\widetilde{AD}=\sin(AD)\cdot d$ $\widetilde{BC}=\sin(BC)\cdot d$

$\widetilde{AB}=\sin(AB)\cdot d$ $\widetilde{CD}=\sin(CD)\cdot d$

$\sin(AC)=S_C$

$\sin(BD)=S_D\cdot C_B-S_B\cdot C_D$

$\sin(AD)=S_D$

$\sin(BC)=S_C\cdot C_B-S_B\cdot C_C$

$\sin(AB)=S_B$

$\sin(CD)=S_D\cdot C_C-S_C\cdot C_D$

Example 6.279 (Brahmagupta's Formula) (0.267, 5, 16) *Let $ABCD$ be a cyclic quadrilateral. Then*

$$S^2 = (p-\widetilde{AB})(p-\widetilde{BC})(p-\widetilde{CD})(p-\widetilde{AD}) = \frac{1}{16}(4\overline{AC}^2\overline{BD}^2 - P_{ABCD}^2)$$

where $p = \frac{1}{2}(\widetilde{AB}+\widetilde{BC}+\widetilde{CD}+\widetilde{AD})$.

Constructive description
((circle $A\ B\ C\ D$) $(16S_{ABCD}^2 = 4\widetilde{AC}^2\cdot\widetilde{BD}^2 - P_{CBAD}^2)$)

Example 6.280 (0.033, 1, 10) *Show that in a cyclic quadrilateral the distances of the point of intersection of the diagonals from two opposite sides are proportional to these sides.*

Constructive description
((circle $A\ B\ C\ D$)
(inter I (l $B\ D$) (l $A\ C$))
(foot $Q\ I\ B\ C$)
(foot $S\ I\ A\ D$)
(eq-product $I\ S\ B\ C\ I\ Q\ A\ D$))

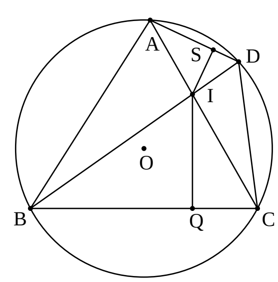


Figure 6-280

The machine proof

$$\frac{P_{ISI}\cdot P_{BCB}}{P_{IQI}\cdot P_{ADA}}$$

$$\overset{S}{=} \frac{(16S_{ADI}^2)\cdot P_{BCB}}{P_{IQI}\cdot P_{ADA}^2}$$

$$\overset{Q}{=} \frac{(16)\cdot S_{ADI}^2\cdot P_{BCB}^2}{(16S_{BCI}^2)\cdot P_{ADA}^2}$$

$$\overset{I}{=} \frac{(-S_{ACD}\cdot S_{ABD})^2\cdot P_{BCB}^2\cdot S_{ABCD}^2}{S_{BCD}\cdot S_{ABC}^2\cdot P_{ADA}^2\cdot S_{ABCD}^2}$$

$$\overset{simplify}{=} \frac{S_{ACD}\cdot S_{ABD}\cdot S_{ACD}\cdot S_{ABD}\cdot P_{BCB}^2}{S_{BCD}\cdot S_{ABC}\cdot S_{BCD}\cdot S_{ABC}\cdot P_{ADA}^2}$$

$$\overset{co-cir}{=} \frac{(-\widetilde{CD}\cdot\widetilde{AD}\cdot\widetilde{AC})\cdot(-\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB})\cdot(-\widetilde{CD}\cdot\widetilde{AD}\cdot\widetilde{AC})\cdot(-\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB})\cdot(2\widetilde{BC}^2)^2\cdot(2d)^4}{(-\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC})\cdot(-\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB})\cdot(-\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC})\cdot(-\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB})\cdot(2\widetilde{AD}^2)^2\cdot(2d)^4}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{ISI} \overset{S}{=} \frac{(16)\cdot S_{ADI}^2}{P_{ADA}}$$

$$P_{IQI} \overset{Q}{=} \frac{(16)\cdot S_{BCI}^2}{P_{BCB}}$$

$$S_{BCI} \overset{I}{=} \frac{S_{BCD}\cdot S_{ABC}}{S_{ABCD}}$$

$$S_{ADI} \overset{I}{=} \frac{-S_{ACD}\cdot S_{ABD}}{S_{ABCD}}$$

$$P_{ADA} = 2(\widetilde{AD}^2)$$

$$S_{ABC} = \frac{\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{BCD} = \frac{\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC}}{(-2)\cdot d}$$

$$P_{BCB} = 2(\widetilde{BC}^2)$$

$$S_{ABD} = \frac{\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{ACD} = \frac{\widetilde{CD}\cdot\widetilde{AD}\cdot\widetilde{AC}}{(-2)\cdot d}$$

Example 6.281 (0.016, 1, 6) *In the cyclic quadrilateral ABCD the perpendicular to AB at A meets CD in A_1, and the perpendicular to CD at C meets AB in C_1. Show that the line A_1C_1 is parallel to the diagonal BD.*

Constructive description

((circle A B C D)
(inter A_1 (l C D) (t A A B))
(inter C_1 (l A B) (t C C D))
(parallel A_1 C_1 B D))

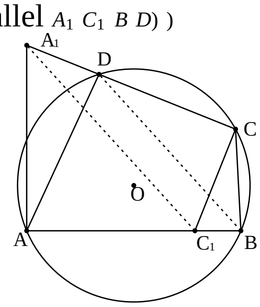


Figure 6-281

The machine proof

$$\frac{S_{BDA_1}}{S_{BDC_1}}$$

$$\overset{C_1}{=} \frac{S_{BDA_1}\cdot(-P_{BCAD})}{-P_{BCD}\cdot S_{ABD}}$$

$$\overset{A_1}{=} \frac{P_{BAD}\cdot S_{BCD}\cdot P_{BCAD}}{P_{BCD}\cdot S_{ABD}\cdot(-P_{BCAD})}$$

$$\overset{simplify}{=} \frac{-P_{BAD}\cdot S_{BCD}}{P_{BCD}\cdot S_{ABD}}$$

$$\overset{co-cir}{=} \frac{-(2\widetilde{AD}\cdot\widetilde{AB}\cdot\cos(BD))\cdot(-\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC})\cdot(2d)}{(-2\widetilde{CD}\cdot\widetilde{BC}\cdot\cos(BD))\cdot(-\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB})\cdot(2d)}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{BDC_1} \overset{C_1}{=} \frac{P_{BCD}\cdot S_{ABD}}{P_{BCAD}}$$

$$S_{BDA_1} \overset{A_1}{=} \frac{P_{BAD}\cdot S_{BCD}}{-P_{BCAD}}$$

$$S_{ABD} = \frac{\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$P_{BCD} = -2(\widetilde{CD}\cdot\widetilde{BC}\cdot\cos(BD))$$

$$S_{BCD} = \frac{\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC}}{(-2)\cdot d}$$

$$P_{BAD} = 2(\widetilde{AD}\cdot\widetilde{AB}\cdot\cos(BD))$$

Definition. *The symmetric of the circumcenter of a cyclic quadrilateral with respect to the centroid is called the anticenter of the cyclic quadrilateral.*

Example 6.282 (0.083, 3, 9) *The perpendicular from the midpoint of each diagonal upon the other diagonal also passes through the anticenter of a cyclic quadrilateral.*

Constructive description

((circle A B C D)
(circumcenter O A B C)
(midpoint T A D)
(midpoint P A B)
(midpoint Q B C)
(midpoint R C D)
(inter M (p P O R) (p Q O T))
(inter I (l P R) (l O M))
(midpoint I P R))

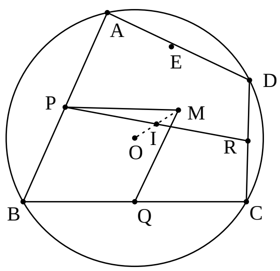


Figure 6-282

The machine proof

$$-\frac{\overline{PI}}{\overline{RI}}$$

$$\stackrel{I}{=} \frac{S_{OPM}}{-S_{ORM}}$$

$$\stackrel{M}{=} \frac{-(-S_{OPTQ}\cdot S_{OPR})}{(-S_{OPR})\cdot(-S_{OTR})}$$

$$\stackrel{simplify}{=} \frac{S_{OPTQ}}{S_{OTR}}$$

$$\stackrel{R}{=} \frac{S_{OPTQ}}{\frac{1}{2}S_{DOT}+\frac{1}{2}S_{COT}}$$

$$\stackrel{Q}{=} \frac{(2)\cdot(-S_{OTP}+\frac{1}{2}S_{COT}+\frac{1}{2}S_{BOT})}{S_{DOT}+S_{COT}}$$

$$\stackrel{P}{=} \frac{-(-S_{COT}+S_{AOT})}{S_{DOT}+S_{COT}}$$

$$\stackrel{T}{=} \frac{-\frac{1}{2}S_{CDO}+\frac{1}{2}S_{ADO}+\frac{1}{2}S_{ACO}}{-\frac{1}{2}S_{CDO}+\frac{1}{2}S_{ADO}+\frac{1}{2}S_{ACO}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{PI}}{\overline{RI}} \stackrel{I}{=} \frac{S_{OPM}}{S_{ORM}}$$

$$S_{ORM} \stackrel{M}{=} -(S_{OPR})$$

$$S_{OPM} \stackrel{M}{=} \frac{S_{OPTQ}\cdot S_{OPR}}{S_{OTR}}$$

$$S_{OTR} \stackrel{R}{=} \frac{1}{2}(S_{DOT}+S_{COT})$$

$$S_{OPTQ} \stackrel{Q}{=} \frac{(2S_{OTP}-S_{COT}-S_{BOT})}{-2}$$

$$S_{OTP} \stackrel{P}{=} \frac{1}{2}(S_{BOT}+S_{AOT})$$

$$S_{DOT} \stackrel{T}{=} \frac{1}{2}(S_{ADO})$$

$$S_{AOT} \stackrel{T}{=} -\frac{1}{2}(S_{ADO})$$

$$S_{COT} \stackrel{T}{=} -\frac{1}{2}(S_{CDO}-S_{ACO})$$

Example 6.283 (0.066, 4, 5) *Show that the anticenter of a cyclic quadrilateral is the orthocenter of the triangle having for vertices the midpoints of the diagonals and the point of intersection of those two lines.*

Constructive description

((circle A B C D)
(circumcenter O A B C)
(inter I (l A D) (l B C))
(midpoint Q B C)
(midpoint S A D)
(midpoint J S Q)
(lratio M J O -1)
(perpendicular M S B C))

The machine proof

$$\frac{P_{CBM}}{P_{CBS}}$$

$$\stackrel{M}{=} \frac{2P_{CBJ}-P_{CBO}}{P_{CBS}}$$

$$\stackrel{J}{=} \frac{(2)\cdot(\frac{1}{2}P_{CBS}+\frac{1}{2}P_{CBQ}-\frac{1}{2}P_{CBO})}{P_{CBS}}$$

$$\stackrel{S}{=} \frac{P_{CBQ}-P_{CBO}+\frac{1}{2}P_{CBD}+\frac{1}{2}P_{ABC}}{\frac{1}{2}P_{CBD}+\frac{1}{2}P_{ABC}}$$

$$\stackrel{Q}{=} \frac{(2)\cdot(-P_{CBO}+\frac{1}{2}P_{CBD}+\frac{1}{2}P_{BCB}+\frac{1}{2}P_{ABC})}{P_{CBD}+P_{ABC}}$$

$$\stackrel{O}{=} \frac{(-2)\cdot(-P_{CBD}-P_{ABC})}{(P_{CBD}+P_{ABC})\cdot(2)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{CBM} \stackrel{M}{=} 2(P_{CBJ}-\tfrac{1}{2}P_{CBO})$$

$$P_{CBJ} \stackrel{J}{=} \frac{1}{2}(P_{CBS}+P_{CBQ})$$

$$P_{CBS} \stackrel{S}{=} \frac{1}{2}(P_{CBD}+P_{ABC})$$

$$P_{CBQ} \stackrel{Q}{=} \frac{1}{2}(P_{BCB}),\quad P_{CBO} \stackrel{O}{=} \frac{1}{2}(P_{BCB})$$

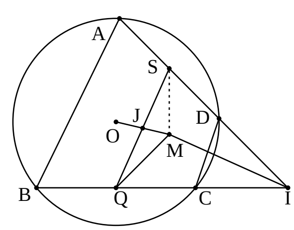


Figure 6-283

Example 6.284 (0.050, 2, 5) *Show that the anticenter of a cyclic quadrilateral is collinear with the two symmetrics of the circumcenter of the quadrilateral with respect to a pair of opposite sides.*

Constructive description

((circle $A\ B\ C\ D$) (circumcenter $O\ A\ B\ C$)
(midpoint $Q\ B\ C$) (midpoint $S\ A\ D$) (midpoint $J\ S\ Q$)
(lratio $Q_1\ Q\ O\ -1$) (lratio $S_1\ S\ O\ -1$) (inter M (l $O\ J$) (l $Q_1\ S_1$)) (midpoint $J\ O\ M$))

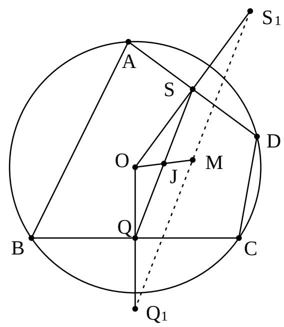


Figure 6-284

The machine proof

$$\frac{\overline{OJ}}{\overline{JM}}$$

$$\overset{M}{=} \frac{S_{OQ_1JS_1}}{S_{JQ_1S_1}}$$

$$\overset{S_1}{=} \frac{-S_{OJQ_1}-2S_{OSJ}}{2S_{SJQ_1}-S_{OJQ_1}}$$

$$\overset{Q_1}{=} \frac{-(2S_{OSJ}-2S_{OQJ})}{-2S_{OSJ}+2S_{OQJ}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{OJ}}{\overline{JM}} \overset{M}{=} \frac{S_{OQ_1JS_1}}{S_{JQ_1S_1}}$$

$$S_{JQ_1S_1} \overset{S_1}{=} 2S_{SJQ_1}-S_{OJQ_1}$$

$$S_{OQ_1JS_1} \overset{S_1}{=} -(S_{OJQ_1}+2S_{OSJ})$$

$$S_{SJQ_1} \overset{Q_1}{=} -(S_{OSJ})$$

$$S_{OJQ_1} \overset{Q_1}{=} -2(S_{OQJ})$$

Example 6.285 (0.050, 1, 12) *Show that the product of the distances of two opposite sides of a cyclic quadrilateral from a point on the circumcircles is equal to the product of the distances of the other two sides from the same point.*

Constructive description
((circle $A\ B\ C\ D\ E$)
(foot $P\ E\ A\ B$)
(foot $Q\ E\ B\ C$)
(foot $R\ E\ C\ D$)
(foot $S\ E\ A\ D$)
(eq-product $E\ S\ E\ Q\ E\ P\ E\ R$))

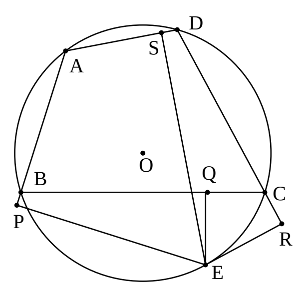


Figure 6-285

The machine proof

$$\frac{P_{ESE}\cdot P_{EQE}}{P_{EPE}\cdot P_{ERE}}$$

$$\overset{S}{=} \frac{(16S_{ADE}^2)\cdot P_{EQE}}{P_{EPE}\cdot P_{ERE}\cdot P_{ADA}}$$

$$\overset{R}{=} \frac{(16)\cdot S_{ADE}^2\cdot P_{EQE}\cdot P_{CDC}}{P_{EPE}\cdot (16S_{CDE}^2)\cdot P_{ADA}}$$

$$\overset{Q}{=} \frac{S_{ADE}^2\cdot (16S_{BCE}^2)\cdot P_{CDC}}{P_{EPE}\cdot S_{CDE}^2\cdot P_{ADA}\cdot P_{BCB}}$$

$$\overset{P}{=} \frac{(16)\cdot S_{ADE}^2\cdot S_{BCE}^2\cdot P_{CDC}\cdot P_{ABA}}{(16S_{ABE}^2)\cdot S_{CDE}^2\cdot P_{ADA}\cdot P_{BCB}}$$

$$\overset{co-cir}{=} \frac{(-\widetilde{DE}\cdot\widetilde{AE}\cdot\widetilde{AD})^2\cdot(-\widetilde{CE}\cdot\widetilde{BE}\cdot\widetilde{BC})^2\cdot(2\widetilde{CD}^2)\cdot(2\widetilde{AB}^2)\cdot(2d)^4}{(-\widetilde{BE}\cdot\widetilde{AE}\cdot\widetilde{AB})^2\cdot(-\widetilde{DE}\cdot\widetilde{CE}\cdot\widetilde{CD})^2\cdot(2\widetilde{AD}^2)\cdot(2\widetilde{BC}^2)\cdot(2d)^4}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{ESE} \overset{S}{=} \frac{(16)\cdot S_{ADE}^2}{P_{ADA}},\quad P_{ERE} \overset{R}{=} \frac{(16)\cdot S_{CDE}^2}{P_{CDC}}$$

$$P_{EQE} \overset{Q}{=} \frac{(16)\cdot S_{BCE}^2}{P_{BCB}},\quad P_{EPE} \overset{P}{=} \frac{(16)\cdot S_{ABE}^2}{P_{ABA}}$$

$$P_{BCB} = 2(\widetilde{BC}^2),\quad P_{ADA} = 2(\widetilde{AD}^2)$$

$$S_{CDE} = \frac{\widetilde{DE}\cdot\widetilde{CE}\cdot\widetilde{CD}}{(-2)\cdot d}$$

$$S_{ABE} = \frac{\widetilde{BE}\cdot\widetilde{AE}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$P_{ABA} = 2(\widetilde{AB}^2)$$

$$P_{CDC} = 2(\widetilde{CD}^2)$$

$$S_{BCE} = \frac{\widetilde{CE}\cdot\widetilde{BE}\cdot\widetilde{BC}}{(-2)\cdot d}$$

$$S_{ADE} = \frac{\widetilde{DE}\cdot\widetilde{AE}\cdot\widetilde{AD}}{(-2)\cdot d}$$

Example 6.286 (0.050, 1, 12) *Show that the projections of a point of the circumcircle of a cyclic quadrilateral upon the sides divide the sides into eight segments such that the product of four nonconsecutive segments is equal to the product of the remaining four.*

Constructive description
((circle $A\ B\ C\ D\ P$)
(circumcenter $O\ A\ B\ C$)
(foot $A_1\ P\ A\ B$) (foot $B_1\ P\ B\ C$)
(foot $C_1\ P\ C\ D$) (foot $D_1\ P\ D\ A$)
($\frac{\overline{AA_1}}{\overline{A_1B}} \frac{\overline{BB_1}}{\overline{B_1C}} \frac{\overline{CC_1}}{\overline{C_1D}} \frac{\overline{DD_1}}{\overline{D_1A}} = 1$))

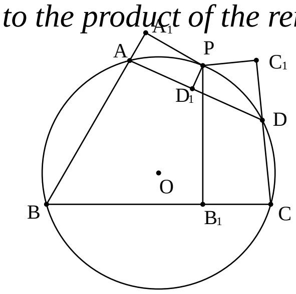


Figure 6-286

The machine proof

$$\frac{\overline{DD_1}}{\overline{AD_1}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}} \stackrel{D_1}{=} \frac{P_{ADP}}{-P_{DAP}} \cdot \frac{\overline{CC_1}}{\overline{DC_1}} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\stackrel{C_1}{=} \frac{-P_{ADP} \cdot P_{DCP}}{P_{DAP} \cdot (-P_{CDP})} \cdot \frac{\overline{BB_1}}{\overline{CB_1}} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\stackrel{B_1}{=} \frac{P_{ADP} \cdot P_{DCP} \cdot P_{CBP}}{P_{DAP} \cdot P_{CDP} \cdot (-P_{BCP})} \cdot \frac{\overline{AA_1}}{\overline{BA_1}}$$

$$\stackrel{A_1}{=} \frac{-P_{ADP} \cdot P_{DCP} \cdot P_{CBP} \cdot P_{BAP}}{P_{DAP} \cdot P_{CDP} \cdot P_{BCP} \cdot (-P_{ABP})}$$

$$\stackrel{co-cir}{=} \frac{(-2\widetilde{DP} \cdot \widetilde{AD} \cdot \cos(AP)) \cdot (2\widetilde{CP} \cdot \widetilde{CD} \cdot \cos(DP))}{(2\widetilde{AP} \cdot \widetilde{AD} \cdot \cos(DP)) \cdot (-2\widetilde{DP} \cdot \widetilde{CD} \cdot \cos(CP))} \cdot$$

$$\frac{(2\widetilde{BP} \cdot \widetilde{BC} \cdot \cos(CP)) \cdot (2\widetilde{AP} \cdot \widetilde{AB} \cdot \cos(BP))}{(-2\widetilde{CP} \cdot \widetilde{BC} \cdot \cos(BP)) \cdot (-2\widetilde{BP} \cdot \widetilde{AB} \cdot \cos(AP))} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{DD_1}}{\overline{AD_1}} \stackrel{D_1}{=} \frac{P_{ADP}}{-P_{DAP}}, \quad \frac{\overline{CC_1}}{\overline{DC_1}} \stackrel{C_1}{=} \frac{P_{DCP}}{-P_{CDP}}$$

$$\frac{\overline{BB_1}}{\overline{CB_1}} \stackrel{B_1}{=} \frac{P_{CBP}}{-P_{BCP}}$$

$$\frac{\overline{AA_1}}{\overline{BA_1}} \stackrel{A_1}{=} \frac{P_{BAP}}{-P_{ABP}}$$

$$P_{ABP} = -2(\widetilde{BP} \cdot \widetilde{AB} \cdot \cos(AP))$$

$$P_{BCP} = -2(\widetilde{CP} \cdot \widetilde{BC} \cdot \cos(BP))$$

$$P_{CDP} = -2(\widetilde{DP} \cdot \widetilde{CD} \cdot \cos(CP))$$

$$P_{DAP} = 2(\widetilde{AP} \cdot \widetilde{AD} \cdot \cos(DP))$$

$$P_{BAP} = 2(\widetilde{AP} \cdot \widetilde{AB} \cdot \cos(BP))$$

$$P_{CBP} = 2(\widetilde{BP} \cdot \widetilde{BC} \cdot \cos(CP))$$

$$P_{DCP} = 2(\widetilde{CP} \cdot \widetilde{CD} \cdot \cos(DP))$$

$$P_{ADP} = -2(\widetilde{DP} \cdot \widetilde{AD} \cdot \cos(AP))$$

Example 6.287 (0.866, 4, 29) *The four lines obtained by joining each vertex of a cyclic quadrilateral to the orthocenter of the triangle formed by the remaining three vertices bisect each other.*

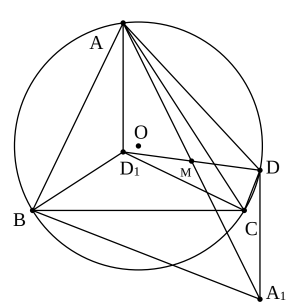


Figure 6-287

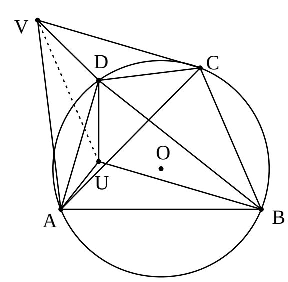


Figure 6-288

Constructive description ((circle $A\ B\ C\ D$) (orthocenter $D_1\ A\ B\ C$)
(orthocenter $A_1\ B\ C\ D$) (inter M (l $D\ D_1$) (l $A\ A_1$)) (midpoint $M\ A\ A_1$))

Example 6.288 (1.750, 11, 35) *A line AD through the vertex A meets the circumcircle of the triangle ABC in D. If U, V are the orthocenters of the triangle ABD, ACD, respectively, prove that UV is equal and parallel to BC.*

Constructive description
((circle $A\ B\ C\ D$) (orthocenter $U\ A\ B\ D$) (orthocenter $V\ A\ C\ D$) (eqdistance $V\ U\ B\ C$))

Example 6.289 (1.633, 12, 35) *The sum of the squares of the distances of the anticenter of a cyclic quadrilateral from the four vertices is equal to the square of the circumdiameter of the quadrilateral.*

Constructive description
((circle $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $A_1\ B\ C\ D$)
(midpoint $M\ A\ A_1$)
($\overline{MA}^2+\overline{MB}^2+\overline{MC}^2+\overline{MD}^2 = 4\overline{OB}^2$))

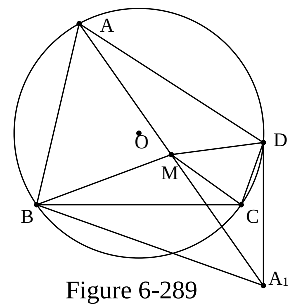


Figure 6-289

Example 6.290 (0.750, 6, 41) *The line joining the centroid of a triangle to a point P on the circumcircle bisects the line joining the diametric opposite of P to the orthocenter.*

Constructive description
((circle $A\ B\ C\ P$)
(circumcenter $O\ A\ B\ C$)
(orthocenter $H\ A\ B\ C$)
(centroid $G\ A\ B\ C$)
(lratio $Q\ O\ P\ -1$)
(inter I (l $H\ Q$) (l $P\ G$))
(midpoint $I\ Q\ H$))

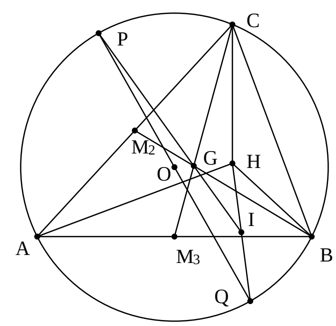


Figure 6-290

Example 6.291 (0.616, 9, 48) *Show that the perpendicular from the point of intersection of two opposite sides, produced, of a cyclic quadrilateral upon the line joining the midpoints of the two sides considered passes through the anticenter of the quadrilateral.*

Constructive description
((circle $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ C$)
(inter I (l $A\ D$) (l $B\ C$))
(midpoint $Q\ B\ C$)
(midpoint $S\ A\ D$)
(midpoint $J\ S\ Q$)
(lratio $M\ J\ O\ -1$)
(perpendicular $I\ M\ S\ Q$))

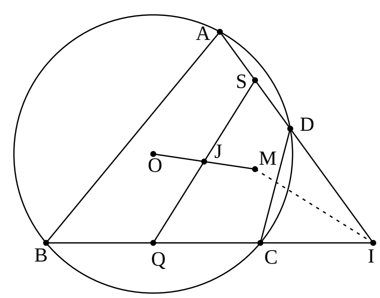


Figure 6-291

Example 6.292 (0.450, 3, 31) *If H_a, H_b, H_c, H_d are the orthocenters of the four triangles determined by the vertices of the cyclic quadrilateral $ABCD$, show that the vertices of $ABCD$ are the orthocenters of the four triangles determined by the points H_a, H_b, H_c, H_d.*

Constructive description
((circle $A\ B\ C\ D$)
(orthocenter $H_D\ A\ B\ C$)
(orthocenter $H_A\ B\ C\ D$)
(orthocenter $H_C\ A\ B\ D$)
(perpendicular $H_C\ H_A\ B\ H_D$))

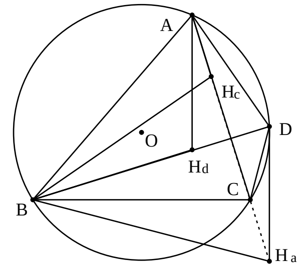


Figure 6-292

6.4.6 Orthodiagonal Quadrilaterals

Definition *A quadrilateral is said to be orthodiagonal if its diagonals are perpendicular to each other.*

Example 6.293 (Theorem of Brahmagupta) (0.133, 4, 15) *In a quadrilateral which is both orthodiagonal and cyclic the perpendicular from the point of intersection of the diagonals to a side bisects the side opposite.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(foot $E\ B\ A\ C$)
(inter D (l $B\ E$) (cir $O\ B$))
(foot $F_F\ E\ C\ D$)
(inter F (l $A\ B$) (l $E\ F_F$))
(midpoint $F\ A\ B$))

The eliminants

$\frac{\overline{AF}}{\overline{BF}} \overset{F}{=} \frac{S_{AEF_F}}{S_{BEF_F}}$ $S_{BEF_F} \overset{F_F}{=} \frac{-P_{CDE}\cdot S_{BCE}}{P_{CDC}}$

$S_{AEF_F} \overset{F_F}{=} \frac{P_{ECD}\cdot S_{AED}}{P_{CDC}}$

$P_{CDE} \overset{D}{=} \frac{-(2P_{OBE}-P_{CBE})\cdot(P_{OEO}-P_{BOB})}{P_{BEB}}$

$S_{AED} \overset{D}{=} \frac{-(P_{OEO}-P_{BOB})\cdot S_{ABE}}{P_{BEB}}$

$P_{ECD} \overset{D}{=} P_{BCE}$

$S_{BCE} \overset{E}{=} \frac{P_{ACB}\cdot S_{ABC}}{P_{ACA}}$

$P_{CBE} \overset{E}{=} \frac{P_{BCB}\cdot P_{BAC}+P_{ACB}\cdot P_{ABC}}{P_{ACA}}$

$P_{OBE} \overset{E}{=} \frac{P_{CBO}\cdot P_{BAC}+P_{ACB}\cdot P_{ABO}}{P_{ACA}}$

$S_{ABE} \overset{E}{=} \frac{P_{BAC}\cdot S_{ABC}}{P_{ACA}}$ $P_{BCE} \overset{E}{=} \frac{P_{ACB}^2}{P_{ACA}}$

$P_{ABO} \overset{O}{=} \frac{1}{2}\left(P_{ABA}\right)$ $P_{CBO} \overset{O}{=} \frac{1}{2}\left(P_{BCB}\right)$

$P_{ABC} = \frac{1}{2}\left(P_{BCB}-P_{ACA}+P_{ABA}\right)$

$P_{BAC} = -\frac{1}{2}\left(P_{BCB}-P_{ACA}-P_{ABA}\right)$

The machine proof

$-\frac{\overline{AF}}{\overline{BF}}$

$\overset{F}{=} \frac{-S_{AEF_F}}{S_{BEF_F}}$

$\overset{F_F}{=} \frac{-P_{ECD}\cdot S_{AED}\cdot P_{CDC}}{(-P_{CDE}\cdot S_{BCE})\cdot P_{CDC}}$

$\overset{simplify}{=} \frac{P_{ECD}\cdot S_{AED}}{P_{CDE}\cdot S_{BCE}}$

$\overset{D}{=} \frac{P_{BCE}\cdot(-P_{OEO}\cdot S_{ABE}+P_{BOB}\cdot S_{ABE})\cdot P_{BEB}}{(-2P_{OEO}\cdot P_{OBE}+P_{OEO}\cdot P_{CBE}+2P_{OBE}\cdot P_{BOB}-P_{CBE}\cdot P_{BOB})\cdot S_{BCE}\cdot P_{BEB}}$

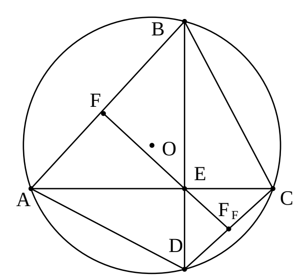


Figure 6-293

$$\stackrel{simplify}{=} \frac{P_{BCE}\cdot S_{ABE}}{(2P_{OBE}-P_{CBE})\cdot S_{BCE}}$$

$$\stackrel{E}{=} \frac{P_{ACB}^2\cdot P_{BAC}\cdot S_{ABC}\cdot P_{ACA}^3}{(2P_{CBO}\cdot P_{BAC}\cdot P_{ACA}-P_{BCB}\cdot P_{BAC}\cdot P_{ACA}+2P_{ACB}\cdot P_{ACA}\cdot P_{ABO}-P_{ACB}\cdot P_{ACA}\cdot P_{ABC})\cdot P_{ACB}\cdot S_{ABC}\cdot P_{ACA}^2}$$

$$\stackrel{simplify}{=} \frac{P_{ACB}\cdot P_{BAC}}{2P_{CBO}\cdot P_{BAC}-P_{BCB}\cdot P_{BAC}+2P_{ACB}\cdot P_{ABO}-P_{ACB}\cdot P_{ABC}}$$

$$\stackrel{O}{=} \frac{P_{ACB}\cdot P_{BAC}\cdot (2)^2}{-4P_{ACB}\cdot P_{ABC}+4P_{ACB}\cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{-P_{BAC}}{P_{ABC}-P_{ABA}} \stackrel{py}{=} \frac{-(-P_{BCB}+P_{ACA}+P_{ABA})\cdot (2)}{(P_{BCB}-P_{ACA}-P_{ABA})\cdot (2)} \stackrel{simplify}{=} 1$$

Example 6.294 (0.333, 4, 34) *Let E be the intersection of the two diagonals AC and BD of cyclic quadrilateral $ABCD$. Let I be the center of circumcircle of ABE. Show the $IE \perp DC$.*

Constructive description
((circle *A B C D*)
(inter *E* (l *B D*) (l *A C*))
(circumcenter *I A B E*)
(perpendicular *I E C D*))

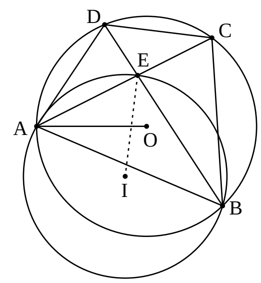


Figure 6-294

Example 6.295 (0.600, 7, 16) *In an orthodiagonal quadrilateral the two lines joining the midpoints of the pairs of opposite sides are equal.*

Constructive description
((points *A B C*)
(foot F_D *B A C*) (on *D* (l *B* F_D))
(midpoint *P A B*)
(midpoint *Q B C*)
(midpoint *R C D*)
(midpoint *S D A*)
(eqdistance *S Q P R*))

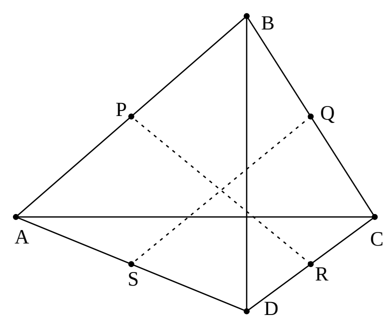


Figure 6-295

Example 6.296 (0.850, 7, 30) *In an orthodiagonal quadrilateral the midpoints of the sides lie on a circle having for center the centroid of the quadrilateral.*

Constructive description
((points *A B C*)
(foot F_D *B A C*)
(on *D* (l *B* F_D))
(midpoint *P A B*)
(midpoint *Q B C*)

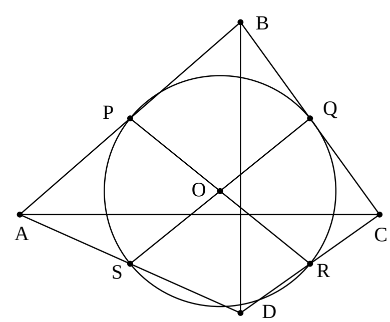


Figure 6-296

(midpoint R C D)
(midpoint S D A)
(inter O (l Q S) (l P R))
(perp-biesct O S R))

Example 6.297 (3.300, 36, 29) *If an orthodiagonal quadrilateral is cyclic, the anticenter coincides with the point of intersection of its diagonals.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot M B A C)
(inter D (l B M) (cir O B))
(midpoint J O M)
(midpoint R C D)
(inter P (l A B) (l J R))
(midpoint P A B))

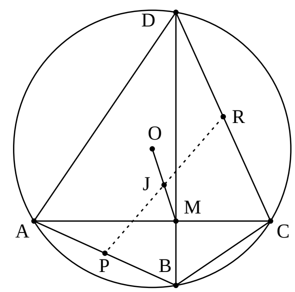


Figure 6-297

Example 6.298 (0.967, 14, 15) *In a cyclic orthodiagonal quadrilateral the distance of a side from the circumcenter of the quadrilateral is equal to half the opposite side.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint P A B)
(foot F_D B A C)
(inter D (l B F_D) (cir O B))
($P_{CDC} = 4P_{OPO}$))

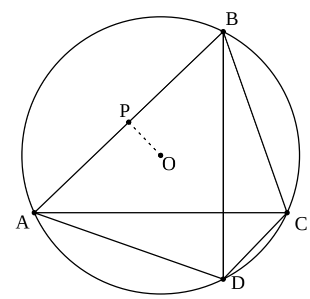


Figure 6-298

Example 6.299 (0.967, 15, 14) *If a quadrilateral is both cyclic and orthodiagonal, the sum of the squares of two opposite sides is equal to the square of the circumdiameter of the quadrilateral.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot F_D B A C)
(inter D (l B F_D) (cir O B))
($\overline{AB}^2+\overline{DC}^2 = 4\overline{OA}^2$))

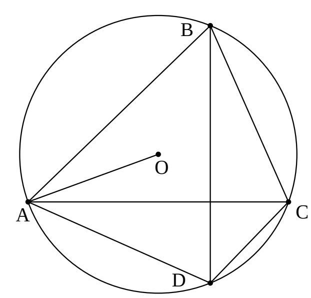


Figure 6-299

Example 6.300 (2.117, 26, 20) *Show that the line joining the midpoints of the diagonals of a cyclic orthodiagonal quadrilateral is equal to the distance of the point of intersection of the diagonals from the circumcenter of the quadrilateral.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot E B A C)
(inter D (l B E) (cir O B))
(midpoint U A C)
(midpoint V B D)
(eqdistance U V O E))

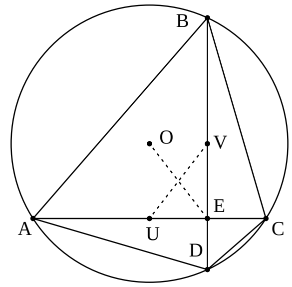


Figure 6-300

Example 6.301 (1.050, 16, 16) *If the diagonals of a cyclic quadrilateral $ABCD$ are orthogonal, and E is the diametric opposite of D on its circumcircle, show that $AE = CB$.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot F_D B A C)
(inter D (l B F_D) (cir O B))
(inter E (l D O) (cir O D))
(eqdistance A E C B))

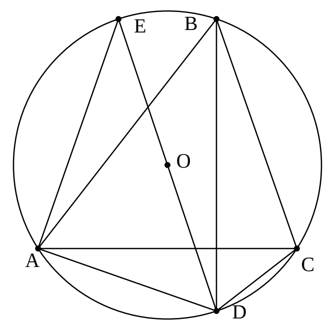


Figure 6-301

Example 6.302 (1.250, 4, 84) *$A_1B_1C_1D_1$ is a quadrilateral with an inscribed circle O. Then O is on the line joining the midpoints of A_1C_1 and B_1D_1.*

Constructive description
((circle A B C D) (circumcenter O A B C)
(on B_T (t B B O)) (on A_T (t A A O))
(on C_T (t C C O)) (on D_T (t D D O))
(inter A_1 (l B B_T) (l A A_T))
(inter B_1 (l C C_T) (l B B_T))
(inter C_1 (l D D_T) (l C C_T))
(inter D_1 (l A A_T) (l D D_T))
(midpoint M A_1 C_1)
(inter N (l B_1 D_1) (l O M))
(midpoint N B_1 D_1))

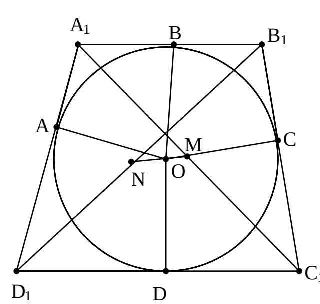


Figure 6-302

6.4.7 The Butterfly Theorems

For the machine proof of the general butterfly theorem for a circle, See Example 3.81 on page 146

Example 6.303 (0.516, 4, 31) *The Butterfly theorem for a circle.*

Constructive description
((circle A B C D)
(circumcenter O A B C)
(inter E (l A B) (l C D))
(inter G (l B C) (t E E O))
(inter F (l A D) (t E E O))
($\frac{\overline{GF}}{\overline{EF}}$ = 2))

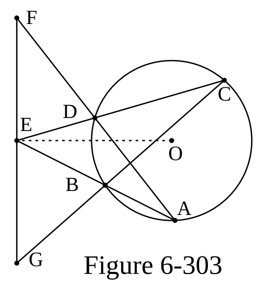


Figure 6-303

Example 6.304 (Butterfly Theorem for Quadrilaterals) (0.166, 3, 19) *Let $ABCD$ be a quadrilateral such that $AB = BC$ and $AD = CD$. M is the intersection of AC and BD. Passing through M two lines are drawn which meet the sides of $ABCD$ at P, Q, S, R. Let $G = PR \cap AC$, $H = SQ \cap AC$. Show that $GM = MH$.*

Constructive description
((points A C)
(on B (b A C))
(midpoint M A C)
(on D (l B M))
(on P (l A B))
(inter Q (l C D) (l P M))
(on S (l B C))
(inter R (l A D) (l S M))
(inter G (l A C) (l P R))
(inter H (l A C) (l S Q))
($\frac{\overline{MG}}{\overline{GA}}$ = $\frac{\overline{MH}}{\overline{HC}}$))

The eliminants

$$\frac{\overline{MH}}{\overline{CH}} \stackrel{H}{=} \frac{S_{MQS}}{S_{CQS}}, \quad \frac{\overline{MG}}{\overline{AG}} \stackrel{G}{=} \frac{S_{MPR}}{S_{APR}}$$

$$S_{APR} \stackrel{R}{=} \frac{-S_{ADP} \cdot S_{AMS}}{S_{AMDS}}, \quad S_{MPR} \stackrel{R}{=} \frac{S_{MPS} \cdot S_{AMD}}{S_{AMDS}}$$

$$S_{AMS} \stackrel{S}{=} (\frac{\overline{BS}}{\overline{BC}} - 1) \cdot S_{ABM}, \quad S_{CQS} \stackrel{S}{=} (\frac{\overline{BS}}{\overline{BC}} - 1) \cdot S_{CBQ}$$

$$S_{MQS} \stackrel{S}{=} -\left(S_{BMQ} \cdot \frac{\overline{BS}}{\overline{BC}} - S_{BMQ} - S_{CMQ} \cdot \frac{\overline{BS}}{\overline{BC}}\right)$$

$$S_{MPS} \stackrel{S}{=} -\left(S_{BMP} \cdot \frac{\overline{BS}}{\overline{BC}} - S_{BMP} - S_{CMP} \cdot \frac{\overline{BS}}{\overline{BC}}\right)$$

$$S_{CMQ} \stackrel{Q}{=} \frac{S_{CMP} \cdot S_{CMD}}{S_{CMDP}}, \quad S_{BMQ} \stackrel{Q}{=} \frac{S_{BMP} \cdot S_{CMD}}{S_{CMDP}}$$

$$S_{CBQ} \stackrel{Q}{=} \frac{S_{CMP} \cdot S_{CBD}}{S_{CMDP}}, \quad S_{ADP} \stackrel{P}{=} -\left(S_{ABD} \cdot \frac{\overline{AP}}{\overline{AB}}\right)$$

$$S_{CMP} \stackrel{P}{=} -\left(S_{CBM} \cdot \frac{\overline{AP}}{\overline{AB}}\right), \quad S_{ABD} \stackrel{D}{=} S_{ABM} \cdot \frac{\overline{BD}}{\overline{BM}}$$

$$S_{CMD} \stackrel{D}{=} (\frac{\overline{BD}}{\overline{BM}} - 1) \cdot S_{CBM}, \quad S_{CBD} \stackrel{D}{=} S_{CBM} \cdot \frac{\overline{BD}}{\overline{BM}}$$

$$S_{AMD} \stackrel{D}{=} (\frac{\overline{BD}}{\overline{BM}} - 1) \cdot S_{ABM}, \quad S_{ABM} \stackrel{M}{=} -\frac{1}{2}(S_{ACB})$$

$$S_{CBM} \stackrel{M}{=} \frac{1}{2}(S_{ACB})$$

The machine proof

$$(\frac{\overline{MG}}{\overline{AG}})/(\frac{\overline{MH}}{\overline{CH}}) \stackrel{H}{=} \frac{-S_{CQS}}{-S_{MQS}} \cdot \frac{\overline{MG}}{\overline{AG}} \stackrel{G}{=} \frac{S_{MPR} \cdot S_{CQS}}{S_{MQS} \cdot S_{APR}}$$

$$\stackrel{R}{=} \frac{(-S_{MPS} \cdot S_{AMD}) \cdot S_{CQS} \cdot S_{AMDS}}{S_{MQS} \cdot (-S_{ADP} \cdot S_{AMS}) \cdot (-S_{AMDS})}$$

$$\stackrel{simplify}{=} \frac{-S_{MPS} \cdot S_{AMD} \cdot S_{CQS}}{S_{MQS} \cdot S_{ADP} \cdot S_{AMS}}$$

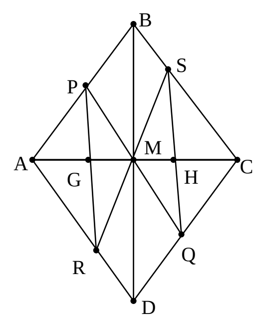


Figure 6-304

$$\stackrel{S}{=} \frac{-(-S_{BMP}\cdot\frac{\overline{BS}}{\overline{BC}}+S_{BMP}+S_{CMP}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{AMD}\cdot(S_{CBQ}\cdot\frac{\overline{BS}}{\overline{BC}}-S_{CBQ})}{(-S_{BMQ}\cdot\frac{\overline{BS}}{\overline{BC}}+S_{BMQ}+S_{CMQ}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{ADP}\cdot(S_{ABM}\cdot\frac{\overline{BS}}{\overline{BC}}-S_{ABM})}$$

$$\stackrel{simplify}{=} \frac{-(S_{BMP}\cdot\frac{\overline{BS}}{\overline{BC}}-S_{BMP}-S_{CMP}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{AMD}\cdot S_{CBQ}}{(S_{BMQ}\cdot\frac{\overline{BS}}{\overline{BC}}-S_{BMQ}-S_{CMQ}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{ADP}\cdot S_{ABM}}$$

$$\stackrel{Q}{=} \frac{-(S_{BMP}\cdot\frac{\overline{BS}}{\overline{BC}}-S_{BMP}-S_{CMP}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{AMD}\cdot S_{CMP}\cdot S_{CBD}\cdot S_{CMDP}\cdot(-S_{CMDP})}{(-S_{CMDP}\cdot S_{BMP}\cdot S_{CMD}\cdot\frac{\overline{BS}}{\overline{BC}}+S_{CMDP}\cdot S_{BMP}\cdot S_{CMD}+S_{CMDP}\cdot S_{CMP}\cdot S_{CMD}\cdot\frac{\overline{BS}}{\overline{BC}})\cdot S_{ADP}\cdot S_{ABM}\cdot S_{CMDP}}$$

$$\stackrel{simplify}{=} \frac{-S_{AMD}\cdot S_{CMP}\cdot S_{CBD}}{S_{CMD}\cdot S_{ADP}\cdot S_{ABM}}$$

$$\stackrel{P}{=} \frac{-S_{AMD}\cdot(-S_{CBM}\cdot\frac{\overline{AP}}{\overline{AB}})\cdot S_{CBD}}{S_{CMD}\cdot(-S_{ABD}\cdot\frac{\overline{AP}}{\overline{AB}})\cdot S_{ABM}}$$

$$\stackrel{simplify}{=} \frac{-S_{AMD}\cdot S_{CBM}\cdot S_{CBD}}{S_{CMD}\cdot S_{ABD}\cdot S_{ABM}}$$

$$\stackrel{D}{=} \frac{-(S_{ABM}\cdot\frac{\overline{BD}}{\overline{BM}}-S_{ABM})\cdot S_{CBM}\cdot S_{CBM}\cdot\frac{\overline{BD}}{\overline{BM}}}{(S_{CBM}\cdot\frac{\overline{BD}}{\overline{BM}}-S_{CBM})\cdot S_{ABM}\cdot\frac{\overline{BD}}{\overline{BM}}\cdot S_{ABM}}$$

$$\stackrel{simplify}{=} \frac{-S_{CBM}}{S_{ABM}} \stackrel{M}{=} \frac{-(\frac{1}{2}S_{ACB})}{-\frac{1}{2}S_{ACB}} \stackrel{simplify}{=} 1$$

Example 6.305 (Butterfly Theorem for Quadrilaterals) (0.066, 2, 14) *Let $ABCD$ be a quadrilateral such that the intersection M of its diagonals is the midpoint of AC. Passing through M two lines are drawn which meet the sides of $ABCD$ at P, Q, S, R. Let $G = PR \cap AC$, $H = SQ \cap AC$. Show that $GM = MH$.*

Constructive description

((points $A\ C\ B$)
(midpoint $M\ A\ C$)
(lratio $D\ M\ B\ r_1$)
(lratio $P\ A\ B\ r_2$)
(lratio $R\ A\ D\ r_3$)
(inter Q (l $P\ M$) (l $C\ D$))
(inter S (l $R\ M$) (l $B\ C$))
(inter G (l $A\ C$) (l $P\ R$))
(inter H (l $A\ C$) (l $S\ Q$))
($\frac{\overline{MG}}{\overline{GA}} = \frac{\overline{MH}}{\overline{HC}}$))

The machine proof

$$(\frac{\overline{MG}}{\overline{AG}})/(\frac{\overline{MH}}{\overline{CH}})$$

$$\stackrel{H}{=} \frac{-S_{CQS}}{-S_{MQS}} \cdot \frac{\overline{MG}}{\overline{AG}}$$

$$\stackrel{G}{=} \frac{S_{MPR} \cdot S_{CQS}}{S_{MQS} \cdot S_{APR}}$$

$$\stackrel{S}{=} \frac{S_{MPR} \cdot (-S_{CMR} \cdot S_{CBQ}) \cdot (-S_{CMBR})}{(-S_{MRQ} \cdot S_{CBM}) \cdot S_{APR} \cdot S_{CMBR}}$$

$$\stackrel{simplify}{=} \frac{-S_{MPR} \cdot S_{CMR} \cdot S_{CBQ}}{S_{MRQ} \cdot S_{CBM} \cdot S_{APR}}$$

$$\stackrel{Q}{=} \frac{-S_{MPR} \cdot S_{CMR} \cdot S_{CMP} \cdot S_{CBD} \cdot (-S_{CMDP})}{S_{MPR} \cdot S_{CMD} \cdot S_{CBM} \cdot S_{APR} \cdot S_{CMDP}}$$

$$\stackrel{simplify}{=} \frac{S_{CMR} \cdot S_{CMP} \cdot S_{CBD}}{S_{CMD} \cdot S_{CBM} \cdot S_{APR}}$$

$$\stackrel{R}{=} \frac{S_{CMD} \cdot r_3 \cdot S_{CMP} \cdot S_{CBD}}{S_{CMD} \cdot S_{CBM} \cdot (-S_{ADP} \cdot r_3)}$$

$$\stackrel{simplify}{=} \frac{-S_{CMP} \cdot S_{CBD}}{S_{CBM} \cdot S_{ADP}}$$

$$\stackrel{P}{=} \frac{-(-S_{CBM} \cdot r_2) \cdot S_{CBD}}{S_{CBM} \cdot (-S_{ABD} \cdot r_2)}$$

$$\stackrel{simplify}{=} \frac{-S_{CBD}}{S_{ABD}}$$

$$\stackrel{D}{=} \frac{-(-S_{CBM} \cdot r_1 + S_{CBM})}{-S_{ABM} \cdot r_1 + S_{ABM}}$$

$$\stackrel{simplify}{=} \frac{-S_{CBM}}{S_{ABM}}$$

$$\stackrel{M}{=} \frac{-(\frac{1}{2} S_{ACB})}{-\frac{1}{2} S_{ACB}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{MH}}{\overline{CH}} \stackrel{H}{=} \frac{S_{MQS}}{S_{CQS}}$$

$$\frac{\overline{MG}}{\overline{AG}} \stackrel{G}{=} \frac{S_{MPR}}{S_{APR}}$$

$$S_{MQS} \stackrel{S}{=} \frac{S_{MRQ} \cdot S_{CBM}}{S_{CMBR}}$$

$$S_{CQS} \stackrel{S}{=} \frac{-S_{CMR} \cdot S_{CBQ}}{S_{CMBR}}$$

$$S_{MRQ} \stackrel{Q}{=} \frac{S_{MPR} \cdot S_{CMD}}{-S_{CMDP}}$$

$$S_{CBQ} \stackrel{Q}{=} \frac{S_{CMP} \cdot S_{CBD}}{S_{CMDP}}$$

$$S_{APR} \stackrel{R}{=} -\left(S_{ADP} \cdot r_3\right)$$

$$S_{CMR} \stackrel{R}{=} S_{CMD} \cdot r_3$$

$$S_{ADP} \stackrel{P}{=} -\left(S_{ABD} \cdot r_2\right)$$

$$S_{CMP} \stackrel{P}{=} -\left(S_{CBM} \cdot r_2\right)$$

$$S_{ABD} \stackrel{D}{=} -\left((r_1 - 1) \cdot S_{ABM}\right)$$

$$S_{CBD} \stackrel{D}{=} -\left((r_1 - 1) \cdot S_{CBM}\right)$$

$$S_{ABM} \stackrel{M}{=} -\frac{1}{2}\left(S_{ACB}\right)$$

$$S_{CBM} \stackrel{M}{=} \frac{1}{2}\left(S_{ACB}\right)$$

Example 6.306 (0.100, 2, 15) *Let ABCD be a quadrilateral and M the intersection of its diagonals. Passing through M two lines are drawn which meet the sides of ABCD at* P, Q, S, R. *Let* $G = PR \cap AC$, $H = SQ \cap AC$. *Show that*

$$\frac{\overline{MG}}{\overline{AG}} = \frac{\overline{MH}}{\overline{CH}} \frac{\overline{CM}}{\overline{MA}}.$$

Constructive description

((points A B C)
(lratio M A C r_0)
(lratio D M B r_1)
(lratio P A B r_2)
(lratio R A D r_3)
(inter Q (l P M) (l C D))
(inter S (l R M) (l B C))
(inter G (l A C) (l P R))
(inter H (l A C) (l S Q))
($\frac{\overline{MG}}{\overline{AG}} = \frac{\overline{MH}}{\overline{CH}}\frac{\overline{CM}}{\overline{MA}}$))

The machine proof

$$\frac{\frac{\overline{MG}}{\overline{AG}}}{-\frac{\overline{MH}}{\overline{CH}}\cdot\frac{\overline{CM}}{\overline{AM}}}$$

$$\overset{H}{=} \frac{-S_{CQS}}{-(-S_{MQS})\cdot\frac{\overline{CM}}{\overline{AM}}}\cdot\frac{\overline{MG}}{\overline{AG}}$$

$$\overset{G}{=} \frac{-S_{MPR}\cdot S_{CQS}}{S_{MQS}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{APR}}$$

$$\overset{S}{=} \frac{-S_{MPR}\cdot(-S_{CMR}\cdot S_{BCQ})\cdot(-S_{BMCR})}{(-S_{MRQ}\cdot S_{BCM})\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{APR}\cdot S_{BMCR}}$$

$$\overset{simplify}{=} \frac{S_{MPR}\cdot S_{CMR}\cdot S_{BCQ}}{S_{MRQ}\cdot S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{APR}}$$

$$\overset{Q}{=} \frac{S_{MPR}\cdot S_{CMR}\cdot S_{CMP}\cdot S_{BCD}\cdot(-S_{CMDP})}{S_{MPR}\cdot S_{CMD}\cdot S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{APR}\cdot S_{CMDP}}$$

$$\overset{simplify}{=} \frac{-S_{CMR}\cdot S_{CMP}\cdot S_{BCD}}{S_{CMD}\cdot S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{APR}}$$

$$\overset{R}{=} \frac{-S_{CMD}\cdot r_3\cdot S_{CMP}\cdot S_{BCD}}{S_{CMD}\cdot S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot(-S_{ADP}\cdot r_3)}$$

$$\overset{simplify}{=} \frac{S_{CMP}\cdot S_{BCD}}{S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot S_{ADP}}$$

$$\overset{P}{=} \frac{S_{BCM}\cdot r_2\cdot S_{BCD}}{S_{BCM}\cdot\frac{\overline{CM}}{\overline{AM}}\cdot(-S_{ABD}\cdot r_2)}$$

$$\overset{simplify}{=} \frac{-S_{BCD}}{\frac{\overline{CM}}{\overline{AM}}\cdot S_{ABD}}$$

$$\overset{D}{=} \frac{-(-S_{BCM}\cdot r_1+S_{BCM})}{\frac{\overline{CM}}{\overline{AM}}\cdot(-S_{ABM}\cdot r_1+S_{ABM})}$$

$$\overset{simplify}{=} \frac{-S_{BCM}}{\frac{\overline{CM}}{\overline{AM}}\cdot S_{ABM}}$$

$$\overset{M}{=} \frac{-(-S_{ABC}\cdot r_0+S_{ABC})\cdot r_0}{(r_0-1)\cdot S_{ABC}\cdot r_0}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{MH}}{\overline{CH}}\overset{H}{=}\frac{S_{MQS}}{S_{CQS}}$$

$$\frac{\overline{MG}}{\overline{AG}}\overset{G}{=}\frac{S_{MPR}}{S_{APR}}$$

$$S_{MQS}\overset{S}{=}\frac{S_{MRQ}\cdot S_{BCM}}{S_{BMCR}}$$

$$S_{CQS}\overset{S}{=}\frac{-S_{CMR}\cdot S_{BCQ}}{S_{BMCR}}$$

$$S_{MRQ}\overset{Q}{=}\frac{S_{MPR}\cdot S_{CMD}}{-S_{CMDP}}$$

$$S_{BCQ}\overset{Q}{=}\frac{S_{CMP}\cdot S_{BCD}}{S_{CMDP}}$$

$$S_{APR}\overset{R}{=}-\left(S_{ADP}\cdot r_3\right)$$

$$S_{CMR}\overset{R}{=}S_{CMD}\cdot r_3$$

$$S_{ADP}\overset{P}{=}-\left(S_{ABD}\cdot r_2\right)$$

$$S_{CMP}\overset{P}{=}S_{BCM}\cdot r_2$$

$$S_{ABD}\overset{D}{=}-\left((r_1-1)\cdot S_{ABM}\right)$$

$$S_{BCD}\overset{D}{=}-\left((r_1-1)\cdot S_{BCM}\right)$$

$$S_{ABM}\overset{M}{=}S_{ABC}\cdot r_0$$

$$\frac{\overline{CM}}{\overline{AM}}\overset{M}{=}\frac{r_0-1}{r_0}$$

$$S_{BCM}\overset{M}{=}-\left((r_0-1)\cdot S_{ABC}\right)$$

6.5 Circles

6.5.1 Chords, Secants, and Tangents

Example 6.307 (The Secant Theorem) (0.016, 1, 8) *The product of the distances of a given point, from any two points which are collinear with the given point and lie on the circle, is a constant. This constant is called the power of the point with respect to the circle.*

Constructive description

((circle A B C D)
(inter E (l B D) (l A C))
(eq-product
A E C E B E D E)).

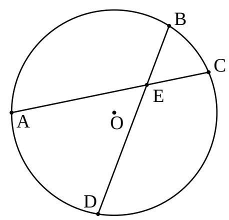


Figure 6-307

The machine proof

$$\frac{P_{AEC}}{P_{BED}}$$

$$\overset{E}{=} \frac{(-P_{ACA}\cdot S_{BCD}\cdot S_{ABD})\cdot S_{ABCD}^2}{(-P_{BDB}\cdot S_{ACD}\cdot S_{ABC})\cdot S_{ABCD}^2}$$

$$\overset{simplify}{=} \frac{P_{ACA}\cdot S_{BCD}\cdot S_{ABD}}{P_{BDB}\cdot S_{ACD}\cdot S_{ABC}}$$

$$= \frac{(2\widetilde{AC}^2)\cdot(-\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC})\cdot(-\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB})}{(2\widetilde{BD}^2)\cdot(-\widetilde{CD}\cdot\widetilde{AD}\cdot\widetilde{AC})\cdot(-\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{BED} \overset{E}{=} \frac{-P_{BDB}\cdot S_{ACD}\cdot S_{ABC}}{S_{ABCD}^2}$$

$$P_{AEC} \overset{E}{=} \frac{-P_{ACA}\cdot S_{BCD}\cdot S_{ABD}}{S_{ABCD}^2}$$

$$S_{ABC} = \frac{\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{ACD} = \frac{\widetilde{CD}\cdot\widetilde{AD}\cdot\widetilde{AC}}{(-2)\cdot d}$$

$$P_{BDB} = 2(\widetilde{BD}^2)$$

$$S_{ABD} = \frac{\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB}}{(-2)\cdot d}$$

$$S_{BCD} = \frac{\widetilde{CD}\cdot\widetilde{BD}\cdot\widetilde{BC}}{(-2)\cdot d}$$

$$P_{ACA} = 2(\widetilde{AC}^2)$$

Example 6.308 (0.050, 4, 3) *The power of a point with respect to a circle is equal, both in magnitude and in sign, to the square of the distance of the point from the center of the circle diminished by the square of the radius of the circle.*

Constructive description

((points A B)
(on O (b A B))
(on E (l A B))
($\frac{1}{2}P_{AEB} = \overline{OE}^2 - \overline{OA}^2$))

The eliminants

$$P_{OEO} \overset{E}{=} P_{BOB}\cdot\frac{\overline{AE}}{\overline{AB}} - P_{AOA}\cdot\frac{\overline{AE}}{\overline{AB}} + P_{AOA} + P_{ABA}\cdot\left(\frac{\overline{AE}}{\overline{AB}}\right)^2 - P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}}$$

$$P_{AEB} \overset{E}{=} \left(\frac{\overline{AE}}{\overline{AB}} - 1\right)\cdot P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}}$$

$$P_{BOB} \overset{O}{=} P_{AOA}$$

The machine proof

$$\frac{P_{AEB}}{P_{OEO} - P_{AOA}}$$

$$\overset{E}{=} \frac{P_{ABA}\cdot\left(\frac{\overline{AE}}{\overline{AB}}\right)^2 - P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}}}{P_{BOB}\cdot\frac{\overline{AE}}{\overline{AB}} - P_{AOA}\cdot\frac{\overline{AE}}{\overline{AB}} + P_{ABA}\cdot\left(\frac{\overline{AE}}{\overline{AB}}\right)^2 - P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}}}$$

$$\overset{simplify}{=} \frac{\left(\frac{\overline{AE}}{\overline{AB}} - 1\right)\cdot P_{ABA}}{P_{BOB} - P_{AOA} + P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}} - P_{ABA}} \overset{O}{=} \frac{P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}} - P_{ABA}}{P_{ABA}\cdot\frac{\overline{AE}}{\overline{AB}} - P_{ABA}} \overset{simplify}{=} 1$$

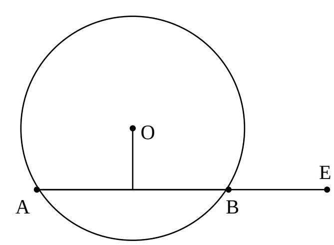


Figure 6-308

Example 6.309 (The Tangent Theorem) (0.017 3 4) *The product of the distances of a given point, from any two points which are collinear with the given point and lie on the circle, is equal to the square of the tangent line from the given point to the circle.*

Constructive description

((points *A T*)
(on *O* (b *A T*)) (tratio *P T O r*)
(inter *B* (l *A P*) (cir *O A*))
(eq-product *P T P T P A P B*))

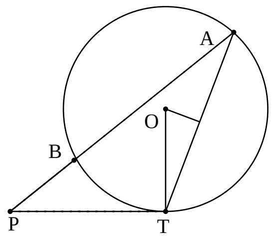


Figure 6-309

The machine proof

$$\frac{P_{TPT}}{P_{APB}}$$

$$\overset{B}{=} \frac{P_{TPT}\cdot P_{APA}}{P_{OPO}\cdot P_{APA}-P_{APA}\cdot P_{AOA}}$$

$$\overset{simplify}{=} \frac{P_{TPT}}{P_{OPO}-P_{AOA}}$$

$$\overset{P}{=} \frac{P_{TOT}\cdot r^2}{P_{TOT}\cdot r^2+P_{TOT}-P_{AOA}}$$

$$\overset{O}{=} \frac{P_{AOA}\cdot r^2}{P_{AOA}\cdot r^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$P_{APB} \overset{B}{=} P_{OPO}-P_{AOA}$

$P_{OPO} \overset{P}{=} (r^2+1)\cdot P_{TOT}$

$P_{TPT} \overset{P}{=} P_{TOT}\cdot (r)^2$

$P_{TOT} \overset{O}{=} P_{AOA}$

Example 6.310 (0.066, 3, 5) *From the ends D and C of a diameter of circle (O) perpendiculars are drawn to chord AB. Let E and F be the feet of the perpendiculars. Show that $OE = OF$.*

Constructive description

((points *A B D*)
(circumcenter *O A B D*)
(lratio *C O D* −1)
(foot *E C A B*)
(foot *F D A B*)
(midpoint *M A B*)
(midpoint *M E F*))

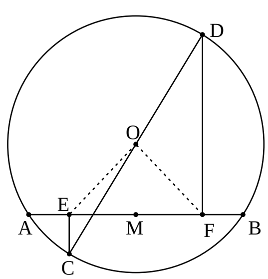


Figure 6-310

The machine proof

$$-\frac{\overline{EM}}{\overline{FM}}$$

$$\overset{M}{=} \frac{\frac{\overline{AE}}{\overline{AB}}-\frac{1}{2}}{-\frac{\overline{AF}}{\overline{AB}}+\frac{1}{2}}$$

$$\overset{F}{=} \frac{-(\frac{\overline{AE}}{\overline{AB}}-\frac{1}{2})\cdot P_{ABA}}{P_{BAD}-\frac{1}{2}P_{ABA}}$$

$$\overset{E}{=} \frac{-(P_{BAC}-\frac{1}{2}P_{ABA})\cdot P_{ABA}}{(P_{BAD}-\frac{1}{2}P_{ABA})\cdot P_{ABA}}$$

$$\overset{simplify}{=} \frac{-(P_{BAC}-\frac{1}{2}P_{ABA})}{P_{BAD}-\frac{1}{2}P_{ABA}}$$

$$\overset{C}{=} \frac{-(2P_{BAO}-P_{BAD}-\frac{1}{2}P_{ABA})}{P_{BAD}-\frac{1}{2}P_{ABA}}$$

$$\overset{O}{=} \frac{(-2)\cdot(-P_{BAD}+\frac{1}{2}P_{ABA})}{(P_{BAD}-\frac{1}{2}P_{ABA})\cdot(2)}$$

$$\overset{simplify}{=} 1$$

The eliminants

$\frac{\overline{EM}}{\overline{FM}} \overset{M}{=} \frac{\frac{\overline{AE}}{\overline{AB}}-\frac{1}{2}}{\frac{\overline{AF}}{\overline{AB}}-\frac{1}{2}}$

$\frac{\overline{AF}}{\overline{AB}} \overset{F}{=} \frac{P_{BAD}}{P_{ABA}}$

$\frac{\overline{AE}}{\overline{AB}} \overset{E}{=} \frac{P_{BAC}}{P_{ABA}}$

$P_{BAC} \overset{C}{=} 2(P_{BAO}-\frac{1}{2}P_{BAD})$

$P_{BAO} \overset{O}{=} \frac{1}{2}(P_{ABA})$

Example 6.311 *Let A, B, C, D, E and P be six cyclic points. From P perpendicular lines are drawn to AB, BC, CD, DE, and EA respectively. Let the foot be $L, M, N,, K$, and S. Show that* $\frac{\overline{AL}}{\overline{LB}} \cdot \frac{\overline{BM}}{\overline{MC}} \cdot \frac{\overline{CN}}{\overline{ND}} \cdot \frac{\overline{DK}}{\overline{KE}} \cdot \frac{\overline{ES}}{\overline{SA}} = -1.$

Constructive description
((circle $A\ B\ C\ D\ E\ P$)
(foot $L\ P\ A\ B$)
(foot $M\ P\ B\ C$)
(foot $N\ P\ C\ D$)
(foot $K\ P\ D\ E$)
(foot $S\ P\ E\ A$)
$(\frac{\overline{AL}}{\overline{LB}} \frac{\overline{BM}}{\overline{MC}} \frac{\overline{CN}}{\overline{ND}} \frac{\overline{DK}}{\overline{KE}} \frac{\overline{ES}}{\overline{SA}} = -1)$)

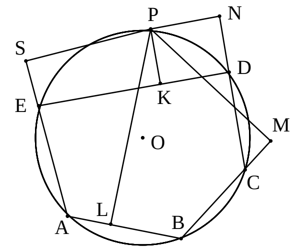


Figure 6-311

The machine proof

$$\frac{\overline{ES}}{\overline{AS}}\cdot\frac{\overline{DK}}{\overline{EK}}\cdot\frac{\overline{CN}}{\overline{DN}}\cdot\frac{\overline{BM}}{\overline{CM}}\cdot\frac{\overline{AL}}{\overline{BL}}$$

$$\stackrel{S}{=} \frac{P_{AEP}}{-P_{EAP}} \cdot \frac{\overline{DK}}{\overline{EK}}\cdot\frac{\overline{CN}}{\overline{DN}}\cdot\frac{\overline{BM}}{\overline{CM}}\cdot\frac{\overline{AL}}{\overline{BL}}$$

$$\stackrel{K}{=} \frac{-P_{AEP}\cdot P_{EDP}}{P_{EAP}\cdot(-P_{DEP})} \cdot \frac{\overline{CN}}{\overline{DN}}\cdot\frac{\overline{BM}}{\overline{CM}}\cdot\frac{\overline{AL}}{\overline{BL}}$$

$$\stackrel{N}{=} \frac{P_{AEP}\cdot P_{EDP}\cdot P_{DCP}}{P_{EAP}\cdot P_{DEP}\cdot(-P_{CDP})} \cdot \frac{\overline{BM}}{\overline{CM}}\cdot\frac{\overline{AL}}{\overline{BL}}$$

$$\stackrel{M}{=} \frac{-P_{AEP}\cdot P_{EDP}\cdot P_{DCP}\cdot P_{CBP}}{P_{EAP}\cdot P_{DEP}\cdot P_{CDP}\cdot(-P_{BCP})} \cdot \frac{\overline{AL}}{\overline{BL}}$$

$$\stackrel{L}{=} \frac{P_{AEP}\cdot P_{EDP}\cdot P_{DCP}\cdot P_{CBP}\cdot P_{BAP}}{P_{EAP}\cdot P_{DEP}\cdot P_{CDP}\cdot P_{BCP}\cdot(-P_{ABP})}$$

$$\stackrel{co-cir}{=} \frac{-(-2\widetilde{EP}\cdot\widetilde{AE}\cdot\cos(AP))\cdot(2\widetilde{DP}\cdot\widetilde{DE}\cdot\cos(EP))}{(2\widetilde{AP}\cdot\widetilde{AE}\cdot\cos(EP))\cdot(-2\widetilde{EP}\cdot\widetilde{DE}\cdot\cos(DP))}$$
$$\frac{(2\widetilde{CP}\cdot\widetilde{CD}\cdot\cos(DP))\cdot(2\widetilde{BP}\cdot\widetilde{BC}\cdot\cos(CP))\cdot(2\widetilde{AP}\cdot\widetilde{AB}\cdot\cos(BP))}{(-2\widetilde{DP}\cdot\widetilde{CD}\cdot\cos(CP))\cdot(-2\widetilde{CP}\cdot\widetilde{BC}\cdot\cos(BP))\cdot(-2\widetilde{BP}\cdot\widetilde{AB}\cdot\cos(AP))}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{ES}}{\overline{AS}} \stackrel{S}{=} \frac{P_{AEP}}{-P_{EAP}}, \quad \frac{\overline{DK}}{\overline{EK}} \stackrel{K}{=} \frac{P_{EDP}}{-P_{DEP}}$$
$$\frac{\overline{CN}}{\overline{DN}} \stackrel{N}{=} \frac{P_{DCP}}{-P_{CDP}}, \quad \frac{\overline{BM}}{\overline{CM}} \stackrel{M}{=} \frac{P_{CBP}}{-P_{BCP}}$$
$$\frac{\overline{AL}}{\overline{BL}} \stackrel{L}{=} \frac{P_{BAP}}{-P_{ABP}}$$
$$P_{ABP} = -2(\widetilde{BP}\cdot\widetilde{AB}\cdot\cos(AP))$$
$$P_{BCP} = -2(\widetilde{CP}\cdot\widetilde{BC}\cdot\cos(BP))$$
$$P_{CDP} = -2(\widetilde{DP}\cdot\widetilde{CD}\cdot\cos(CP))$$
$$P_{DEP} = -2(\widetilde{EP}\cdot\widetilde{DE}\cdot\cos(DP))$$
$$P_{EAP} = 2(\widetilde{AP}\cdot\widetilde{AE}\cdot\cos(EP))$$
$$P_{BAP} = 2(\widetilde{AP}\cdot\widetilde{AB}\cdot\cos(BP))$$
$$P_{CBP} = 2(\widetilde{BP}\cdot\widetilde{BC}\cdot\cos(CP))$$
$$P_{DCP} = 2(\widetilde{CP}\cdot\widetilde{CD}\cdot\cos(DP))$$
$$P_{EDP} = 2(\widetilde{DP}\cdot\widetilde{DE}\cdot\cos(EP))$$
$$P_{AEP} = -2(\widetilde{EP}\cdot\widetilde{AE}\cdot\cos(AP))$$

Example 6.312 (0.416, 4, 22) *If $ABCD$ is a rectangle inscribed in a circle, center O, and if PX, PX_1, PY PY_1 are the perpendiculars from any point P upon the sides AB, CD, AD, BC, prove that $PX \cdot PX_1 + PY \cdot PY_1$ is equal to the power of P with respect to the circle O.*

Constructive description
((points $A\ B\ P$) (on D (t $A\ A\ B$))
(inter C (p $B\ A\ D$) (p $D\ A\ B$))
(inter O (l $B\ D$) (l $A\ C$))
(foot $X\ P\ A\ B$)
(inter X_1 (l $C\ D$) (l $P\ X$))
(foot $Y\ P\ A\ D$)
(inter Y_1 (l $B\ C$) (l $Y\ P$))
$(\overline{OP}^2 - \overline{OA}^2 = \frac{1}{2}P_{XPX_1} + \frac{1}{2}P_{YPY_1})$)

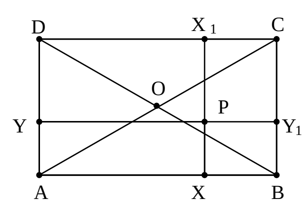


Figure 6-312

Example 6.313 (0.150, 3, 15) *Let D be a point on the side CB of a right triangle ABC such that the circle (O) with diameter CD touches the hypotenuse AB at E. Let $F = AC \cap DE$. Show that $AF = AE$.*

Constructive description
((points C E)
(on D (t E E C))
(midpoint O D C)
(inter A (t E E O) (t C C O))
(inter B (l D C) (l A E))
(inter F (l D E) (l A C))
(eqdistance A F A E))

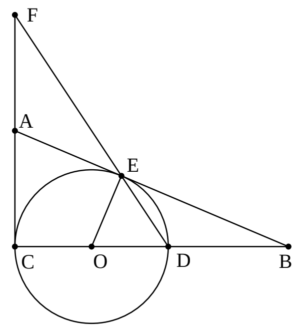


Figure 6-313

The eliminants

$P_{AFA} \stackrel{F}{=} \frac{P_{CAC}\cdot(S_{EDA})^2}{(S_{CEAD})^2}$

$S_{CEAD} \stackrel{A}{=} \frac{P_{DEO}\cdot P_{ECO}-16S_{CEO}\cdot S_{CED}}{(-16)\cdot S_{CEO}}$

$P_{EAE} \stackrel{A}{=} \frac{P_{EOE}\cdot(P_{ECO})^2}{(16)\cdot(S_{CEO})^2}$

$S_{EDA} \stackrel{A}{=} \frac{P_{DEO}\cdot P_{ECO}}{(16)\cdot S_{CEO}}$

$P_{CAC} \stackrel{A}{=} \frac{P_{COC}\cdot(P_{CEO})^2}{(16)\cdot(S_{CEO})^2}$

$S_{CEO} \stackrel{O}{=} \frac{1}{2}(S_{CED})$

$P_{ECO} \stackrel{O}{=} \frac{1}{2}(P_{ECD})$

$P_{EOE} \stackrel{O}{=} \frac{1}{4}(2P_{EDE}-P_{CDC}+2P_{CEC})$

$P_{DEO} \stackrel{O}{=} \frac{1}{2}(P_{EDE})$

$P_{CEO} \stackrel{O}{=} \frac{1}{2}(P_{CEC})$

$P_{COC} \stackrel{O}{=} \frac{1}{4}(P_{CDC})$

$S_{CED} \stackrel{D}{=} -\frac{1}{4}(P_{CEC}\cdot\frac{\overline{DE}}{\overline{EC}})$

$P_{ECD} \stackrel{D}{=} P_{CEC}$

$P_{EDE} \stackrel{D}{=} P_{CEC}\cdot(\frac{\overline{DE}}{\overline{EC}})^2$

$P_{CDC} \stackrel{D}{=} (\frac{\overline{DE}}{\overline{EC}}^2+1)\cdot P_{CEC}$

The machine proof

$$\frac{P_{AFA}}{P_{EAE}}$$

$$\stackrel{F}{=} \frac{P_{CAC}\cdot S_{EDA}^2}{P_{EAE}\cdot S_{CEAD}^2}$$

$$\stackrel{A}{=} \frac{P_{COC}\cdot P_{CEO}^2\cdot((-\frac{1}{4}P_{DEO}\cdot P_{ECO}))^2\cdot((-4S_{CEO}))^2\cdot(16S_{CEO}^2)}{P_{EOE}\cdot P_{ECO}^2\cdot(\frac{1}{4}P_{DEO}\cdot P_{ECO}-4S_{CEO}\cdot S_{CED})^2\cdot((-4S_{CEO}))^2\cdot(16S_{CEO}^2)}$$

$$\stackrel{simplify}{=} \frac{P_{COC}\cdot(P_{CEO})^2\cdot(P_{DEO})^2}{P_{EOE}\cdot(P_{DEO}\cdot P_{ECO}-16S_{CEO}\cdot S_{CED})^2}$$

$$\stackrel{O}{=} \frac{(\frac{1}{4}P_{CDC})\cdot((\frac{1}{2}P_{CEC}))^2\cdot((\frac{1}{2}P_{EDE}))^2}{(\frac{1}{2}P_{EDE}-\frac{1}{4}P_{CDC}+\frac{1}{2}P_{CEC})\cdot(\frac{1}{4}P_{EDE}\cdot P_{ECD}-8S_{CED}^2)^2}$$

$$\stackrel{D}{=} \frac{(P_{CEC}\cdot\frac{\overline{DE}}{\overline{EC}}^2+P_{CEC})\cdot(P_{CEC})^2\cdot(P_{CEC}\cdot\frac{\overline{DE}}{\overline{EC}}^2)^2}{(P_{CEC}\cdot\frac{\overline{DE}}{\overline{EC}}^2+P_{CEC})\cdot((-P_{CEC}^2\cdot\frac{\overline{DE}}{\overline{EC}}^2))^2} \stackrel{simplify}{=} 1$$

Example 6.314 (3.250, 21, 23) *Show that the sum of the powers, with respect to the circumcircle of a triangle, of the symmetries of the orthocenter with respect to the vertices of the triangle is equal to the sum of the squares of the sides of the triangle.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot D A B C)
(foot F C A B)
(foot E B A C)

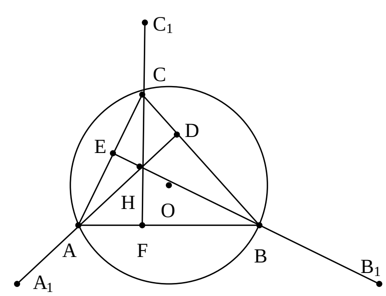


Figure 6-314

(inter H (l B E) (l A D))
(lratio A_1 A H -1)
(lratio B_1 B H -1)
(lratio C_1 C H -1)
($\overline{A_1O}^2 - \overline{OB}^2 + \overline{B_1O}^2 - \overline{OB}^2 + \overline{C_1O}^2 - \overline{OB}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2$))

Example 6.315 (0.700, 17, 17) *Two unequal circles are tangent internally at A. The tangent to the smaller circle at a point B meets the larger circle in C, D. Show that AB bisects the angle CAD.*

Constructive description
((points A B C)
(inter O_1 (t B B C) (b A B))
(inter O_2 (l A O_1) (b A C))
(inter D (l B C) (cir O_2 C))
(eqangle C A B B A D))

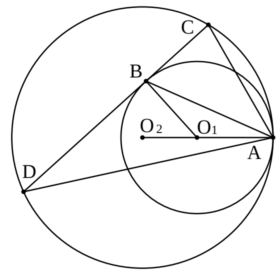


Figure 6-315

Example 6.316 (1.850, 65, 19) *If G is the centroid of a triangle ABC, show that the powers of the vertices A, B, C for the circles GBC, GCA, GAB, respectively, are equal.*

Constructive description
((points A B C)
(centroid G A B C)
(circumcenter O_1 B C G)
(circumcenter O_2 A C G)
($\overline{AO_1}^2 - \overline{O_1C}^2 = \overline{BO_2}^2 - \overline{O_2C}^2$))

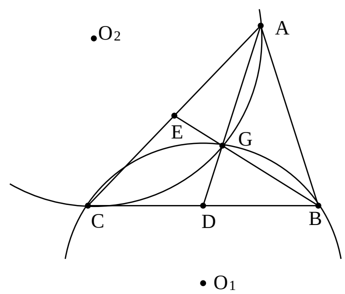


Figure 6-316

Example 6.317 (0.083, 4, 10) *Let C be a point on a chord AB of circle O. Let D and E be the intersections of perpendicular of OC through C with the two tangents of the circle at A and B, respectively. Show that $CE = CD$.*

Constructive description
((points A B)
(on O (b A B))
(lratio C A B r)
(tratio X B O s)
(inter E (l B X) (t C C O))
(inter D (l C E) (t A A O))
(midpoint C D E))

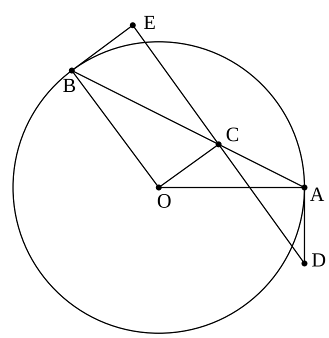


Figure 6-317

The machine proof

$-\frac{\overline{CD}}{\overline{CE}}$

$\overset{D}{=} \frac{-P_{OAC}}{-P_{OCAE}}$

$\overset{E}{=} \frac{P_{OAC} \cdot P_{CBOX}}{-P_{CBOX} \cdot P_{OAC} - P_{OCX} \cdot P_{BAO} + P_{OAX} \cdot P_{BCO}}$

$\overset{X}{=} \frac{P_{OAC} \cdot (4S_{BOC} \cdot s)}{-(4P_{OAC} \cdot S_{BOC} \cdot s + 4P_{BCO} \cdot S_{ABO} \cdot s - 4P_{BAO} \cdot S_{BOC} \cdot s)}$

$\overset{simplify}{=} \frac{-P_{OAC} \cdot S_{BOC}}{P_{OAC} \cdot S_{BOC} + P_{BCO} \cdot S_{ABO} - P_{BAO} \cdot S_{BOC}}$

$\overset{C}{=} \frac{-P_{BAO} \cdot r \cdot (-S_{ABO} \cdot r + S_{ABO})}{-P_{BAO} \cdot S_{ABO} \cdot r^2 + P_{BAO} \cdot S_{ABO} \cdot r + P_{ABA} \cdot S_{ABO} \cdot r^2 - P_{ABA} \cdot S_{ABO} \cdot r}$

$\overset{simplify}{=} \frac{-P_{BAO}}{P_{BAO} - P_{ABA}}$

$\overset{py}{=} \frac{-(-P_{BOB} + P_{AOA} + P_{ABA}) \cdot (2)}{(-P_{BOB} + P_{AOA} - P_{ABA}) \cdot (2)}$

$\overset{O}{=} \frac{P_{ABA}}{P_{ABA}}$

$\overset{simplify}{=} 1$

The eliminants

$\frac{\overline{CD}}{\overline{CE}} \overset{D}{=} \frac{P_{OAC}}{-P_{OCAE}}$

$P_{OCAE} \overset{E}{=} \frac{-(P_{CBOX} \cdot P_{OAC} + P_{OCX} \cdot P_{BAO} - P_{OAX} \cdot P_{BCO})}{P_{CBOX}}$

$P_{OAX} \overset{X}{=} P_{BAO} - 4S_{ABO} \cdot s$

$P_{OCX} \overset{X}{=} P_{BCO} - 4S_{BOC} \cdot s$

$P_{CBOX} \overset{X}{=} 4(S_{BOC} \cdot s)$

$P_{BCO} \overset{C}{=} -((r-1) \cdot (P_{BAO} - P_{ABA} \cdot r))$

$S_{BOC} \overset{C}{=} -((r-1) \cdot S_{ABO})$

$P_{OAC} \overset{C}{=} P_{BAO} \cdot r, \; P_{BOB} \overset{O}{=} P_{AOA}$

$P_{BAO} = -\frac{1}{2}(P_{BOB} - P_{AOA} - P_{ABA})$

Example 6.318 (0.083, 5, 14) *Let G be a point on the circle (O) with diameter BC, A be the midpoint of the arc BG. $AD \perp BC$. $E = AD \cap BG$ and $F = AC \cap BG$. Show that $AE = BE (= EF)$.*

Constructive description
((points G B)
(midpoint M B G)
(tratio A M B r)
(inter O (l A M) (b A B))
(lratio C O B -1)
(foot D A B C)
(inter E (l A D) (l B G))
(inter F (l A C) (l B G))
(eqdistance A E B E))

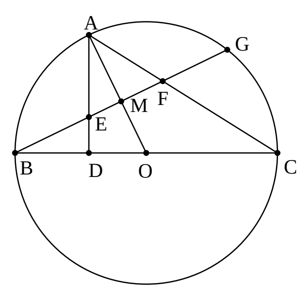


Figure 6-318

Example 6.319 (0.050, 2, 6) *Let D be the intersection of one of the bisectors of $\angle A$ of triangle ABC with side BC, E be the intersection of AD with the circumcircle of ABC. Show that $AB \cdot AC = AD \cdot AE$.*

Constructive description

((circle $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(lratio $D\ B\ C\ \frac{\widetilde{AB}}{\widetilde{AB}+\widetilde{AC}}$)
(inter E (l $A\ D$) (cir $O\ A$))
($P_{DAAE} = 2\widetilde{AB}\cdot\widetilde{AC}$))

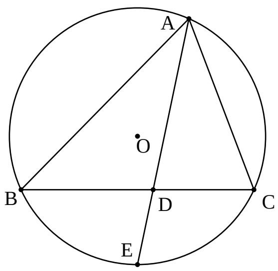


Figure 6-319

The machine proof

$$\frac{P_{DAE}}{(2)\cdot\widetilde{AC}\cdot\widetilde{AB}}$$
$$\stackrel{E}{=} \frac{2P_{OAD}\cdot P_{ADA}}{(2)\cdot\widetilde{AC}\cdot\widetilde{AB}\cdot P_{ADA}}$$
$$\stackrel{simplify}{=} \frac{P_{OAD}}{\widetilde{AC}\cdot\widetilde{AB}}$$
$$\stackrel{D}{=} \frac{\widetilde{AC}\cdot P_{BAO}+\widetilde{AB}\cdot P_{CAO}}{\widetilde{AC}\cdot\widetilde{AB}\cdot(\widetilde{AC}+\widetilde{AB})}$$
$$\stackrel{O}{=} \frac{2\widetilde{AC}\cdot P_{ABA}+2\widetilde{AB}\cdot P_{ACA}}{\widetilde{AC}\cdot\widetilde{AB}\cdot(\widetilde{AC}+\widetilde{AB})\cdot(2)^2}$$
$$\stackrel{co-cir}{=} \frac{2\widetilde{AC}^2\cdot\widetilde{AB}+2\widetilde{AC}\cdot\widetilde{AB}^2}{(2)\cdot\widetilde{AC}\cdot\widetilde{AB}\cdot(\widetilde{AC}+\widetilde{AB})}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{DAE} \stackrel{E}{=} (2)\cdot P_{OAD}$$
$$P_{OAD} \stackrel{D}{=} \frac{\widetilde{AC}\cdot P_{BAO}+\widetilde{AB}\cdot P_{CAO}}{\widetilde{AC}+\widetilde{AB}}$$
$$P_{CAO} \stackrel{O}{=} \frac{1}{2}(P_{ACA})$$
$$P_{BAO} \stackrel{O}{=} \frac{1}{2}(P_{ABA})$$
$$P_{ACA} = 2(\widetilde{AC}^2)$$
$$P_{ABA} = 2(\widetilde{AB}^2)$$

Example 6.320 (0.067, 5, 12) *Let ABC be a triangle. Through A a line is drawn tangent to the circle with diameter BC at D. Let E be a point on AB such that $AD = AE$. The perpendicular to AB at E meets AC at F. Show that $AE/AB = AC/AF$.*

Constructive description

((points $E\ D$)
(on A (b $E\ D$))
(on B (l $A\ E$))
(tratio $O_O\ D\ A\ 1$)
(inter O (l $O_O\ D$) (b $B\ D$))
(lratio $C\ O\ B\ -1$)
(inter F (l $A\ C$) (t $E\ E\ A$))
(eq-product $A\ B\ A\ C\ A\ E\ A\ F$))

The eliminants

$$\frac{\overline{AF}}{\overline{AC}} \stackrel{F}{=} \frac{P_{EAE}}{P_{EAC}}$$
$$P_{EAC} \stackrel{C}{=} 2P_{EAO}-P_{EAB}$$
$$P_{EAO} \stackrel{O}{=} \frac{2P_{DBO_O}\cdot P_{EAD}-P_{DBD}\cdot P_{EAO_O}-P_{DBD}\cdot P_{EAD}}{(-2)\cdot P_{BDO_O}}$$
$$P_{EAO_O} \stackrel{O_O}{=} P_{EAD}+4S_{EDA}$$
$$P_{DBO_O} \stackrel{O_O}{=} P_{DBD}-4S_{DAB}$$
$$P_{BDO_O} \stackrel{O_O}{=} 4(S_{DAB})$$
$$S_{DAB} \stackrel{B}{=} S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}}$$
$$P_{EAB} \stackrel{B}{=} P_{EAE}\cdot\frac{\overline{AB}}{\overline{AE}}$$
$$P_{DBD} \stackrel{B}{=} -\left(P_{DAD}\cdot\frac{\overline{AB}}{\overline{AE}}-P_{DAD}-P_{EAE}\cdot\frac{\overline{AB}}{\overline{AE}}^2+P_{EAE}\cdot\frac{\overline{AB}}{\overline{AE}}-P_{EDE}\cdot\frac{\overline{AB}}{\overline{AE}}\right)$$
$$\frac{\overline{AB}}{\overline{EA}} \stackrel{B}{=} -\left(\frac{\overline{AB}}{\overline{AE}}\right)$$
$$P_{EAD} = \frac{1}{2}(P_{DAD}+P_{EAE}-P_{EDE})$$
$$P_{DAD} \stackrel{A}{=} P_{EAE}$$

The machine proof

$$\left(-\frac{\overline{AB}}{\overline{EA}}\right)/\left(\frac{\overline{AF}}{\overline{AC}}\right)$$
$$\stackrel{F}{=} \frac{P_{EAC}}{P_{EAE}}\cdot -\frac{\overline{AB}}{\overline{EA}}$$
$$\stackrel{C}{=} \frac{-(2P_{EAO}-P_{EAB})}{P_{EAE}}\cdot\frac{\overline{AB}}{\overline{EA}}$$
$$\stackrel{O}{=} \frac{-(P_{BDO_O}\cdot P_{EAB}+2P_{DBO_O}\cdot P_{EAD}-P_{DBD}\cdot P_{EAO_O}-P_{DBD}\cdot P_{EAD})}{P_{EAE}\cdot(-P_{BDO_O})}\cdot\frac{\overline{AB}}{\overline{EA}}$$

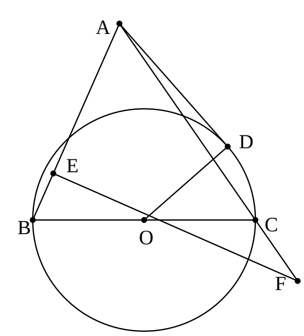


Figure 6-320

$\stackrel{O_O}{=} \frac{-4P_{DBD}\cdot S_{EDA}+4P_{EAB}\cdot S_{DAB}-8P_{EAD}\cdot S_{DAB}}{P_{EAE}\cdot(4S_{DAB})} \cdot \frac{\overline{AB}}{\overline{EA}}$

$\stackrel{B}{=} \frac{-\frac{\overline{AB}}{\overline{AE}}\cdot(-P_{DAD}\cdot S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}}+P_{DAD}\cdot S_{EDA}+2P_{EAD}\cdot S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}}-P_{EAE}\cdot S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}}+P_{EDE}\cdot S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}})}{P_{EAE}\cdot S_{EDA}\cdot\frac{\overline{AB}}{\overline{AE}}\cdot(-1)}$

$\stackrel{simplify}{=} \frac{-(P_{DAD}\cdot\frac{\overline{AB}}{\overline{AE}}-P_{DAD}-2P_{EAD}\cdot\frac{\overline{AB}}{\overline{AE}}+P_{EAE}\cdot\frac{\overline{AB}}{\overline{AE}}-P_{EDE}\cdot\frac{\overline{AB}}{\overline{AE}})}{P_{EAE}}$

$\stackrel{py}{=} \frac{-(-2P_{DAD})}{P_{EAE}\cdot(2)}$

$\stackrel{A}{=} \frac{P_{EAE}}{P_{EAE}}$

$\stackrel{simplify}{=} 1$

Example 6.321 (0.600, 2, 40) *The cross ratio of four points on a circles with respect to any points on the circle is constant.*

Constructive description
(circle $A\ E\ E_1\ B\ C\ D$)
(inter G (l $A\ B$) (l $C\ E$))
(inter F (l $D\ E$) (l $A\ B$))
(inter G_1 (l $E_1\ C$) (l $A\ B$))
(inter F_1 (l $E_1\ D$) (l $A\ B$))
($\frac{\overline{AF}}{\overline{BF}}\cdot\frac{\overline{BG}}{\overline{AG}} = \frac{\overline{AF_1}}{\overline{BF_1}}\cdot\frac{\overline{BG_1}}{\overline{AG_1}}$))

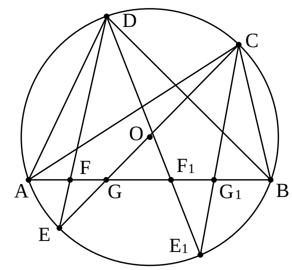


Figure 6-321

The machine proof

$\frac{\frac{\overline{BG}}{\overline{AG}}\cdot\frac{\overline{AF}}{\overline{BF}}}{\frac{\overline{BG_1}}{\overline{AG_1}}\cdot\frac{\overline{AF_1}}{\overline{BF_1}}}$

$\stackrel{F_1}{=} \frac{-S_{E_1BD}}{\frac{\overline{BG_1}}{\overline{AG_1}}\cdot S_{AE_1D}} \cdot \frac{\overline{BG}}{\overline{AG}}\cdot\frac{\overline{AF}}{\overline{BF}}$

$\stackrel{G_1}{=} \frac{-S_{E_1BD}\cdot S_{AE_1C}}{(-S_{E_1BC})\cdot S_{AE_1D}} \cdot \frac{\overline{BG}}{\overline{AG}}\cdot\frac{\overline{AF}}{\overline{BF}}$

$\stackrel{F}{=} \frac{(-S_{AED})\cdot S_{E_1BD}\cdot S_{AE_1C}}{S_{E_1BC}\cdot S_{AE_1D}\cdot S_{EBD}} \cdot \frac{\overline{BG}}{\overline{AG}}$

$\stackrel{G}{=} \frac{-S_{EBC}\cdot S_{AED}\cdot S_{E_1BD}\cdot S_{AE_1C}}{S_{E_1BC}\cdot S_{AE_1D}\cdot S_{EBD}\cdot(-S_{AEC})}$

$\stackrel{co-cir}{=} \frac{(-\widetilde{BC}\cdot\widetilde{EC}\cdot\widetilde{EB})\cdot(-\widetilde{ED}\cdot\widetilde{AD}\cdot\widetilde{AE})\cdot(-\widetilde{BD}\cdot\widetilde{E_1D}\cdot\widetilde{E_1B})\cdot(-\widetilde{E_1C}\cdot\widetilde{AC}\cdot\widetilde{AE_1})\cdot((2d))^4}{(-\widetilde{BC}\cdot\widetilde{E_1C}\cdot\widetilde{E_1B})\cdot(-\widetilde{E_1D}\cdot\widetilde{AD}\cdot\widetilde{AE_1})\cdot(-\widetilde{BD}\cdot\widetilde{ED}\cdot\widetilde{EB})\cdot(-\widetilde{EC}\cdot\widetilde{AC}\cdot\widetilde{AE})\cdot((2d))^4}$

$\stackrel{simplify}{=} 1$

The eliminants

$\frac{\overline{AF_1}}{\overline{BF_1}} \stackrel{F_1}{=} \frac{S_{AE_1D}}{-S_{E_1BD}}$

$\frac{\overline{BG_1}}{\overline{AG_1}} \stackrel{G_1}{=} \frac{-S_{E_1BC}}{S_{AE_1C}}$

$\frac{\overline{AF}}{\overline{BF}} \stackrel{F}{=} \frac{-S_{AED}}{S_{EBD}}$

$\frac{\overline{BG}}{\overline{AG}} \stackrel{G}{=} \frac{S_{EBC}}{-S_{AEC}}$

$S_{AEC}=\frac{\widetilde{EC}\cdot\widetilde{AC}\cdot\widetilde{AE}}{(-2)\cdot d}$

$S_{EBD}=\frac{\widetilde{BD}\cdot\widetilde{ED}\cdot\widetilde{EB}}{(-2)\cdot d}$

$S_{AE_1D}=\frac{\widetilde{E_1D}\cdot\widetilde{AD}\cdot\widetilde{AE_1}}{(-2)\cdot d}$

$S_{E_1BC}=\frac{\widetilde{BC}\cdot\widetilde{E_1C}\cdot\widetilde{E_1B}}{(-2)\cdot d}$

$S_{AE_1C}=\frac{\widetilde{E_1C}\cdot\widetilde{AC}\cdot\widetilde{AE_1}}{(-2)\cdot d}$

$S_{E_1BD}=\frac{\widetilde{BD}\cdot\widetilde{E_1D}\cdot\widetilde{E_1B}}{(-2)\cdot d}$

$S_{AED}=\frac{\widetilde{ED}\cdot\widetilde{AD}\cdot\widetilde{AE}}{(-2)\cdot d}$

$S_{EBC}=\frac{\widetilde{BC}\cdot\widetilde{EC}\cdot\widetilde{EB}}{(-2)\cdot d}$

Example 6.322 (0.083, 1, 17) *The tangent to a circle at the point C meets the diameter AB, produced, in T; Prove that the other tangent from T to the circle is divided harmonically by CA, CB, CT and its point of contact.*

Constructive description
((points A C)
(on B (t C C A))
(midpoint O A B)
(foot E C A O)
(lratio D E C -1)
(inter T (l A O) (t C C O))
(inter A_1 (l T D) (l C A))
(inter B_1 (l T D) (l C B))
(harmonic T D A_1 B_1))

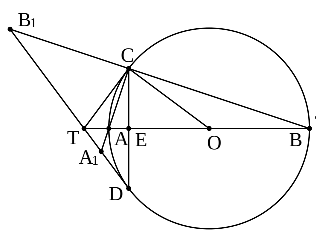


Figure 6-322

The machine proof

$$(-\frac{\overline{TA_1}}{\overline{DA_1}})/(\frac{\overline{TB_1}}{\overline{DB_1}})$$

$$\stackrel{B_1}{=} \frac{S_{CBD}}{S_{CBT}} \cdot -\frac{\overline{TA_1}}{\overline{DA_1}}$$

$$\stackrel{A_1}{=} \frac{-(-S_{ACT}) \cdot S_{CBD}}{S_{CBT} \cdot (-S_{ACD})}$$

$$\stackrel{T}{=} \frac{-P_{ACO} \cdot S_{ACO} \cdot S_{CBD} \cdot (-P_{AOC})}{(-P_{BCO} \cdot S_{ACO}) \cdot S_{ACD} \cdot (-P_{AOC})}$$

$$\stackrel{simplify}{=} \frac{P_{ACO} \cdot S_{CBD}}{P_{BCO} \cdot S_{ACD}}$$

$$\stackrel{D}{=} \frac{P_{ACO} \cdot (2S_{CBE})}{P_{BCO} \cdot (2S_{ACE})}$$

$$\stackrel{E}{=} \frac{P_{ACO} \cdot P_{ABC} \cdot S_{ACB} \cdot P_{AOA}}{P_{BCO} \cdot P_{CAO} \cdot S_{ACO} \cdot P_{ABA}}$$

$$\stackrel{O}{=} \frac{(\frac{1}{2}P_{ACA}) \cdot P_{ABC} \cdot S_{ACB} \cdot (\frac{1}{4}P_{ABA})}{(\frac{1}{2}P_{CBC}) \cdot (\frac{1}{2}P_{CAB}) \cdot (\frac{1}{2}S_{ACB}) \cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{P_{ACA} \cdot P_{ABC}}{P_{CBC} \cdot P_{CAB}}$$

$$\stackrel{py}{=} \frac{P_{ACA} \cdot (P_{CBC}+P_{ABA}-P_{ACA}) \cdot (2)}{P_{CBC} \cdot (-P_{CBC}+P_{ABA}+P_{ACA}) \cdot (2)}$$

$$\stackrel{B}{=} \frac{-2P_{CBC} \cdot P_{ACA}}{-2P_{CBC} \cdot P_{ACA}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{TB_1}}{\overline{DB_1}} \stackrel{B_1}{=} \frac{S_{CBT}}{S_{CBD}}$$
$$\frac{\overline{TA_1}}{\overline{DA_1}} \stackrel{A_1}{=} \frac{S_{ACT}}{S_{ACD}}$$
$$S_{CBT} \stackrel{T}{=} \frac{P_{BCO} \cdot S_{ACO}}{P_{AOC}}$$
$$S_{ACT} \stackrel{T}{=} \frac{P_{ACO} \cdot S_{ACO}}{-P_{AOC}}$$
$$S_{ACD} \stackrel{D}{=} 2(S_{ACE})$$
$$S_{CBD} \stackrel{D}{=} 2(S_{CBE})$$
$$S_{ACE} \stackrel{E}{=} \frac{P_{CAO} \cdot S_{ACO}}{P_{AOA}}$$
$$S_{CBE} \stackrel{E}{=} \frac{P_{ABC} \cdot S_{ACB}}{P_{ABA}}$$
$$S_{ACO} \stackrel{O}{=} \frac{1}{2}(S_{ACB})$$
$$P_{CAO} \stackrel{O}{=} \frac{1}{2}(P_{CAB})$$
$$P_{BCO} \stackrel{O}{=} \frac{1}{2}(P_{CBC})$$
$$P_{AOA} \stackrel{O}{=} \frac{1}{4}(P_{ABA})$$
$$P_{ACO} \stackrel{O}{=} \frac{1}{2}(P_{ACA})$$
$$P_{CAB} = \frac{(P_{CBC}-P_{ABA}-P_{ACA})}{-2}$$
$$P_{ABC} = \frac{1}{2}(P_{CBC}+P_{ABA}-P_{ACA})$$
$$P_{ABA} \stackrel{B}{=} P_{CBC}+P_{ACA}$$

Example 6.323 (0.650, 10, 19) *Let R be a point on the circle with diameter AB. At a point P of AB a perpendicular is drawn meeting BR at N, AR at M, and meeting the circle at Q. Show that $PQ^2 = PM \cdot PN$.*

Constructive description
((points A Q R)
(circumcenter O A Q R)
(lratio B O A -1)
(foot P Q A O)
(inter M (l P Q) (l A R))
(inter N (l P Q) (l R B))
(eq-product P M P N P Q P Q))

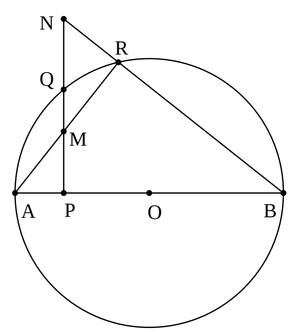


Figure 6-323

Example 6.324 (0.066, 3, 12) *Through point F on the circle with diameter AB a tangent to the circle is drawn meeting the two lines, perpendicular to AB at A and B, at D and E. Show the $OA^2 = DF \cdot EF$.*

Constructive description

((points A F)
(on B (t F F A))
(midpoint O A B)
(on M (t F F O))
(inter D (l F M) (t A A B))
(inter E (l F M) (t B B A))
(eq-product D F F E O A O A))

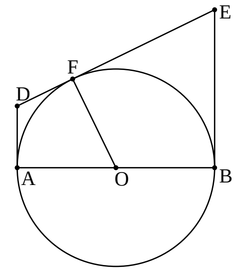


Figure 6-324

The machine proof

$$\frac{-P_{DFE}}{P_{AOA}}$$
$$\overset{E}{=} \frac{-P_{MFD}\cdot P_{ABF}}{P_{AOA}\cdot P_{BFAM}}$$
$$\overset{D}{=} \frac{-P_{FMF}\cdot P_{FAB}\cdot P_{ABF}}{P_{AOA}\cdot P_{BFAM}\cdot(-P_{BFAM})}$$
$$\overset{M}{=} \frac{P_{FOF}\cdot \frac{\overline{MF}}{\overline{FO}}^2\cdot P_{FAB}\cdot P_{ABF}}{P_{AOA}\cdot((-4S_{AFB}\cdot\frac{\overline{MF}}{\overline{FO}}))^2}$$
$$\overset{simplify}{=} \frac{P_{FOF}\cdot P_{FAB}\cdot P_{ABF}}{(16)\cdot P_{AOA}\cdot(S_{AFB})^2}$$
$$\overset{O}{=} \frac{(\frac{1}{2}P_{FBF}-\frac{1}{4}P_{ABA}+\frac{1}{2}P_{AFA})\cdot P_{FAB}\cdot P_{ABF}}{(16)\cdot(\frac{1}{4}P_{ABA})\cdot(S_{AFB})^2}$$
$$\overset{coor}{=} \frac{(2u_2^2+2u_1^2)\cdot(2u_1^2)\cdot(2u_2^2)\cdot((2))^2}{(16)\cdot(2u_2^2+2u_1^2)\cdot((-u_2\cdot u_1))^2}$$
$$\overset{simplify}{=} 1$$

The eliminants

$P_{DFE} \overset{E}{=} \frac{P_{MFD}\cdot P_{ABF}}{P_{BFAM}}$
$P_{MFD} \overset{D}{=} \frac{P_{FMF}\cdot P_{FAB}}{-P_{BFAM}}$
$P_{BFAM} \overset{M}{=} -4(S_{AFB}\cdot\frac{\overline{MF}}{\overline{FO}})$
$P_{FMF} \overset{M}{=} P_{FOF}\cdot(\frac{\overline{MF}}{\overline{FO}})^2$
$P_{AOA} \overset{O}{=} \frac{1}{4}(P_{ABA})$
$P_{FOF} \overset{O}{=} \frac{1}{4}(2P_{FBF}-P_{ABA}+2P_{AFA})$
$S_{AFB} = -\frac{1}{2}(u_2\cdot u_1)$
$P_{ABF} = 2((u_2)^2)$
$P_{FAB} = 2((u_1)^2)$
$P_{AFA} = 2((u_1)^2)$
$P_{ABA} = 2(u_2^2+u_1^2)$
$P_{FBF} = 2((u_2)^2)$

In the above machine proof, $F = (0,0), A = (u_1, 0), B = (0, u_2)$.

Example 6.325 (0.066, 2, 12) *Let AD be the altitude on the hypotenuse BC of right triangle ABC. A circle passing through C and D meets AC at E. BE meets the circle at another point F. Show that $AF \perp BE$.*

Constructive description

((points A B)
(tratio C A B H)
(foot D A B C)
(midpoint M C D)
(tratio O M C I)
(inter E (l C A) (cir O C))
(inter F (l E B) (cir O E))
(perpendicular A F B E))

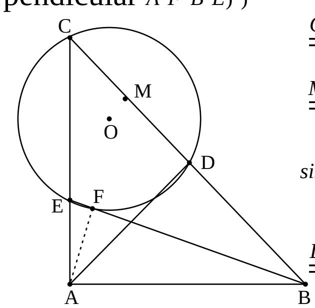


Figure 6-325

The machine proof

$$\frac{P_{ABE}}{P_{EBF}}$$
$$\overset{F}{=} \frac{P_{ABE}\cdot P_{BEB}}{-P_{OEO}\cdot P_{BEB}+P_{BEB}\cdot P_{BOB}}$$
$$\overset{simplify}{=} \frac{P_{ABE}}{-(P_{OEO}-P_{BOB})}$$
$$\overset{E}{=} \frac{P_{ABC}}{-(P_{COC}-P_{BOB})}$$
$$\overset{O}{=} \frac{-P_{ABC}}{P_{CMC}-P_{BMB}}$$
$$\overset{M}{=} \frac{-P_{ABC}}{-\frac{\overline{BC}}{\overline{CD}}^2\cdot P_{CDC}-\frac{\overline{BC}}{\overline{CD}}\cdot P_{CDC}}$$
$$\overset{simplify}{=} \frac{P_{ABC}}{(\frac{\overline{BC}}{\overline{CD}}+1)\cdot\frac{\overline{BC}}{\overline{CD}}\cdot P_{CDC}}$$
$$\overset{D}{=} \frac{P_{ABC}\cdot P_{BCB}\cdot(P_{ACB})^2}{(-P_{BCB}+P_{ACB})\cdot(-P_{BCB})\cdot P_{ACB}^2}$$
$$\overset{simplify}{=} \frac{P_{ABC}}{P_{BCB}-P_{ACB}} \overset{C}{=} \frac{P_{ABA}}{P_{ABA}} \overset{simplify}{=} 1$$

The eliminants

$P_{EBF} \overset{F}{=} -(P_{OEO}-P_{BOB})$
$P_{OEO} \overset{E}{=} P_{COC}$
$P_{ABE} \overset{E}{=} P_{ABC}$
$P_{BOB} \overset{O}{=} P_{CMC}\cdot I^2+P_{BMB}$
$P_{COC} \overset{O}{=} (I^2+1)\cdot P_{CMC}$
$P_{BMB} \overset{M}{=} \frac{1}{4}((2\frac{\overline{BC}}{\overline{CD}}+1)^2\cdot P_{CDC})$
$P_{CMC} \overset{M}{=} \frac{1}{4}(P_{CDC})$
$P_{CDC} \overset{D}{=} \frac{(P_{ACB})^2}{P_{BCB}}$
$\frac{\overline{BC}}{\overline{CD}} \overset{D}{=} \frac{-P_{BCB}}{P_{ACB}}$
$P_{ACB} \overset{C}{=} P_{ABA}\cdot(r)^2$
$P_{BCB} \overset{C}{=} (r^2+1)\cdot P_{ABA}$
$P_{ABC} \overset{C}{=} P_{ABA}$

Example 6.326 (0.067, 5, 9) *Let M be the midpoint of the arc AB of circle (O), D be the midpoint of AB. The perpendicular through M is drawn to the tangent of the circle at A meeting that tangent at E. Show $ME = MD$.*

Constructive description
((points A D)
(on M (t D D A))
(lratio B D A -1)
(inter O (l M D) (b A M))
(inter E (p M O A) (t A O A))
(perp-biesct M D E))

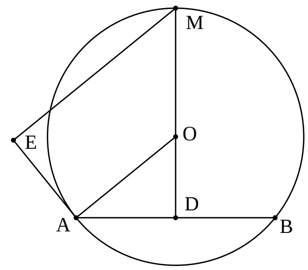


Figure 6-326

Example 6.327 (0.200, 4, 11) *The circle with the altitude AD of triangle ABC as a diameter meets AB and AC at E and F, respectively. Show that B, C, E and F are on the same circle.*

Constructive description
((points A B C)
(foot D A B C)
(midpoint O A D)
(inter E (l A B) (cir O A))
(inter F (l A C) (cir O A))
(cocircle E F C B))

The eliminants

$$S_{BCF} \stackrel{F}{=} \frac{(P_{COC}-P_{AOA})\cdot S_{ABC}}{P_{ACA}}$$
$$P_{BFC} \stackrel{F}{=} \frac{-(2P_{CAO}-P_{BAC})\cdot(P_{COC}-P_{AOA})}{P_{ACA}}$$
$$P_{BEC} \stackrel{E}{=} \frac{-(2P_{BAO}-P_{BAC})\cdot(P_{BOB}-P_{AOA})}{P_{ABA}}$$
$$S_{BCE} \stackrel{E}{=} \frac{(P_{BOB}-P_{AOA})\cdot S_{ABC}}{P_{ABA}}$$
$$P_{BAO} \stackrel{O}{=} \frac{1}{2}(P_{BAD})$$
$$P_{CAO} \stackrel{O}{=} \frac{1}{2}(P_{CAD})$$
$$P_{BAD} \stackrel{D}{=} \frac{P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA}}{P_{BCB}}$$
$$P_{CAD} \stackrel{D}{=} \frac{P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC}}{P_{BCB}}$$
$$P_{ABC} = \frac{1}{2}(P_{BCB}-P_{ACA}+P_{ABA})$$
$$P_{ACB} = \frac{1}{2}(P_{BCB}+P_{ACA}-P_{ABA})$$
$$P_{BAC} = -\frac{1}{2}(P_{BCB}-P_{ACA}-P_{ABA})$$

The machine proof

$$\frac{S_{BCE}\cdot P_{BFC}}{S_{BCF}\cdot P_{BEC}}$$
$$\stackrel{F}{=} \frac{S_{BCE}\cdot(-2P_{COC}\cdot P_{CAO}+P_{COC}\cdot P_{BAC}+2P_{CAO}\cdot P_{AOA}-P_{BAC}\cdot P_{AOA})\cdot P_{ACA}}{(P_{COC}\cdot S_{ABC}-P_{AOA}\cdot S_{ABC})\cdot P_{BEC}\cdot P_{ACA}}$$
$$\stackrel{simplify}{=} \frac{-S_{BCE}\cdot(2P_{CAO}-P_{BAC})}{S_{ABC}\cdot P_{BEC}}$$
$$\stackrel{E}{=} \frac{-(P_{BOB}\cdot S_{ABC}-P_{AOA}\cdot S_{ABC})\cdot(2P_{CAO}-P_{BAC})\cdot P_{ABA}}{S_{ABC}\cdot(-2P_{BOB}\cdot P_{BAO}+P_{BOB}\cdot P_{BAC}+2P_{BAO}\cdot P_{AOA}-P_{BAC}\cdot P_{AOA})\cdot P_{ABA}}$$
$$\stackrel{simplify}{=} \frac{2P_{CAO}-P_{BAC}}{2P_{BAO}-P_{BAC}}$$

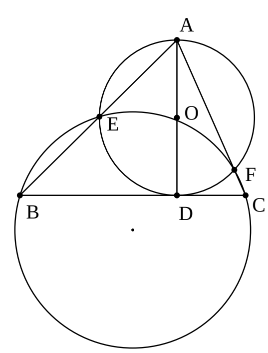


Figure 6-327

$$\overset{O}{=} \frac{P_{CAD}-P_{BAC}}{P_{BAD}-P_{BAC}}$$
$$\overset{D}{=} \frac{(-P_{BCB}\cdot P_{BAC}+P_{BAC}\cdot P_{ACB}+P_{ACA}\cdot P_{ABC})\cdot P_{BCB}}{(-P_{BCB}\cdot P_{BAC}+P_{BAC}\cdot P_{ABC}+P_{ACB}\cdot P_{ABA})\cdot P_{BCB}}$$
$$\overset{simplify}{=} \frac{P_{BCB}\cdot P_{BAC}-P_{BAC}\cdot P_{ACB}-P_{ACA}\cdot P_{ABC}}{P_{BCB}\cdot P_{BAC}-P_{BAC}\cdot P_{ABC}-P_{ACB}\cdot P_{ABA}}$$
$$\overset{py}{=} \frac{(-2P_{BCB}^2+2P_{ACA}^2-4P_{ACA}\cdot P_{ABA}+2P_{ABA}^2)\cdot((2))^3}{(-2P_{BCB}^2+2P_{ACA}^2-4P_{ACA}\cdot P_{ABA}+2P_{ABA}^2)\cdot((2))^3} \overset{simplify}{=} 1$$

Example 6.328 (0.033, 1, 11) *Let A, B, C, D be four points on circle (O). $E = CD \cap AB$. CB meets the line passing through E and parallel to AD at F. GF is tangent to circle (O) at G. Show that $FG = FE$.*

By Example 6.309, we only need to prove the following statement.

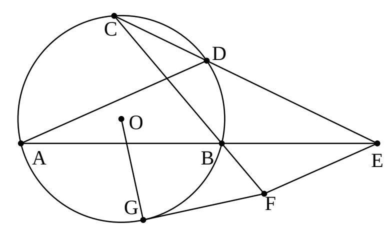


Figure 6-328

Constructive description
((circle A B D C)
(inter E (l A B) (l C D))
(inter F (l B C) (p E A D))
(eq-product F B F C E F E F))

The machine proof

$$\frac{P_{BFC}}{P_{EFE}}$$
$$\overset{F}{=} \frac{P_{BCB}\cdot S_{BDE}\cdot S_{ACE}\cdot S_{ABDC}^2}{P_{ADA}\cdot S_{BCE}^2\cdot S_{ABDC}^2}$$
$$\overset{simplify}{=} \frac{P_{BCB}\cdot S_{BDE}\cdot S_{ACE}}{P_{ADA}\cdot (S_{BCE})^2}$$
$$\overset{E}{=} \frac{P_{BCB}\cdot(-S_{BDC}\cdot S_{ABD})\cdot(-S_{ADC}\cdot S_{ABC})\cdot(S_{ADBC})^2}{P_{ADA}\cdot((-S_{BDC}\cdot S_{ABC}))^2\cdot(S_{ADBC})^2}$$
$$\overset{simplify}{=} \frac{P_{BCB}\cdot S_{ABD}\cdot S_{ADC}}{P_{ADA}\cdot S_{BDC}\cdot S_{ABC}}$$
$$\overset{co-cir}{=} \frac{(2\widetilde{BC}^2)\cdot(-\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB})\cdot(-\widetilde{DC}\cdot\widetilde{AC}\cdot\widetilde{AD})\cdot((2d))^2}{(2\widetilde{AD}^2)\cdot(-\widetilde{DC}\cdot\widetilde{BC}\cdot\widetilde{BD})\cdot(-\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB})\cdot((2d))^2}$$
$$\overset{simplify}{=} 1$$

The eliminants

$$P_{EFE} \overset{F}{=} \frac{P_{ADA}\cdot(S_{BCE})^2}{(S_{ABDC})^2}$$
$$P_{BFC} \overset{F}{=} \frac{P_{BCB}\cdot S_{BDE}\cdot S_{ACE}}{(S_{ABDC})^2}$$
$$S_{BCE} \overset{E}{=} \frac{-S_{BDC}\cdot S_{ABC}}{S_{ADBC}}$$
$$S_{ACE} \overset{E}{=} \frac{-S_{ADC}\cdot S_{ABC}}{S_{ADBC}}$$
$$S_{BDE} \overset{E}{=} \frac{-S_{BDC}\cdot S_{ABD}}{S_{ADBC}}$$
$$S_{ABC} = \frac{\widetilde{BC}\cdot\widetilde{AC}\cdot\widetilde{AB}}{(-2)\cdot d}$$
$$S_{BDC} = \frac{\widetilde{DC}\cdot\widetilde{BC}\cdot\widetilde{BD}}{(-2)\cdot d}$$
$$P_{ADA} = 2((\widetilde{AD})^2)$$
$$S_{ADC} = \frac{\widetilde{DC}\cdot\widetilde{AC}\cdot\widetilde{AD}}{(-2)\cdot d}$$
$$S_{ABD} = \frac{\widetilde{BD}\cdot\widetilde{AD}\cdot\widetilde{AB}}{(-2)\cdot d}$$
$$P_{BCB} = 2((\widetilde{BC})^2)$$

Example 6.329 (1.783 25 21) *The bisector of triangle ABC at vertex C bisects the arc AB of the circumcircle of triangle ABC.*

Constructive description
((points A B C)
(circumcenter O A B C)

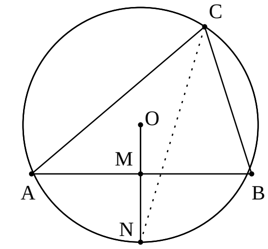


Figure 6-329

(midpoint M A B)
(inter N (l O M) (cir O A))
(eqangle A C N N C B))

Example 6.330 (0.066, 2, 6) *Let PT and PB be two tangents to a circle, AB the diameter through B, and TH the perpendicular from T to AB. Then AP bisects TH.*

Constructive description
((points B P)
(on O (t B B P))
(inter T (cir O B) (cir P B))
(lratio A O B -1)
(inter H (l B O) (p T B P))
(inter I (l T H) (l A P))
(midpoint I T H))

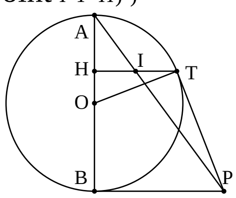


Figure 6-330

The machine proof

$$-\frac{\overline{TI}}{\overline{HI}}$$

$$\stackrel{I}{=} \frac{-S_{PTA}}{-S_{PAH}}$$

$$\stackrel{H}{=} \frac{S_{PTA}\cdot(-S_{BPO})}{-S_{BTPA}\cdot S_{BPO}}$$

$$\stackrel{simplify}{=} \frac{S_{PTA}}{S_{BTPA}}$$

$$\stackrel{A}{=} \frac{-2S_{POT}-S_{BPT}}{-S_{BPT}+2S_{BPO}}$$

$$\stackrel{T}{=} \frac{(2P_{POP}\cdot S_{BPO}-2P_{BPO}\cdot S_{BPO})\cdot P_{POP}}{(-2P_{POP}\cdot S_{BPO}+2P_{BPO}\cdot S_{BPO})\cdot P_{POP}\cdot(-1)}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{TI}}{\overline{HI}} \stackrel{I}{=} \frac{S_{PTA}}{-S_{PAH}}$$

$$S_{PAH} \stackrel{H}{=} S_{BTPA}$$

$$S_{BTPA} \stackrel{A}{=} -\left(S_{BPT}-2S_{BPO}\right)$$

$$S_{PTA} \stackrel{A}{=} -\left(2S_{POT}+S_{BPT}\right)$$

$$S_{BPT} \stackrel{T}{=} \frac{(2)\cdot P_{BPO}\cdot S_{BPO}}{P_{POP}}$$

$$S_{POT} \stackrel{T}{=} -\left(S_{BPO}\right)$$

Example 6.331 [5] (0.083 3 15) *Let M be a point on line AB. Two squares $AMCD$ and $BMEF$ are drawn on the same side of AB. Let U and V be the center of the squares $AMCD$ and $BMEF$. Line BC and circle VB meet in N. Show that when point M moving on line AB, the line MN passes through a fixed point.*

[5] This is a problem from the 1959 International Mathematical Olympia.

Constructive description
((points A B)
(on M (l A B))
(tratio C M A 1)
(midpoint U A C)
(tratio E M B -1)
(midpoint V E B)
(midpoint O A B)
(tratio R O A -1)
(inter N (l B C) (cir V B))
(inter T (l B C) (l M R))
($\frac{\overline{BN}}{\overline{CN}} = \frac{\overline{BT}}{\overline{CT}}$))

The machine proof

$$(\frac{\overline{BN}}{\overline{CN}})/(\frac{\overline{BT}}{\overline{CT}})$$

$$\stackrel{T}{=} \frac{-S_{MCR}}{S_{BMR}} \cdot \frac{\overline{BN}}{\overline{CN}}$$

$$\stackrel{N}{=} \frac{-P_{CBV} \cdot S_{MCR}}{S_{BMR} \cdot (P_{CBV} - \frac{1}{2}P_{BCB})}$$

$$\stackrel{R}{=} \frac{-P_{CBV} \cdot S_{MCO}}{(\frac{1}{4}P_{MABO}) \cdot (P_{CBV} - \frac{1}{2}P_{BCB})}$$

$$\stackrel{O}{=} \frac{(-4) \cdot P_{CBV} \cdot (\frac{1}{2}S_{BMC} + \frac{1}{2}S_{AMC})}{(-\frac{1}{2}P_{ABM}) \cdot (P_{CBV} - \frac{1}{2}P_{BCB})}$$

$$\stackrel{V}{=} \frac{(4) \cdot (\frac{1}{2}P_{CBE}) \cdot (S_{BMC} + S_{AMC})}{P_{ABM} \cdot (\frac{1}{2}P_{CBE} - \frac{1}{2}P_{BCB})}$$

$$\stackrel{E}{=} \frac{(4) \cdot (P_{MBC} + 4S_{BMC}) \cdot (S_{BMC} + S_{AMC})}{P_{ABM} \cdot (P_{MBC} - P_{BCB} + 4S_{BMC})}$$

$$\stackrel{C}{=} \frac{(4) \cdot (P_{BMB} - P_{AMB}) \cdot (-\frac{1}{4}P_{AMB} - \frac{1}{4}P_{AMA})}{P_{ABM} \cdot (-P_{AMB} - P_{AMA})}$$

$$\stackrel{simplify}{=} \frac{P_{BMB} - P_{AMB}}{P_{ABM}}$$

$$\stackrel{M}{=} \frac{-P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}} + P_{ABA}}{-P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}} + P_{ABA}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BT}}{\overline{CT}} \stackrel{T}{=} \frac{S_{BMR}}{-S_{MCR}}$$

$$\frac{\overline{BN}}{\overline{CN}} \stackrel{N}{=} \frac{P_{CBV}}{(\frac{1}{2}) \cdot (2P_{CBV} - P_{BCB})}$$

$$S_{BMR} \stackrel{R}{=} \frac{1}{4}(P_{MABO})$$

$$S_{MCR} \stackrel{R}{=} S_{MCO}$$

$$P_{MABO} \stackrel{O}{=} -\frac{1}{2}(P_{ABM})$$

$$S_{MCO} \stackrel{O}{=} \frac{1}{2}(S_{BMC} + S_{AMC})$$

$$P_{CBV} \stackrel{V}{=} \frac{1}{2}(P_{CBE})$$

$$P_{CBE} \stackrel{E}{=} P_{MBC} + 4S_{BMC}$$

$$P_{BCB} \stackrel{C}{=} P_{BMB} + P_{AMA}$$

$$S_{AMC} \stackrel{C}{=} -\frac{1}{4}(P_{AMA})$$

$$S_{BMC} \stackrel{C}{=} -\frac{1}{4}(P_{AMB})$$

$$P_{MBC} \stackrel{C}{=} P_{BMB}$$

$$P_{ABM} \stackrel{M}{=} -((\frac{\overline{AM}}{\overline{AB}} - 1) \cdot P_{ABA})$$

$$P_{AMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}} - 1) \cdot P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}}$$

$$P_{BMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}} - 1)^2 \cdot P_{ABA}$$

Example 6.332 (0.616, 22, 31) *Let M be the midpoint of the hypotenuse of the right triangle ABC. A circle passing through A and M meet AB at E. F is the point on the circle such that $EF \parallel BC$. Show that $BC = 2EF$.*

Constructive description
((points A B)
(on C (t A A B))
(midpoint M B C)
(midpoint X A M)
(on O (t X X A))
(inter E (l A B) (cir O A))
(inter F (p E B C) (cir O E))
($\frac{\overline{EF}}{\overline{BC}} = 1/2$))

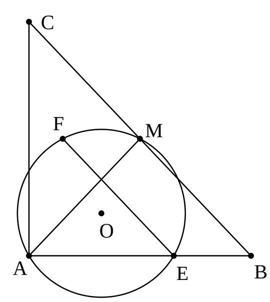


Figure 6-332

Example 6.333 (0.767, 14, 21) *Let PA tangent to circle (O) at point A. M is the midpoint of PA. C is a point on the circle. PC and MC meet the circle at points E and B, respectively. PB meets the circle at D. Show that ED is parallel to AP.*

Constructive description
((points $A\ B\ C$) (circumcenter $O\ A\ B\ C$)
(inter M (l $B\ C$) (t $A\ A\ O$))
(lratio $P\ M\ A\ -1$)
(inter D (l $B\ P$) (cir $O\ B$))
(inter E (l $C\ P$) (cir $O\ C$))
(parallel $E\ D\ A\ M$))

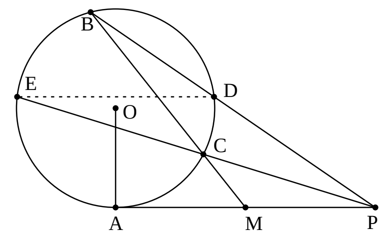


Figure 6-333

Example 6.334 (0.366, 6, 23) *If P is any point on a semicircle, diameter AB, and BC, CD are two equal arcs, then if $E = CA \cap PB$, $F = AD \cap PC$, prove that AD is perpendicular to EF.*

Constructive description
((points $A\ P\ C$)
(circumcenter $O\ A\ P\ C$)
(lratio $B\ O\ A\ -1$)
(inter D (cir $C\ B$) (cir $O\ B$))
(inter E (l $P\ B$) (l $C\ A$))
(inter F (l $P\ C$) (l $A\ D$))
(perpendicular $E\ F\ A\ D$))

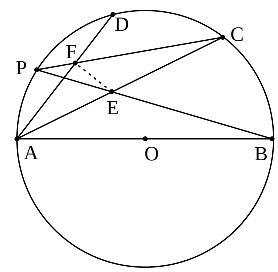


Figure 6-334

Example 6.335 (1.900, 49, 18) *From the point S the two tangents SA, SB and the secant SPQ are drawn to the same circle. Prove that $AP/AQ = BP/BQ$.*

Constructive description
((points $A\ B\ P$) (circumcenter $O\ A\ B\ P$)
(midpoint $M\ A\ B$)
(inter S (l $O\ M$) (t $A\ A\ O$))
(inter Q (l $S\ P$) (cir $O\ P$))
(eq-product $A\ Q\ B\ P\ A\ P\ B\ Q$))

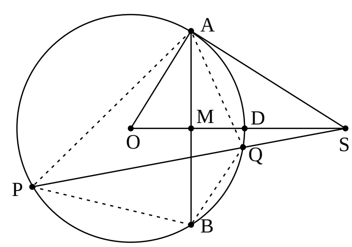


Figure 6-335

Example 6.336 (0.466, 6, 19) *Show that the lines joining a point of a circle to the ends of a chord divide harmonically the diameter perpendicular to the chord.*

Constructive description
((points $A\ D\ P$)
(circumcenter $O\ A\ D\ P$)
(lratio $B\ O\ A\ -1$)
(foot $F\ D\ O\ A$)
(lratio $E\ F\ D\ -1$)
(inter Q (l $A\ O$) (l $P\ E$))
(inter S (l $A\ O$) (l $P\ D$))
(harmonic $A\ B\ Q\ S$))

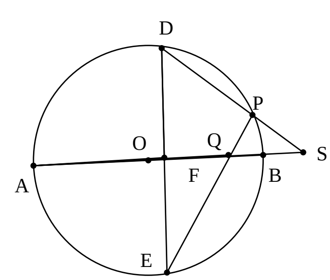


Figure 6-336

Example 6.337 (0.133, 3, 16) *From a point A two lines are drawn tangent to circle (O) at B and C. From a point P on the circle perpendiculars are drawn to BC, AB, and AC. Let D, F, E be the feet. Show that $PD^2 = PE \cdot PF$.*

Constructive description
((points B C P)
(circumcenter O B C P)
(inter A (t B B O) (t C C O))
(foot D P B C)
(foot E P A C)
(foot F P A B)
(eq-product P D P D P E P F))

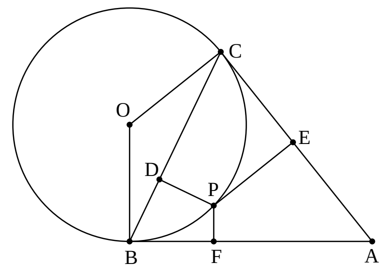


Figure 6-337

Example 6.338 (1.113, 14, 24) *Through P a tangent PE and a secant PAB of circle (O) are drawn. The bisector of angle APE meets AE and BE at C and D. Show that $EC = ED$.*

Constructive description
((points P E C)
(on Y (a P C E P C))
(inter A (l E C) (l P Y))
(midpoint M A E) (tratio Z E P H)
(inter O (t M M E) (l E Z))
(inter B (l P A) (cir O A))
(inter D (l P C) (l B E))
(eqdistance E D E C))

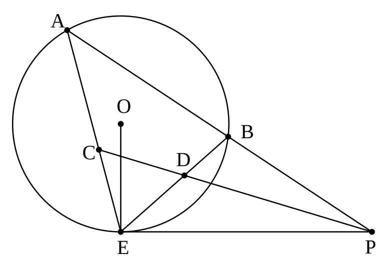


Figure 6-338

Example 6.339 (0.216, 3, 18) *Let N be the traces of the internal bisectors of the triangle ABC on the circumscribed circle (O). Show that the Simson line of N is the external bisector of the medial triangle of ABC.*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint D B C)
(midpoint E A C) (midpoint F B A)
(inter N (l O F) (cir O A))
(foot K N A C)
(eqangle E F K K F D))

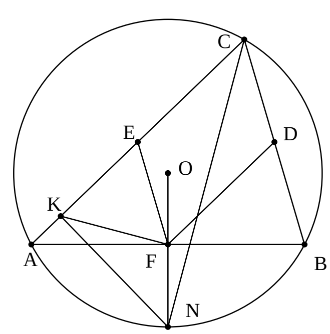


Figure 6-339

6.5.2 Intersectional Circles

Example 6.340 (0.216, 3, 18) *Through the two common points A, B of two circles (O) and (O_1) two lines are drawn meeting the circles at C and D, E and F, respectively. Show that $CE \parallel DF$.*

For a proof of this theorem based on full-angles, see Example 1.115 on page 47.

Constructive description
((circle B A C E)
(midpoint X A B)
(tratio O X B H)
(foot M O A C)
(foot N O B E)
(lratio D M A -1)
(lratio F N B -1)
($S_{CED} = S_{CEF}$))

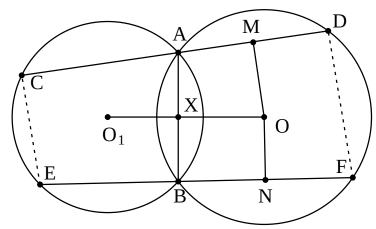


Figure 6-340

The machine proof

$$\frac{S_{CED}}{S_{CEF}}$$

$$\overset{F}{=} \frac{S_{CED}}{2S_{CEN}-S_{BCE}}$$

$$\overset{D}{=} \frac{2S_{CEM}-S_{ACE}}{2S_{CEN}-S_{BCE}}$$

$$\overset{N}{=} \frac{(2S_{CEM}-S_{ACE})\cdot P_{BEB}}{2P_{BEO}\cdot S_{BCE}-P_{BEB}\cdot S_{BCE}}$$

$$\overset{simplify}{=} \frac{(2S_{CEM}-S_{ACE})\cdot P_{BEB}}{(2P_{BEO}-P_{BEB})\cdot S_{BCE}}$$

$$\overset{M}{=} \frac{(2P_{ACO}\cdot S_{ACE}-P_{ACA}\cdot S_{ACE})\cdot P_{BEB}}{(2P_{BEO}-P_{BEB})\cdot S_{BCE}\cdot P_{ACA}}$$

$$\overset{simplify}{=} \frac{(2P_{ACO}-P_{ACA})\cdot S_{ACE}\cdot P_{BEB}}{(2P_{BEO}-P_{BEB})\cdot S_{BCE}\cdot P_{ACA}}$$

$$\overset{O}{=} \frac{(2P_{ACX}-P_{ACA}-8S_{BCX}\cdot r)\cdot S_{ACE}\cdot P_{BEB}}{(2P_{BEX}-P_{BEB}-8S_{BEX}\cdot r)\cdot S_{BCE}\cdot P_{ACA}}$$

$$\overset{X}{=} \frac{(P_{BCA}+4S_{BAC}\cdot r)\cdot S_{ACE}\cdot P_{BEB}}{(P_{BEA}+4S_{BAE}\cdot r)\cdot S_{BCE}\cdot P_{ACA}}$$

$$\overset{co-cir}{=} \frac{(-4\widetilde{AC}\cdot\widetilde{BC}\cdot\widetilde{BA}\cdot r+4\widetilde{AC}\cdot\widetilde{BC}\cdot\cos(BA)\cdot d)\cdot(-\widetilde{CE}\cdot\widetilde{AE}\cdot\widetilde{AC})\cdot(2\widetilde{BE}^2)\cdot((2d))^2}{(-4\widetilde{AE}\cdot\widetilde{BE}\cdot\widetilde{BA}\cdot r+4\widetilde{AE}\cdot\widetilde{BE}\cdot\cos(BA)\cdot d)\cdot(-\widetilde{CE}\cdot\widetilde{BE}\cdot\widetilde{BC})\cdot(2\widetilde{AC}^2)\cdot((2d))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{CEF} \overset{F}{=} 2S_{CEN}-S_{BCE}$$
$$S_{CED} \overset{D}{=} 2S_{CEM}-S_{ACE}$$
$$S_{CEN} \overset{N}{=} \frac{P_{BEO}\cdot S_{BCE}}{P_{BEB}}$$
$$S_{CEM} \overset{M}{=} \frac{P_{ACO}\cdot S_{ACE}}{P_{ACA}}$$
$$P_{BEO} \overset{O}{=} P_{BEX}-4S_{BEX}\cdot r$$
$$P_{ACO} \overset{O}{=} P_{ACX}-4S_{BCX}\cdot r$$
$$S_{BEX} \overset{X}{=} -\frac{1}{2}(S_{BAE})$$
$$P_{BEX} \overset{X}{=} \frac{1}{2}(P_{BEA}+P_{BEB})$$
$$S_{BCX} \overset{X}{=} -\frac{1}{2}(S_{BAC})$$
$$P_{ACX} \overset{X}{=} \frac{1}{2}(P_{ACA}+P_{BCA})$$
$$P_{ACA} = 2((\widetilde{AC})^2)$$
$$S_{BCE} = \frac{\widetilde{CE}\cdot\widetilde{BE}\cdot\widetilde{BC}}{(-2)\cdot d}$$
$$S_{BAE} = \frac{\widetilde{AE}\cdot\widetilde{BE}\cdot\widetilde{BA}}{(-2)\cdot d}$$
$$P_{BEA} = 2(\widetilde{AE}\cdot\widetilde{BE}\cdot\cos(BA))$$
$$P_{BEB} = 2((\widetilde{BE})^2)$$
$$S_{ACE} = \frac{\widetilde{CE}\cdot\widetilde{AE}\cdot\widetilde{AC}}{(-2)\cdot d}$$
$$S_{BAC} = \frac{\widetilde{AC}\cdot\widetilde{BC}\cdot\widetilde{BA}}{(-2)\cdot d}$$
$$P_{BCA} = 2(\widetilde{AC}\cdot\widetilde{BC}\cdot\cos(BA))$$

Example 6.341 (1.583, 39, 25) *Let A and B the two common points of two circles (O) and (O_1). Through A a line is drawn meeting the circles at C and D respectively. G is the midpoint of CD. Line BG intersects circles (O) and (O_1) at E and F, respectively. Show that $EG = GF$.*

Constructive description
((points A B C)
(midpoint X A B)
(on O_1 (t X X A))
(inter O (l X O_1) (b A C))
(inter D (l C A) (cir O_1 A))
(midpoint G C D)
(inter E (l B G) (cir O B))
(inter F (l B G) (cir O_1 B))
(midpoint G F E))

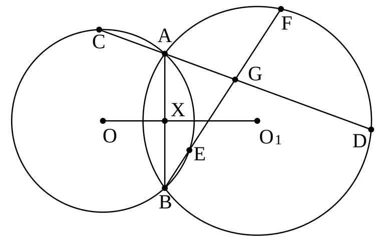


Figure 6-341

Example 6.342 (0.133, 6, 14) *Three circles, centers A, B, C, have a point D in common and intersect two-by-two in the points A_1, B_1, C_1. The common chord DC_1 of the first two circles meets the third in C_2. Let A_2, B_2 be the analogous points on the other two circles. Prove that the segments A_1A_2, B_1B_2, C_1C_2 are twice as long as the altitudes of the triangle ABC.*

Constructive description
((points A B C D)
(foot F C A B)
(inter C_1 (cir B D) (cir A D))
(inter C_2 (l D C_1) (cir C D))
($\frac{\overline{C_2C_1}}{\overline{CF}} = 2$))

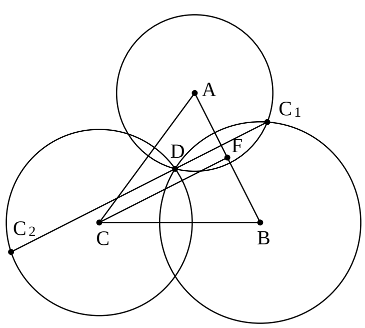


Figure 6-342

The eliminants

$$\frac{\overline{C_1C_2}}{\overline{CF}} \overset{C_2}{=} \frac{-(P_{CC_1C}-P_{CDC})}{P_{DC_1D}} \cdot \frac{\overline{DC_1}}{\overline{CF}}$$

$$P_{DC_1D} \overset{C_1}{=} \frac{(4)\cdot(P_{BDB}\cdot P_{BAD}-P_{BAD}\cdot P_{ABD}+P_{ADA}\cdot P_{ABD})}{P_{ABA}}$$

$$\frac{\overline{DC_1}}{\overline{CF}} \overset{C_1}{=} \frac{(2)\cdot S_{ABD}}{S_{ABC}}$$

$$P_{CC_1C} \overset{C_1}{=} -(P_{CDC}\cdot P_{ABA}-2P_{BDB}\cdot P_{BAD}-2P_{BCB}\cdot P_{BAD}$$
$$+4P_{BAD}\cdot P_{ABD}-2P_{ADA}\cdot P_{ABD}-2P_{ACA}\cdot P_{ABD})/P_{ABA}$$

$$S_{ABC}=\frac{1}{2}(y_1\cdot x_0)$$

$$S_{ABD}=\frac{1}{2}(y_2\cdot x_0)$$

$$P_{ABD}=(-2(x_2-x_0)\cdot x_0)$$

$$P_{ACA}=2(y_1^2+x_1^2)$$

$$P_{ADA}=2(y_2^2+x_2^2)$$

$$P_{BAD}=2(x_2\cdot x_0)$$

$$P_{BCB}=2(y_1^2+x_1^2-2x_1\cdot x_0+x_0^2)$$

$$P_{BDB}=2(y_2^2+x_2^2-2x_2\cdot x_0+x_0^2)$$

$$P_{ABA}=2(x_0^2)$$

$$P_{CDC}=2(y_2^2-2y_2\cdot y_1+y_1^2+x_2^2-2x_2\cdot x_1+x_1^2)$$

The machine proof

$$\frac{1}{2}\left(-\frac{\overline{C_1C_2}}{\overline{CF}}\right)$$

$$\overset{C_2}{=} \frac{-(-\frac{\overline{DC_1}}{\overline{CF}}\cdot P_{CC_1C}+\frac{\overline{DC_1}}{\overline{CF}}\cdot P_{CDC})}{(2)\cdot P_{DC_1D}}$$

$$\overset{simplify}{=} \frac{P_{CC_1C}-P_{CDC}}{(2)\cdot P_{DC_1D}} \cdot \frac{\overline{DC_1}}{\overline{CF}}$$

$$\overset{C_1}{=} \frac{(-2P_{CDC}\cdot P_{ABA}+2P_{BDB}\cdot P_{BAD}+2P_{BCB}\cdot P_{BAD}-4P_{BAD}\cdot P_{ABD}+2P_{ADA}\cdot P_{ABD}+2P_{ACA}\cdot P_{ABD})\cdot(2S_{ABD})\cdot P_{ABA}}{(2)\cdot(4P_{BDB}\cdot P_{BAD}-4P_{BAD}\cdot P_{ABD}+4P_{ADA}\cdot P_{ABD})\cdot S_{ABC}\cdot P_{ABA}}$$

$$\stackrel{simplify}{=} \frac{-(P_{CDC}\cdot P_{ABA}-P_{BDB}\cdot P_{BAD}-P_{BCB}\cdot P_{BAD}+2P_{BAD}\cdot P_{ABD}-P_{ADA}\cdot P_{ABD}-P_{ACA}\cdot P_{ABD})\cdot S_{ABD}}{(2)\cdot(P_{BDB}\cdot P_{BAD}-P_{BAD}\cdot P_{ABD}+P_{ADA}\cdot P_{ABD})\cdot S_{ABC}}$$

$$\stackrel{coor}{=} \frac{-(-8y_2\cdot y_1\cdot x_0^2)\cdot y_2\cdot x_0\cdot(2)}{(2)\cdot(4y_2^2\cdot x_0^2)\cdot y_1\cdot x_0\cdot(2)} \stackrel{simplify}{=} 1$$

In the above proof, we have $A=(0,0), B=(x_0,0), C=(x_1,y_1)$, and $D=(x_2,y_2)$.

Example 6.343 (0.050, 3, 8) *Show that an altitude of a triangle is the radical axis of the two circles having for diameters the medians issued from the other two vertices.*

Constructive description
((points A B C)
(midpoint B_1 A C)
(midpoint C_1 A B)
(midpoint M B B_1)
(midpoint N C C_1)
(on-radical A M B N C))

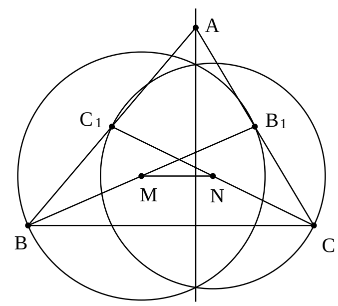


Figure 6-343

The machine proof

$$\frac{-P_{BMB}+P_{AMA}}{-P_{CNC}+P_{ANA}}$$

$$\stackrel{N}{=} \frac{-P_{BMB}+P_{AMA}}{-\frac{1}{2}P_{CC_1C}+\frac{1}{2}P_{AC_1A}+\frac{1}{2}P_{ACA}}$$

$$\stackrel{M}{=} \frac{(2)\cdot(\frac{1}{2}P_{BB_1B}-\frac{1}{2}P_{AB_1A}-\frac{1}{2}P_{ABA})}{P_{CC_1C}-P_{AC_1A}-P_{ACA}}$$

$$\stackrel{C_1}{=} \frac{P_{BB_1B}-P_{AB_1A}-P_{ABA}}{\frac{1}{2}P_{BCB}-\frac{1}{2}P_{ACA}-\frac{1}{2}P_{ABA}}$$

$$\stackrel{B_1}{=} \frac{(2)\cdot(\frac{1}{2}P_{BCB}-\frac{1}{2}P_{ACA}-\frac{1}{2}P_{ABA})}{P_{BCB}-P_{ACA}-P_{ABA}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$P_{ANA} \stackrel{N}{=} -\frac{1}{4}(P_{CC_1C}-2P_{AC_1A}-2P_{ACA})$
$P_{CNC} \stackrel{N}{=} \frac{1}{4}(P_{CC_1C})$
$P_{AMA} \stackrel{M}{=} -\frac{1}{4}(P_{BB_1B}-2P_{AB_1A}-2P_{ABA})$
$P_{BMB} \stackrel{M}{=} \frac{1}{4}(P_{BB_1B})$
$P_{AC_1A} \stackrel{C_1}{=} \frac{1}{4}(P_{ABA})$
$P_{CC_1C} \stackrel{C_1}{=} \frac{1}{4}(2P_{BCB}+2P_{ACA}-P_{ABA})$
$P_{AB_1A} \stackrel{B_1}{=} \frac{1}{4}(P_{ACA})$
$P_{BB_1B} \stackrel{B_1}{=} \frac{1}{4}(2P_{BCB}-P_{ACA}+2P_{ABA})$

Example 6.344 (0.800, 12, 18) *Let A and B be the two common points of two circles (O) and (O_1). Through B a line is drawn meeting the circles at C and D respectively. Show $AC/AD = OA/O_1A$.*

Constructive description
((points A B C)
(midpoint X A B)
(on O_1 (t X X A))
(inter O (l X O_1) (b A C))
(inter D (l C B) (cir O_1 B))
(eq-product A C O_1 A A D O A))

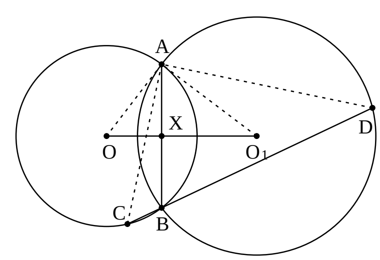


Figure 6-344

Example 6.345 (0.783, 16, 25) *From a point P on the line joining the two common points A and B of two circles (O) and (O_1) two secants PCE and PFD are drawn to the circles respectively. Show that $PC \cdot PE = PF \cdot PD$.*

Constructive description
((points *A C F X*)
(on *Y* (t *X X A*))
(inter *O* (l *X Y*) (b *A C*))
(inter O_1 (l *X Y*) (b *A F*))
(lratio *B X A* −1) (on *P* (l *A X*))
(inter *E* (l *P C*) (cir *O C*))
(inter *D* (l *P F*) (cir O_1 *F*))
(eq-product *P C P E P D P F*))

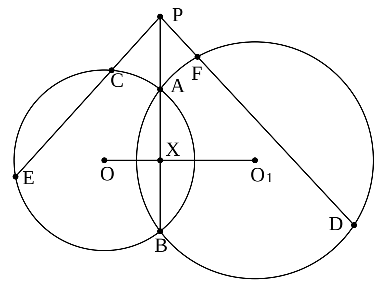


Figure 6-345

Example 6.346 (1.283, 60, 67) *If three chords drawn through a point of a circle are taken for diameters of three circles, these circles intersect, in pairs, in three new points, which are collinear.*

Constructive description
((circle *A B C D*) (midpoint M_1 *A D*)
(midpoint M_2 *B D*) (midpoint M_3 *C D*)
(inter *E* (cir M_2 *D*) (cir M_1 *D*))
(inter *F* (cir M_3 *D*) (cir M_1 *D*))
(inter *G* (cir M_3 *D*) (cir M_2 *D*))
(collinear *E F G*))

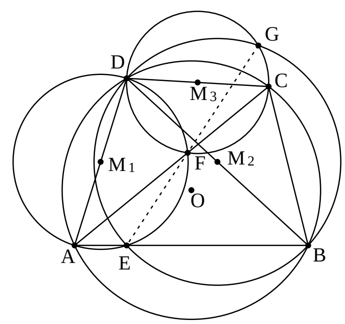


Figure 6-346

Example 6.347 (0.733, 11, 38) *If three circles pass through the same point of the circumcircle of the triangle of their centers, these circles intersect, in pairs, in three collinear points.*

Constructive description
((circle *A B C P*)
(inter A_1 (cir *B P*) (cir *C P*))
(inter B_1 (cir *C P*) (cir *A P*))
(inter C_1 (cir *B P*) (cir *A P*))
(collinear A_1 B_1 C_1))

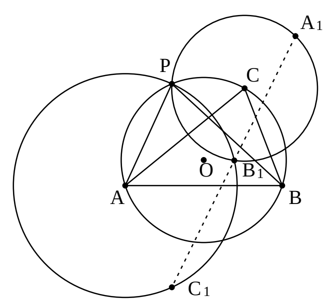


Figure 6-347

Example 6.348 (1.083, 24, 66) *If three circles having a point in common intersect in pairs in three collinear points, their common point is cyclic with their centers.*

Constructive description
((points A_1 B_1 P)
(on C_1 (l B_1 A_1)) (midpoint M_1 P B_1)
(on T_1 (t M_1 M_1 P) 1) (midpoint M_2 P C_1)
(on T_2 (t M_2 M_2 P) 1) (midpoint M_3 P A_1)
(on T_3 (t M_3 M_3 P) 1)
(inter A (l M_1 T_1) (l M_2 T_2))
(inter B (l M_3 T_3) (l M_2 T_2))
(inter C (l M_1 T_1) (l M_3 T_3))
(cocircle A B C P))

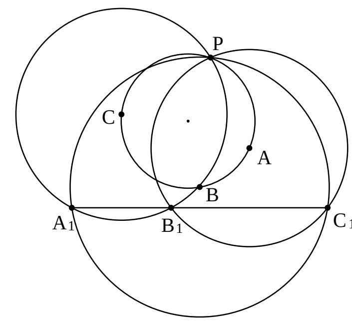


Figure 6-348

Example 6.349 (0.333, 7, 15) *Show that the radical axis of the two circles having for diameters the diagonals AC, BD of a trapezoid ABCD passes through the point of intersection E of the nonparallel sides BC, AD.*

Constructive description
((points A B C)
(on D (p C A B))
(inter E (l A D) (l B C))
(midpoint N_1 A C)
(midpoint N_2 B D)
(on-radical E N_1 A N_2 B))

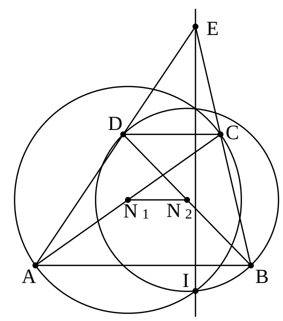


Figure 6-349

Example 6.350 (0.250, 8, 13) *Given two circles* (A), (B) *intersecting in* E, F, *show that the chord* E_1F_1 *determined in* (A) *by the lines* MEE_1, MFF_1 *joining* E, F *to any point* M *of* (B) *is perpendicular to* MB.

Constructive description
((points E F M)
(circumcenter B E F M)
(midpoint D E F) (lratio A D B r)
(inter E_1 (l M E) (cir A E))
(inter F_1 (l M F) (cir A F))
(perpendicular E_1 F_1 M B))

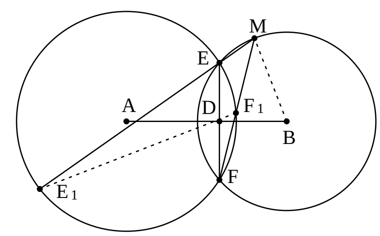


Figure 6-350

Example 6.351 (0.667, 4, 44) *From the midpoint C of arc AB of a circle, two secants are drawn meeting line AB at F, G, and the circle at D and E. Show that F, D, E, and G are on the same circle.*

Constructive description

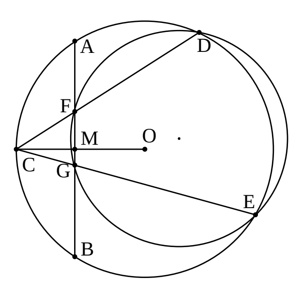


Figure 6-451

((circle $A\ C\ D\ E$)
(circumcenter $O\ A\ C\ D$)
(foot $M\ A\ O\ C$) (lratio $B\ M\ A\ -1$)
(inter F (l $A\ M$) (l $D\ C$))
(inter G (l $A\ M$) (l $C\ E$))
(cocircle $D\ E\ F\ G$))

Example 6.352 (1.067 39 18) *From point P two tangent lines PA and PB of a circle are drawn. D is the middle point of segment AB. Through D a secant EF is drawn. Then $\angle EPD = \angle FPD$.*

Constructive description
((points $A\ B\ E$)
(circumcenter $O\ A\ B\ E$)
(midpoint $D\ A\ B$)
(inter P (l $O\ D$) (t $A\ A\ O$))
(inter F (l $E\ D$) (cir $O\ E$))
(eqangle $E\ P\ O\ O\ P\ F$))

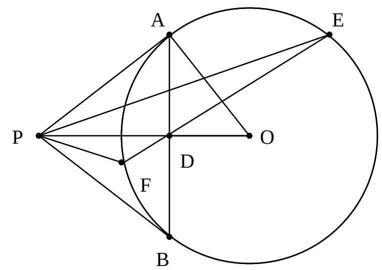


Figure 6-352

Example 6.353 (0.100 4 13) *Let D and E be two points on sides AB and AC of triangle ABC such that $DE \parallel BC$. Show that the circumcircles of triangle ABC and ADE are tangent.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(on D (l $A\ B$))
(inter E (l $A\ C$) (p $D\ B\ C$))
(circumcenter $N\ A\ D\ E$)
($S_{ANO} = 0$))

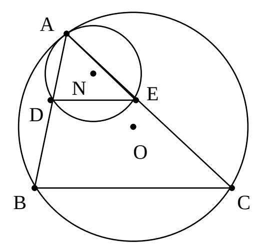


Figure 6-353

6.5.3 The Inversion

Definition *Suppose that we have given a circle whose center is O and the radius has the length $r \neq 0$. Let P and Q be any two points collinear with O such that*

$$\overline{OP} \cdot \overline{OQ} = r^2.$$

Then P is said to be the inverse of Q with regard to the circle, and the transformation from P to Q is called an inversion. The point O is called the center of inversion, the given circle the circle of inversion, and its radius the radius of inversion.

Example 6.354 (0.001, 2, 3) *Two inverse points divide the corresponding diameter harmonically.*

Constructive description

((points B A)
 (midpoint O A B)
 (lratio P O A r)
 (lratio Q O A $\frac{1}{r}$)
 (harmonic A B P Q))

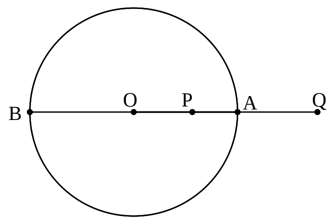


Figure 6-354

The machine proof

$$\left(-\frac{\overline{AP}}{\overline{BP}}\right)/\left(\frac{\overline{AQ}}{\overline{BQ}}\right)$$

$$\stackrel{Q}{=} \frac{-\frac{\overline{BO}}{\overline{AO}}\cdot r+1}{-r+1}\cdot -\frac{\overline{AP}}{\overline{BP}}$$

$$\stackrel{P}{=} \frac{-(r-1)\cdot(\frac{\overline{BO}}{\overline{AO}}\cdot r-1)}{(r-1)\cdot(-\frac{\overline{BO}}{\overline{AO}}+r)}$$

$$\stackrel{simplify}{=} \frac{\frac{\overline{BO}}{\overline{AO}}\cdot r-1}{\frac{\overline{BO}}{\overline{AO}}-r}$$

$$\stackrel{O}{=} \frac{(-\frac{1}{2}r-\frac{1}{2})\cdot(\frac{1}{2})}{(-\frac{1}{2}r-\frac{1}{2})\cdot(\frac{1}{2})} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AQ}}{\overline{BQ}} \stackrel{Q}{=} \frac{r-1}{\frac{\overline{BO}}{\overline{AO}}\cdot r-1}$$

$$\frac{\overline{AP}}{\overline{BP}} \stackrel{P}{=} \frac{r-1}{-(\frac{\overline{BO}}{\overline{AO}}-r)}$$

$$\frac{\overline{BO}}{\overline{AO}} \stackrel{O}{=} -(1)$$

Example 6.355 *From a point P outside a given circle, center O, the tangents are drawn to the circle. Show that P is the inverse of the point of the intersection of OP with the chord of contact.*

Constructive description

((points A C)
 (on B (t C C A))
 (midpoint O A B)
 (inter P (l A O) (t C C O))
 (foot E C A O)
 (inversion O A P E))

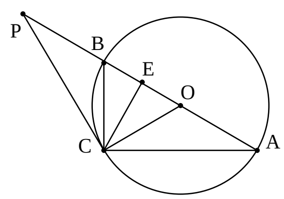


Figure 6-355

The machine proof

$$\left(-\frac{\overline{OP}}{\overline{AO}}\right)/\left(-\frac{\overline{AO}}{\overline{OE}}\right)$$

$$\stackrel{E}{=} \frac{P_{AOC}}{P_{AOA}}\cdot -\frac{\overline{OP}}{\overline{AO}}$$

$$\stackrel{P}{=} \frac{-P_{COC}\cdot P_{AOC}}{P_{AOA}\cdot(-P_{AOC})}$$

$$\stackrel{simplify}{=} \frac{P_{COC}}{P_{AOA}}$$

$$\stackrel{O}{=} \frac{\frac{1}{2}P_{CBC}-\frac{1}{4}P_{ABA}+\frac{1}{2}P_{ACA}}{\frac{1}{4}P_{ABA}}$$

$$\stackrel{B}{=} \frac{P_{CBC}+P_{ACA}}{P_{CBC}+P_{ACA}} \stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AO}}{\overline{OE}} \stackrel{E}{=} \frac{-P_{AOA}}{P_{AOC}}$$

$$\frac{\overline{OP}}{\overline{AO}} \stackrel{P}{=} \frac{P_{COC}}{-P_{AOC}}$$

$$P_{AOA} \stackrel{O}{=} \frac{1}{4}(P_{ABA})$$

$$P_{COC} \stackrel{O}{=} \frac{1}{4}(2P_{CBC}-P_{ABA}+2P_{ACA})$$

$$P_{ABA} \stackrel{B}{=} P_{CBC}+P_{ACA}$$

Example 6.356 (0.433, 10, 11) *Prove that two pairs of inverse points with respect to the same circle are cyclic, or collinear.*

Constructive description

((points A B)
(on O (b A B))
(lratio P O A r_1)
(lratio Q O A $\frac{1}{r_1}$)
(lratio R O B r_2)
(lratio S O B $\frac{1}{r_2}$)
(cocircle P Q R S))

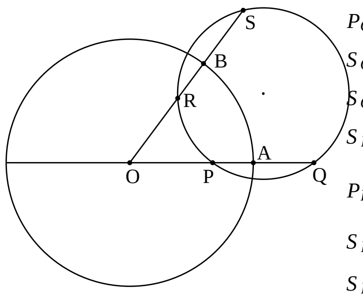


Figure 6-356

The eliminants

$S_{PQS} \stackrel{S}{=} \frac{S_{BPQ}}{r_2}$

$P_{RSR} \stackrel{S}{=} \frac{P_{ORO}\cdot r_2^2 - P_{ORO}\cdot r_2 + P_{BRB}\cdot r_2 - P_{BOB}\cdot r_2 + P_{BOB}}{r_2^2}$

$S_{PRS} \stackrel{S}{=} \frac{S_{OPR}\cdot r_2 - S_{OPR} + S_{BPR}}{r_2}$

$S_{QRS} \stackrel{S}{=} \frac{S_{OQR}\cdot r_2 - S_{OQR} + S_{BQR}}{r_2}$

$S_{PQR} \stackrel{R}{=} S_{BPQ}\cdot r_2 \quad P_{BRB} \stackrel{R}{=} (r_2-1)^2\cdot P_{BOB}$

$P_{ORO} \stackrel{R}{=} P_{BOB}\cdot r_2^2 \quad S_{BPR} \stackrel{R}{=} (r_2-1)\cdot S_{BOP}$

$S_{OPR} \stackrel{R}{=} S_{BOP}\cdot r_2 \quad S_{BQR} \stackrel{R}{=} (r_2-1)\cdot S_{BOQ}$

$S_{OQR} \stackrel{R}{=} S_{BOQ}\cdot r_2$

$S_{BPQ} \stackrel{Q}{=} \frac{-(S_{BOP}\cdot r_1 - S_{BOP} - S_{ABP})}{r_1}$

$P_{PQP} \stackrel{Q}{=} \frac{P_{OPO}\cdot r_1^2 - P_{OPO}\cdot r_1 + P_{APA}\cdot r_1 - P_{AOA}\cdot r_1 + P_{AOA}}{r_1^2}$

$S_{BOQ} \stackrel{Q}{=} \frac{S_{ABO}}{r_1} \quad S_{ABP} \stackrel{P}{=} -\left((r_1-1)\cdot S_{ABO}\right)$

$S_{BOP} \stackrel{P}{=} S_{ABO}\cdot r_1 \quad P_{APA} \stackrel{P}{=} (r_1-1)^2\cdot P_{AOA}$

$P_{OPO} \stackrel{P}{=} P_{AOA}\cdot r_1^2 \quad P_{BOB} \stackrel{O}{=} P_{AOA}$

The machine proof

$$\frac{P_{PQP}\cdot(-S_{QRS})\cdot S_{PRS}}{P_{RSR}\cdot S_{PQS}\cdot(-S_{PQR})}$$

$$\stackrel{S}{=} \frac{P_{PQP}\cdot(-S_{OQR}\cdot r_2 + S_{OQR} - S_{BQR})\cdot(S_{OPR}\cdot r_2 - S_{OPR} + S_{BPR})\cdot r_2\cdot r_2^2}{(P_{ORO}\cdot r_2^2 - P_{ORO}\cdot r_2 + P_{BRB}\cdot r_2 - P_{BOB}\cdot r_2 + P_{BOB})\cdot S_{BPQ}\cdot(-S_{PQR})\cdot r_2^2}$$

$$\stackrel{simplify}{=} \frac{P_{PQP}\cdot(S_{OQR}\cdot r_2 - S_{OQR} + S_{BQR})\cdot(S_{OPR}\cdot r_2 - S_{OPR} + S_{BPR})\cdot r_2}{(P_{ORO}\cdot r_2^2 - P_{ORO}\cdot r_2 + P_{BRB}\cdot r_2 - P_{BOB}\cdot r_2 + P_{BOB})\cdot S_{BPQ}\cdot S_{PQR}}$$

$$\stackrel{R}{=} \frac{P_{PQP}\cdot(S_{BOQ}\cdot r_2^2 - S_{BOQ})\cdot(S_{BOP}\cdot r_2^2 - S_{BOP})\cdot r_2}{(P_{BOB}\cdot r_2^4 - 2P_{BOB}\cdot r_2^2 + P_{BOB})\cdot S_{BPQ}\cdot S_{BPQ}\cdot r_2}$$

$$\stackrel{simplify}{=} \frac{P_{PQP}\cdot S_{BOQ}\cdot S_{BOP}}{P_{BOB}\cdot S_{BPQ}^2}$$

$$\stackrel{Q}{=} \frac{(P_{OPO}\cdot r_1^2 - P_{OPO}\cdot r_1 + P_{APA}\cdot r_1 - P_{AOA}\cdot r_1 + P_{AOA})\cdot S_{ABO}\cdot S_{BOP}\cdot r_1^2}{P_{BOB}\cdot(-S_{BOP}\cdot r_1 + S_{BOP} + S_{ABP})^2\cdot r_1\cdot r_1^2}$$

$$\stackrel{simplify}{=} \frac{(P_{OPO}\cdot r_1^2 - P_{OPO}\cdot r_1 + P_{APA}\cdot r_1 - P_{AOA}\cdot r_1 + P_{AOA})\cdot S_{ABO}\cdot S_{BOP}}{P_{BOB}\cdot(S_{BOP}\cdot r_1 - S_{BOP} - S_{ABP})^2\cdot r_1}$$

$$\stackrel{P}{=} \frac{(P_{AOA}\cdot r_1^4 - 2P_{AOA}\cdot r_1^2 + P_{AOA})\cdot S_{ABO}\cdot S_{ABO}\cdot r_1}{P_{BOB}\cdot(S_{ABO}\cdot r_1^2 - S_{ABO})^2\cdot r_1}$$

$$\stackrel{simplify}{=} \frac{P_{AOA}}{P_{BOB}} \stackrel{O}{=} \frac{P_{AOA}}{P_{AOA}} \stackrel{simplify}{=} 1$$

Example 6.357 (0.067, 6, 9) *The inverse of a line not passing through the center of inversion is a circle through that point.*

Constructive description

((points O A)
(lratio Q O A $\frac{1}{r}$)
(tratio R Q O r_1)
(lratio P O A r)
(midpoint U P O)
(lratio G O R $\frac{P_{OAO}}{P_{ORO}}$)
(eqdistance G U U O))

The eliminants

$P_{UGU} \overset{G}{=} P_{RUR} \cdot P_{OAO} + P_{OUO} \cdot P_{ORO} - P_{OUO} \cdot P_{OAO} - P_{ORO} \cdot P_{OAO} + P_{OAO}^2 / P_{ORO}$

$P_{OUO} \overset{U}{=} \frac{1}{4}(P_{OPO})$

$P_{RUR} \overset{U}{=} \frac{1}{4}(2P_{RPR} - P_{OPO} + 2P_{ORO})$

$P_{OPO} \overset{P}{=} P_{OAO} \cdot (r)^2$

$P_{RPR} \overset{P}{=} P_{ARA} \cdot r - P_{ORO} \cdot r + P_{ORO} + P_{OAO} \cdot r^2 - P_{OAO} \cdot r$

$P_{ORO} \overset{R}{=} (r_1^2 + 1) \cdot P_{OQO}$

$P_{ARA} \overset{R}{=} P_{AQA} + P_{OQO} \cdot r_1^2$

$P_{OQO} \overset{Q}{=} \frac{P_{OAO}}{(r)^2}$

$P_{AQA} \overset{Q}{=} \frac{(r-1)^2 \cdot P_{OAO}}{(r)^2}$

The machine proof

$$\frac{P_{UGU}}{P_{OUO}}$$

$$= \frac{P_{RUR} \cdot P_{ORO} \cdot P_{OAO} + P_{OUO} \cdot P_{ORO}^2 - P_{OUO} \cdot P_{ORO} \cdot P_{OAO} - P_{ORO}^2 \cdot P_{OAO} + P_{ORO} \cdot P_{OAO}^2}{P_{OUO} \cdot P_{ORO}^2}$$

$$\overset{simplify}{=} \frac{P_{RUR} \cdot P_{OAO} + P_{OUO} \cdot P_{ORO} - P_{OUO} \cdot P_{OAO} - P_{ORO} \cdot P_{OAO} + P_{OAO}^2}{P_{OUO} \cdot P_{ORO}}$$

$$\overset{U}{=} \frac{\frac{1}{2} P_{RPR} \cdot P_{OAO} + \frac{1}{4} P_{OPO} \cdot P_{ORO} - \frac{1}{2} P_{OPO} \cdot P_{OAO} - \frac{1}{2} P_{ORO} \cdot P_{OAO} + P_{OAO}^2}{(\frac{1}{4} P_{OPO}) \cdot P_{ORO}}$$

$$\overset{P}{=} \frac{2P_{ARA} \cdot P_{OAO} \cdot r + P_{ORO} \cdot P_{OAO} \cdot r^2 - 2P_{ORO} \cdot P_{OAO} \cdot r - 2P_{OAO}^2 \cdot r + 4P_{OAO}^2}{P_{OAO} \cdot r^2 \cdot P_{ORO}}$$

$$\overset{simplify}{=} \frac{2P_{ARA} \cdot r + P_{ORO} \cdot r^2 - 2P_{ORO} \cdot r - 2P_{OAO} \cdot r + 4P_{OAO}}{(r)^2 \cdot P_{ORO}}$$

$$\overset{R}{=} \frac{2P_{AQA} \cdot r + P_{OQO} \cdot r_1^2 \cdot r^2 + P_{OQO} \cdot r^2 - 2P_{OQO} \cdot r - 2P_{OAO} \cdot r + 4P_{OAO}}{(r)^2 \cdot (P_{OQO} \cdot r_1^2 + P_{OQO})}$$

$$\overset{simplify}{=} \frac{2P_{AQA} \cdot r + P_{OQO} \cdot r_1^2 \cdot r^2 + P_{OQO} \cdot r^2 - 2P_{OQO} \cdot r - 2P_{OAO} \cdot r + 4P_{OAO}}{(r)^2 \cdot (r_1^2 + 1) \cdot P_{OQO}}$$

$$\overset{Q}{=} \frac{(P_{OAO} \cdot r_1^2 \cdot r^4 + P_{OAO} \cdot r^4) \cdot r^2}{(r)^2 \cdot (r_1^2 + 1) \cdot P_{OAO} \cdot (r^2)^2} \overset{simplify}{=} 1$$

Example 6.358 (0.133, 5, 11) *If the circle U passes trough two inverse points of circle O. Then the inverse of any point on circle U with regard to circle O is on circle U. In other words, the inverse of circle U with regard to circle O is itself.*

Constructive description

((points A O X)
(lratio P O A r) (lratio Q O A $\frac{1}{r}$)
(midpoint M_U P Q)
(tratio U M_U P r_U)
(inter E (l P X) (cir U P))
(inter F (l E O) (cir U E))
(inversion O A E F))

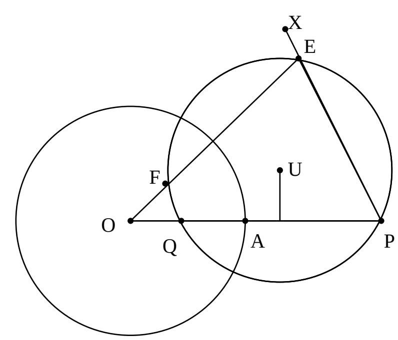


Figure 6-358

The machine proof

$$\frac{P_{AOA}}{P_{EOF}} \overset{F}{=} \frac{P_{AOA}\cdot P_{OEO}}{-P_{UEU}\cdot P_{OEO}+P_{OEO}\cdot P_{OUO}}$$

$$\overset{simplify}{=} \frac{P_{AOA}}{-(P_{UEU}-P_{OUO})} \overset{E}{=} \frac{P_{AOA}}{-(P_{PUP}-P_{OUO})}$$

$$\overset{U}{=} \frac{-P_{AOA}}{P_{PM_UP}-P_{OM_UO}} \overset{M_U}{=} \frac{-P_{AOA}}{-\frac{\overline{OP}}{\overline{PQ}}^{2}\cdot P_{PQP}-\frac{\overline{OP}}{\overline{PQ}}\cdot P_{PQP}}$$

$$\overset{simplify}{=} \frac{P_{AOA}}{(\frac{\overline{OP}}{\overline{PQ}}+1)\cdot\frac{\overline{OP}}{\overline{PQ}}\cdot P_{PQP}}$$

$$\overset{Q}{=} \frac{P_{AOA}\cdot r^2\cdot(\frac{\overline{OP}}{\overline{AO}}\cdot r+1)^2}{(-\frac{\overline{OP}}{\overline{AO}}\cdot r)\cdot(P_{OPO}\cdot r^2-P_{OPO}\cdot r+P_{APA}\cdot r-P_{AOA}\cdot r+P_{AOA})}$$

$$\overset{simplify}{=} \frac{P_{AOA}\cdot r\cdot(\frac{\overline{OP}}{\overline{AO}}\cdot r+1)^2}{-\frac{\overline{OP}}{\overline{AO}}\cdot(P_{OPO}\cdot r^2-P_{OPO}\cdot r+P_{APA}\cdot r-P_{AOA}\cdot r+P_{AOA})}$$

$$\overset{P}{=} \frac{P_{AOA}\cdot r\cdot(r^2-1)^2\cdot(-1)}{-r\cdot(P_{AOA}\cdot r^4-2P_{AOA}\cdot r^2+P_{AOA})\cdot((-1))^2} \overset{simplify}{=} 1$$

The eliminants

$$P_{EOF}\overset{F}{=}-(P_{UEU}-P_{OUO})$$

$$P_{UEU}\overset{E}{=}P_{PUP}$$

$$P_{OUO}\overset{U}{=}P_{PM_UP}\cdot r_U^2+P_{OM_UO}$$

$$P_{PUP}\overset{U}{=}(r_U^2+1)\cdot P_{PM_UP}$$

$$P_{PQP}\overset{Q}{=}\frac{P_{OPO}\cdot r^2-P_{OPO}\cdot r+P_{APA}\cdot r-P_{AOA}\cdot r+P_{AOA}}{(r)^2}$$

$$P_{OM_UO}\overset{M_U}{=}\frac{1}{4}\left((2\frac{\overline{OP}}{\overline{PQ}}+1)^2\cdot P_{PQP}\right)$$

$$P_{PM_UP}\overset{M_U}{=}\frac{1}{4}\left(P_{PQP}\right)$$

$$\frac{\overline{OP}}{\overline{PQ}}\overset{Q}{=}\frac{-r}{\frac{\overline{OP}}{\overline{AO}}\cdot r+1}\cdot\frac{\overline{OP}}{\overline{AO}}$$

$$P_{APA}\overset{P}{=}(r-1)^2\cdot P_{AOA}$$

$$P_{OPO}\overset{P}{=}P_{AOA}\cdot(r)^2,\ \frac{\overline{OP}}{\overline{AO}}\overset{P}{=}-\left(r\right)$$

Example 6.359 (10.817, 267, 29) *The inverse of a circle not passing through the center of inversion is a circle.*

Constructive description
((points O A X)
(lratio P O A r_1)
(lratio Q O A r_2)
(midpoint U P Q)
(lratio E O A $\frac{1}{r_1}$)
(lratio F O A $\frac{1}{r_2}$)
(midpoint I E F)
(inter R (l P X) (cir U P))
(lratio G O R $\frac{P_{OAO}}{P_{ORO}}$)
(eqdistance G I I E))

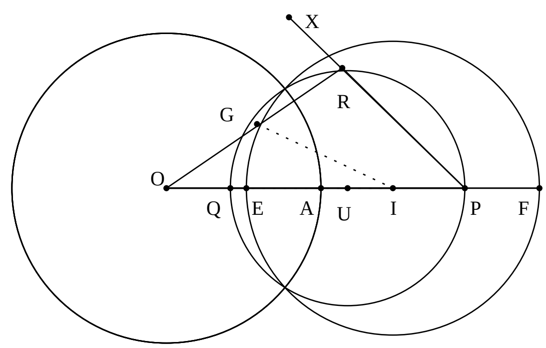


Figure 6-359

Example 6.360 (0.033, 4, 24) *The harmonic conjugate of a fixed point with respect to a variable pair of points which lie on a given circle and are collinear with the fixed point, describes a straight line. This line is called the polar of the fixed point with the circle, and the fixed point is said to be the pole of the line.*

Constructive description
((circle E F A) (circumcenter O E F A)
(inter P (l O A) (l E F))
(lratio Q O A $\frac{\overline{OA}}{\overline{OP}}$)
(inter M (l E F) (t Q Q A))
(harmonic E F M P))

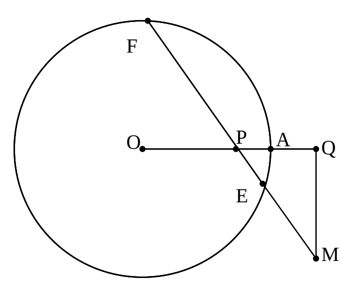


Figure 6-360

Example 6.361 (0.216, 5, 6) *Let P and Q be inverse points with regard to circle OA. Then for any point C on circle O, we have $CP \cdot AQ = CQ \cdot AP$.*

Constructive description
((points $A\ B\ C$)
(circumcenter $O\ A\ B\ C$)
(lratio $P\ O\ A\ r$)
(lratio $Q\ O\ A\ \frac{1}{r}$)
(eq-product $C\ P\ A\ Q\ C\ Q\ A\ P$))

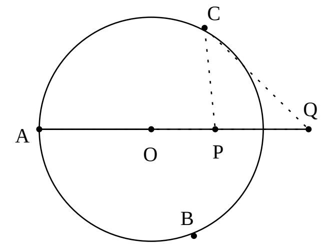


Figure 6-361

Example 6.362 (0.400, 8, 16) *If the circle (B) passes through the center A of the circle (A), and a diameter of (A) meets the common chord of the two circles in F and the circle (B) again in G, show that the points F, G are inverse for the circle (A).*

Constructive description
((points $D\ E\ C$)
(circumcenter $A\ D\ E\ C$) (midpoint $I\ E\ C$)
(inter B (l $A\ I$) (b $A\ E$))
(inter F (l $E\ C$) (l $A\ D$))
(inter G (l $A\ D$) (cir $B\ A$))
(inversion $A\ D\ G\ F$))

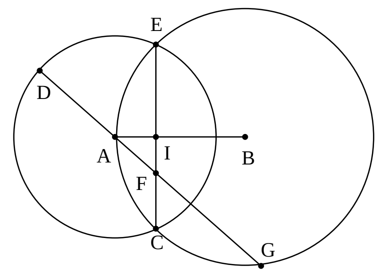


Figure 6-362

Example 6.363 (0.466, 6, 15) *Show that the two lines joining any point of a circle to the ends of a given chord meet the diameter perpendicular to that chord in two inverse points.*

Constructive description
((points $A\ E\ D$)
(circumcenter $O\ A\ E\ D$)
(foot $I\ E\ A\ O$) (lratio $F\ I\ E\ -1$)
(inter P (l $O\ A$) (l $D\ E$))
(inter Q (l $O\ A$) (l $D\ F$))
(inversion $O\ A\ P\ Q$))

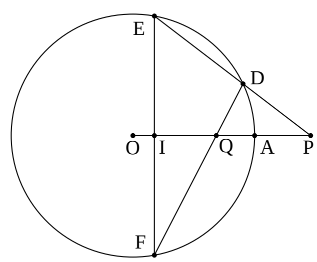


Figure 6-363

Example 6.364 (0.700, 8, 25) *TP, TQ are the tangents at the extremities of a chord PQ of a circle. The tangent at any point R of the circle meets PQ in S; prove that TR is the polar of S.*

Constructive description
((points $P\ Q\ R$)
(circumcenter $O\ P\ Q\ R$)

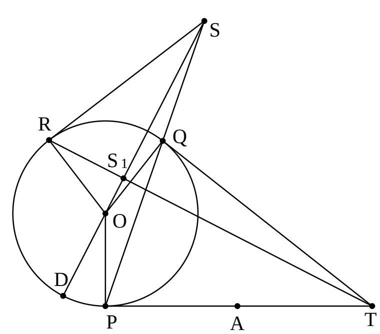


Figure 6-364

(on A (t P P O))
(inter T (l A P) (t Q Q O))
(inter S (l P Q) (t R R O))
(inter S_1 (l T R) (l O S))
(inversion O P S S_1))

6.5.4 Orthogonal Circles

Definition. *Two circles O_1 and O_2 with a common point A are said to be orthogonal if $O_1A \perp O_2A$.*

Example 6.365 (0.050, 4, 7) *If two circles are orthogonal, any two diametrically opposite points of one circle are conjugate with respect to the other circle.*

Constructive description
((points D E)
(on O_1 (b D E))
(lratio F O_1 E -1)
(on O (t D D O_1))
(foot G F O E)
(inversion O D G E))

The eliminants

$$P_{EOG} \stackrel{G}{=} P_{EOF}$$
$$P_{EOF} \stackrel{O}{=} P_{EDF}+P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2 - 4S_{DO_1F}\cdot \frac{\overline{OD}}{\overline{DO_1}} + 4S_{DEO_1}\cdot \frac{\overline{OD}}{\overline{DO_1}}$$
$$P_{DOD} \stackrel{O}{=} P_{DO_1D}\cdot (\frac{\overline{OD}}{\overline{DO_1}})^2$$
$$S_{DO_1F} \stackrel{F}{=} S_{DEO_1}$$
$$P_{EDF} \stackrel{F}{=} 2P_{EDO_1} - P_{DED}$$
$$P_{EDO_1} = -\frac{1}{2}(P_{EO_1E} - P_{DO_1D} - P_{DED})$$
$$P_{EO_1E} \stackrel{O_1}{=} P_{DO_1D}$$

The machine proof

$$\frac{P_{DOD}}{P_{EOG}} \stackrel{G}{=} \frac{P_{DOD}}{P_{EOF}}$$
$$\stackrel{O}{=} \frac{P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2}{P_{EDF}+P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2 - 4S_{DO_1F}\cdot \frac{\overline{OD}}{\overline{DO_1}} + 4S_{DEO_1}\cdot \frac{\overline{OD}}{\overline{DO_1}}}$$
$$\stackrel{F}{=} \frac{P_{DO_1D}\cdot (\frac{\overline{OD}}{\overline{DO_1}})^2}{2P_{EDO_1}+P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2 - P_{DED}}$$
$$\stackrel{py}{=} \frac{P_{DO_1D}\cdot (\frac{\overline{OD}}{\overline{DO_1}})^2\cdot(2)}{-2P_{EO_1E}+2P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2 + 2P_{DO_1D}} \stackrel{O_1}{=} \frac{-P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2}{-P_{DO_1D}\cdot \frac{\overline{OD}}{\overline{DO_1}}^2} \stackrel{simplify}{=} 1$$

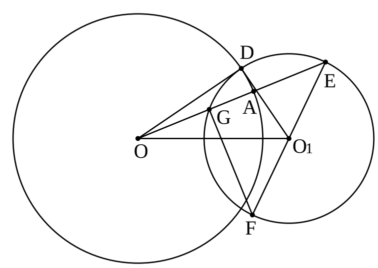


Figure 6-365

Example 6.366 (0.016, 1, 3) *Show that the two poles of the common chord of two orthogonal circles with respect to these circles coincide with the centers of the given circles.*

Constructive description

((points A R)
(on O_1 (b A R))
(foot D A R O_1)
(lratio B D A -1)
(inter O_2 (l O_1 D) (t A A O_1))
(inversion O_1 R D O_2))

The machine proof

$$\left(-\frac{\overline{O_1D}}{\overline{RO_1}}\right)/\left(-\frac{\overline{RO_1}}{\overline{O_1O_2}}\right)$$

$$\stackrel{O_2}{=} \frac{P_{AO_1A}}{P_{AO_1R}} \cdot -\frac{\overline{O_1D}}{\overline{RO_1}}$$

$$\stackrel{D}{=} \frac{-P_{AO_1R}\cdot P_{AO_1A}}{P_{AO_1R}\cdot(-P_{RO_1R})}$$

$$\stackrel{simplify}{=} \frac{P_{AO_1A}}{P_{RO_1R}}$$

$$\stackrel{O_1}{=} \frac{P_{AO_1A}}{P_{AO_1A}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{RO_1}}{\overline{O_1O_2}} \stackrel{O_2}{=} \frac{-P_{AO_1R}}{P_{AO_1A}}$$

$$\frac{\overline{O_1D}}{\overline{RO_1}} \stackrel{D}{=} \frac{P_{AO_1R}}{-P_{RO_1R}},\ P_{RO_1R} \stackrel{O_1}{=} P_{AO_1A}$$

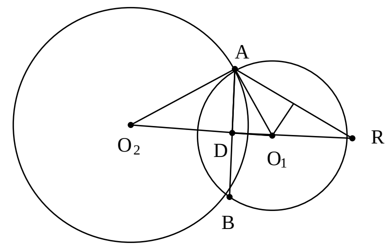


Figure 6-366

Example 6.367 (0.001, 1, 5) *A circle orthogonal to two given circles has its center on the radical axis of the two circles.*

Constructive description: ((points D_1 D_2) (on O (b D_1 D_2)) (on O_1 (t D_1 D_1 O)) (on O_2 (t D_2 D_2 O)) (on-radical O O_1 D_1 O_2 D_2))

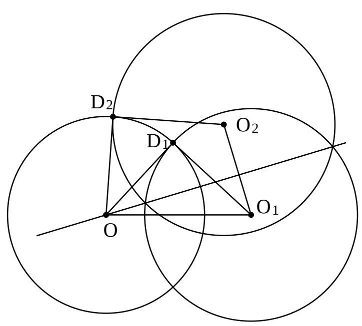


Figure 6-367

The machine proof

$$\frac{P_{OO_1O}-P_{D_1O_1D_1}}{P_{OO_2O}-P_{D_2O_2D_2}}$$

$$\stackrel{O_2}{=} \frac{P_{OO_1O}-P_{D_1O_1D_1}}{P_{D_2OD_2}}$$

$$\stackrel{O_1}{=} \frac{P_{D_1OD_1}}{P_{D_2OD_2}}$$

$$\stackrel{O}{=} \frac{P_{D_1OD_1}}{P_{D_1OD_1}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{D_2O_2D_2} \stackrel{O_2}{=} P_{D_2OD_2}\cdot\left(\frac{\overline{O_2D_2}}{\overline{D_2O}}\right)^2$$

$$P_{OO_2O} \stackrel{O_2}{=} \left(\frac{\overline{O_2D_2}}{\overline{D_2O}}^2+1\right)\cdot P_{D_2OD_2}$$

$$P_{D_1O_1D_1} \stackrel{O_1}{=} P_{D_1OD_1}\cdot\left(\frac{\overline{O_1D_1}}{\overline{D_1O}}\right)^2$$

$$P_{OO_1O} \stackrel{O_1}{=} \left(\frac{\overline{O_1D_1}}{\overline{D_1O}}^2+1\right)\cdot P_{D_1OD_1}$$

$$P_{D_2OD_2} \stackrel{O}{=} P_{D_1OD_1}$$

Example 6.368 (0.283, 8, 12) *If two circles are orthogonal, any two points of one of them collinear with the center of the second circle are inverse for that second circle.*

Constructive description

((points M E F)
(circumcenter B M E F)

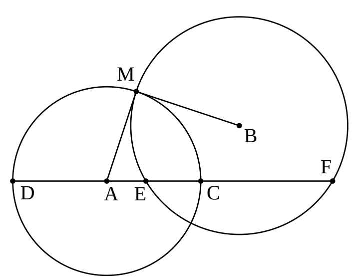


Figure 6-368

(inter A (l E F) (t M M B))
(inversion A M E F))

Example 6.369 (0.350, 4, 18) *The two lines joining the points of intersection of two orthogonal circles to a point on one of the circles meet the other circle in two diametrically opposite points.*

Constructive description
((points C E F)
(circumcenter Q C E F)
(lratio D Q C -1)
(midpoint I E F)
(inter P (l Q I) (t E E Q))
(inter A (l E C) (cir P E))
(inter A_1 (l F D) (l C E))
($\frac{\overline{CA}}{\overline{EA}} = \frac{\overline{CA_1}}{\overline{EA_1}}$))

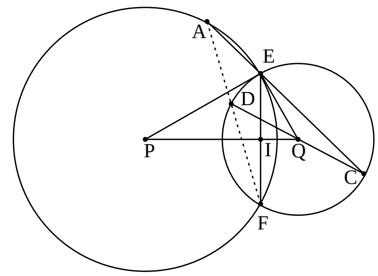


Figure 6-369

Example 6.370 (0.850, 11, 19) *Show that in a triangle ABC the circles on AH and BC as diameters are orthogonal.*

Constructive description
((points A B C)
(orthocenter H A B C)
(midpoint M B C)
(midpoint N A H)
(inter I (l A B) (cir N A))
(perpendicular M I N I))

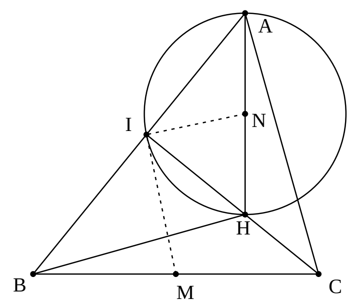


Figure 6-370

Example 6.371 (1.183, 10, 12) *The circle IBC is orthogonal to the circle on I_bI_c as diameter.*

Constructive description
((points B C I)
(incenter A I B C)
(inter I_B (t C C I) (l B I))
(inter I_C (t B B I) (l C I))
(circumcenter O B C I)
(midpoint M I_B I_C)
(perpendicular M B O B))

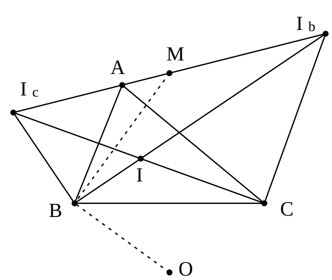


Figure 6-371

Example 6.372 (0.533, 16, 32) *Show that given two perpendicular diameters of two orthogonal circles, the lines joining an end of one of these diameters to the ends of the other pass through the points common to the two circles.*

Constructive description
```
( (points P D I)
 (circumcenter A P D I)
 (lratio E A D −1)
 (midpoint X P I)
 (inter B (l A X)  (t P P A))
 (inter G (l I E)  (cir B I))
 (lratio F B G −1)
 (perpendicular F G D E) )
```

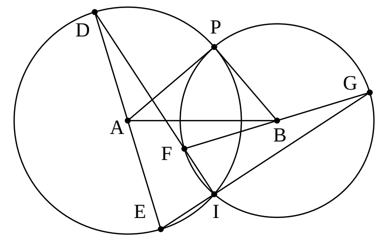


Figure 6-372

Example 6.373 (0.083, 4, 15) *Show that if AB is a diameter and M any point of a circle, center O, the two circles AMO, BMO are orthogonal.*

Constructive description
```
( (points A M)
 (on B (t M M A))
 (midpoint O A B)
 (circumcenter I A M O)
 (circumcenter J B M O)
 (perpendicular I M M J) )
```

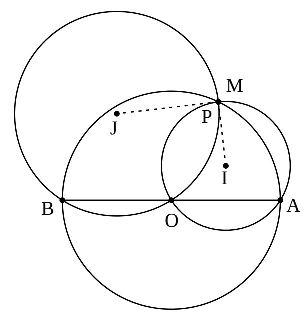


Figure 6-373

Example 6.374 (0.783, 11, 23) *If the line joining the ends A, B of a diameter AB of a given circle* (O) *to a given point P meets* (O) *again in A_1, B_1, show that the circle PA_1B_1 is orthogonal to* (O).

Constructive description
```
( (points A1 B1 A)
 (circumcenter O A1 B1 A)
 (lratio B O A −1)
 (inter P (l A A1)  (l B B1))
 (circumcenter O1 A1 B1 P)
 (perpendicular O1 A1 A1 O) )
```

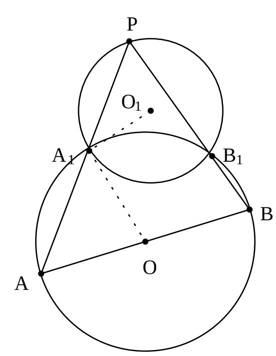


Figure 6-374

6.5.5 The Simson Line

Let D be a point on the circumscribed circle of triangle ABC. From D three perpendiculars are drawn to the three sides BC, AC, and AB of triangle ABC. Let E, F, and G be the three feet respectively. Then E,F and G are collinear. The line EFG is called the *Simson line* of the point D with respect to the triangle ABC, and D is called the *pole* of the simson line. For machine proofs of Simson's theorem see Example 3.79 on page 144, Example 3.106 on page 164 and Example 5.55 on page 248.

Example 6.375 (0.933, 3, 37) *The Simson line bisects the line joining its pole to the orthocenter of the triangle.*

Constructive description
((circle A B C D)
(orthocenter H A B C)
(foot F D A C)
(foot G D A B)
(inter N (l G F) (l D H))
(midpoint N D H))

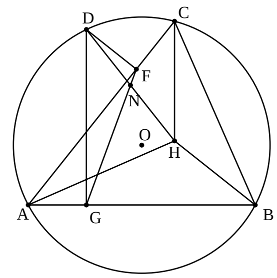


Figure 6-375

Example 6.376 (0.050, 1, 8) *Let D be a point on the circumcircle of triangle ABC. If line DA is parallel to BC, show that the Simson line $D(ABC)$ is parallel to the circumradius OA.*

Constructive description
((points A B C)
(circumcenter O A B C)
(pratio P_D A B C 1)
(inter D (l A P_D) (cir O A))
(foot F D A C)
(foot G D A B)
(parallel G F O A))

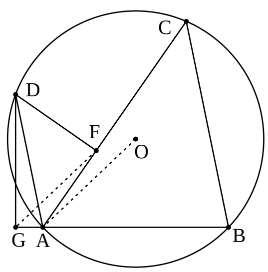


Figure 6-376

The machine proof

$$\frac{-S_{AOG}}{-S_{AOF}}$$

$$\overset{G}{=} \frac{P_{BAD}\cdot S_{ABO}}{(-S_{AOF})\cdot P_{ABA}}$$

$$\overset{F}{=} \frac{-P_{BAD}\cdot S_{ABO}\cdot P_{ACA}}{(-P_{CAD}\cdot S_{ACO})\cdot P_{ABA}}$$

$$\overset{D}{=} \frac{(2P_{OAP_D}\cdot P_{BAP_D})\cdot S_{ABO}\cdot P_{ACA}\cdot P_{AP_DA}}{(2P_{OAP_D}\cdot P_{CAP_D})\cdot S_{ACO}\cdot P_{ABA}\cdot P_{AP_DA}}$$

$$\overset{simplify}{=} \frac{P_{BAP_D}\cdot S_{ABO}\cdot P_{ACA}}{P_{CAP_D}\cdot S_{ACO}\cdot P_{ABA}}$$

$$\overset{P_D}{=} \frac{P_{ABC}\cdot S_{ABO}\cdot P_{ACA}}{P_{ACB}\cdot S_{ACO}\cdot P_{ABA}\cdot(-1)}$$

$$\overset{O}{=} \frac{-P_{ABC}\cdot P_{ACB}\cdot P_{ABA}\cdot P_{ACA}\cdot(-32S_{ABC})}{P_{ACB}\cdot P_{ACA}\cdot P_{ABC}\cdot P_{ABA}\cdot(32S_{ABC})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$S_{AOG} \overset{G}{=} \frac{-P_{BAD}\cdot S_{ABO}}{P_{ABA}}$$

$$S_{AOF} \overset{F}{=} \frac{-P_{CAD}\cdot S_{ACO}}{P_{ACA}}$$

$$P_{CAD} \overset{D}{=} \frac{(2)\cdot P_{OAP_D}\cdot P_{CAP_D}}{P_{AP_DA}}$$

$$P_{BAD} \overset{D}{=} \frac{(2)\cdot P_{OAP_D}\cdot P_{BAP_D}}{P_{AP_DA}}$$

$$P_{CAP_D} \overset{P_D}{=} P_{ACB}$$

$$P_{BAP_D} \overset{P_D}{=} -\left(P_{ABC}\right)$$

$$S_{ACO} \overset{O}{=} \frac{P_{ACA}\cdot P_{ABC}}{(-32)\cdot S_{ABC}}$$

$$S_{ABO} \overset{O}{=} \frac{P_{ACB}\cdot P_{ABA}}{(32)\cdot S_{ABC}}$$

Example 6.377 (0.033, 1, 8) *If E, F, G are the feet of the perpendiculars from a point D of the circumcircle of a triangle ABC upon its sides BC, CA, AB, prove that the triangle DFG, DBC are similar.*

Constructive description
((circle A B C D)
(foot F D A C)
(foot G D A B)
(eq-product D B D F D C D G))

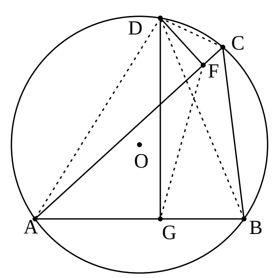


Figure 6-377

The machine proof

$$\frac{P_{BDB}\cdot P_{DFD}}{P_{CDC}\cdot P_{DGD}}$$

$$\overset{G}{=} \frac{P_{BDB}\cdot P_{DFD}\cdot P_{ABA}}{P_{CDC}\cdot(16S_{ABD}^2)}$$

$$\overset{F}{=} \frac{P_{BDB}\cdot(16S_{ACD}^2)\cdot P_{ABA}}{(16)\cdot P_{CDC}\cdot S_{ABD}^2\cdot P_{ACA}}$$

$$\overset{co-cir}{=} \frac{(2\overline{BD}^2)\cdot(-\overline{CD}\cdot\overline{AD}\cdot\overline{AC})^2\cdot(2\overline{AB}^2)\cdot(2d)^2}{(2\overline{CD}^2)\cdot(-\overline{BD}\cdot\overline{AD}\cdot\overline{AB})^2\cdot(2\overline{AC}^2)\cdot(2d)^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{DGD}\overset{G}{=}\frac{(16)\cdot S_{ABD}^2}{P_{ABA}}$$
$$P_{DFD}\overset{F}{=}\frac{(16)\cdot S_{ACD}^2}{P_{ACA}}$$
$$P_{ACA}=2(\overline{AC}^2)$$
$$S_{ABD}=\frac{\overline{BD}\cdot\overline{AD}\cdot\overline{AB}}{(-2)\cdot d}$$
$$P_{CDC}=2(\overline{CD}^2)$$
$$P_{ABA}=2(\overline{AB}^2)$$
$$S_{ACD}=\frac{\overline{CD}\cdot\overline{AD}\cdot\overline{AC}}{(-2)\cdot d}$$
$$P_{BDB}=2(\overline{BD}^2)$$

Example 6.378 (0.050, 3, 5) *Simson lines corresponding to pairs of diametrically opposite points on the circumcircle of a triangle meet a side of the triangle in two isotomic points.*

Constructive description
((circle $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ D$)
(lratio $D_1\ O\ D\ -1$)
(foot $G\ D\ A\ B$)
(foot $G_1\ D_1\ A\ B$)
(midpoint $C_1\ A\ B$)
(midpoint $C_1\ G\ G_1$))

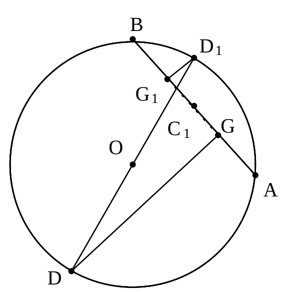


Figure 6-378

The machine proof

$$-\frac{\overline{GC_1}}{\overline{G_1C_1}}$$

$$\overset{C_1}{=} \frac{\frac{\overline{AG}}{\overline{AB}}-\frac{1}{2}}{-\frac{\overline{AG_1}}{\overline{AB}}+\frac{1}{2}}$$

$$\overset{G_1}{=} \frac{-(\frac{\overline{AG}}{\overline{AB}}-\frac{1}{2})\cdot P_{ABA}}{P_{BAD_1}-\frac{1}{2}P_{ABA}}$$

$$\overset{G}{=} \frac{-(P_{BAD}-\frac{1}{2}P_{ABA})\cdot P_{ABA}}{(P_{BAD_1}-\frac{1}{2}P_{ABA})\cdot P_{ABA}}$$

$$\overset{simplify}{=} \frac{-(P_{BAD}-\frac{1}{2}P_{ABA})}{P_{BAD_1}-\frac{1}{2}P_{ABA}}$$

$$\overset{D_1}{=} \frac{-(P_{BAD}-\frac{1}{2}P_{ABA})}{2P_{BAO}-P_{BAD}-\frac{1}{2}P_{ABA}}$$

$$\overset{O}{=} \frac{-(P_{BAD}-\frac{1}{2}P_{ABA})\cdot(2)}{(2)\cdot(-P_{BAD}+\frac{1}{2}P_{ABA})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{GC_1}}{\overline{G_1C_1}}\overset{C_1}{=}\frac{\frac{\overline{AG}}{\overline{AB}}-\frac{1}{2}}{\frac{\overline{AG_1}}{\overline{AB}}-\frac{1}{2}}$$
$$\frac{\overline{AG_1}}{\overline{AB}}\overset{G_1}{=}\frac{P_{BAD_1}}{P_{ABA}}$$
$$\frac{\overline{AG}}{\overline{AB}}\overset{G}{=}\frac{P_{BAD}}{P_{ABA}}$$
$$P_{BAD_1}\overset{D_1}{=}2(P_{BAO}-\frac{1}{2}P_{BAD})$$
$$P_{BAO}\overset{O}{=}\frac{1}{2}(P_{ABA})$$

Example 6.379 (1.817, 4, 35) *If the perpendicular from a point D of the circumcircle (O) of a triangle ABC to the sides BC, CA, AB meet (O) again in the points N, M, L, the three lines AN, BM, CL are parallel to the simson of D for ABC.*

Constructive description
((circle $A\ B\ C\ D$)
(circumcenter $O\ A\ B\ C$)
(foot $F\ D\ A\ C$)

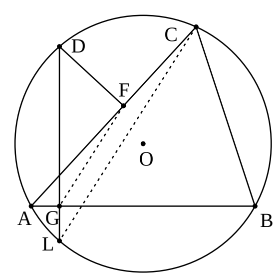


Figure 6-379

(foot G D A B)
(inter L (l D G) (cir O D))
(parallel C L F G))

Example 6.380 (4.283, 26, 25) *Show that the simson of the point where an altitude cuts the circumcircle again passes through the foot of the altitude and is antiparallel to the corresponding side of the triangle with respect to the other two sides.*

Constructive description
((points A B C)
(circumcenter O A B C)
(foot G C A B)
(inter D (l C G) (cir O C))
(foot E D A C)
(foot F D B C)
(inter G_1 (l A B) (l E F))
($\frac{\overline{AG}}{\overline{BG}} = \frac{\overline{AG_1}}{\overline{BG_1}}$))

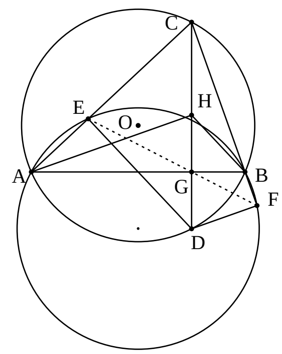


Figure 6-380

Example 6.381 (1.633, 4, 39) *If the Simson line $D(ABC)$ meets BC in E and the altitude from A in K, show that the line DK is parallel to EH, where H is the orthocenter of ABC.*

Constructive description
((circle A B C D)
(foot E D B C)
(foot G D A B)
(orthocenter H A B C)
(inter K (l E G) (l A H))
(parallel D K E H))

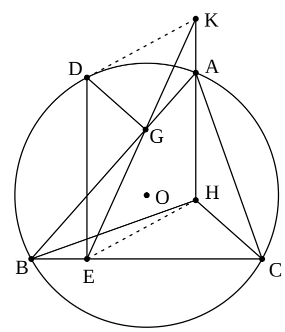


Figure 6-381

Example 6.382 (1.483, 7, 44) *Show that the symmetries, with respect to the sides of a triangle, of a point on its circumcircle lie on a line passing through the orthocenter of the triangle.*

Constructive description
((circle A B C D)
(orthocenter H A B C)
(foot F D A C)
(foot G D A B)
(lratio G_1 G D -1)
(lratio F_1 F D -1)
(collinear F_1 G_1 H)

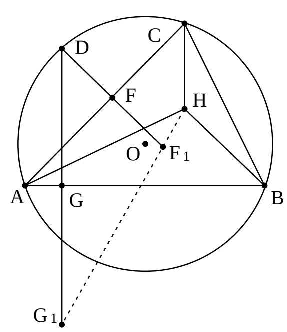


Figure 6-382

Example 6.383 (4.133, 4, 53) *The four Simson lines of four points of a circle, each taken for the triangle formed by the remaining three points, are concurrent.*

Constructive description
((circle A B C D)
(circumcenter O A B C)
(foot E D A B) (foot F D A C)
(foot G C A B) (foot H C A D)
(foot I A D B) (foot J A B C)
(inter K (l H G) (l E F))
(inter M (l E F) (l I J)) ($\frac{\overline{EK}}{\overline{FK}} = \frac{\overline{EM}}{\overline{FM}}$))

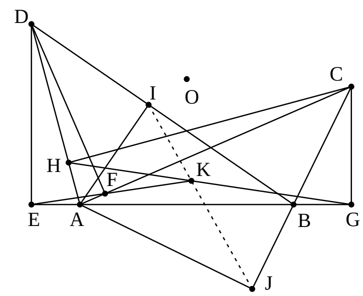


Figure 6-383

Example 6.384 (5.333, 58, 76) *If two triangles are inscribed in the same circle and are symmetrical with respect to the center of that circle, show that the two simsons of any point of the circle for these triangles are rectangular.*

Constructive description
((circle A B C D)
(circumcenter O A B C)
(lratio A_1 O A -1)
(lratio B_1 O B -1) (lratio C_1 O C -1)
(foot G D A B) (foot F D A C)
(foot G_1 D A_1 B_1)
(foot F_1 D A_1 C_1)
(perpendicular G_1 F_1 G F))

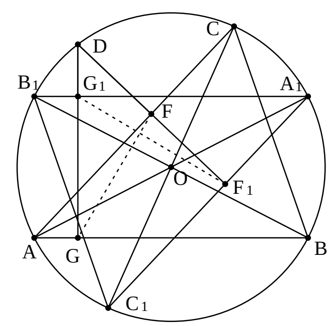


Figure 6-384

Example 6.385 (1.683, 15, 48) *Show that the Simson lines of the three points where the altitudes of a triangle cut the circumcircle again form a triangle homothetic to the orthic triangle, and its circumcenter coincides with orthocenter of the orthic triangle.*

Constructive description
((points A B C)
(circumcenter O A B C) (orthocenter H A B C)
(foot D A B C) (foot E B A C)
(foot F C A B) (orthocenter G D E F)
(lratio D_1 D H -1) (lratio E_1 E H -1)
(lratio F_1 F H -1) (foot D_2 D_1 A B)
(foot E_2 E_1 A B) (foot F_2 F_1 A C)
(inter A_1 (l E E_2) (l D D_2))
(inter B_1 (l D D_2) (l F F_2)) (inter C_1 (l E E_2) (l F F_2))
(inter I (l A_1 F) (l B_1 E)) ($\frac{\overline{IB_1}}{\overline{IE}} = \frac{\overline{IA_1}}{\overline{IF}}$))

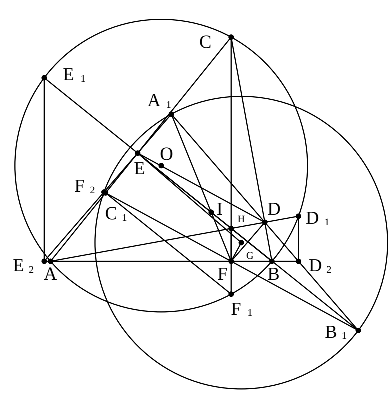


Figure 6-385

Example 6.386 (0.633, 3, 40) *The perpendiculars dropped upon the sides BC, CA, AB of the triangle from a point P on its circumcircle meet these sides in L, M, N and the circle in* A_1, B_1, C_1. *The Simson line LMN meets* B_1C_1, C_1A_1, A_1B_1 *in* L_1, M_1, N_1. *Prove that the lines* AL_1, BM_1, CN_1 *are concurrent.*

Constructive description
((circle $A\ B\ C\ P$) (circumcenter $O\ A\ B\ C$)
(midpoint $X\ A\ B$) (foot $L\ P\ B\ C$)
(foot $M\ P\ A\ C$) (foot $N\ P\ A\ B$)
(foot $D\ O\ P\ L$) (foot $E\ O\ P\ M$)
(foot $F\ O\ P\ N$) (lratio $A_1\ D\ P\ -1$)
(lratio $B_1\ E\ P\ -1$) (lratio $C_1\ F\ P\ -1$)
(inter L_1 (l $B_1\ C_1$) (l $M\ N$))
(inter M_1 (l $C_1\ A_1$) (l $M\ N$))
(inter N_1 (l $A_1\ B_1$) (l $M\ N$)) (inter I (l $B\ M_1$) (l $A\ L_1$))
(inter J (l $A\ L_1$) (l $C\ N_1$)) ($\frac{\overline{AI}}{\overline{L_1I}} = \frac{\overline{AJ}}{\overline{L_1J}}$))

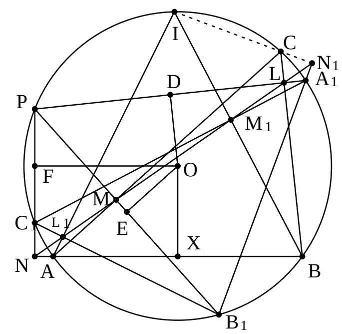


Figure 6-386

Example 6.387 (0.467, 5, 38) *The circumradius OP of the triangle ABC meets the sides of the triangle in the points* A_1, B_1, C_1. *Show that the projections* A_2, B_2, C_2 *of the points* A_1, B_1, C_1 *upon the lines AP, BP, CP lie on the Simson line of P for ABC.*

Constructive description
((circle $A\ B\ C\ P$)
(circumcenter $O\ A\ B\ C$)
(inter A_1 (l $B\ C$) (l $P\ O$))
(foot $A_2\ A_1\ A\ P$)
(foot $G\ P\ A\ B$)
(foot $F\ P\ A\ C$)
(inter F_1 (l $A\ C$) (l $G\ A_2$))
($\frac{\overline{AF_1}}{\overline{CF_1}} = \frac{\overline{AF}}{\overline{CF}}$))

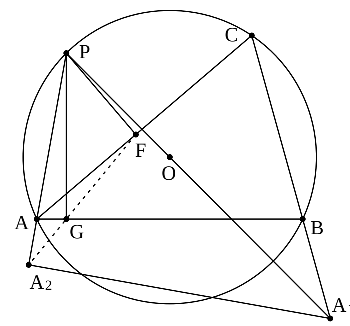


Figure6-387

Example 6.388 (0.950, 3, 37) *Let L, M, N be the projections of the point P of the circumcircle of the triangle ABC upon the sides BC, CA, AB, and let the Simson line LMN meet the altitudes AD, BE, CF in the points* L_1, M_1, N_1. *Show that the segments LM,* L_1M_1 *are equal to the projection of the sides upon the Simson line.*

Constructive description
((circle $A\ B\ C\ P$)
(circumcenter $O\ A\ B\ C$) (orthocenter $H\ A\ B\ C$)
(foot $L\ P\ B\ C$) (foot $M\ P\ A\ C$)
(foot $N\ P\ A\ B$)
(inter L_1 (l $A\ H$) (l $N\ M$))

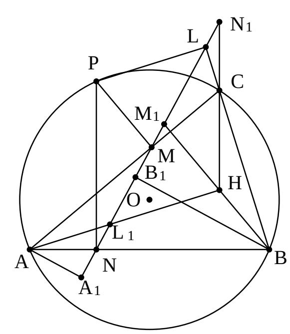


Figure 6-388

(inter M_1 (l B H) (l N M))
(inter N_1 (l C H) (l N M))
(foot A_1 A M N)
(foot B_1 B M N)
($\frac{\overline{L_1M_1}}{\overline{ML}} = 1$))

6.5.6 The Pascal Configuration

For a machine proof of Pascal's theorem, see Example 3.80 on page 145.

Example 6.389 (The Converse of Pascal's Theorem On a Circle) (0.933, 7, 44)

Constructive description
((circle A E B D C)
(circumcenter O A B C)
(inter P (l A B) (l E D))
(lratio Q B C r)
(inter S (l D C) (l P Q))
(inter F (l E Q) (l A S))
(cocircle F A B C))

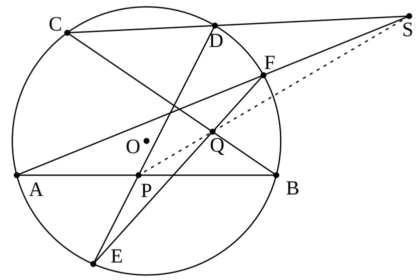


Figure 6-389

Example 6.390 (Pascal's Theorem: The General Case) (0.666, 3, 23) *Let A, B, C, D, F and E be six points with $P = AB \cap DE$, $Q = BC \cap EF$ and $S = CD \cap FA$ collinear. Then $P_1 = AC \cap DE$, $Q_1 = BE \cap CF$ and $S_1 = AB \cap FD$ are collinear.*

Constructive description
((points B A D E S)
(on C (l D S))
(inter P (l B A) (l D E))
(inter Q (l B C) (l S P))
(inter F (l E Q) (l A S))
(inter S_1 (l D F) (l B A))
(inter P_1 (l A C) (l D E))
(inter Z_2 (l B E) (l S_1 P_1))
(inter Z_1 (l C F) (l S_1 P_1))
($\frac{\overline{S_1Z_1}}{\overline{P_1Z_1}} \cdot \frac{\overline{P_1Z_2}}{\overline{S_1Z_2}} = 1$))

The eliminants

$\frac{\overline{S_1Z_1}}{\overline{P_1Z_1}} \stackrel{Z_1}{=} \frac{S_{CFS_1}}{S_{CFP_1}}$

$\frac{\overline{P_1Z_2}}{\overline{S_1Z_2}} \stackrel{Z_2}{=} \frac{S_{BEP_1}}{S_{BES_1}}$

$S_{CFP_1} \stackrel{P_1}{=} \frac{-S_{DEC} \cdot S_{ACF}}{S_{ADCE}}$

$S_{BEP_1} \stackrel{P_1}{=} \frac{S_{AEC} \cdot S_{BDE}}{S_{ADCE}}$

$S_{BES_1} \stackrel{S_1}{=} \frac{-S_{BDF} \cdot S_{BAE}}{S_{BDAF}}$

$S_{CFS_1} \stackrel{S_1}{=} \frac{S_{DCF} \cdot S_{BAF}}{S_{BDAF}}$

$S_{BDF} \stackrel{F}{=} \frac{S_{ASQ} \cdot S_{BDE} + S_{AES} \cdot S_{BDQ}}{S_{AESQ}}$

$S_{ACF} \stackrel{F}{=} \frac{-S_{ASC} \cdot S_{AEQ}}{S_{AESQ}}$

$S_{BAF} \stackrel{F}{=} \frac{S_{AEQ} \cdot S_{BAS}}{S_{AESQ}}$

$S_{DCF} \stackrel{F}{=} \frac{S_{ESQ} \cdot S_{ADC}}{S_{AESQ}}$

$S_{BDQ} \stackrel{Q}{=} \frac{S_{BSP} \cdot S_{BDC}}{S_{BSCP}}$

The eliminants

$S_{ASQ} \stackrel{Q}{=} \frac{S_{ASP} \cdot S_{BSC}}{S_{BSCP}}$

$S_{ESQ} \stackrel{Q}{=} \frac{S_{ESP} \cdot S_{BSC}}{S_{BSCP}}$

$S_{BSP} \stackrel{P}{=} \frac{-S_{BDE} \cdot S_{BAS}}{S_{BDAE}}$

$S_{ASP} \stackrel{P}{=} \frac{-S_{ADE} \cdot S_{BAS}}{S_{BDAE}}$

$S_{ESP} \stackrel{P}{=} \frac{S_{DES} \cdot S_{BAE}}{S_{BDAE}}$

$S_{BDC} \stackrel{C}{=} S_{BDS} \cdot \frac{\overline{DC}}{\overline{DS}}$

$S_{ASC} \stackrel{C}{=} (\frac{\overline{DC}}{\overline{DS}} - 1) \cdot S_{ADS}$

$S_{DEC} \stackrel{C}{=} S_{DES} \cdot \frac{\overline{DC}}{\overline{DS}}$

$S_{AEC} \stackrel{C}{=} S_{AES} \cdot \frac{\overline{DC}}{\overline{DS}} + S_{ADE} \cdot \frac{\overline{DC}}{\overline{DS}} - S_{ADE}$

$S_{ADC} \stackrel{C}{=} S_{ADS} \cdot \frac{\overline{DC}}{\overline{DS}}$

$S_{BSC} \stackrel{C}{=} (\frac{\overline{DC}}{\overline{DS}} - 1) \cdot S_{BDS}$

The machine proof

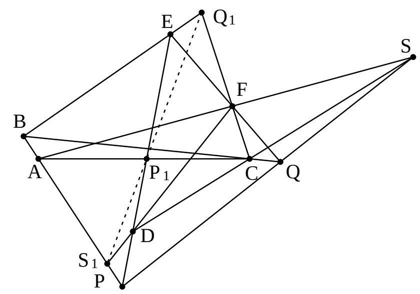


Figure 6-390

$$\frac{\overline{S_1Z_1}}{\overline{P_1Z_1}}\cdot\frac{\overline{P_1Z_2}}{\overline{S_1Z_2}} \overset{Z_1}{=} \frac{S_{CFS_1}}{S_{CFP_1}}\cdot\frac{\overline{P_1Z_2}}{\overline{S_1Z_2}} \overset{Z_2}{=} \frac{S_{CFS_1}\cdot S_{BEP_1}}{S_{CFP_1}\cdot S_{BES_1}}$$

$$\overset{P_1}{=} \frac{S_{CFS_1}\cdot(-S_{AEC}\cdot S_{BDE})\cdot S_{ADCE}}{(-S_{DEC}\cdot S_{ACF})\cdot S_{BES_1}\cdot(-S_{ADCE})}$$

$$\overset{simplify}{=} \frac{-S_{CFS_1}\cdot S_{AEC}\cdot S_{BDE}}{S_{DEC}\cdot S_{ACF}\cdot S_{BES_1}}$$

$$\overset{S_1}{=} \frac{-(-S_{DCF}\cdot S_{BAF})\cdot S_{AEC}\cdot S_{BDE}\cdot S_{BDAF}}{S_{DEC}\cdot S_{ACF}\cdot(-S_{BDF}\cdot S_{BAE})\cdot(-S_{BDAF})}$$

$$\overset{simplify}{=} \frac{S_{DCF}\cdot S_{BAF}\cdot S_{AEC}\cdot S_{BDE}}{S_{DEC}\cdot S_{ACF}\cdot S_{BDF}\cdot S_{BAE}}$$

$$\overset{F}{=} \frac{S_{ESQ}\cdot S_{ADC}\cdot S_{AEQ}\cdot S_{BAS}\cdot S_{AEC}\cdot S_{BDE}\cdot(-S_{AESQ})\cdot S_{AESQ}}{S_{DEC}\cdot(-S_{ASC}\cdot S_{AEQ})\cdot(-S_{ASQ}\cdot S_{BDE}-S_{AES}\cdot S_{BDQ})\cdot S_{BAE}\cdot(S_{AESQ})^2}$$

$$\overset{simplify}{=} \frac{-S_{ESQ}\cdot S_{ADC}\cdot S_{BAS}\cdot S_{AEC}\cdot S_{BDE}}{S_{DEC}\cdot S_{ASC}\cdot(S_{ASQ}\cdot S_{BDE}+S_{AES}\cdot S_{BDQ})\cdot S_{BAE}}$$

$$\overset{Q}{=} \frac{-(-S_{ESP}\cdot S_{BSC})\cdot S_{ADC}\cdot S_{BAS}\cdot S_{AEC}\cdot S_{BDE}\cdot S_{BSCP}\cdot(-S_{BSCP})}{S_{DEC}\cdot S_{ASC}\cdot(-S_{BSCP}\cdot S_{ASP}\cdot S_{BSC}\cdot S_{BDE}-S_{BSCP}\cdot S_{AES}\cdot S_{BSP}\cdot S_{BDC})\cdot S_{BAE}\cdot(-S_{BSCP})}$$

$$\overset{simplify}{=} \frac{-S_{ESP}\cdot S_{BSC}\cdot S_{ADC}\cdot S_{BAS}\cdot S_{AEC}\cdot S_{BDE}}{S_{DEC}\cdot S_{ASC}\cdot(S_{ASP}\cdot S_{BSC}\cdot S_{BDE}+S_{AES}\cdot S_{BSP}\cdot S_{BDC})\cdot S_{BAE}}$$

$$\overset{P}{=} \frac{-(-S_{DES}\cdot S_{BAE})\cdot S_{BSC}\cdot S_{ADC}\cdot S_{BAS}\cdot S_{AEC}\cdot S_{BDE}\cdot(S_{BDAE})^2}{S_{DEC}\cdot S_{ASC}\cdot(-S_{BDAE}\cdot S_{AES}\cdot S_{BDC}\cdot S_{BDE}\cdot S_{BAS}-S_{BDAE}\cdot S_{ADE}\cdot S_{BSC}\cdot S_{BDE}\cdot S_{BAS})\cdot S_{BAE}\cdot(-S_{BDAE})}$$

$$\overset{simplify}{=} \frac{S_{DES}\cdot S_{BSC}\cdot S_{ADC}\cdot S_{AEC}}{S_{DEC}\cdot S_{ASC}\cdot(S_{AES}\cdot S_{BDC}+S_{ADE}\cdot S_{BSC})}$$

$$\overset{C}{=} \frac{S_{DES}\cdot(S_{BDS}\cdot\frac{\overline{DC}}{\overline{DS}}-S_{BDS})\cdot S_{ADS}\cdot\frac{\overline{DC}}{\overline{DS}}\cdot(S_{AES}\cdot\frac{\overline{DC}}{\overline{DS}}+S_{ADE}\cdot\frac{\overline{DC}}{\overline{DS}}-S_{ADE})}{S_{DES}\cdot\frac{\overline{DC}}{\overline{DS}}\cdot(S_{ADS}\cdot\frac{\overline{DC}}{\overline{DS}}-S_{ADS})\cdot(S_{AES}\cdot S_{BDS}\cdot\frac{\overline{DC}}{\overline{DS}}+S_{ADE}\cdot S_{BDS}\cdot\frac{\overline{DC}}{\overline{DS}}-S_{ADE}\cdot S_{BDS})} \overset{simplify}{=} 1$$

Example 6.391 (Brianchon's Theorem) (1.750, 2, 137) *The dual of Pascal theorem.*

Constructive description

((circle A B C D E F) (circumcenter O A B C)
(on B_T (t B B O)) (on A_T (t A A O))
(on C_T (t C C O)) (on D_T (t D D O))
(on E_T (t E E O)) (on F_T (t F F O))
(inter A_1 (l B B_T) (l A A_T)) (inter D_1 (l E E_T) (l D D_T))
(inter B_1 (l C C_T) (l B B_T)) (inter E_1 (l F F_T) (l E E_T))
(inter C_1 (l D D_T) (l C C_T)) (inter F_1 (l A A_T) (l F F_T))
(inter I (l B_1 E_1) (l A_1 D_1)) (inter J (l A_1 D_1) (l C_1 F_1))
($\frac{\overline{A_1I}}{\overline{D_1I}} = \frac{\overline{A_1J}}{\overline{D_1J}}$))

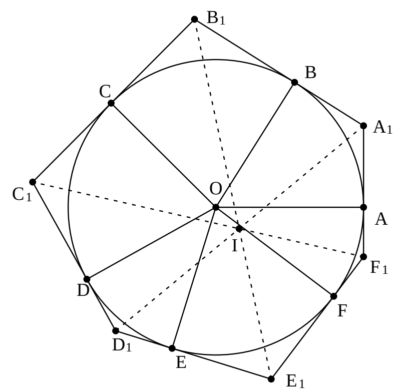


Figure 6-391

Example 6.392 (Kirkman's theorem) (0.650, 2, 50) *Given six points A, B, C, D, E, and F on a circle (or a conic), the three Pascal lines* [$BAECDF$], [$CDBFEA$], [$FECABD$] *are concurrent.*

There are 60 Kirkman points for one Pascal configuration.

Constructive description
((circle *A B C D E F*)
(inter *P* (l *C D*) (l *A B*))
(inter *Q* (l *D F*) (l *A E*))
(inter *S* (l *A C*) (l *B F*))
(inter *T* (l *B D*) (l *A E*))
(inter *X* (l *A B*) (l *E F*))
(inter *Y* (l *D F*) (l *A C*))
(inter *I* (l *S T*) (l *P Q*))
(inter *J* (l *P Q*) (l *X Y*)) ($\frac{\overline{PI}}{\overline{QI}} = \frac{\overline{PJ}}{\overline{QJ}}$))

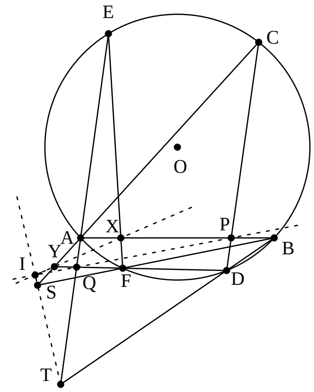


Figure 6-392

Example 6.393 (Steiner's theorem) (0.700, 2, 63) *Given six points A, B, C, D, E, and F on a circle (or a conic), the three Pascal lines* [*ABEDCF*], [*CDAFEB*], [*EFCBAD*] *are concurrent.*

There are 20 steiner points for one Pascal configuration.

Constructive description
((circle *A B C D E F*) (inter *P* (l *C D*) (l *A B*))
(inter *Q* (l *F A*) (l *D E*))
(inter *S* (l *B C*) (l *F A*))
(inter *T* (l *A D*) (l *B E*))
(inter *X* (l *A B*) (l *E F*))
(inter *Y* (l *C F*) (l *A D*))
(inter *I* (l *S T*) (l *P Q*))
(inter *J* (l *P Q*) (l *X Y*)) ($\frac{\overline{PI}}{\overline{QI}} = \frac{\overline{PJ}}{\overline{QJ}}$))

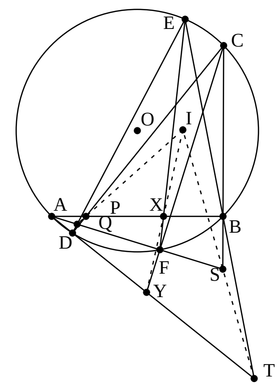


Figure 6-393

Example 6.394 (0.967, 2, 30) *Given five points* A_0, A_1, A_2, A_3 *and* A_4, *then points* $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_2A_4$, $A_0A_2 \cap A_1A_3$, $A_0A_2 \cap A_1A_4$, $A_0A_3 \cap A_1A_2$, $A_0A_4 \cap A_1A_2$ *are on the same conic. (There are 60 such conics for five points.)*

Constructive description
((points A_0 A_1 A_2 A_3 A_4) (inter P_0 (l A_2 A_3) (l A_0 A_1))
(inter P_1 (l A_2 A_4) (l A_0 A_1))
(inter P_2 (l A_1 A_3) (l A_0 A_2))
(inter P_3 (l A_1 A_4) (l A_0 A_2))
(inter P_4 (l A_1 A_2) (l A_0 A_3))
(inter P_5 (l A_1 A_2) (l A_0 A_4))
(inter *P* (l P_3 P_4) (l P_0 P_1))
(inter *Q* (l P_4 P_5) (l P_1 P_2))
(inter *S* (l P_5 P_0) (l P_2 P_3)) (collinear *P Q S*))

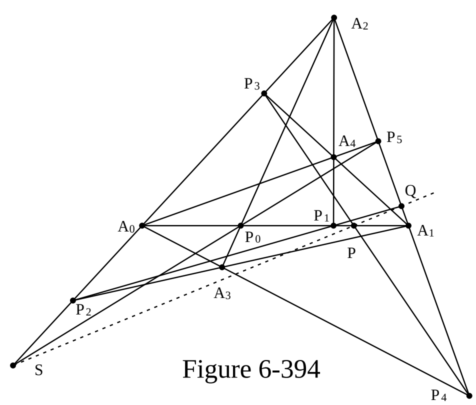


Figure 6-394

Example 6.395 (4.383, 3, 84) *Given six points A_0, A_1, A_2, A_3, A_4 and A_5 on one conic, then points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_1A_3$, $A_0A_3 \cap A_1A_2$, $A_0A_4 \cap A_1A_5$, $A_0A_5 \cap A_1A_4$ are on the same conic. (There are 45 such conics for one Pascal configuration.)*

Constructive description
((circle A_0 A_1 A_2 A_3 A_4 A_5)
(inter P_0 (l A_2 A_3) (l A_0 A_1))
(inter P_1 (l A_4 A_5) (l A_0 A_1))
(inter P_2 (l A_1 A_3) (l A_0 A_2))
(inter P_3 (l A_1 A_2) (l A_0 A_3))
(inter P_4 (l A_1 A_5) (l A_0 A_4))
(inter P_5 (l A_1 A_4) (l A_0 A_5))
(inter X (l P_3 P_4) (l P_0 P_1))
(inter Y (l P_4 P_5) (l P_1 P_2)) (inter Z (l P_5 P_0) (l P_2 P_3))
(inter Z_1 (l P_2 P_3) (l X Y)) ($\frac{\overline{P_2Z}}{\overline{P_3Z}} = \frac{\overline{P_2Z_1}}{\overline{P_3Z_1}}$))

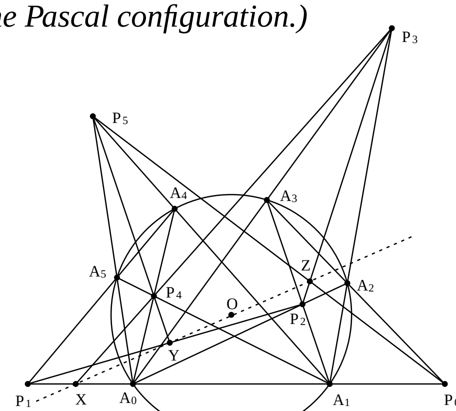


Figure 6-395

Example 6.396 (4.566, 4, 75) *Given six points A_0, A_1, A_2, A_3, A_4 and A_5 on one conic, then points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_1A_4$, $A_0A_3 \cap A_1A_5$, $A_0A_4 \cap A_1A_2$, $A_0A_5 \cap A_1A_3$ are on the same conic.*

There are 90 such conics for one Pascal configuration.

Constructive description
((circle A_0 A_1 A_2 A_3 A_4 A_5)
(inter P_0 (l A_2 A_3) (l A_0 A_1))
(inter P_1 (l A_4 A_5) (l A_0 A_1))
(inter P_2 (l A_1 A_4) (l A_0 A_2))
(inter P_3 (l A_1 A_5) (l A_0 A_3))
(inter P_4 (l A_1 A_2) (l A_0 A_4))
(inter P_5 (l A_1 A_3) (l A_0 A_5))
(inter Y (l P_4 P_5) (l P_1 P_2))
(inter Z (l P_5 P_0) (l P_2 P_3))
(inter X (l P_3 P_4) (l P_0 P_1))
(inter W (l P_0 P_1) (l Y Z)) ($\frac{\overline{P_1X}}{\overline{P_0X}} = \frac{\overline{P_1W}}{\overline{P_0W}}$))

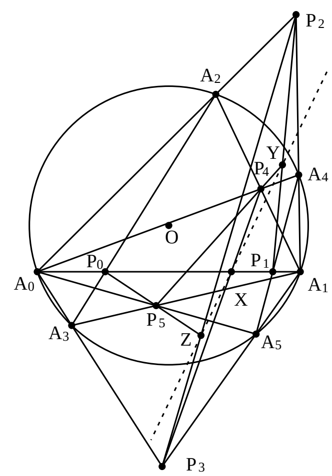


Figure 6-396

Example 6.397 [6] (0.866, 2, 55) *Given six points A_0, A_1, A_2, A_3, A_4 and A_5 on one conic, then points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_3A_4$, $A_0A_3 \cap A_2A_5$, $A_1A_4 \cap A_2A_5$, $A_1A_5 \cap A_3A_4$ are on the same conic.*

There are 60 such conics for one Pascal configuration.

[6] Example 6.397 was a new theorem found by Wu in 1980 [36]. Examples 6.395 and 6.396 were found by us.

Constructive description
((circle A_0 A_1 A_2 A_3 A_4 A_5)
(inter P_0 (l A_2 A_3) (l A_0 A_1))
(inter P_1 (l A_4 A_5) (l A_0 A_1))
(inter P_2 (l A_3 A_4) (l A_0 A_2))
(inter P_3 (l A_2 A_5) (l A_0 A_3))
(inter P_4 (l A_2 A_5) (l A_1 A_4))
(inter P_5 (l A_3 A_4) (l A_1 A_5))
(inter X (l P_4 P_2) (l A_0 A_1))
(inter Y (l A_3 A_4) (l P_1 P_3))
(inter Z (l P_5 P_0) (l A_2 A_5))
(inter W (l P_5 P_0) (l X Y)) ($\frac{\overline{P_5Z}}{\overline{P_0Z}} = \frac{\overline{P_5W}}{\overline{P_0W}}$))

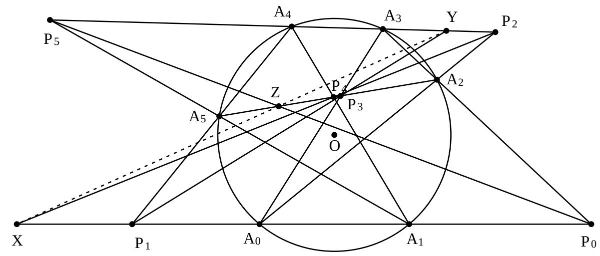


Figure 6-397

6.5.7 Cantor's Theorems

Example 6.398 (0.083, 3, 8) *The perpendiculars from the midpoints of the sides of a triangle to the tangent lines of the circumcircle at the third vertex of the triangle are concurrent, and this concurrent point is the center of the nine point circle of the triangle*

Constructive description
((points A B C)
(circumcenter O A B C)
(midpoint L B C)
(midpoint M A C)
(inter N (p L O A) (p M O B))
(eqdistance N L N M))

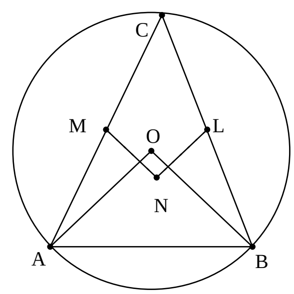


Figure 6-398

The machine proof

$$\frac{P_{LNL}}{P_{MNM}} \overset{N}{=} \frac{S_{BLOM}^2 \cdot P_{AOA} \cdot S_{ABO}^2}{S_{ALOM}^2 \cdot P_{BOB} \cdot S_{ABO}^2}$$

$$\overset{simplify}{=} \frac{S_{BLOM}^2 \cdot P_{AOA}}{S_{ALOM}^2 \cdot P_{BOB}}$$

$$\overset{M}{=} \frac{(-S_{BOL} - \frac{1}{2} S_{BCO} + \frac{1}{2} S_{ABO})^2 \cdot P_{AOA}}{(-S_{AOL} - \frac{1}{2} S_{ACO})^2 \cdot P_{BOB}}$$

$$\overset{L}{=} \frac{(-\frac{1}{2} S_{ABO})^2 \cdot P_{AOA}}{(-\frac{1}{2} S_{ABO})^2 \cdot P_{BOB}}$$

$$\overset{simplify}{=} \frac{P_{AOA}}{P_{BOB}}$$

$$\overset{O}{=} \frac{P_{BCB} \cdot P_{ACA} \cdot P_{ABA} \cdot (64 S_{ABC}^2)}{P_{BCB} \cdot P_{ACA} \cdot P_{ABA} \cdot (64 S_{ABC}^2)} \overset{simplify}{=} 1$$

The eliminants

$$P_{MNM} \overset{N}{=} \frac{S_{ALOM}^2 \cdot P_{BOB}}{S_{ABO}^2} \quad P_{LNL} \overset{N}{=} \frac{S_{BLOM}^2 \cdot P_{AOA}}{S_{ABO}^2}$$

$$S_{ALOM} \overset{M}{=} -\left(S_{AOL} + \frac{1}{2} S_{ACO}\right)$$

$$S_{BLOM} \overset{M}{=} -\left(S_{BOL} + \frac{1}{2} S_{BCO} - \frac{1}{2} S_{ABO}\right)$$

$$S_{AOL} \overset{L}{=} -\frac{1}{2}\left(S_{ACO} + S_{ABO}\right)$$

$$S_{BOL} \overset{L}{=} -\frac{1}{2}\left(S_{BCO}\right)$$

$$P_{BOB} \overset{O}{=} \frac{P_{BCB} \cdot P_{ACA} \cdot P_{ABA}}{(64) \cdot S_{ABC}^2}$$

$$P_{AOA} \overset{O}{=} \frac{P_{BCB} \cdot P_{ACA} \cdot P_{ABA}}{(64) \cdot S_{ABC}^2}$$

Example 6.399 (0.017, 5, 5) *Let A, B, C, D be four points on a circle O. The perpendiculars from the centroids of the four triangles ABC, ABD, ACD, and BCD to the tangent lines of* *circle O at points D, C, B, A are concurrent.*

Constructive description
((circle $A\ B\ C\ D$) (circumcenter $O\ A\ B\ C$)
(centroid $M\ A\ B\ C$) (centroid $N\ A\ B\ D$)
(centroid $L\ B\ C\ D$)
(inter P (p $M\ O\ D$) (p $N\ O\ C$))
(parallel $P\ L\ O\ A$))

The eliminants

$$S_{AOP} \stackrel{P}{=} \frac{S_{CMON} \cdot S_{ADO} + S_{CDO} \cdot S_{AOM}}{S_{CDO}}$$
$$S_{AOL} \stackrel{L}{=} -\frac{1}{3}(S_{ADO}+S_{ACO}+S_{ABO})$$
$$S_{CMON} \stackrel{N}{=} -\left(S_{COM}+\frac{1}{3}S_{CDO}-\frac{1}{3}S_{BCO}-\frac{1}{3}S_{ACO}\right)$$
$$S_{AOM} \stackrel{M}{=} -\frac{1}{3}(S_{ACO}+S_{ABO})$$
$$S_{COM} \stackrel{M}{=} \frac{1}{3}(S_{BCO}+S_{ACO})$$

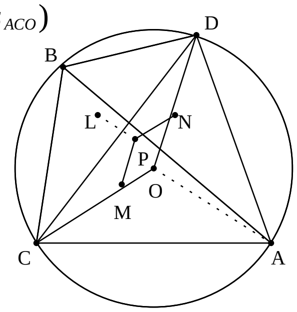


Figure 6-399

The machine proof

$$\frac{-S_{AOP}}{-S_{AOL}} \stackrel{P}{=} \frac{S_{CMON} \cdot S_{ADO}+S_{CDO} \cdot S_{AOM}}{(-S_{AOL}) \cdot (-S_{CDO})}$$
$$\stackrel{L}{=} \frac{(S_{CMON} \cdot S_{ADO}+S_{CDO} \cdot S_{AOM}) \cdot (3)}{(-S_{ADO}-S_{ACO}-S_{ABO}) \cdot S_{CDO}}$$
$$\stackrel{N}{=} \frac{(-3) \cdot (-3S_{COM} \cdot S_{ADO}+3S_{CDO} \cdot S_{AOM}-S_{CDO} \cdot S_{ADO}+S_{BCO} \cdot S_{ADO}+S_{ADO} \cdot S_{ACO})}{(S_{ADO}+S_{ACO}+S_{ABO}) \cdot S_{CDO} \cdot (3)}$$
$$\stackrel{M}{=} \frac{(3) \cdot (3S_{CDO} \cdot S_{ADO}+3S_{CDO} \cdot S_{ACO}+3S_{CDO} \cdot S_{ABO})}{(S_{ADO}+S_{ACO}+S_{ABO}) \cdot S_{CDO} \cdot (3)^2} \stackrel{simplify}{=} 1$$

Example 6.400 (1.900, 42, 81) *Let A, B, C, D, E be five points on a circle O. The perpendiculars from the centroids of the triangles whose vertices are from A, B, C, D, E to the lines joining the remaining two points are concurrent.*

Constructive description
((circle $A\ B\ C\ D\ E$) (circumcenter $O\ A\ B\ C$)
(centroid $A_1\ B\ C\ D$) (centroid $B_1\ A\ C\ D$)
(centroid $C_1\ A\ B\ D$) (midpoint $A_2\ A\ E$)
(midpoint $B_2\ B\ E$) (midpoint $C_2\ C\ E$)
(inter N (p $A_1\ A_2\ O$) (p $B_1\ B_2\ O$)) (perpendicular $N\ C_1\ C\ E$))

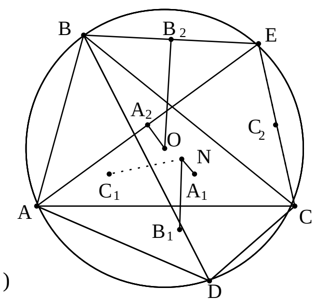


Figure 6-400

For more results related to Cantor's theorems, see Example 3.82 on page 147, Example 5.50 on page 245, Example 5.51 on page 245, and Example 5.56 on page 248.

6.6 A Summary

We have given 400 machine proved geometry theorems in Sections 6.2–6.5 and 78 machine solved geometry problems in Part I of the book. Thus totally there are 478 machine solved geometry problems in this book, including 280 proofs produced automatically by a computer program.

In order to access the overall performance of the algorithm/program, we will first list the machine computation times and proof lengths of the examples in Part I.

No.	page	time	maxt	lems
2.35	73	0.017	1	3
2.36	74	0.117	2	9
2.37	75	0.067	1	7
2.41	78	0.050	1	5
2.42	80	0.133	1	14
2.46	83	0.067	2	4
2.47	84	0.033	1	3
2.52	86	0.083	4	5
2.53	87	0.250	6	8
2.54	88	0.300	9	13
2.55	89	0.300	10	7
2.56	91	0.167	2	17
2.58	92	0.517	17	18
2.59	93	1.033	15	18
2.62	94	0.050	1	4
2.65	96	0.117	1	10
2.66	98	0.117	1	10
3.36	120	0.001	1	2
3.40	123	0.050	3	3
3.41	124	0.017	1	4
3.42	124	0.750	3	15
3.43	125	0.083	2	10
3.44	126	0.033	1	7
3.45	127	0.250	6	16
3.51	130	0.067	1	6
3.52	131	0.050	3	6
3.53	132	1086.8	3125	65
3.68	139	0.050	2	2
3.69	140	0.067	2	5
3.70	140	0.033	1	5
3.71	141	0.033	3	3
3.79	144	0.033	1	12
3.80	145	0.083	1	14
3.81	146	0.050	1	14
3.82	147	0.067	4	5
3.102	156	1.050	48	15
3.105	162	0.867	5	8
3.106	164	0.117	2	7
3.107	164	0.033	2	4

No.	page	time	maxt	lems
3.108	165	0.200	5	8
3.109	165	0.350	4	7
3.110	166	0.050	3	7
3.111	167	1.383	4	11
3.112	167	1.100	5	10
4.23	180	0.033	1	4
4.24	180	0.017	1	2
4.25	180	0.033	2	7
4.40	189	0.067	2	6
4.41	190	0.883	9	10
4.47	192	0.083	3	7
4.53	194	0.017	1	4
4.54	195	0.033	1	3
4.61	198	0.083	4	8
4.62	198	0.133	3	8
4.63	199	0.083	2	5
4.64	200	0.067	3	6
4.65	201	0.083	3	6
4.87	210	0.100	4	7
4.88	211	0.117	5	6
4.89	211	0.067	3	6
4.90	212	0.133	4	6
4.91	213	0.167	4	10
4.92	215	99.200	140	78
5.38	237	0.083	4	8
5.46	243	0.017	2	4
5.47	243	0.100	3	7
5.48	244	0.117	4	5
5.49	244	0.083	5	5
5.55	248	0.017	1	9
5.56	248	0.717	4	5
5.57	250	0.083	5	7
5.58	252	0.083	6	5
5.59	252	0.117	9	6
5.61	253	0.067	4	3
5.62	254	0.133	5	3
5.63	255	0.017	3	2
5.64	255	0.050	3	3
5.65	256	0.267	7	9

Table 1. Statistics for the Examples in Part I

We use a triple *(time,maxt,lems)* to measure how difficult a machine proof is:

1. *time* is the time needed to complete the machine proof. The program is implemented on a NexT Turbo workstation (25 MIPS) using AKCL (Austin-Kyoto Common Lisp).

2. *maxt* is the number of terms of the maximal polynomial occurring in the machine proof. Thus maxt measures the amount of computation needed in the proof.

3. *lems* is the number of elimination lemmas used to eliminate points from geometry quantities. In other words, lems is the number of deduction steps in the proof.

The following table contains some statistics for the timings and proof lengths of the 478 machine solved problems in this book.

Proving Time		Proof Length		Deduction Step	
Time (secs)	% of Thm.	Maxterm	% of Thm.	Lemmas	% of Thm.
$t \leq 0.1$	45.3%	$m = 1$	16.9%	$l \leq 3$	7.1%
$t \leq 0.5$	68.8%	$m \leq 2$	33.0%	$l \leq 5$	16.7%
$t \leq 1$	85.5%	$m \leq 5$	66.9%	$l \leq 10$	42.6%
$t \leq 5$	97.45%	$m \leq 10$	81.7%	$l \leq 20$	73.2%
$t \leq 10$	98.9%	$m \leq 100$	98.7%	$l \leq 50$	95.1%
$t < 1087$	100%	$m \leq 3125$	100%	$l \leq 137$	100%

Table 2. Statistics for the 478 Theorems

Remark.

1. We can see that our program is very fast and can produce short proofs for many difficult geometry theorems. Eighty-five percent of the 478 proofs can be completed within one second, and the average maximal term and deduction step for the 478 examples are 14.87[7]and 17.35 steps respectively.

2. If we set a standard that a machine proof is *readable* if one of the following conditions holds

   (1) the maximal term in the proof is less than or equal to 5;

   (2) the deduction step of the proof is less than or equal to 10; or

   (3) the maximal term in the proof is less than or equal to 10, and the deduction step is less than or equal to 20,

[7]If not considering Morley's theorem (see 3 of the this remark), the average maximal term is 8.37.

then 76.9 percent (or 368) of the proofs of the 478 theorems produced by our prover are readable. The machine proofs for 59.2% or 283 of the 478 theorems are actually presented in this book.

3. In spite of this success, there are still some geometry theorems for which the method/program performs badly. For instance, for Morley's theorem (on page 132), we have *time*= *1086.8, maxt = 3125,* and *lems = 65*. There are two main factors in the description of the geometry statements that affect the machine proof: the *number of points* in the statement and the *type of constructions* needed to describe it. Generally speaking, the number of points in a geometry statement is fixed and reflects the difficulty level of the statement in nature. On the other hand, the difficulty related to the type of constructions is somehow artificial. According to our experience, the geometry relations can be listed in ascending order of difficulties as follows:

   collinear, parallel, ratios, perpendicular, circles, angles.

   Morley's theorem involves information mainly about angles, and most of the other "difficult geometry theorems" involve perpendiculars, circles, or angles.

   We have two ideas for further improvement of the method/program. First, we can put elimination results for more constructions into the program instead of dividing these constructions into other simple constructions. Second, we can use new geometry quantities to produce short and readable proofs. In our method, we mainly use areas and Pythagoras differences, which deal perfectly with geometry statements about collinear and parallel, but do not always work well for a geometry statement about perpendicular, circles, and angles. For instance, to express the fact that the sum of two angles is equal to another angle, we have to use a complicated equation of areas and Pythagoras differences.

   Besides area and Pythagoras difference, we also discussed how to use other geometry quantities such as the vector, the complex number, and the full-angle, to produce short and readable proofs. The approach based on full-angles presented in Section 3.8 is quite promising. It uses the angle as the basic geometry quantity, and hence might produce short proofs for difficult geometry problems about angles, circles, and perpendiculars.

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List of Symbols

$\triangle ABC$	5	triangle ABC
$\triangledown ABC$	5	the area of triangle ABC
$(ABCD)$	270	the cross-ratio
$\measuredangle ABCD$	37	the oriented angle from AB to CD
$\angle[AB, CD]$	44	the full-angle from AB to CD
$[\overrightarrow{AB}, \overrightarrow{CD}]$	231	the exterior product
$\langle\overrightarrow{AB}, \overrightarrow{CD}\rangle$	230	the inner product
$[\boldsymbol{x}, \boldsymbol{y}]$	226	the exterior product
$\langle\boldsymbol{x}, \boldsymbol{y}\rangle$	223	the inner product
$\overrightarrow{A}$	229	the vector from the origin to A
$\overrightarrow{AB}$	229	the vector from A to B
$\overline{AB}$	3	the directed line segment from A to B
$\widetilde{AB}$	39	the oriented chord
$\widehat{AB}$	142	the cochord of $\widetilde{AB}$
$AB \cap CD$	9	the intersection of AB and CD
$AB \parallel CD$	17	AB is parallel to CD
$AB \perp CD$	30	AB is perpendicular to CD
$\mathbf{C}_{\mathrm{H}}$	61	the Hilbert intersection point statements in the plane
$\mathbf{C}_{\mathrm{L}}$	109	the linear constructive statements in the plane
C_{ABC}	27	the co-area of triangle ABC
$O(ABCD)$	271	the cross-ratio for five points
P_{ABC}	28	the Pythagoras difference of triangle ABC for point B
P_{ABCD}	30	the Pythagoras difference of quadrilateral $ABCD$
S_{ABC}	6	the signed area of triangle ABC
S_{ABCD}	8	the signed area of quadrilateral $ABCD$
$\mathbf{S}_{\mathrm{H}}$	182	the Hilbert intersection point statements in the space
V_{ABCD}	171	the signed volume of tetrahedron $ABCD$
V_{ABCDE}	173	the signed volume of polyhedron A-BCD-E
$K[x]$	150	the ring of polynomials
$class(P)$	150	the class of polynomial P
$init(P)$	150	the initial of polynomial P
$ld(P)$	150	the leading degree of polynomial P
$lv(P)$	150	the leading variable of polynomial P

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