On the Dimension of an Arbitrary Ascending Chain

Xiao-Shan Gao
Institute of Systems Science, Academia Sinica, Beijing 100080
Shang-Ching Chou
Department of Computer Sciences
University of Texas at Austin, Austin Texas 78712, USA

1993

Abstract

We show that the length of an arbitrary ascending chain has geometric meaning and hence describes certain natural properties for the ascending chain. The results proved in this paper can be used to enhance the efficiency of the Ritt-Wu’s decomposition algorithm and to obtain an unmixed decomposition for an algebraic variety more easily.

Keywords. Ascending chain, dimension of a variety, Ritt-Wu’s decomposition, unmixed decomposition.

We know that the dimension for an irreducible ascending chain ASC is a crucial concept in Ritt-Wu’s constructive theory of algebraic geometry [WU1]. We can also define the dimension for an arbitrary ascending chain similarly. But one may say that this definition has no geometric meaning. In this paper, we will show that the dimension of an arbitrary ascending chain does have geometric meaning and hence describes certain natural properties for the ascending chain. The results proved in this paper can be used to enhance the efficiency of the Ritt-Wu’s decomposition algorithm and to obtain an unmixed decomposition for an algebraic variety more easily.

1. The Dimension of an Arbitrary Ascending Chain

Let $K$ be a field of characteristic zero and $K[y_1, ..., y_n]$ or $K[y]$ be the ring of polynomials for the variables $y_1, ..., y_n$. All polynomials in this paper are in $K[y]$ unless explicitly mentioned otherwise. Let $P$ be a polynomial. The class of $P$, denoted by $class(P)$, is the largest $p$ such that some $y_p$ actually occurs in $P$. If $P \in K$, $class(P) = 0$. Let a polynomial $P$ be of class $p > 0$. The coefficient of the highest power of $x_p$ in $P$ considered as a polynomial of $x_p$ is called the initial of $P$. For polynomials $P$ and $G$ with $class(P) > 0$, let $prem(G; P)$ be the pseudo remainder of $G$ wrpt (ab. with respect to) $P$ [WU1].
A sequence of polynomials $ASC = A_1, ..., A_p$ is said to be a quasi ascending chain, if either $r = 1$ and $A_1 \neq 0$ or $0 < \text{class}(A_i) < \text{class}(A_j)$ for $1 \leq i < j$. $ASC$ is called nontrivial if $\text{class}(A_1) > 0$. A quasi ascending chain $ASC = A_1, ..., A_p$ is called an ascending chain if either $ASC$ is trivial or $A_j$ is of higher degree than $A_i$ ($i = j + 1, ..., p$) in $y_{n_j}$ where $n_j = \text{class}(A_j)$.

For a non-trivial quasi ascending chain $ASC = A_1, ..., A_p$ and a polynomial $G$, we define the pseudo remainder of $G$ wrpt $ASC$ inductively as

$$\text{prem}(G; ASC) = \text{prem}(\text{prem}(G; A_p); A_1, ..., A_{p-1}).$$

Let $R = \text{prem}(G; ASC)$, then we have the following important remainder formula [WU1]:

$$JG - R \in \text{Ideal}(A_1, ..., A_p) \quad (1.1)$$

where $J$ is a product of the initials of the polynomials in $ASC$ and $\text{Ideal}(A_1, ..., A_p)$ is the ideal generated by $A_1, ..., A_p$.

**Definition 1.1.** The dimension of a quasi ascending chain $ASC = A_1, ..., A_p$ is defined to be $\text{DIM}(ASC) = n - p$.

For a quasi ascending chain $ASC = A_1, ..., A_p$, let $A_i$ be of class $m_i$, then we call $\{y_1, ..., y_n\} - \{y_{m_1}, ..., y_{m_p}\}$ the parameter set of $ASC$. Thus $\text{DIM}(ASC)$ is equal to the number of parameters of $ASC$.

**Definition 1.2.** For a quasi ascending chain $ASC$, we define

$$QD(ASC) = \{g \mid \exists J, Jg \in \text{Ideal}(ASC)\}$$

where $J$ is a product of powers of the initials of the polynomials in $ASC$.

It is obvious that $QD(ASC)$ is an ideal. By (1.1), we have that if $\text{prem}(P; ASC) = 0$ then $P \in QD(ASC)$. Let $PS$ and $DS$ be polynomial sets. For an algebraic closed extension field $E$ of $K$, let

$$\text{Zero}(PS) = \{x = (x_1, ..., x_n) \in E^n \mid \forall P \in PS, P(x) = 0\}$$

and

$$\text{Zero}(PS/DS) = \text{Zero}(PS) - \bigcup_{g \in DS} \text{Zero}(g).$$

**Theorem 1.3.** Let $ASC = \{A_1, ..., A_r\}$ be a non-trivial quasi ascending chain, $J = \{I_1, ..., I_r\}$ where $I_i$ are the initials of $A_i$. Then either $\text{Zero}(ASC/J)$ is empty or

$$\text{Zero}(ASC/J) = \bigcup_{1 \leq i \leq r} \text{Zero}(QD(ASC_i)/J)$$

where each $ASC_i$ is irreducible and with the same parameter set as $ASC$. (For the concept of irreducible ascending chain, see [WU1]).

**Proof.** It is a direct consequence of Theorem 4.4 in [CG1] and the affine dimension theorem (p48, [HA1]).

If $ASC$ is an irreducible ascending, then it is known that $QD(ASC)$ is a prime ideal of dimension $\text{DIM}(ASC)$ [WU1]. A variety whose irredundant components have the same dimension is called an unmixed or pure variety. Theorem 1.3 means that $\text{Zero}(ASC/J)$
is contained in an unmixed variety of dimension $DIM(ASC)$. Moreover, this unmixed variety satisfies a property that all its components have the same parameter set. We call such a variety a parameter unmixed variety.

2. An Unmixed Decomposition for $QD(ASC)$

For a polynomial set $PS$, let $M(PS)$ be the multiplicative set generated by $PS$, i.e. the products of the powers of finite polynomials in $PS$. For two polynomial sets $PS$ and $DS$, let

$$QD(PS : DS) = \{ g \in K[y] \mid \exists J \in M(DS), Jg \in Ideal(PS) \}$$

then it is obvious that $QD(ASC) = QD(ASC : J)$ where $J$ is the initial set of $ASC$.

**Theorem 2.1.** For two polynomial sets $PS = \{ f_1, \ldots, f_k \}$ and $DS = \{ d_1, \ldots, d_s \}$ in $K[y]$, let $PD = Ideal(PS, d_1z_1 - 1, \ldots, d_sz_s - 1)$ in $K[y, z_1, \ldots, z_s]$ where $z_i$ are new variables. Then $QD(PS : DS) = PD \cap K[y]$.

Proof. For $P \in QD(PS : DS)$, there is a $J = d_1^{n_1} \cdots d_s^{n_s} \in M(DS)$ such that $JP \in Ideal(PS)$. Note that $(z_i f_i)^m \equiv 1 \mod PD$, then we have $z_1^{n_1} \cdots z_s^{n_s}JP = \prod_{i=1}^s (z_i f_i)^m P \equiv P \equiv 0 \mod PD$, i.e., $P \in PD$. We have proved $QD(PS : DS) \subseteq PD \cap K[y]$. For the other direction, let $P \in PD \cap K[y]$, then $P = \sum B_i f_i + \sum C_i (z_i f_i - 1)$ for some polynomials $B_i$ and $C_i$ in $K[y, z]$. Set $z_i = 1/d_i$ and clear the denominators. We have $JP = \sum B'_i f_i$ where $J \in M(DS)$, i.e., $P \in QD(PS : DS)$.

Theorem 2.1 together with the following result give a method to compute a basis for $QD(PD : DS)$.

**Lemma 2.2.** (Lemma 6.8 in [BU1]) For an ideal $ID \subset K[x_1, \ldots, x_n, y_1, \ldots, y_k]$, if $GB$ is a Gröbner basis of $ID$ under the pure lexicographic order $x_1 < \ldots < x_n < y_1 < \ldots < y_k$ then $GB \cap K[x_1, \ldots, x_n]$ is a Gröbner basis of $ID \cap K[x_1, \ldots, x_n]$.

Let $G$ be a polynomial ideal and $S$ be a multiplicative polynomial set, the fraction of $G$ by $S$ is defined to be $S^{-1}G = S \times G/ \sim$ where $\sim$ satisfies that $(s, a) \sim (s', a')$ iff $sa' = s'a$. Certain elements of $S^{-1}G$ can be treated like polynomials. Here, we always treat $(s, sa)$ and $a$ as the same element (for more details, see [AM1]).

**Theorem 2.3.** Let $S = M(DS)$, then $QD(PS : DS) = (S^{-1}Ideal(PS)) \cap K[y]$.

Proof. Define a map $\phi : QD(PS : DS) \to (S^{-1}Ideal(PS)) \cap K[y]$ by setting $\phi(P) = (J, JP)$ where $J \in S$ satisfies $JP \in Ideal(PS)$. It is easy to see that $\phi$ is a well defined injective map. By the explanation in the previous paragraph, an elements in $(S^{-1}Ideal(PS)) \cap K[y]$ must have the form $(J, JP)$ where $J \in S$ and $JP \in Ideal(PS)$. Thus $P \in QD(PS : DS)$, and therefore $\phi$ is also surjective.

The following result permits us to obtain the irreducible components of $QD(PS : DS)$ from $PS$ and $DS$ directly using Ritt-Wu’s decomposition algorithm.

**Theorem 2.4.** For polynomial sets $PS$ and $DS$ in $K[y]$,

$$Zero(PS/DS) = \cup_{i=1}^m Zero(QD(ASC_i)/DS)$$

is an irreducible irredundant decomposition for $Zero(PS/DS)$ iff

$$Zero(QD(PS : DS)) = \cup_{i=1}^m Zero(QD(ASC_i))$$
is an irreducible irredundant decomposition for \( \text{Zero}(QD(PS : DS)) \).

Proof. This theorem comes from the following lemma and Theorem 2.1 immediately.

**Lemma 2.5.** For polynomial sets \( PS \) and \( DS = \{d_1, ..., d_s\} \) in \( K[y] \), let \( PD = \text{Ideal}(PS, d_1z_1 - 1, ..., d_sz_s - 1) \) where \( z_i \) are new variables. If

\[
\text{Zero}(PS/DS) = \bigcup_{i=1}^m \text{Zero}(QD(ASC_i)/DS)
\]

(A.1)
is an irreducible irredundant decomposition for \( \text{Zero}(PS/DS) \), then we have an irredundant decomposition for \( \text{Zero}(PD) \)

\[
\text{Zero}(PD) = \bigcup_{i=1}^m \text{Zero}(QD(ASC_i'))
\]

(A.2)
where \( ASC_i' = ASC_i, d_1z_1 - 1, ..., d_sz_s - 1 \), and vice versa.

Proof. Suppose we have (A.1). Note that the pseudo remainders of the generators of \( PD \) w.r.p \( ASC_i' \) are zero, then by (1.1) we have \( PD \subset QD(ASC_i') \), \( i = 1, ..., m \). One direction of (A.2) is proved. For the other direction, let \( \eta = (x', z_1', ..., z_s') \) be a zero of \( \text{Zero}(PD) \), then \( d_i(x')z_i' - 1 = 0 \) which implies \( d_i(x') \neq 0 \), \( i = 1, ..., s \). Thus \( x' \in \text{Zero}(PS/DS) \), and hence \( x' \in \text{Zero}(QD(ASC_i')/DS) \) for some \( i \), say \( i = 1 \). We will prove \( \eta \in \text{Zero}(QD(ASC_1')) \).

Let \( h \in QD(ASC_1') \), then \( Jh = P + \sum C_i(z_i - 1) \), where \( P \in QD(ASC_1') \) and \( J \) is a product of the the powers of some \( d_i \). As the \( d_i(x') \neq 0 \), then \( \eta \) is a zero of \( h \). Hence (A.2) is true. It is similar to derive (A.1) from (A.2).

We now give some properties for \( QD(ASC) \). First, the algorithm to compute a finite basis of the prime ideal \( QD(ASC) \) for an irreducible ascending chain \( ASC \) in \( \text{p}85 [CH1] \) can be generalized to the following form.

**Theorem 2.6.** For a quasi ascending chain \( ASC \) in \( K[y] \), let \( ID = \text{Ideal}(ASC, I_1z_1 - 1, ..., I_pz_p - 1) \) in \( K[y, z] \), where \( I_i \) are the initials of the polynomials in \( ASC \) and \( z_i \) are new variables, then \( QD(ASC) = ID \cap K[y] \).

Proof. Since \( QD(ASC) = QD(ASC : J) \), this is a direct consequence of Theorem 2.1.

A finite basis of \( QD(ASC) \) can be found by Lemma 2.2.

**Theorem 2.7.** For a quasi ascending chain \( ASC \) in \( K[y] \), either \( \text{Zero}(QD(ASC)) \) is empty, or we have

\[
\text{Zero}(QD(ASC)) = \cup_i \text{Zero}(QD(ASC_i))
\]

where each \( ASC_i \) is irreducible and has the same parameter set as \( ASC \).

Proof. This is a consequence of Theorem 1.3 and Theorem 2.4.

Theorem 2.7 shows that \( \text{Zero}(QD(ASC)) \) is a parameter unmixed variety for an arbitrary ascending chain \( ASC \).

3. Applications

Theorem 2.6 and Theorem 2.7 provide more information for Ritt-Wu’s decomposition algorithm. For example, if we need an unmixed decomposition for a variety as in [CA1], i.e., to decompose a variety into the union of some unmixed varieties, then we need only to get a coarse decomposition using Ritt-Wu’s decomposition algorithm.
Theorem 3.1. [WU2] (Ritt-Wu’s Zero Decomposition Algorithm: the Coarse Form)

For two finite sets of polynomials $PS$ and $DS$, we can either detect the emptiness of $Zero(PS/DS)$ or furnish a decomposition of the following forms:

$$Zero(PS/DS) = \bigcup_{i=1}^{l} Zero(ASC_i/DS \cup J_i)$$ (3.1)

$$Zero(PS/DS) = \bigcup_{i=1}^{l} Zero(QD(ASC_i)/DS)$$ (3.2)

where for each $i \leq l$, $ASC_i$ is an ascending chain such that $\text{prem}(G, ASC_i) = 0$ for $\forall G \in PS$, $\text{prem}(P, ASC_i) \neq 0$ for $P \in DS$ and $J_i$ is the initial set of $ASC_i$.

By the affine dimension theorem (p48 [HA1]) and Theorem 2.7, we have two conclusions:

1. Since each $Zero(QD(ASC_i)/DS)$ is either empty or an unmixed variety, (3.2) actually provides an unmixed decomposition for $Zero(PS/DS)$.

2. Let $PS$ contain $m$ polynomials then the components $Zero(QD(ASC_i)/DS)$ with more then $m$ polynomials in $ASC_i$ are redundant and can be deleted from (3.2).

As another application, we give a new proof for a nontrivial theorem in algebraic geometry. An irreducible variety $V$ over $K$ may become reducible over an extension field $K^*$ of $K$. Such a variety is called a relatively irreducible variety [HP1]. We have the following refinement for a result about a relatively irreducible variety.

\textbf{Theorem 3.2.} If $V$ is an irreducible variety of dimension $d$ over the ground field $K$ then over any extension $K^*$ of $K$, $V$ is an (parameter) unmixed variety of dimension $d$.

\textbf{Proof.} As $V$ is irreducible, we have $V = Zero(QD(ASC))$ for an irreducible ascending chain $ASC$ in $K[y]$ with $\text{DIM}(ASC) = d$ [WU1]. Now the result comes from Theorem 2.7.

\textbf{Reference}


Appendix. A Proof of the Dimension Theorem

Theorem (4.1). Let \( n \) be the number of polynomials in \( S \), \( \text{length}(ASC_i) \) be the number of polynomials in \( ASC_i \). Those components \( \text{Zero}(PD(ASC_i)/G) \) in (1.2) (or \( \text{Zero}(QD(ASC_i)/G) \) in (1.3)) for which \( \text{length}(ASC_i) > n \) are redundant, thus can be removed from (1.2) (or from (1.3)).

Proof of the Theorem (4.1). First we assume \( E \) is algebraically closed and \( G = \{1\} \). If \( \text{Zero}(S) \) is empty, then nothing is needed to prove. Assume \( \text{Zero}(S) \) is non-empty. Then we can rearrange the order on the right side of (1.2) as follows:

\[
\text{Zero}(S) = \bigcup_{1 \leq i \leq l} \text{Zero}(ASC_i/I_i) \cup \bigcup_{l < i \leq k} \text{Zero}(ASC_i/I_i)
\]

where \( \text{length}(ASC_i) \leq n \) for \( i \leq l \) and \( \text{length}(ASC_i) > n \) for \( i > l \). By the Affine Dimension Theorem (page 48 in ), the dimensions of all irredundant (irreducible) components of \( \text{Zero}(S) \) are greater than or equal to \( m - n \). (Remember that \( m \) is the number of variables \( y_1, ..., y_m \).) By Lemma (2.2) below, \( \text{Zero}(ASC_i/I_n) \) is contained in the union of irreducible varieties of \( \text{Zero}(PD(ASC_i)) \) with dimension \( \leq m - \text{length}(ASC_i) \). Thus, if \( i > l, m - \text{length}(ASC_i) < m - n \) and each such irreducible variety of \( \text{Zero}(PD(ASC_i)) \) with dimension \( < m - n \) must be in one of the components of \( \text{Zero}(S) \). Therefore, \( l > 0 \) and each component of \( \text{Zero}(S) \) must be contained in some \( \text{Zero}(PD(ASC_i)) \) for \( i \leq l \). Hence,

\[
(2.1) \quad \text{Zero}(S) = \bigcup_{1 \leq i \leq l} \text{Zero}(PD(ASC_i)).
\]

Since any extension \( E \) of \( K \) is contained in an algebraically closed extension of \( K \), (2.1) is valid for any extension \( E \) of \( K \). Hence for any polynomial set \( G \), Theorem (1.4) follows from (2.1).

Lemma (2.2). Let \( ASC = f_1, ..., f_r \) be a non-trivial quasi ascending chain, \( I_i \) be the initials of \( f_i \), and \( J = \{I_1, ..., I_r\} \). Then \( \text{Zero}(ASC/J) \) is contained in the union of irreducible varieties \( \subset \text{Zero}(PD(ASC)) \) with dimensions \( \leq m - r \).

Proof. We use induction on \( m - r \).

(1) Base case: \( m - r = 0 \). In that case, the parameter set of \( ASC \) is empty.

Case (1.1) \( ASC \) is not in weak sense, i.e., \( \text{prem}(I_j; f_1, ..., f_{j-1}) = 0 \) for some \( j > 1 \), then \( \text{Zero}(ASC/J) \) is empty.

Case (1.2) \( ASC \) is irreducible. Then \( \text{Zero}(ASC/J) \) is contained in the (irreducible) variety \( \text{Zero}(PD(ASC)) \), the dimension of which is \( m - r = 0 \). The theorem is true.

Case (1.3) \( ASC \) is reducible. Suppose \( f_1, ..., f_{k-1} \) is irreducible, and \( f_1, ..., f_k \) is reducible \( (1 \leq k) \). For simplicity and without loss of generality, we can assume \( f_k \) has only two irreducible factors, i.e., there are two polynomials \( f_k^1 \) and \( f_k^2 \) with the same class as \( \text{class}(f_k) \) such that \( f_1, ..., f_k^1 \) and \( f_1, ..., f_k^2 \) are irreducible, \( f_k^1, f_k^2 \in \text{Ideal}(f_1, ..., f_k) \),
\( \text{prem}(f_k; f_1, ..., f_r^i) = 0 \) and \( \text{prem}(f_k; f_1, ..., f_r^m) = 0 \). Furthermore, we can chose \( f_k^i \) and \( f_k^m \) in such a way that the initials \( I_k' = \text{lcm}(f_k^i) \) and \( I_k'' = \text{lcm}(f_k^m) \) contain parameters only. Thus

\[
\text{Zero}(\text{ASC} / J) = \text{Zero}(\text{ASC}' / J \cup \{I_k'\}) \cup \text{Zero}(\text{ASC}'' / J \cup \{I_k''\}) \subseteq \text{Zero}(\text{ASC} \cup \{I_k'\} / J) \cup \text{Zero}(\text{ASC} \cup \{I_k''\} / J), (2.2.1)
\]

where

\[
\begin{align*}
\text{ASC}' &= f_1, ..., f_{k-1}, f_k^i, f_{k+1}, ..., f_r, \\
\text{ASC}'' &= f_1, ..., f_{k-1}, f_k^m, f_{k+1}, ..., f_r.
\end{align*}
\]

In this base case, since parameter set is empty, \( I_k' \) and \( I_k'' \) are constants. Thus (2.2.1) actually is

\[
(2.2.2) \quad \text{Zero}(\text{ASC} / J) = \text{Zero}(\text{ASC}' / J \cup \{I_k'\}) \cup \text{Zero}(\text{ASC}'' / J \cup \{I_k''\}).
\]

For quasi ascending \( \text{ASC}' \) (or \( \text{ASC}'' \)) we have three cases:

1. Case (1.3.1) \( \text{ASC}' \) is not in the weak sense, i.e., \( \text{prem}(I_j; \text{ASC}') = 0 \) for some \( j > k \), then \( \text{Zero}(\text{ASC}' / J \cup \{I_k'\}) \) is empty. We can delete it from the union (2.2.2).

2. Case (1.3.2) \( \text{ASC}' \) is irreducible. Then

\[
(2.2.3) \quad \text{prem}(f_j; \text{ASC}') = 0 \text{ for all } i = 1, ..., r.
\]

Thus \( PD(\text{ASC}) \subseteq PD(\text{ASC}') \) by Lemma (2.3) below. Hence

\[
\text{Zero}(\text{ASC}' / J \cup \{I_k'\}) \subseteq \text{Zero}(PD(\text{ASC}')) \subset \text{Zero}(PD(\text{ASC})).
\]

\( \text{Zero}(PD(\text{ASC}')) \) is a variety of dimension \( m - r \).

3. Case (1.3.3) \( \text{ASC}' \) is reducible. We recursively repeat the same procedure of \( \text{Zero}(\text{ASC} / J) \) as for \( \text{Zero}(\text{ASC}' / J') \), until either case (1.3.1) or case (1.3.2) happen, here \( J' = \{I_1, ..., I_k', ..., I_r\} \). When case (1.3.2) happens, (2.2.3) is still valid.

Thus we conclude that \( \text{Zero}(\text{ASC} / J) \) is contained in the union of those components of the algebraic set \( \text{Zero}(PD(\text{ASC})) \) whose dimension is \( m - r = 0 \).

(2) Induction case: suppose the theorem is true for quasi ascending chains \( g_1, ..., g_d \) with \( m - d < m - r \). We want to show it is also true for \( f_1, ..., f_r \). We can use the same argument as in the base case.

Case (2.1) \( \text{ASC} \) is not in weak sense, then \( \text{Zero}(\text{ASC} / J) \) is empty.

Case (2.2) \( \text{ASC} \) is irreducible. Then as before, the theorem is true.

Case (2.3) \( \text{ASC} \) is reducible. We can repeat the same argument as in case (1.3) and also have 3 cases for each of ascending chains \( \text{ASC}' \) and \( \text{ASC}'' \). Here we emphasize that \( I_k' \) and \( I_k'' \) contain only the parameters of \( \text{ASC} \). Decomposition (2.2.1) is valid, but (2.2.2) is no longer valid. Instead, we can decompose (Ritt–Wu’s Algorithm again), say, \( \text{Zero}(\{I_k'\}) \), into

\[
\text{Zero}(\{I_k'\}) = \bigcup_i \text{Zero}(\text{ASC}_i'/I_{k,i}).
\]
Here for each $i$, $I_{k,i}$ is the initial set of the ascending chain $ASC'_i$. Then

$$\text{Zero}(ASC \cup \{I_k^i\}/J) = \bigcup_i \text{Zero}(ASC'_i \cup ASC/I_{k,i} \cup J).$$

Note that $ASC'_i \cup ASC$ forms another quasi ascending chain since $ASC'_i$ involves only the parameters of $ASC$. For each $\text{Zero}(ASC'_i \cup ASC/I_{k,i} \cup J)$, we now can use the induction hypothesis to conclude that it is contained in the union of varieties (with dimension $\leq m-r+1$) $\subset Zero(PD(ASC'_i \cup ASC)) \subset Zero(PD(ASC))$. Thus the proof is completed.

**Lemma (2.3).** Let $ASC_1$ and $ASC_2$ be two irreducible asc chains. $PD(ASC_1) \subset PD(ASC_2)$ only if $\text{prem}(p, ASC_2) = 0$ for all $p \in ASC_1$. If this is the case and $\text{prem}(\text{lc}(p), ASC_2) \neq 0$ for all $p \in ASC_1$, then $PD(ASC_1) \subset PD(ASC_2)$.

**Proof.** It is trivial.