Chapter 10

Automated reasoning in geometry

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Contents

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In the past 20 years highly successful methods for geometry theorem proving and discovering have been developed. This chapter gives a brief account of these successful methods. We will use elementary and understandable examples to show the essence of the techniques, letting the reader consult the related references for more detailed issues underlying these techniques.

1. A history review of automated reasoning in geometry

Generally, there are two approaches to proving geometry theorems using computers: the artificial intelligence (AI) approach and the algebraic computation approach. The earliest work in geometry theorem proving by computer programs was done by Gelernter and his collaborators [Gelernter 1959]. It was based on the human simulation approach and has been considered a landmark in the AI area for its time. Wos and his collaborators used their powerful general-purpose resolution theorem prover to experiment with proving theorems in Tarski's axioms for elementary geometry [McCharen, Overbeek and Wos 1976]. In spite of the success and significant improvements [Gilmore 1970, Nevins 1976, Coelho and Perceira 1986, Koedinger and Anderson 1990, Quaife 1989, Balbiani and del Cerro 1995] with these methods, the results did not lead to the development of a *powerful* geometry theorem prover.

In the area of algebraic computation approach, the earliest work dates back to Hilbert. In his classic book [Hilbert 1971], Hilbert outlined a decision method for a class of constructive geometry statements in *affine geometry*. As Tarski pointed out, Hilbert's result "is closely connected with the decision method for elementary geometry, but has a rather restricted character".

In 1951, Tarski published a decision method for the theory of real closed fields, thus giving a decision method for what he called elementary geometry [Tarski 1951]. In spite of subsequent improvements by Seidenberg [Seidenberg 1954] and others, for years variations of Tarski's method remained impractical for proving non-trivial theorems in geometry. In 1974, Collins made an important contribution along the Tarski line [Collins 1975]. His cylindrical algebraic decomposition (CAD) algorithm is currently the best general algorithm of Tarski type. This method was implemented by Arnon, and several difficult algebra-geometry related problems were solved by Arnon's program [Arnon and Mignotte 1988, Arnon 1988]. Another major practical improvement of Collins' method has been made in [Collins and Hong 1991].

Practically, Davis appears to be the first to explore the algebraic approach to proving geometry theorems using the computer [Davis and Cerutti 1969]. His approach for the computer proof of Pappus' theorem is essentially the one described by Hilbert, but he did not provide a unifying mechanical way to do it.

A breakthrough in automated geometry theorem proving (AGTP) is made by Wu. Restricting himself to a class of geometry statements of *equality type*, Wu introduced a method in 1977 which can be used to prove quite difficult geometry theorems efficiently [Wu 1978]. Wu's work became known outside China mainly through the papers [Wu 1984*c*, Chou 1984], and the fact that over 130 theorems were proved by the method in [Chou 1984] was quite encouraging. Ko and Hussain [Ko and Hussain

1985], Wang and Hu [Wang and Hu 1987, Wang and Gao 1987], Gao [Gao 1990], Kapur and Wan [Kapur and Wan 1990] also succeeded in implementing theorem provers based on various modified version of Wu's method. Later it was clarified [Wu 1984*a*] that the algebraic tools needed in Wu's approach can be developed from Ritt's work in [Ritt 1950]. The algebraic aspect of this approach is now known as the Wu-Ritt's characteristic set (CS) method. It is now the case that hundreds of theorems in Euclidean and non-Euclidean geometries can be proved automatically by computer programs with Wu's method.

The success of Wu's method has revived interest in proving geometry theorems by computers. In particular, the application of the Gröbner basis (GB) method [Buchberger 1985] to the same class of geometry theorems that Wu's method addresses has been investigated. In 1985–1986, three groups ([Chou and Schelter 1986, Kapur 1986, Kutzler and Stifter 1986]) reported practical successes. A recent tutorial on the Gröbner basis method can be found in [Wang 1998*b*]. Other successful elimination methods for automated geometry theorem proving (AGTP) include the resultant approach [Yang, Zhang and Hou 1992], the gcd computation approach [Kalkbrener 1995], the numerical example checking approach [Hong 1986, Zhang, Yang and Deng 1990, Wang 1988], the Brauer-Seidenberg-Wang approach [Wang 1995*b*] and the Dixon resultant approach [Kapur 1997].

Here we would like to remind the reader that Wu's method and the GB method can only deal with theorems involving equalities, but not inequalities. Theoretically, Collins' method can prove (or disprove) any elementary sentences in the Tarski geometry. Many researchers focused on developing more efficient algorithms for special classes of problems involving inequalities. Wu proposed a method to find the maximal or minimal values for a polynomial (pol) function under certain conditions using the CS method and the Lagrangian multiplier method [Wu 1992*a*]. The work in [McPhee, Chou and Gao 1994] is based on a combination of Wu's method and the CAD method. The work in [Dolzmann, Sturm and Weispfenning 1996] is based on quantifier elimination methods for equations with low degrees. Recently, Yang et al proposed the complete discriminant theory which is quite efficient in finding real roots classifications for univariate pol equations [Yang, Zhang and Hou 1996]. Yang also developed an inequality prover which has been used to prove more than one thousand interesting geometric inequalities including many new ones [Yang 1998].

At the same time, automated derivation of geometric locus equations and other geometric formulas was investigated [Wu 1986*a*, Wang and Gao 1987, Chou 1987, Chou and Gao 1990*a*, Wang 1991, Wang 1995*c*]. About 120 problems in geometry were solved in [Chou and Gao 1989]. Dixon resultant computation is used to derive geometric formulas in [Kapur, Saxena and Yang 1994]. Formula derivation is actually to find the manifold solutions of equation systems.

The above work is concerned with *elementary geometries* in Wu's sense, i.e., geometries in which no differentiation is involved. The CS method is also applicable to *differential pols* [Ritt 1938]. In [Wu 1979], Wu extended the CS method to prove theorems in *differential geometry*. Extensive computer experiments with this method for the theory of space curves were done in [Wu 1987*c*, Chou and Gao 1991]. The results are encouraging, and nearly one hundred non-trivial theorems in space curve theory have been proved [Chou and Gao 1991]. In [Li 1995*b*], Wu's method was used to prove theorems of space surfaces. In [Ferro and Gallo 1990, Ferro and Gallo 1994], new methods for proving theorems in differential geometry based on the computation of the dimensions of zero sets were proposed. In [Wang 1995*b*], Brauer-Seidenberg's elimination theory is modified to prove theorems in space curve theory [Wang 1995*b*]. In [Li and Cheng 1998], a method based on vector calculation for AGTP in differential geometry is proposed, which is capable of producing proofs like those in the textbooks. There have also been several successful applications of the CS method to mechanics [Wu 1987*b*, Chou and Gao 1993*b*, Chou and Gao 1993*c*]; notably, automated proofs of Newton's laws from Kepler's laws were given. Computer experiments in automated formula derivation in differential geometry and mechanics were also discussed in these pieces of work.

All the above methods have the same character that they first transform geometric properties into equations in coordinates of the related points and then deal with these equations. The search for a vector based method for AGTP began in the mid-eighties, because it is believed that such a method would produce more elegant proofs. In the early-nineties, several successful vector approaches were proposed: the area method [Chou, Gao and Zhang 1993*a*], the vector method for constructive statements [Chou, Gao and Zhang 1993*b*], the Gröbner basis method [Stifter 1993], the bracket algebra method [Richter-Gebert 1995] and the Clifford algebra method [Li and Cheng 1996, Fèvre and Wang 1997, Wang 1998*a*]. Experiments show that proofs produced by these methods are generally short than those given by the coordinate methods. Of these methods, the area method uses pure geometric invariants, such as area, ratio of segments, Pythagorean difference, etc. The main advantage of this method is that each step of the generated proof has clear geometric meanings. The computer program based on the area method has produced proofs of more than 500 geometry theorems, some of which are even shorter than those given by geometry experts.

More recently, the AI approach has been revived to such an extent that it can solve hundreds of difficult geometry problems and produce multiple and shortest proofs for geometry theorems efficiently [Chou, Gao and Zhang 1996*a*, Chou, Gao and Zhang 1996*c*]. The AI approach is also used for automated generation of construction steps of geometric diagrams and successfully applied to many difficult geometric construction problems (e.g., the Appolonius Problem) [Gao and Chou 1998*a*].

Methods of automated reasoning in geometry have a wide range of applications, including kinetic analysis of robotics [Wu 1989*b*, Huang and Wu 1992, Kapur 1997, Yang, Fu and Zheng 1997], linkage design [Gao, Zhu and Huang 1998*a*], computer vision [Kapur and Mundy 1988, Gao and Cheng 1998, Wang 1998*a*, Bondyfalat, Mourrain and Papadopoulo 1999], intelligent CAD [Gao and Chou 1998*a*, Gao and Chou 1998*b*], intelligent CAI (computer aided instruction) [Gao, Zhu and Huang 1998*b*, Li and Zhang 1998], solid modeling [Wu 1993, Kapur 1997], etc. Several pieces of software originaed from this field have been published [Gao, Zhang and Chou 1998, Li and Zhang 1998, Richter-Gebert and Kortenkamp 1999].

Finally, we want to say that although we hope to cover all the work in the subject, some existing work might be missed. Also, the reader may consult previous surveys on the similar subject [Wu 1992*b*, Buchberger, Collins and Kutzler 1995, Wang 1996*b*, Kapur 1997]. In particular, a report on AGTP provers can be found in [Hong, Wang and Winkler 1995].

The rest of this review is divided into four sections. Section 2 is a review of the algebraic approaches to AGTP. Section 3 is a review of coordinate free approaches to AGTP. Section 4 is a review of AI approaches to AGTP. Section 5 is a summary of this paper.

2. Algebraic approaches to automated reasoning in geometry

2.1. Proving theorems in elementary geometries

This is the most developed and successful area using the characteristic set (CS), the Gröbner basis (GB), and other elimination methods. Roughly speaking, the methods can address those *geometry statements of equality type*, for which, in their algebraic form, the hypotheses can be expressed by a set (conjunction) of equations

$$\begin{array}{c} h_1(y_1,\cdots,y_m)=0 \\ h_2(y_1,\cdots,y_m)=0 \\ \cdots \\ h_r(y_1,\cdots,y_m)=0, \end{array} \tag{2.1}$$

and the conclusion is also a pol equation $c(y_1,\cdots,y_m)=0$, where the h's and c are pols with coefficients in a *base field* K. Usually, we assume, $K=\mathbf{Q}$, the field of rational numbers. Thus the algebraic form of the geometry statement would be

$$\forall y[(h_1=0\wedge\cdots\wedge h_r=0)\Rightarrow c=0].$$

However, such formulas are usually not valid because most geometry statements are only valid under some non-degenerate (ndg) conditions. Let us look at two concrete examples to see the real situations.

2.1.1. Two examples and the simple version of Wu's method

2.1. Example. Let $ABCD$ be a parallelogram, and E the intersection of the two diagonals AC and BD. Show that $AE=CE$ (Fig. 1).

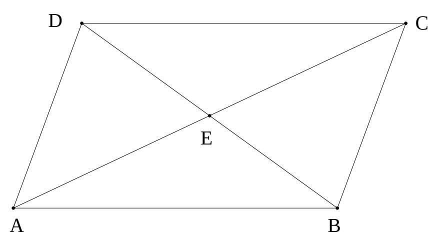

Fig. 1

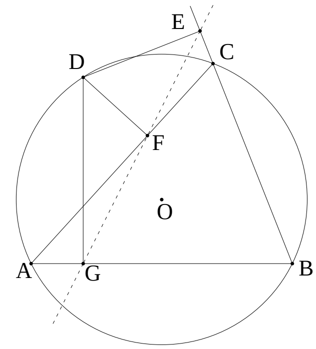

Fig. 2

The first step is to assign coordinates to the points, and then translate the hypotheses and conclusion of the statement into pol equations.

We can choose point A as the origin, and line AB as the x–axis of the coordinate system: $A = (0, 0)$, $B = (u_1, 0)$. We can assign the coordinates to point $C = (u_3, u_2)$. Since points A, B, and C can be arbitrarily chosen, their non-zero coordinates are independent variables or parameters, denoted by u's. Once these three points are fixed, the other points D and E are determined; their coordinates are dependent variables, denoted by x's. Let $E = (x_2, x_1)$, $D = (x_4, x_3)$.

Once we have the coordinate, the conversion of the hypotheses and conclusion to their algebraic forms is straightforward. Thus we have the following equations corresponding to the hypotheses:

$$
\begin{array}{lll}
 & h_1 = u_1x_1 - u_1u_2 = 0 & DC \parallel AB \\
 & h_2 = u_2x_2 - (u_3 - u_1)x_1 = 0 & DA \parallel BC \\
(2.2) & h_3 = u_2x_4 - u_3x_3 = 0 & E \text{ is on } AC \\
 & h_4 = x_1x_4 - (x_2 - u_1)x_3 - u_1x_1 = 0 & E \text{ is on } BD.
\end{array}
$$

The conclusion $AE = CE$ can be expressed by $c = x_4^2 + x_3^2 - [(x_4 - u_3)^2 + (x_3 - u_2)^2] = 2u_3x_4 + 2u_2x_3 - u_3^2 - u_2^2 = 0$.

Thus the tentative algebraic form of the parallelogram theorem would be

$$\forall ux[(h_1 = 0 \wedge h_2 = 0 \wedge h_3 = 0 \wedge h_4 = 0) \Rightarrow c = 0].$$

The above formula is "almost" correct except for a ndg condition: it is not valid when A, B, and C are collinear.

A nice feature of Wu's method is that ndg conditions sufficient for a geometry statement to be valid can be generated automatically during the proof process. The basic operation of the method is pseudo division.

Pseudo Division. Let

$$
\begin{array}{ll}
g = a_ny^n + \cdots + a_1y + a_0 & (a_n \neq 0) \\
f = b_ky^k + \cdots + b_1y + b_0 & (b_k \neq 0 \wedge k > 0)
\end{array}
$$

be two pols in the variable y, where the coefficients a_i, b_i are in $\mathbf{Q}[v_1, \cdots, v_d]$, $y \notin \{v_1, \cdots, v_d\}$. Here a_n (or b_k) and n (or k) are called the leading coefficient and leading degree of g (or f) in the variable y and denoted by $lc(g, y)$ and $ld(g, y)$ (or $lc(f, y)$ and $ld(f, y)$), respectively. We want to divide g by f in the variable y. If b_k

is 1, then we use long division from high schools; otherwise we use pseudo division. For our purpose, all we really care about is the pseudo remainder r, denoted by $r = prem(g, f, y)$, which verifies the following remainder formula:

$$b_k^s \cdot g = q \cdot f + r, \qquad ld(r, y) < k, \tag{2.3}$$

where s ($\leq n - k + 1$) is a non-negative integer.

The first step is to transform the set of hypotheses into a triangular set, where each equation introduces only one new dependent variable. Let us look at (2.2); h_1 introduces x_1 and h_2 introduces x_2; so far so good. But h_3 introduces two new dependent variables x_3 and x_4 at the same time; thus (2.2) is not in triangular form. However, it is easy to transform it into a triangular set by letting $f_1 = h_1$, $f_2 = h_2$, $f_3 = prem(h_4, h_3, x_4)$, $f_4 = h_4$. Then $f_1, \cdots, f_4$ is a triangular set or an ascending chain ASC:[1]

$$\begin{aligned} f_1 &= u_1x_1 - u_1u_2 = 0 \\ f_2 &= u_2x_2 - (u_3 - u_1)x_1 = 0 \\ f_3 &= (-u_2x_2 + u_3x_1 + u_1u_2)x_3 - u_1u_2x_1 = 0 \\ f_4 &= x_1x_4 - (x_2 - u_1)x_3 - u_1x_1 = 0. \end{aligned} \tag{2.4}$$

Now we can perform the key step of the method: the successive (pseudo) division of c, the conclusion pol, with respect to that triangular set, i.e.,

$R_4 = prem(c, f_4, x_4) = (2u_3x_2 + 2u_2x_1 - 2u_1u_3)x_3 - (u_3^2 - 2u_1u_3 + u_2^2)x_1$

$R_3 = prem(R_4, f_3, x_3) = (u_2u_3^2 + u_2^3)x_1x_2 - (u_3^3 - 2u_1u_3^2 + u_2^2u_3 - 2u_1u_2^2)x_1^2 - (u_1u_2u_3^2 + u_1u_2^3)x_1$

$R_2 = prem(R_3, f_2, x_2) = (u_1u_2u_3^2 + u_1u_2^3)x_1^2 - (u_1u_2^2u_3^2 + u_1u_2^4)x_1$

$R_1 = prem(R_2, f_1, x_1) = 0.$

The last remainder R_1 is denoted by $prem(c; f_1, \cdots, f_4)$ or $prem(c; ASC)$. In this particular case it turns out to be zero by computation. That means we have proved the theorem. To see this, first we always have the following *remainder formula* when doing a pseudo successive division of any pol g with respect to a triangular set $ASC = f_1, \cdots, f_p$,

$$I_1^{s_1} \ldots I_p^{s_p} g = Q_1 f_1 + \cdots + Q_p f_p + R_1. \tag{2.5}$$

Here the I_k are leading coefficients of the f_k and $s_k \geq 0$ ($k = 1, \cdots, p$); $R_1 = prem(c; ASC)$ is the final remainder. In this case, $p = 4$ and the I_k are

$$\begin{aligned} I_1 &= u_1 \\ I_2 &= u_2 \\ I_3 &= -u_2x_2 + u_3x_1 + u_1u_2 \\ I_4 &= x_1. \end{aligned}$$

Since $R_1 = 0$ and $f_k = 0$ ($k = 1, \cdots, 4$) by the hypotheses, the right side of (2.5) is zero. Thus the conclusion pol must be zero if we assume all $I_k \neq 0$ ($k = 1, \cdots, 4$).

$I_k \neq 0$ are usually connected with non-degeneracy. Thus the last step is the analysis of subsidiary conditions $I_k \neq 0$. For example, $I_1 \neq 0$ and $I_2 \neq 0$ mean that

[1] Strictly speaking, there are other restrictions for a triangular set to be an ascending chain [Wu 1984*a*].

points A, B, and C are not collinear; $I_3 \neq 0$ means that AC and BD have a normal intersection; $I_4 \neq 0$ means that D is not on line AB.

Before going to the next example, let us summarize this simple version of Wu's method [Wu 1978]:

Step 1. Assign coordinates to the points involved, then translate the geometric hypotheses to a set (conjunction) of pol equations $h_1 = 0, \cdots, h_r = 0$; also the conclusion is a pol equation $c = 0$.

Step 2. Transform the set of hypothesis pols into a triangular set $ASC = f_1, \cdots, f_p$. Generally, we have a complete triangular algorithm presented and referred to as Ritt's principle by Wu [Wu 1984*a*].

Step 3. Divide the conclusion pol c successively with respect to the triangular set $ASC = f_1, .., f_p$ to obtain the final remainder $R_1 = prem(c; ASC)$. If $R_1 = 0$, then the statement is confirmed under subsidiary conditions $I_k \neq 0$, where the I_k are the leading coefficients of the f_k.

Step 4. Analyze the subsidiary conditions $I_k \neq 0$ which are usually connected with non-degeneracy.

For this simple example, we don not have to use this general schema. We can solve the dependent variables $x_1, \cdots, x_4$ successively in terms of the *u's*, then substitute the solutions into the conclusion pol c to see whether it is identical to zero. This is exactly what *Hilbert's mechanical proof method* for constructive affine geometry statements does. However, Hilbert's original method does not provide ndg conditions which are important for a geometry statement to be valid. In the general case, we cannot solve dependent variables explicitly and have to use the above general schema. The following problem is such an example.

2.2. Example *(Simson's Theorem).* D is a point on the circumscribed circle of triangle ABC. From D, perpendiculars are drawn to three sides BC, CA and AB. Let E, F and G be the three feet. Show that E, F and G are collinear (Fig. 2).

Step 1. Let $A = (0, 0)$, $B = (u_1, 0)$, $C = (u_3, u_2)$, $O = (x_1, x_2)$, $D = (x_3, u_4)$, $E = (x_5, x_4)$, $F = (x_7, x_6)$, and $G = (x_9, x_8)$. We then have a set of hypothesis equations H:

$$
\begin{array}{lr}
h_1 = 2u_1x_1 - u_1^2 = 0 & OA = OB \\
h_2 = 2u_2x_2 + 2u_3x_1 - u_3^2 - u_2^2 = 0 & OA = OC \\
h_3 = x_3^2 - 2x_1x_3 + u_4^2 - 2x_2u_4 = 0 & DO = OA \\
h_4 = u_2x_5 - (u_3 - u_1)x_4 - u_1u_2 = 0 & E \text{ is on } BC \\
h_5 = (u_3 - u_1)x_5 + u_2x_4 - (u_3 - u_1)x_3 - u_2u_4 = 0 & ED \perp BC \\
h_6 = u_2x_7 - u_3x_6 = 0 & F \text{ is on } AC \\
h_7 = u_3x_7 + u_2x_6 - u_3x_3 - u_2u_4 = 0 & FD \perp AC \\
h_8 = u_1x_8 = 0 & G \text{ is on } AB \\
h_9 = u_1x_9 - u_1x_3 = 0 & GD \perp AB.
\end{array}
\tag{2.6}
$$

The conclusion that E, F, and G are collinear can be expressed by the equation $c = (x_6 - x_4)x_9 - (x_7 - x_5)x_8 + x_4x_7 - x_5x_6 = 0$.

Step 2. Let $f_4 = prem(h_5, h_4, x_5)$, $f_6 = prem(h_7, h_6, x_7)$, $f_i = h_i$ for $i \neq 4, 6$. We thus have a triangular set $ASC = f_1, \cdots, f_9$:

$$
\begin{aligned}
f_1 &= 2u_1x_1 - u_1^2 \\
f_2 &= 2u_2x_2 + 2u_3x_1 - u_3^2 - u_2^2 \\
f_3 &= x_3^2 - 2x_1x_3 + u_4^2 - 2x_2u_4 \\
f_4 &= (u_3^2 - 2u_1u_3 + u_2^2 + u_1^2)x_4 - (u_2u_3 - u_1u_2)x_3 - u_2^2u_4 + u_1u_2u_3 - u_1^2u_2 \\
f_5 &= u_2x_5 - (u_3 - u_1)x_4 - u_1u_2 \\
f_6 &= (u_3^2 + u_2^2)x_6 - u_2u_3x_3 - u_2^2u_4 \\
f_7 &= u_2x_7 - u_3x_6 \\
f_8 &= u_1x_8 \\
f_9 &= u_1x_9 - u_1x_3.
\end{aligned}
\tag{2.7}
$$

Step 3. Now use successive pseudo division to compute the final remainder $R_1 = prem(c; ASC) = 0$.
Since the final remainder R_1 is 0, by the remainder formula (2.5), $c = 0$ follows from $h_i = 0$ and subsidiary conditions $I_i \neq 0$, where the I_i are the leading coefficients of the f_i.

Step 4. Analysis of subsidiary conditions $I_k \neq 0$. Here the non-zeroness of I_1, I_2, I_8, and I_9 mean that A, B, and C are not collinear. $I_7 \neq 0$ and $I_5 \neq 0$ are not necessary by a more careful analysis (Section 2.1.4). The role of the conditions $I_4 \neq 0$ and $I_6 \neq 0$ is very subtle, and they are *necessary* when using Wu's method or the GB method. See Example 2.3 below.

2.1.2. Geometry statements of constructive type

This simple use of Wu's method is powerful enough to prove hundreds of nontrivial geometry statements. But its application is restrictive. First, the ndg conditions $I_i \neq 0$ depend on the choices of the coordinates and this may cause problems. For instance, we may "prove" the 8_3 configuration problem using this method, which is actually invalid [Chou, Gao and Mcphee 1989]. Secondly, $I_i \neq 0$ are in algebraic form and there is no general method to transform these conditions into geometric form.
The great success of Wu's method is closely connected to some special geometry statements: the class of *constructive statements* [Wu 1984*a*, Wang and Hu 1987, Chou and Gao 1992]. Actually, the statements considered by Hilbert are constructive ones, but it has a rather restricted character: they are about those configurations formed by straight lines in a constructive way. Hilbert's method was clarified and revitalized by Wu [Wu 1982*b*], and extended later by Wang and Hu [Wang and Hu 1987] for a wider domain of application. In [Chou 1984, Chou and Gao 1992], a class of constructive geometry statements (henceforth referred to as Class C) is considered. Class C is actually the statements about the configurations which can be drawn by rulers and compasses. For instance, Examples 2.1 and 2.2 are in this class. About 85 percent of the 512 theorems in [Chou 1988] belong to this class.
The basic method introduced in Section 2.1.2 can be developed into a complete

method for constructive statements. For a constructive statement, point coordinates can be chosen automatically and coordinate-independent ndg conditions in geometric form can be generated. Furthermore, these ndg conditions are sufficient, i.e., a geometry statement is true in a complex geometry iff it is true under these ndg conditions [Chou and Gao 1992]. For geometry statements of constructive type, since new points are introduced one by one, the new dependent variables are introduced at most two by two. Therefore their corresponding algebraic problems are easier to solve than the algebraic problems corresponding to the general geometry problems.
More properties of constructive statements can be found in [Chou and Gao 1992]. In [Chou and Gao 1993*e*], most of the results about constructive statements have been extended to solid and Riemannian geometries.

2.1.3. Formulation problem

The above way of proving theorems is not the one adopted by most provers based on the logic approach. We start with a set of hypotheses which do not necessarily imply the conclusion and end up with the confirmation of the conclusion by adding some subsidiary conditions. Would it be possible that the original geometry statement is substantially weakened or changed? To answer this question we need to address the formulation problem – in what sense do we prove geometry theorems? Generally, for a geometry statement, the equality part of the hypotheses (i.e., $H = \{h_1 = 0, \cdots, h_r = 0\}$) is easy to identify. However, the inequation part of the hypotheses, which is usually connected with non-degeneracy, is not so simple to identify. For a given geometry statement, a ndg condition that is obvious to one person might not be obvious to a second, and a third person might refuse to accept the condition as relevant or appropriate. The key issue here is how to understand and handle these ndg conditions.
Generally, for a geometric configuration defined by a set of equations

$$H = \{h_1(y_1, \cdots, y_m) = 0, \cdots, h_r(y_1, \cdots, y_m) = 0\},$$

we want to decide whether an assertion $c(y_1, \cdots, y_m) = 0$ on this configuration is valid. We define the notation

$$Zero(h_1, \cdots, h_r) = \{(y_1, \cdots, y_m) \in E^m \; | h_i(y_1, \cdots, y_m) = 0 \text{ for } i = 1, \cdots, r\},$$

where E is an extension of the base field $\mathbf{Q}$; usually, $E = \mathbf{C}$ (the field of complex numbers; later we will discuss the case when $E = \mathbf{R}$, the field of really numbers). As we know the formula $\forall y \in E[H \Rightarrow c = 0]$, or equivalently

$$Zero(H) \subset Zero(c)$$

is usually not valid because of missing ndg conditions.
It is now accepted by most researchers that there are two different but related formulations for dealing with ndg conditions [Chou and Yang 1989, Kapur 1997].
Formulation F1. Identify some variables among $y_1, \cdots, y_m$ as parameters, then decide whether the conclusion $c = 0$ follows from the hypothesis H generically with

respect to those parameters. The zeros of the hypothesis pols can be decomposed into irreducible components as follows:

$$Zero(H) = V_1^* \cup \cdots V_d^* \cup V_1^{\rm dege} \cup \cdots \cup V_l^{\rm dege},$$

where $V_1^*, \cdots, V_d^*$ are all those components on which the selected parameters are exactly those algebraically independent variables, corresponding to ndg cases; the others correspond to degenerate cases. If the conclusion is valid in all ndg cases, then we say the statement is *generically true*. If it is valid on none of the ndg cases, then it is *generically false*.

The concept of generically true was proposed by Wu [Wu 1984*a*]. The above description was given in [Chou 1988].

Formulation F2. Explicitly specify the ndg condition $D = \{d_1 \neq 0, \cdots, d_q \neq 0\}$ as a part of the hypotheses. Then the aim is to decide whether the statement

$$\forall y[(H \wedge D) \Rightarrow c = 0] \tag{2.8}$$

is valid without adding any additional conditions.

Formulation F2 was proposed by Kapur using GB approach [Kapur 1988] and was adopted in [Chou and Gao 1990*b*, Kapur and Wan 1990, Ko 1988, Wang 1996*a*] using the CS approach.

In [Wang 1995*c*, Winkler 1990, Winkler 1992], other formulations are proposed.

F1 can help to find the missing ndg conditions. Furthermore, it addresses the nature of the statement: if a statement is proved to be generically false, it cannot be a theorem no matter how many reasonable additional ndg conditions are added. However, ndg conditions are usually in algebraic form and the currently used methods based on F1 usually generate ndg conditions more than needed. On the other hand, F2 is easier to understand. The geometry statement is exactly specified, and the user can select ndg conditions he/she thinks suitable. However, if the statement is proved to be false, we don't know the nature of the statement: whether it is generically false or the proof failed due to missing ndg conditions. The ndg conditions are often implicit in a statement in geometry textbooks and identifying them is sometimes very hard and subtle, even in Euclidean geometry.

Now we come to the question why E was chosen to be $\mathbf{C}$, the field of complex numbers, instead of $\mathbf{R}$, the field of real numbers. First, we emphasize that if the CS or GB method confirms a geometry statement according to either F1 or F2, then it is valid in *all* fields, including $\mathbf{R}$. However, if the statement is disproved by one of the two methods, then it is not valid in $\mathbf{C}$, but may be valid in $\mathbf{R}$. In axiomatic geometry, there are several systems of axioms. There is a system of axioms for *unordered metric geometry* which involves the relations incidence, perpendicularity, segment congruence, and angle congruence; $\mathbf{C}^2$ (or $\mathbf{R}^2$) is a *model* for the theory of this metric geometry [Wu 1984*b*]. If we want to decide whether a geometry statement (with universal quantifiers outside) is a *logical consequence* of the theory of metric geometry, then the CS and GB methods are complete, i.e., they are a *decision procedure* for such a theory.

2.3. EXAMPLE *(Simson's theorem continued).* Suppose that the only ndg condition is "A, B, and C are not collinear", i.e., $d_1 = u_1u_2 \neq 0$ for Simson's theorem. Then we ask (according to F2) whether the statement is a logical consequence of the theory of metric geometry, or equivalently, in its algebraic form, whether

$$\forall ux \in \mathbf{C}[(H \wedge d_1 \neq 0) \Rightarrow c = 0]$$

is valid. Then the CS or GB method proves it is not the case. That means the statement, as it is, is *not a theorem* (logical consequence) in the theory of metric geometry. Or putting it another way, the statement cannot be proved *without using axioms of order.* However, if we add another ndg condition that "the three sides of the triangle are non-isotropic (with non-zero the length)", the statement is true as verified by both methods. In Euclidean geometry, isotropic lines do not exist, thus Simson's theorem is proved by the CS or GB method under the sole ndg condition that A, B and C are not collinear.

2.1.4. The CS and GB methods

From the above discussion, we have four approaches: CS(F1), CS(F2), GB(F1), and GB(F2). According to Kapur, one can use direct or refutational approaches [Kapur 1997]. Thus altogether there are possibly $4 \times 2 = 8$ approaches. Here we cite the presentations for those approaches: for CS(F1), see [Wu 1984*a*, Chou 1988]; for CS(F2), see [Ko 1988, Chou and Gao 1990*b*, Kapur and Wan 1990, Wang 1995*c*]; for GB(F1), see [Chou and Schelter 1986, Kutzler and Stifter 1986, Chou 1988, Wang 1998*b*]; for GB(F2), see [Chou and Schelter 1986, Kapur 1988, Chou 1988, Kapur 1986, Wang 1998*b*]. We now briefly present a representative of each of the four approaches.

CS(F1). The method we just used for the two examples in Section 2.1.2 is actually based on Formulation F1. In both examples, the triangular sets ASC are irreducible and represent the only ndg component, V_1^*, and $prem(c; ASC) = 0$ means that $c = 0$ is valid on V_1^*, i.e., the statements are generically true.

CS(F2). Let S and G be two pol sets. Denote $Zero(S/G)$ the set difference $Zero(S) - \bigcup_{d \in G} Zero(d)$. Thus according to F2, the goal is to prove (2.8), i.e., to prove

$$Zero(H/D) \subset Zero(c),$$

where $H = \{h_1 = 0, \cdots, h_r = 0\}$ and $D = \{d_1 \neq 0, \cdots, d_q \neq 0\}$ are the equality part and the inequation part of the hypotheses, respectively.

Using Wu-Ritt's decomposition algorithm [Wu 1984*a*],

$$Zero(H/D) = \bigcup_{1 \leq i \leq k} Zero(PD(ASC_i)/D)$$

where the ASC_i are irreducible ascending chains;

$$PD(ASC) = \{g \mid prem(g, ASC) = 0\}.$$

To decide whether the statement is true or not (in **C**), we only need to verify whether $prem(c; ASC_i) = 0$ for all i. If we chose $D = \{u_1u_2 \neq 0\}$ for the parallelogram example and D to be the same as in Example 2.3 for Simson's theorem, then $k = 1$, and ASC_1 are just the triangular sets ASC in (2.4) and (2.7). Since $prem(c; ASC_1) = 0$ in both examples (Step 3: successive division), they have been confirmed under the specified ndg conditions without adding any other conditions.
GB(F1). The conclusion $c = 0$ follows from the hypotheses $h_1 = 0, \cdots, h_r = 0$ generically iff there is a non–zero pol U containing only the parameters such that $U \cdot c \in Radical(h_1, \cdots, h_r)$. This is in turn equivalent to c being in the radical generated by $h_1, \cdots, h_r$ in the ring $\mathbf{Q}(u)[x]$, where the u are parameters, and the x are the dependent variables. This is equivalent to a Gröbner basis of $h_1, \cdots, h_r, cz-1$ in $\mathbf{Q}(u)[x]$ containing 1, where z is a new variable. This is the case for Simson's theorem as confirmed by a computer program based on this approach, and the pol U was also found during computing the GB. Thus under $U \neq 0$, $(H \Rightarrow c = 0)$. It is generically hard to interpret the geometric meaning of $U \neq 0$ automatically. However, for a constructive statement, if $(H \Rightarrow c = 0)$ is confirmed to be generically true, then $(H \Rightarrow c = 0)$ is valid under the *geometric* ndg conditions generated by the algorithm in [Chou and Gao 1992]. Based on this theorem, Simson's theorem has been proved by GB(F1) under the ndg conditions that the three vertices are not collinear and the three sides are non-isotropic.
GB(F2). First we observe that in any field, $d \neq 0$ iff $\exists z(zd - 1 = 0)$. Thus (2.8) is equivalent to

$$\forall y[\exists z_1 \cdots z_q(H \wedge d_1z_1 - 1 = 0 \wedge \cdots \wedge d_qz_q - 1 = 0) \Rightarrow c = 0],$$

which is in turn equivalent (because c is free of z_i) to

$$\forall yz_1 \cdots z_q[(H \wedge d_1z_1 - 1 = 0 \wedge \cdots \wedge d_qz_q - 1 = 0) \Rightarrow c = 0].$$

In algebraically closed fields, the above formula is equivalent to whether $H' = \{h_1, \cdots, h_r,\ d_1z_1 - 1, \cdots, d_qz_q - 1, zc - 1\}$ generates the unit ideal, where the z are new variables. Thus the method is to compute a Gröbner basis of H' to see whether it contains 1. It is the case as confirmed by computers for the two examples. The use of new variables z_i and z was first introduced by Rabinowitsch in connection with a proof of Hilbert's Nullstellensatz. It has been used extensively in computer algebra, e.g., in [Gianni, Trager and Zacharias 1988]. In geometry theorem proving, it was first used in [Chou and Schelter 1986, Kapur 1986].
A large number of geometric problems are solved by programs based on these methods [Chou 1988, Wang and Gao 1987, Kutzler and Stifter 1986, Kapur 1986]. In [Chou 1988], extensive experiments were carried out for methods CS(F1), CS(F2) and GB(F1) using 512 geometric problems. It is the case that most of the theorems can be proved within seconds, and for most of the theorems ndg conditions in geometric form could be generated.
A typical example is Morley's trisector theorem [Wu 1984*a*, Chou 1988, Wang 1998*b*, Wang and Zhi 1998]: "The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle (Fig. 3)."

CS(F1) and GB(F1) confirmed it to be generically true, but with ndg conditions in algebraic form. This theorem has been proved under the ndg conditions that the three vertices are not collinear and the three sides are non-isotropic using CS(F2).

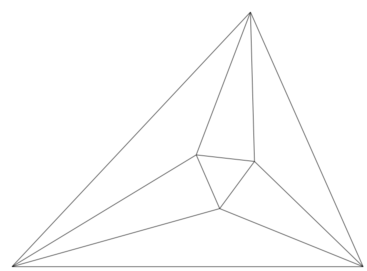

Fig. 3

The key factor to speed up the proving process is to have efficient implementations for the CS and GB methods. Many modifications of the original CS and GB methods are proposed for this purpose [Chou and Gao 1990*b*, Kapur and Wan 1990, Wang 1995*a*]. In particular, factorization of pols is proved to be quite important to enhance the speed for both CS [Chou and Gao 1990*b*] and GB methods [Wang 1998*b*]. For the CS method, factorization over extended field is a necessity. In [Wang 1996*a*, Wang and Zhi 1998], a new method of pol factorization is proposed and used to AGTP.
The CS and GB methods are also used to solve the reducibility problems [Wu 1986*b*], to prove theorems in finite geometries [Lin and Liu 1992], to give transformation theorems of Cayley-Klein geometries [Chou and Ko 1986, Gao and Wang 1995], to prove theorems with complex numbers [Stokes 1990], to analysis robotics [Huang and Wu 1992, Wu 1989*b*, Kapur 1997, Yang et al. 1997], to design linkages [Wu 1989*a*, Gao, Zhu and Huang 1998*a*], to solve problems from computer vision [Kapur and Mundy 1988, Gao and Cheng 1998, Wang 1998*a*, Bondyfalat et al. 1999], to design intelligent CAD systems [Gao and Chou 1998*a*, Gao and Chou 1998*b*], to design intelligent CAI (computer aided instruction) systems [Gao, Zhu and Huang 1998*b*, Li and Zhang 1998], to solve problems from solid modeling [Wu 1993, Kapur 1997], etc. For details, please consult these references.

2.1.5. Other elimination methods and AGTP

Theoretically, any elimination method could be used to prove geometry theorems according to the two approaches. We singled out the CS and GB methods in the preceding section because they are the most extensively studied ones. In this section, we will give a brief introduction to other methods.
Wu's complete method needs pol factorization over algebraic extension fields which is a costly operation. In [Zhang et al. 1990], a complete method without using pol factorization was proposed. The method works as follows. Let g be a pol and $f_1, \cdots, f_r$ be a triangular set of pols. We define the resultant of g wrt $f_1, \cdots, f_r$

inductively as

$$resl(g, f_1, \cdots, f_r) = resl(resultant(g, f_r, lv(f_r)), f_1, \cdots, f_{r-1})$$

where $lv(f_r)$ is the leading variable of f_r. It is known that if $prem(g, f_1, \cdots, f_r) = 0$ then $g = 0$ follows from $f_1 = 0, \cdots, f_r = 0$ generically and if $resl(g, f_1, \cdots, f_r) \neq 0$ then $g = 0$ cannot be deduced from $f_1, \cdots, f_r$. Otherwise, we have $prem(g, f_1, \cdots, f_r) \neq 0$ and $resl(g, f_1, \cdots, f_r) = 0$. In this case $f_1, \cdots, f_r$ must be reducible and the factors can be found in the computation procedure of the resultant. Repeating the above procedure, finally $f_1, \cdots, f_r$ can be factorized into ascending chains $ASC_1, \cdots, ASC_t$ such that for each ASC_i, either $prem(g, ASC_i) = 0$ or $resl(g, ASC_i) \neq 0$, i.e., $g = 0$ is either generically true or generically false. Similar algorithms based on gcd computation were given in [Kalkbrener 1995].

The elimination in the CS method is bottom-up, i.e., eliminating the variables in an increasing order. A top-down elimination method was developed by Brauer in algebraic case [Brauer 1948] and extended to differential case by Seidenberg [Seidenberg 1955]. Recently, Wang showed that this technique can be used to give a zero decomposition of Wu-Ritt type and the efficiency of the method is quite good [Wang 1995*b*]. He also used this method to prove theorems in elementary and differential geometries. We now introduce the key idea of this elimination method. Let

$$P_1 = 0, \cdots, P_r = 0, D \neq 0 \tag{2.9}$$

be a pol equation system in variables $x_1, \cdots, x_n$. Suppose that all of them involves x_n and P_1 has the lowest degree in x_n. Let $P_1 = Ix_n^d + U$ where U is of lower degree than d in x_n. Then (2.9) is equivalent to

$$I = 0, U = 0, P_2 = 0, \cdots, P_r = 0, D \neq 0$$

and

$$P_1 = 0, R_2 = 0 \cdots, R_r = 0, ID \neq 0$$

where $R_i = prem(P_i, P_1)$. Continue the above process, we can eliminate all variables and obtain a series of triangular sets. Both the direct approach and the refutational approach can be used to prove theorems with this method.

Both the CS and the GB methods eliminate variables one by one. In [Kapur et al. 1994], the concept of the Dixon resultant was extended to give an efficient method to eliminate multiple variables simultaneously. The Dixon resultant was used to prove geometry theorems using both direct and refutational approaches according to two formulations [Kapur and Saxena 1995, Kapur 1997]. We now give a brief introduction to the Dixon resultant method. Let $P_1, \cdots, P_{n+1}$ be pols in n variables $x_1, \cdots, x_n$ with coefficients as pols in parameters $u_1, \cdots, u_k$. Let $\bar{x}_1, \cdots, \bar{x}_n$ be n new variables. The *Cancellation matrix* of P_i is defined to be the following matrix:

$$C_P = \begin{bmatrix} P_1(x_1,\cdots,x_n) & \cdots & P_{n+1}(x_1,\cdots,x_n) \\ P_1(\bar{x}_1,\cdots,x_n) & \cdots & P_{n+1}(\bar{x}_1,\cdots,x_n) \\ \vdots & \cdots & \vdots \\ P_1(\bar{x}_1,\cdots,\bar{x}_n) & \cdots & P_{n+1}(\bar{x}_1,\cdots,\bar{x}_n) \end{bmatrix}$$

Since for all $1 \le i \le n$, $x_i - \bar{x}_i$ is a zero of C_p, $\prod_{i=1}^{n}(x_i - \bar{x}_i)$ divides $|C_p|$. The *Dixon pol* of P_i is defined as

$$\delta_P = \frac{|C_P|}{\prod_{i=1}^{n}(x_i - \bar{x}_i)}.$$

The main property of the Dixon resultant is that $\delta_P = 0$ is a necessary condition for $P_1 = 0, \cdots, P_{n+1} = 0$ to have zeros. It often happens that $|C_P| = 0$ and we cannot get any information. In [Kapur et al. 1994], a method of rank submatrix construction is proposed to solve this problem. The details of the method and its application to AGTP can be found in [Kapur et al. 1994, Kapur 1997].

2.1.6. Proving geometry theorems by numerical computation

Based on the CS method, Hong showed that to prove a geometry statement, only a single numerical example is needed to check [Hong 1986]. To understand the method, let us mention the simple fact: a pol $p(x_1) \in Q[x_1]$ is identically zero if $p(x_0) = 0$ for a sufficiently large rational number x_0. Hong's work is actually a generalization of the above result. Hong's work is generalized and used to test whether an algebraic variety is included in another variety in [Wang 1988].

In [Zhang et al. 1990], a method of proving geometry theorems by checking several numerical examples instead of one is presented. This method is similar to Hong's method and is based on a generalization of the following fact: a pol $p(x_1)$ of degree d is identically zero if it has more than d distinct roots.

In both of the above methods, approximate calculations are needed and at last we need to check whether an approximate number is small enough to be zero which is a difficult problem. However, in the case of linear geometry statements, the approximation problem can be avoided by using rational number calculation which is widely available in the symbolic computer software. A prover for linear statements has been developed and has been used to prove many nontrivial examples [Yang et al. 1992].

In [Ferro, Gallo and Gennaro 1998, Rege 1995], probabilistic methods for AGTP were studied.

2.2. Proving theorems involving inequalities

The methods reviewed in Section 2.1 address geometry statements of equality type and are complete only for geometry over complex numbers. If a geometry statement is proved in complex geometry, it is also valid in Euclidean geometry. The converse is not true. An example which is valid in **R**, but not in **C**, the field of complex numbers, is the 8_3 problem [Kutzler 1989, Chou et al. 1989, Conti and Traverso 1995, Wang 1995*c*]. A simple algebraic example is $\forall u_1 x_1 [u_1^2 + x_1^2 = 0 \Rightarrow x_1 = 0]$. Both the CS and GB methods can disprove this formula in **C**, but cannot confirm it in **R**. However, such kinds of formulas rarely appear in Euclidean geometry. If we change the above formula a "little" bit: $\forall u_1 x_1 [u_1^2 + x_1^2 - 1 = 0 \Rightarrow x_1 = 0]$, then for $u_1 \in (-1, 1)$, x_1 always has solutions in **R**. Such kinds of formulas are called **R**–generic. If an **R**–generic formula is not valid in **C**, it is also not valid in **R**. Most geometry statements of equality type are **R**–generic, and for such statements the CS and GB methods are complete for Euclidean geometry [Chou and Yang 1989]. This is *the real reason* that these methods, which are complete only for complex geometry, can prove so many theorems in real geometry. But for theorems involving inequalities, we still need to develop new methods.

2.2.1. Proving theorems by quantifier elimination

Theoretically, Collins' method can prove (or disprove) any first order statements in the Tarski geometry. In [Arnon 1988], the CAD method is used to geometry theorems in an interactive way. Thanks to the generosity of Collins and Hong, we have been able to experiment with proving geometry theorems using Hong's implementation of Collins' CAD algorithm. The results were very encouraging. We use the following example to give some idea about how far Collins' method can reach now.

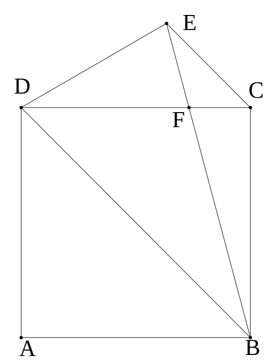

Fig. 4

2.4. Example. Let $ABCD$ be a square. CE is parallel to the BD such that $BE = BD$. F is the intersection of BE and DC. Show that $DF = DE$ (Fig. 4).

Let $A = (0, 0)$, $B = (u_1, 0)$, $C = (u_1, u_1)$, $D = (0, u_1)$, $E = (x_1, x_2)$ and $F = (x_3, u_1)$. Then the hypotheses can be expressed by the following equations:

$$\begin{array}{ll} h_1 = x_2^2 + x_1^2 - 2u_1 x_1 - u_1^2 = 0 & BE = BD \\ h_2 = u_1 x_2 + u_1 x_1 - 2u_1^2 = 0 & CE \parallel BD \\ h_3 = x_2 x_3 - u_1 x_2 - u_1 x_1 + u_1^2 = 0 & F \text{ is on } BE. \end{array}$$

The conclusion $(DF = DE)$ can be expressed by $c = (x_3-0)^2+(u_1-u_1)^2-[(x_1-0)^2+(x_2-u_1)^2] = x_3^2-x_2^2+2u_1x_2-x_1^2-u_1^2 = 0$.
Thus the algebraic form of the above statement is:

$$\forall u_1x_1x_2x_3[(h_1 = 0 \wedge h_2 = 0 \wedge h_3 = 0 \wedge u_1 \neq 0) \Rightarrow c = 0],$$

which was proved to be valid by Hong's program in 16 seconds on a SPARC-20 Station. This theorem has been considered fairly difficult in high school geometry. Also the previous implementation of Collins' method (Arnon's program) was unable to prove this theorem within reasonable computer resources. Collins' method was also used to prove this theorem in [Wang 1991].
In the area of AGTP, Collins' method requires further improvements in order to prove a substantial number of non-trivial theorems in practice.
In [Dolzmann et al. 1996], a quantifier elimination algorithm for linear and quadratic equations is presented. A *generic quantifier elimination method* is also proposed. In this method, variables are divided into parameters and variables, and pure parametric expressions are assumed to be non-zero. These expressions are similar to the ndg conditions in Wu's method. The program based on this method has been used to prove a large number of difficult geometry theorems. One reason behind the success of this method is that algebraic equations for most geometry theorems only involve quadratic equations. In [Weispfenning 1994], this method is used to solve many computational geometry problems. In [Dolzmann 1998], quantifier elimination methods are used to solve the real implicitization of the Enneper surface.

2.2.2. Proving theorems by optimization

Based on his CS method, Wu proposed a method to solve the following problem [Wu 1992*a*].
Problem Ineq. Let $\mathbf{R}^n(\mathbf{X})$ be the *real* Euclidean space of dimension n in the coordinates $\mathbf{X} = (x_1, \ldots, x_n)$ and D a domain in $\mathbf{R}^n$. Let f, h_i $(i \in I = \{1, \ldots, r\}, r < n)$ and g be all real pols in $\mathbf{R}^n[\mathbf{X}]$ over the domain D. To determine for what real value c we shall always have

$$f \geq c \quad \text{or} \quad f > c \quad \text{or} \quad f \leq c \quad \text{or} \quad f < c$$

under the conditions

$$\mathbf{HS} = 0, \quad \text{where} \quad \mathbf{HS} = \{h_i \mid i \in I\} \quad \text{and} \quad g \neq 0.$$

Wu proved the following *finite kernel theorem.* Let HS be an arbitrary pol-set and f an arbitrary pol in $\mathbf{R}[\mathbf{X}]$. Then we can construct a *finite* set of real values K such that the extremal values of f under the constraint $HS = 0$ is contained in K.
To solve this problem, Wu uses the *Lagrangian pol* with *Lagrangian multipliers* $\lambda_j, j \in M$ $L = f + \sum_{j \in M} \lambda_j h_j$. The *Lagrangian pol-set* is

$$\mathbf{LS} = \{\frac{\partial L}{\partial x_i}, h_j |\ i \in N, j \in M\}.$$

The *Jacobian* for $t = (i_1, \ldots, i_m) \in T$ is the determinant $J_t = \frac{\partial(h_1,\ldots,h_m)}{\partial(x_{i_1},\ldots,x_{i_m})} = \left|\frac{\partial h_j}{\partial x_{i_k}}\right|$. The *Jacobian pol-set* is the pol-set

$$\mathbf{JS} = \{J_t, h_j |\ t \in T, j \in M\}.$$

Then the points where extremal values are achieved are contained in the projection of the zeros of the Lagrangian pol-set and the Jacobian pol-set. Based on this observation, using the CS method, we can compute the finite kernel.

2.5. Example. [Wu 1995] Suppose that two cars of given form are moving respectively along the X-axis and the Y-axis in positive directions with known velocities $v_1 > 0$ and $v_2 > 0$. To decide whether the cars will collide or not, and to determine in the colliding case the time and place of first collision.

Let us consider the case in which the two cars are both of elliptical form given by $(c_1 < 0, c_2 < 0)$ $f_1(x, y) = b_1^2(x - c_1)^2 + a_1^2y^2 - a_1^2b_1^2$, $f_2(x, y) = b_2^2x^2 + a_2^2(y - c_2)^2 - a_2^2b_2^2$. Then the collision problem is seen to be a Problem Ineq for which $D = \{t > 0\} \subset \mathbf{R}^3(t, x, y)$, $f = t$, $g = 1, h_1 = f_1(x - v_1t, y)$, $h_2 = f_2(x, y - v_2t)$. In the case of having *generic* values for a_i, b_i, c_i and v_i, Wu's method gives rise to an irreducible pol equation of degree 8 in t with 696 terms. Leaving aside some uninteresting cases, the two cars will collide if and only if this equation has a positive root. The time and place of first collision can be easily determined if numerical values of a_i, b_i, c_i and v_i are substituted.
The method is used to prove geometry theorems involving inequalities [Wu 1992*a*], to prove trigonometric inequalities [Wang 1993], to solve non-linear programming problems [Wu 1995], to solve optimization problems [Wu 1992*a*], etc.

2.2.3. AGTP by combining the CS method and the CAD method
A theorem involving inequalities generally also involves equalities. Since the CS and GB methods work so well for equality problems, we might expect a combination of the CS (or GB) method with Collins' method could solve problems not in the scope of the CS method, but which cannot be solved by Collins' method alone within the available time and space. The work in [Chou, Gao and Arnon 1992] is in this direction, and a number of hard problems were solved with some human interaction. Here we use the following simple example to illustrate the basic idea.

2.6. Example. Let $ABCD$ be a parallelogram. Show that points B and D are on either side of diagonal AC.

This "trivial" fact is repeatedly used in traditional proofs of the parallelogram theorem. However, it seems nontrivial to find a rigorous traditional proof of this fact. (Try it!)
Let $A = (0, 0)$, $B = (u_1, 0)$, $C = (u_2, u_3)$, and $D = (x_2, x_1)$. Then we have two equations for the hypotheses

$h_1 = u_1x_1 - u_1u_3 = 0$ $\qquad$ AB is parallel to CD

$h_2 = u_3x_2 - (u_2 - u_1)x_1 = 0$ AD is parallel to BC.

The conclusion that B and D are on either side of AC is $g < 0$, where $g = (u_3u_1 - u_2 \cdot 0)(u_3x_2 - u_2x_1) = u_1u_3^2x_2 - u_1u_2u_3x_1$. We want to decide whether the following statement is valid under certain ndg conditions: $\forall u_1u_2u_3x_1x_2[(h_1 = 0 \wedge h_2 = 0) \Rightarrow g < 0]$. Here u_1, u_2, u_3 are selected to be parameters, and x_1 and x_2 are selected to be dependent variables. Reducing g to canonical form modulo the ideal (in $\mathbf{Q}(u)[x]$) generated by h_1 and h_2, we obtain $g = -u_1^2u_3^2$. This canonical form of g modulo the ideal is only valid under the conditions $u_1 \neq 0$ and $u_3 \neq 0$; in other words, u_1 and u_3 occur in the denominators of the elements c_1 and c_2 of $Q(u)[x]$ such that $g = c_1h_1 + c_2h_2$. Thus we have $g < 0$, under the condition that $u_1u_3 \neq 0$. Note that $u_1u_3 \neq 0$ is indeed connected with non-degeneracy, i.e. to insure that points A, B and C are not collinear.

A more general scheme has been proposed, and it has been used to solve the 8_3 problem automatically [McPhee et al. 1994]. Recently, N. McPhee gives an automatic solution to the Steiner-Lehmus theorem and the Pompieu's theorem based on a combination of the CS method and Collins' CAD method [McPhee et al. 1994].

2.2.4. Complete discriminant systems and AGTP

In [Yang, Hou and Zeng 1996] a powerful tool, called the *complete discrimination system (CDS)* was introduced, which can be considered as an extension of the Sturm theorem. For a univariate pol equation $P(x) = 0$ of degree n, the CDS can be used to give the conditions that $P(x) = 0$ has p and q distinct real and complex solutions respectively and the multiplicities for these solutions are $r_1, \cdots, r_p$ and $c_1, \cdots, c_p$ such that $\sum_{i=1}^{p} r_i + \sum_{j=1}^{q} c_j = n$.

By means of CDS, together with Wu's method and a partial CAD algorithm, a generic program called "EXPLORER" was implemented in Maple that is able to discover and prove new inequalities [Yang, Hou and Xia 1998]. Using this program, Yang et al have re-discovered 37 inequalities in the first chapter of the monograph on geometric inequalities [Mitrinovic, Pecaric and Volenec 1989]. One of the inequalities "discovered" in this way is about the "basic inequality of triangles." For a triangle, from the basic inequalities about the three sides $a + b < c$, $a + c < b$,and $b + c < a$, the program can discover the basic inequality about the half perimeter s, circumradius R, and inradius r of the triangle:

$$s^4 + 2r^2s^2 - 4R^2s^2 - 20rRs^2 + 12r^3R + 48r^2R^2 + r^4 + 64rR^3 \leq 0.$$

An interesting inequality about triangles is discovered in [Guergueb, Mainguené and Roy 1998]. It would interest to know if this inequality can be discovered automatically.

In most of the geometric inequalities about triangles, there are radicals. A dimension-decreasing algorithm introduced by Yang [Yang 1998] can treat these kinds of inequalities efficiently. Based on this algorithm, a generic program called *BOTTEMA* was implemented on a PC computer. More than 1000 algebraic and geometric inequalities including hundreds of open problems have been verified in this way. The total CPU time spent for proving 120 basic inequalities from Bottema's monograph, "Geometric Inequalities" on a Pentium/200, was 20-odd seconds only.

2.3. Proving theorems in differential geometry

An advantage of the CS method is that it can be extended to cover ordinary and partial algebraic differential equations [Ritt 1950]. As a consequence, theorems from differential geometry and mechanics can be proved automatically [Wu 1987*a*].

2.3.1. Space curves and mechanics

Here we are dealing with differential pol rings over a differential field (usually it is $\mathbf{Q}(t)$) in which there is a third operation, "$'$", compatible with the two operations of a ring, "+" and "$*$":

$$(a+b)' = a' + b', (ab)' = a'b + ab'.$$

The pseudo division, triangular algorithm, and the variety decomposition algorithm can be extended to the differential pol case with minor modifications [Ritt 1950, Wu 1987*a*, Chou and Gao 1993*a*].

The geometry statements addressed are still of *equality type*. In the local theory of space curves, one uses parametric representation of a curve: $C = (x(t), y(t), z(t))$. The practical problems encountered in the curve theory and mechanics are of the similar nature. Thus we use the following elegant example first worked by Wu [Wu 1987*b*] to illustrate the type of problems we address:

2.7. Example *(The Kepler–Newton problem).* Kepler's first two laws are:
(K1) The planets move in elliptic orbits with the sun as a focus (Fig. 5).
(K2) The vector from the sun to the planet sweeps equal areas in equal times.
Newton's law of gravitation (special form):
(N1) The acceleration of a planet is inversely proportional to the square of the distance from the sun to the planet.
Now we want to prove that (K1) and (K2) imply (N1).

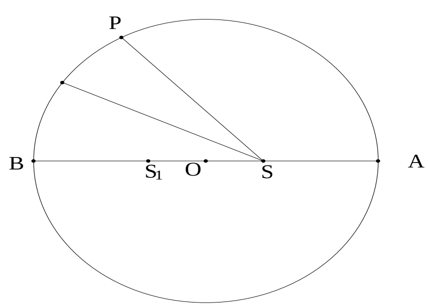

Fig. 5

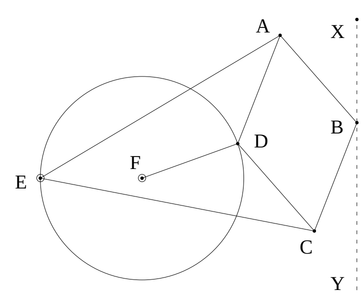

Fig. 6

Choose the sun as the origin of the coordinate system, and let $(x(t), y(t))$ be the position of the planet; $(-c, 0)$ be the center of the ellipse; h be the area velocity of the planet. Then the equation part of the hypotheses is:

$$
\begin{array}{ll}
K_{11}: & \frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\
K_{12}: & a^2 - (b^2 + c^2) = 0 \\
K_2: & x'y - xy' - h = 0, \qquad \text{with} \quad h' = 0. \\
f_1: & r^2 - (x^2 + y^2) = 0 \\
f_2: & A^2 - (x''^2 + y''^2) = 0.
\end{array}
$$

The conclusion N_1 is: $[Ar^2]' = 0$.
Then a tentative algebraic form for this problem would be:

$$\forall v[(K_{11} \wedge K_{12} \wedge K_2 \wedge f_1 \wedge f_2) \Rightarrow N_1].$$

As usual, x' denotes the derivative of x with respect to t. Here, a, b, c, and h are constants (independent of t). Again, the above formula is valid under some ndg conditions. With the simple use of the CS method described in Section 2.1.2, Wu proved a variation of the above specification [Wu 1987*b*] under some ndg conditions. As in elementary geometry, there are also two Formulations F1 and F2 for proving theorems in differential geometry. While these two formulations and the CS methods for them are similar to the case of elementary geometry, Formulation F1 is much more complicated than a simple rewording [Chou and Gao 1993*c*]. Nearly 100 theorems in space curves and 10 statements in mechanics have been proved [Chou and Gao 1991, Chou and Gao 1993*d*].
In [Ferro and Gallo 1990, Ferro and Gallo 1994], methods for proving theorems in differential geometry based on the computation of the dimensions of zero sets were proposed. It tries to find components with the highest dimension and prove the conclusion on these components. One problem with this approach is that the component with the highest dimension might not be the main component.
Another elimination method that works in differential case is the Brauer-Seidenberg technique. Since this technique can be used to eliminate quantifiers, it actually provides a decision method for differential closed field. Wang has modified this method to give a zero decomposition theorem and used the it to prove theorems in differential geometry [Wang 1995*b*].

2.3.2. Space surfaces

This involves partial differential pols (pdp). In this area, Wu has developed tools [Wu 1979, Wu 1982*a*, Wu 1987*a*, Wu 1989*c*] based on the work [Ritt 1950, Thomas 1954, Cartan 1946], where again the method is only for geometry statements of equality type. This is similar to, but much more complicated than the ordinary differential pol case. Some experiment results are given and a new result result is discovered in [Li 1995*b*]. The method currently used is the simple one described in Section 2.1.2, i.e., first triangularize a pdp set to obtain a pdp ascending chain ASC, then do similar successive pseudo divisions of the conclusion pdp with respect to that ASC to see whether the final remainder is zero. In elementary geometry such simple use of the CS method often proves a geometry statement to be generically true because most statements are of constructive types. However, in the pdp case, the situation is unclear. There is no work to formulate "generically true" precisely as in Formulation F1 for elementary geometry. Formulation F2 is straight forward and the CS method for F2 can presumably be given, though no experimental work has been done yet.

2.3.3. Clifford algebra approach for AGTP

In differential geometry textbooks, vector algebra and Frenet moving frames are used to solve problems in the local theory of space curves. Li and Cheng recognized that Clifford algebra is more suitable for symbolic vector equations solving than vector algebra. Based on Wu's method of characteristic sets, they proposed a method that employs Clifford algebraic representation for geometric entities and constraints [Li and Cheng 1998]. A prover based on this method is capable of producing proofs much the same with those used in the textbooks.

The procedure of proving a theorem is composed of three stages: (a) find a reduction set, (b) find a parametric reduction set, and (c) find a characteristic set. To compute a reduction set, for scalar equations, Wu's method is used; for vector equations, the Clifford algebraic reduction method is used. For equations of differential forms, these elimination techniques can be used to compute a triangular set and to prove theorems about surfaces similar as in Section 2.1.1. This vector approach is not a decision procedure. In the final stage, coordinates are needed in order to give a complete method.

To overcome the difficulty of integrability pols in computing a characteristic set in the local theory of space surfaces, Li also proposed a simple method [Li 1995*a*] to integrate the calculus of differential forms with Wu's method.

2.4. Mechanical geometric formula derivation

2.4.1. Elementary geometry

There are two kinds of problems in elementary geometry other than theorem proving. One is finding locus equations, the other is deriving geometry formulas. Automatic derivation of geometry formulas were studied in [Wu 1986*a*, Chou 1984, Wang and Gao 1987, Chou 1987, Chou and Gao 1990*a*, Wang 1995*c*, Kapur et al. 1994]. We use Heron's Formula to illustrate this type of problems.

2.8. Example. Find the formula for the area of a triangle ABC in terms of its three sides.

Let a, b, and c be the three sides, $B = (0, 0)$, $C = (a, 0)$, and $A = (x_1, x_2)$. Then the geometry conditions can be expressed by the following set of pol equations:

$$\begin{aligned} h_1 &= x_2^2 + x_1^2 - 2ax_1 - b^2 + a^2 = 0 && b = AC \\ h_2 &= x_2^2 + x_1^2 - c^2 = 0 && c = AB \\ h_3 &= ax_2 - 2k = 0 && k = \text{the area of } ABC. \end{aligned}$$

The aim is to find a pol equation involving only a, b, c, and k which is a consequence of the above equations and some ndg conditions.

In general, for a geometric configuration given by a set of pol equations $h_1(u_1, \cdots, u_q, x_1, \cdots, x_p) = 0, \cdots, h_r(u_1, \cdots, u_q, x_1, \cdots, x_p) = 0$ (possibly with a set of inequations $\{D = d_1(u,x) \neq 0, \cdots, d_s(u,x) \neq 0\}$), we want to find a relation (formula) between arbitrarily chosen variables $u_1, \cdots, u_q$ (parameters) and a dependent variable, say, x_1. In [Chou and Gao 1990*a*], CS and GB methods are used for formula derivation. In [Wang 1991], CS, GB and Wang's methods are used for formula derivation. In [Kapur et al. 1994], Dixon resultant is used for formula derivation. Heron's formula can be easily derived by any of the above methods:

$$16k^2 + c^4 - (2b^2 + 2a^2)c^2 + b^4 - 2a^2b^2 + a^4 = 0.$$

Here is a more interesting example.

2.9. EXAMPLE *(Peaucellier's Linkage).* Links AD, AB, DC and BC have equal length, as do links EA and EC. We assume $FD = EF$. The locations of joints E and F are fixed points on the plane, but the linkage is allowed to rotate about these points. As it does, what is the locus of the joint B? (Fig. 6)

Let $F = (0,0)$, $E = (r,0)$, $C = (x_2, y_2)$, $D = (x_1, y_1)$, $B = (x, y)$, n and m be the lengths of the projections of CD and BC on BD and AC when E, D, B are collinear. Then the geometry conditions can be expressed by the following set of equations H

$$\begin{array}{ll}
h_1 = y_1^2 + x_1^2 - r^2 = 0 & r = FD \\
h_2 = y_2^2 - 2y_1y_2 + x_2^2 - 2x_1x_2 + y_1^2 + x_1^2 - n^2 - m^2 = 0 & CD = n^2 + m^2 \\
h_3 = y_2^2 - 2yy_2 + x_2^2 - 2xx_2 + x^2 + y^2 - n^2 - m^2 = 0 & CB = n^2 + m^2 \\
h_4 = y_2^2 + x_2^2 - 2rx_2 - n^2 - 4rn - m^2 - 3r^2 = 0 & EC = (n+2r)^2 + m^2 \\
h_5 = (x - r)y_1 - yx_1 + ry = 0 & E \text{ is on } DB,
\end{array}$$

together with the following set of pol inequations D:

$$d_1 = x_1 - x \neq 0 \qquad B \neq D.$$

Selecting m, n, r, and y to be the parameters of the problem, we want to find the relation among m, n, r, y and x. Using the CS method in [Chou and Gao 1990*a*], a relation $x + 2n + r = 0$ is found, which tells us that the locus is a line parallel to the y-axis.

New theorems discovered in this way may be found in [Gao and Wang 1995, Wang 1992, Wu 1986*a*].

This problem can also be formulated as one of finding a quantifier free formula $f(u, x_1)$ such that $f(u, x_1) \iff \exists x_2 \cdots x_p[h_1(u,x) \wedge \cdots \wedge h_r(u,x) \wedge d_1(u,x) \neq 0 \wedge \cdots \wedge d_s(u,x) \neq 0]$ [Chou 1990, Wang 1991]. Formulated in this way, it is actually to calculate the projection of an algebraic set in the affine space E^{q+p} into the subspace E^{q+1}. If E is algebraically closed, there are methods for computing such projections. The method in [Wu 1990] works over the field of complex numbers. If E is a real closed field, Collins' method [Collins 1975] gives a solution to the above problem; so it would also be interesting to examine the connection between the real closed case and algebraically closed case. For example, we expect that Collins' method produces the following form for $f(u, x_1)$ for Peaucellier's Linkage:

$$(x + 2n + r = 0) \wedge (-d \leq y \leq d) \wedge (\text{other nondegenerate conditions})$$

where d is from the 4th (quadratic) equation of the above ASC_1^*.

2.4.2. Differential geometry and mechanics

Formula derivation in differential geometry was initiated by Wu in connection with finding possibly unknown properties on Bertrand curves [Wu 1987*c*]. The approach used by Wu was to look at the differential pols produced during generation of a CS. This involves human assistance. A more automatic method has been proposed in [Chou and Gao 1993*c*, Chou and Gao 1990*a*]. Using this method a complete list of the properties of Bertrand curves in metric and affine geometries has been obtained [Chou and Gao 1993*c*].

3. Coordinate-free approaches to automated reasoning in geometry

Algebraic methods, though powerful, generally can only tell whether a statement is true or not. If one wishes to look at the proofs, he/she will find tedious and formidable computations of pols. After Wu's method, several researchers tried to develop AGTP methods based on vector calculation in the mid-80s in order to find simpler proofs [Havel 1991, White and Mcmillan 1988]. It is well known that incidence geometry relations can be represented by exterior products and measurement geometric relations can be represented by inner products. Therefore, developing vector approach of AGTP is to find algorithms of manipulating exterior and inner products. In the mid-90s, several successful vector approaches were proposed. As expected, these methods can produce shorter proofs than that of the coordinate based methods. But, this advantage comes with a price: these methods are not complete in complex or real geometries as the methods introduced in Section 2 are.

3.1. Area method

The area method was proposed in 1992 from a quite different point of view [Chou et al. 1993*a*, Chou, Gao and Zhang 1994, Zhang, Chou and Gao 1995]. Zhang found many elegant ad hoc methods based on areas of triangles to solve geometric problems when he was a middle school teacher and trainer of the Chinese Mathematical Olympian Team [Zhang 1982]. These ad hoc methods have been developed into a complete method of AGTP, which are surprisingly powerful in that it has been used to prove hundreds of geometry theorems of constructive type and the proofs are generally short and elegant [Chou et al. 1994]. A computer program called *Geometry Expert* based on this method has produced proofs of 500 nontrivial theorems *entirely* automatically [Chou et al. 1994, Gao, Zhang and Chou 1998]. This method seems to be the first to produce human-readable proofs for hard geometry theorems efficiently.

Instead of coordinates, three basic *geometric quantities*: the ratio of parallel line segments, the signed area, and the Pythagorean difference are used. The basic propositions, which formally describe the properties of these quantities, are the deductive basis of the area method. The method involves the elimination of the constructed points from the conclusion using these basic geometry propositions. Two of the basic propositions are given below.

3.1. LEMMA (The Co-side Theorem). *Let M be the intersection of two lines AB and PQ and $Q \neq M$. Then $\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}$, where S_{PAB} and S_{QAB} are the signed area of triangles PAB and QAB.*

3.2. LEMMA. *$PQ \parallel AB$ iff $S_{PAB} = S_{QAB}$.*

We use a simple example to illustrate how the method works. The following proof is essentially the same as the one produced by the prover based on the area method. *Proof of Example 2.1.* By the co-side theorem, $\frac{\overline{AE}}{\overline{EC}} = \frac{S_{ABD}}{S_{DBC}}$. Since $AB \parallel CD$, $S_{ABD} = S_{ABC}$ by Lemma 3.2. Since $AD \parallel BC$, $S_{DBC} = S_{ABC}$ by Lemma 3.2. Then we have

$$\frac{\overline{AO}}{\overline{OC}} = \frac{S_{ABD}}{S_{DBC}} = \frac{S_{ABC}}{S_{ABC}} = 1.$$

The area method has been extended to prove theorems in solid geometry (the volume method), Minkowskian geometry, Bolyai-Lobachevsky geometry, and Riemannian geometry [Chou, Gao and Zhang 1995, Yang, Gao, Chou and Zhang 1998]. In [Chou, Gao and Zhang 1996*b*], a new geometric quantity *the full-angle* is used to prove geometry theorems. This method, though not as powerful as the area method, can produce very elegant proofs for some difficult geometry theorems for which the area method fails to give short proofs. In [Chou et al. 1993*b*, Chou et al. 1994], vector and complex number approaches based on a similar idea is presented. The idea developed in the area method has been used to find locus of robotics arms [Yang et al. 1997] and to prove Newton's basic proposition [Fleuriot and Paulson 1998].

3.2. Bracket algebra methods

One of the earliest effort to develop coordinate free methods of geometric reasoning is to use techniques from the bracket algebra such as Cayley factorization [White and Mcmillan 1988]. The bracket algebra is a non-commutative algebra. There is still no decision method similar to that of the Gröbner basis. Therefore, bracket algebra can only be used to do "computer-aided geometric reasoning" at that time [(ed.) 1987].

In [Richter-Gebert 1995], an algorithm based on bracket algebra for proving projective geometry theorems was given. The basic idea is to represent geometric hypotheses and conclusions as algebraic relations and use simple algebraic computation to deduce the conclusion from the hypotheses. Many difficult theorems from projective geometry have been proved by the method. The proofs thus generated are very short. Based on this technique, a program called *CINDERELLA* has been developed [Richter-Gebert and Kortenkamp 1999]. In [Bondyfalat et al. 1999], similar techniques are used to find unknown geometric properties raised from computer vision.

3.3. Clifford algebra methods

Clifford Algebra is a generalization of the Grassmann algebra. In [Li and Cheng 1996], techniques of Clifford algebra are combined with Wu's elimination method to prove geometry theorems. Many theorems have been proved with this approach. The key idea in [Li and Cheng 1996] is to use several rules of solving vector equations in vector level. But these rules alone are not complete. Complete methods can be achieved by substituting the vector by their coordinates and using Wu's characteristic set method. This method has also been used to formula derivation. A problem proposed by Erdös was partially solved [Li and Shi 1997].
In [Li 1998*b*, Wang 1998*a*], techniques of Clifford algebra are used to prove theorems of constructive type. This approach has the same scope and style as the vector version of the area method [Chou et al. 1993*b*]. This method loses a key feature of the area method: producing proofs with geometric meaning. On the other hand, it may provide a uniform treatment for Euclidean and several non-Euclidean geometries. In [Fèvre and Wang 1997, Fèvre and Wang 1998, Boy de la Tour, Fèvre and Wang 1998], rewrite rules and Clifford algebra are combined to prove theorems from both plane and solid geometries. This may enlarge the scope of the method to cover non-constructive statements and use many techniques from term re-writing to enhance the efficiency. Other approaches based on Clifford algebraic method can be found in [Yang, Zhang and Feng 1998, Fearnley-Sander 1998].

3.4. Gröbner bases methods

We have mentioned that bracket algebra and Clifford algebra are non-commutative algebras. For some non-commutative algebra, methods of generating Gröbner bases have been given In [Stifter 1993], Gröbner bases of vector algebra involving exterior products are used to prove geometry theorems. Theoretically, inner products can also be introduced. This method is actually a combination of the vector approach and coordinates approach, because it introduces many scalar variables to represent geometric relations. In [Wang 1989], Gröbner basis of Clifford algebra is used for AGTP.

4. AI approaches to automated reasoning in geometry

Generally speaking, the algebraic approaches are decision procedures and are more powerful. The AI approaches are not decision procedures and are less powerful. Despite its "weakness", it is still worth improving the AI approach because this may lead to techniques useful to automated reasoning in the general case. Even for automated geometry reasoning alone, AI methods have the following advantages. (1) Proofs produced by the AI method are generally easy to understand than proofs based on algebraic computations. (2) Using predicates only (no algebraic computation) makes the reaching of fixpoint possible. (3) Although algebraic methods can prove a much larger number of theorems, there still exist theorems (Example 4.2) which can be solved by the AI approaches elegantly but can not be solved with the algebraic approaches because to prove them excessively large computer memory is needed.

4.1. Gelernter's geometry machine

Geometry theorem proving on computers began in the 50s with the landmark work of Gelernter et al [Gelernter 1959, Gelernter, Hanson and Loveland 1960]. Several basic ideas of geometric reasoning such as using a numerical model, constructing auxiliary points, and generating geometric lemmas were studied in this work. Most of the other work on AI approach of geometric reasoning can be considered extensions of this work.

Gelernter's geometry machine uses a *backward chaining approach*: that is, it reasons from the conclusion to the hypotheses. Let $H_1, \cdots, H_r$ and G be the hypotheses and conclusion of a geometry statement, i.e., we need to prove

$$\forall \text{ geometric elements}[(H_1 \wedge \cdots \wedge H_r) \Rightarrow G].$$

Then G is the *goal* of the proof procedure. To prove G, we search the *axiom or rule set* to find a rule of the following form

$$[(G_1 \wedge \cdots \wedge G_r) \Rightarrow G].$$

Then for G to be valid, we need only to prove the *subgoals* $G_1, \cdots, G_r$. Now we may repeat the above process for each of the subgoals until the subgoal is one of the hypotheses. In this way, we generate an *and-proof-tree* — meaning that to prove any goal in the tree, we need to prove all of its subgoals. On the other hand, there might be more than one rules that will lead to a goal. In this case, we need only to prove the subgoals generated from one of the rules. In other words, to prove a goal in the tree, we need only prove one of the branches. Therefore, in the general case, the proof process will generate an *and-or-proof-tree*.

Let us consider the following proof of Example 2.1.

The hypotheses are: $AB \parallel CD$, $AD \parallel BC$, $\text{coll}(E, A, C)$ (points E, A, C are collinear), and $\text{coll}(E, B, D)$. The conclusion or goal is $AE = EC$. The proof generated by the geometry machine is the same as the proof given at geometry textbooks. To prove $AE = EC$, we need to show $\triangle ECD \cong \triangle EAB$, which in turn follows from three subgoals: $AB = CD$, $\angle AEB = \angle CED$, and $\angle ECD = \angle EAB$. To prove $AB = CD$, we need to prove $\triangle ABC \cong \triangle CDA$ which follows from three subgoals: $AC = CA$, $\angle ACD = \angle CAB$, and $\angle CAD = \angle ACB$. $\angle ACD = \angle CAB$ follows from $AB \parallel CD$; $\angle CAD = \angle ACB$ follows from $AD \parallel BC$.

Opposite to the backward chaining, *forward chaining* reasons from the hypotheses to the conclusion. Most of the AI approaches to AGTP use backward chaining [Gelernter 1959, Anderson 1981, Coelho and Perceira 1986]. In [Nevins 1976], Nevins used a combination of forward chaining and backward chaining with emphasis on the forward chaining. In doing so, Nevins made many improvements in building a powerful geometry theorem prover. But still, most of the theorems proved by these methods are relatively easy.
Wos and his collaborators used their powerful general-purpose resolution theorem prover to experiment with proving theorems in Tarski's axioms for elementary geometry [McCharen et al. 1976]. The work was continued in [Quaife 1989] using the general purpose prover OTTER. Recently, extensive work were done in [Balbiani and del Cerro 1995, Balbiani 1995] using logic deduction techniques such as term rewriting to AGTP. In these work, some interesting but relatively easy theorems were proved. In [Fèvre 1998], the classical first-order logic and algebraic methods are combined to develop a hybrid deduction system which is used to produce proofs at different levels for better understanding.

4.2. A geometry deductive database

In [Chou et al. 1996*c*], the technique of deductive database [Gallaire, Minker and Nicola 1984] is used to geometric reasoning. The resulted program can be used to find the *fixpoint* for a geometric configuration, i.e., the system can find all the properties of the configuration that can be deduced using a fixed set of geometric rules. This program has been used to prove more than 150 difficult geometry theorems, and most of these theorems are beyond the scope of the previous provers based on AI approaches.
The idea of *structured deductive database* is presented to reduce the size of the database. Experiments with 150 problems show that this technique could reduce the size of the database by one thousand times.

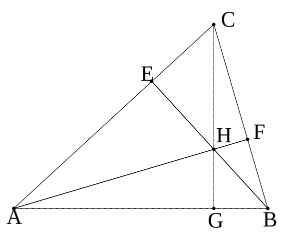

Fig. 7

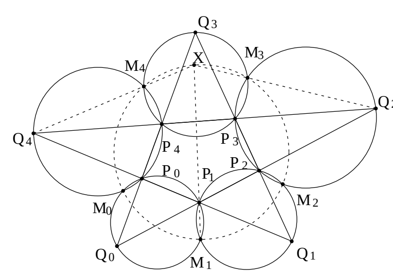

Fig. 8

4.1. Example *(Orthocenter Theorem).* Show that the three altitudes of a triangle are concurrent (Fig. 7).

As in Fig. 7, the hypotheses (extensional database) are: points(A, B, C), coll(E, A, C), perp(B, E, A, C), coll(F, B, C), perp(A, F, B, C), coll(H, A, F),

coll(H, B, E), coll(G, A, B), coll(G, C, H).
Reaching the fixpoint costs the program 0.75 second on a Sparc-20. The size of the fixpoint is 151 if the structured database is used. In predicate form, the size of the fixpoint would be 83,076. The fixpoint contains two of the most often encountered properties of this configuration: perp(C, G, A, B) (the conclusion of the orthocenter theorem) and $\angle[GF, GC] = \angle[GC, GE]$. For each fact in the database, the program can give a synthetic proof. The following is the proof of the Orthocenter theorem generated automatically by the program.

1. $CG \perp AB$, because $AF \perp BC$(hypothesis), (2)$\angle[AF, BC] = \angle[CH, AB]$.
2. $\angle[AF, BC] = \angle[CH, AB]$, because (3)$\angle[AF, CH] = \angle[BC, BA]$.
3. $\angle[AF, CH] = \angle[BC, BA]$,

   because (4)$\angle[AF, CH] = \angle[FE, AC]$, (5)$\angle[BC, BA] = \angle[FE, AC]$.
4. $\angle[AF, CH] = \angle[FE, AC]$, because (6)cyclic:$[C, F, E, H]$.
5. $\angle[BC, BA] = \angle[FE, AC]$, because (7)cyclic:$[A, F, B, E]$.
6. cyclic:$[C, F, E, H]$, because $FH \perp FC$(hypothesis), $EH \perp EC$(hypothesis).
7. cyclic:$[A, F, B, E]$, because $FB \perp FA$(hypothesis), $EB \perp EA$(hypothesis).

4.2. Example. As in Fig. 8, $P_0P_1P_2P_3P_4$ is a pentagon. $Q_i = P_{i-1}P_i \cap P_{i+1}P_{i+2}$, $M_i = circle(Q_{i-1}P_{i-1}P_i) \cap circle(Q_iP_iP_{i+1})$ (the subscripts are understood to be mod 5). Show that points M_0, M_1, M_2, M_3, M_4 are cyclic.

The fixpoint contains 541 (220,680 in predicate form) facts. Besides the fact that M_0, M_1, M_2, M_3, and M_4 are cyclic, the program finds the following new result: the following ten groups of lines

$$\{P_{i+1}M_{i+1}, Q_{i-1}M_{i-1}, Q_{i+2}M_{i-2}\}, \{P_{i-1}M_{i-2}, P_iM_{i+1}, Q_{i-1}M_{i+2}\}, i = 0, 1, 2, 3, 4$$

are concurrent and the ten intersection points of them are on the circle determined by $M_0M_1M_2$, i.e., this circle contains 15 points. The three dotted lines in Fig. 8 represent one group of concurrent lines.

4.3. Automated diagram generating

Most work on automated geometry reasoning focused on theorem proving and discovering. In [Gao and Chou 1998*a*], a *global propagation* method for automated generation of construction steps of diagrams was presented. This method uses a forward chaining to find the information needed in the construction and uses a backward chaining to find the construction sequences. For a diagram described declaratively with geometric constraints, the method may be used to find a sequence of constructing steps of drawing the diagram with ruler and compass.

4.3. Example. In a solid object, there is a hollow triangle tunnel. We want to put a prism with a square cross section into the tunnel in a position as shown in Fig. 9a. We need only to consider the normal cross section. Then the problem is reduced to a plane constraint problem: to put a square into a triangle ABC (Fig. 9a).

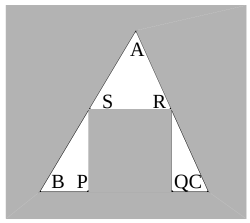

Fig. 9a

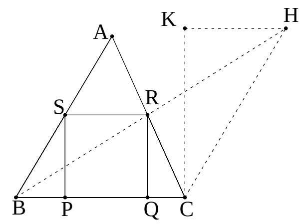

Fig. 9b

In [Gao and Chou 1998a], a solution is generated as follows (Fig. 9b): since $RQ/RS = 1$,$RQ \perp BC$, and $RS \parallel BC$, R is on a line BH, where H is the intersection of the line passing through C and parallel to AB and the line with distance $|BC|$ to line BC.
Methods of automated diagram generation have direct applications. They are the central topic in much of the current work of developing intelligent CAD systems [Brüderlin 1986, Gao and Chou 1998a, Hoffmann 1995, Kramer 1992, Owen 1991]. The main advantage of intelligent CAD system is that the resulting systems accept declarative descriptions of diagrams or mechanical designs, while for conventional CAD systems the users need to specify how to draw the diagrams.
In [Wang 1996c, Gao and Chou 1998b], the CS method is used to generate geometric diagrams automatically.

4.4. Issues in developing a prover based on AI approaches

In this section, we will discuss some of main issues in developing a powerful geometry theorem prover with the AI approaches.
Numerical Model and Generating Diagram Independent Proofs. The use of the numerical diagram as the semantic model has been the cornerstone of most of the AI approaches [Gelernter 1959, Gilmore 1970, Koedinger and Anderson 1990, Coelho and Perceira 1986]. There are two benefits the provers derive from a numerical diagram. (1) The diagram is used as a filter to reject goals not consistent with its numerical representation. (2) The numerical diagram is used to determine order relations which are necessary for the prover to find a proof. In the proof of Example 2.1 in Section 4.1, when deducing $\angle ACB = \angle ADC$ from $AB \parallel CD$ we implicitly assume that points B and D are on the opposite sides of line AC. In Gelernter's geometry machine, this fact is deduced by checking the numerical model, and a formal proof is not given.
While as a counterexample the diagram is used to control the search space successfully, the second benefit of determining order relation has some theoretical problems. Since only one or several numerical examples are checked, the provers have

the risk of proving only some special cases of the theorem. Nevins [Nevins 1976] claimed that he had got rid of this drawback by adding the ordering relations to the hypotheses of the statement. This makes the situation much clear, but still does not solve the problem. First, to prepare for the order relation, people still need to consult a diagram. Second, for some geometry theorems it may happen that the order of points in *different diagrams of the same theorem* may be different.

A key idea in AGTP is clarified by Wu in his algebraic method [Wu 1984*b*]. Wu observed that the validity of most geometry theorems involving equalities only is independent of the relative order positions of the points involved. Such theorems belong to *unordered geometry.* In unordered geometry, the proofs of these theorems can be very simple. However, the ordinary proofs of these theorems involve the order relation, hence are not only complicated, but also not strict.

The algebraic methods are for the unordered geometry and thus capable of producing diagram independent proofs. Among the AI approaches, the deductive database approach seems to be the only one that can produce diagram independent proofs.

Adding Auxiliary Points. Constructing new points or lines is a basic method of solving geometry problems. In logic, this corresponds to the Skolemization of the existential quantifiers [Robinson 1954]. Based on similar ideas, Reiter presented a deductive system that can generate new points [Reiter 1977]. But these ideas are not implemented. The idea of adding auxiliary point has been experimented in [Gelernter 1959, Coelho and Perceira 1986] but in a very limited sense. One of the reasons that previous AI geometry provers did not prove many difficult theorems is: without techniques of adding auxiliary points, the geometric axioms used by them can not prove most of the geometry theorems at all.

Extensive experiments on constructing auxiliary points are done in [Chou et al. 1996*c*]: more than thirty rules of adding auxiliary points are used. The experiments show that generating new points by Skolemization arbitrarily may easily lead to search space explosion. Strategies are used to achieve effectiveness [Chou et al. 1996*c*]. About fourty theorems were proved by adding auxiliary points.

Multiple and Shortest Proof Generation. Since generating proofs for geometry theorems becomes very fast, we may combine search techniques and these proving methods to generate multiple proofs and in particular the shortest proofs for a geometry theorem. The experiments with the area method [Chou et al. 1996*a*] and the full-angle method [Chou et al. 1996*b*] show that by selecting control strategies properly, this approach could be successful.

The basic idea is to use rules from the area method to build a rule-based reasoning system. In other words, we "relax" the deterministic style of the original area method. Using a relaxed search strategy has two positive aspects. First, this allows the program to generate multiple proofs for the same theorem. Second, this may allow us to extend the area method to prove more theorems and to produce shorter proofs.

5. Final remarks

5.1. Purposes of studying geometric reasoning

Geometry has always been a model of precise reasoning. It is quite natural that it is selected as one of the first mathematical branches to be experimented with when the field of AI started in the fifties. There are other reasons leading to the extensive study of geometric reasoning. The existence of a diagram for each geometry theorem makes geometry theorem proving easy to understand by general audiences. There are a huge amount of theorems in geometry and there are always new research topics.

Besides these, are there any practical purposes to study geometric reasoning? The answer is yes.

Study of geometric reasoning has led to the invention of new concepts and new algorithms. For instance, Gelernter's work led to several important ideas in automated reasoning, such as using a model and using lemmas. As another example, Wu's method leads to the rediscovery and improvements of Ritt's work on characteristic sets which has much more applications besides geometry reasoning. Also, as pointed out in [Davis 1995], study of AGTP may lead to the reviving of the classic Euclidean geometry.

Theories of geometric reasoning may have commercial potentials. Various methods developed in automated geometry reasoning can be used to solve problems from robotics [Huang and Wu 1992, Wu 1989*b*, Kapur 1997, Yang et al. 1997], linkage design [Gao, Zhu and Huang 1998*a*], computer vision [Kapur and Mundy 1988, Gao and Cheng 1998, Wang 1998*a*, Bondyfalat et al. 1999], intelligent CAD [Gao and Chou 1998*a*, Gao and Chou 1998*b*], intelligent CAI [Gao, Zhu and Huang 1998*b*, Li and Zhang 1998], solid modeling [Wu 1993, Kapur 1997], etc.

5.2. Further research directions

First, we should say that theories of automated geometry reasoning are quite mature in that we can not only prove most of the geometry theorems efficiently but also produce elegant proofs for most of them. In our point of view the further research should focus on developing more general purpose techniques than proving geometry theorem alone and techniques with industrial application potential. Some possible directions are:

- In the AI setting, there is a need to develop more powerful search strategies, especially strategies used to control redundant deductions. The mechanization of other traditional proof techniques in geometry such as proving by contradiction, proving by coincidence, etc., is also very interesting.
- In the setting of coordinate-free approach, the current research focus has shifted to finding general elimination theories in the vector level and building more powerful geometric models with the Clifford algebra and other algebraic technologies [Havel 1998, Fearnley-Sander and Stokes 1998, Li 1998*a*]. This research

direction is still at the beginning stage.

- In the setting of coordinate approach, the focus should be on powerful elimination techniques for both complex and real number fields in order to solve difficult problems raised in the practical fields such as robotics and mechanical design.
- Automated reasoning in differential geometry, especially in the theory of surfaces, still needs more efficient algorithms and practical programs.
- The field of automated diagram generating is still quite open for development. One interesting question is that can we design a rule-based complete method for ruler and compass construction?

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Bibliography

Anderson J. [1981], Turning the search of the problem space for geometry proofs, *in* 'Proc. IJCAI'81, Vancouver', pp. 165–170.

Arnon D. [1988], 'Geometric reasoning with logic and algebra', *Artificial Intelligence* **37**, 37–60.

Arnon D. and Mignotte M. [1988], 'On mechanical quantifier elimination for elementary algebra and geometry', *J. of Symbolic Computation* **5**, 237–259.

Balbiani P. [1995], Equation solving in geometrical theories, *in* 'LNCS 968', Springer-Verlag, Berlin, pp. 31–55.

Balbiani P. and del Cerro L. F. [1995], Affine geometry of collinearity and conditional term rewriting, *in* 'LNCS 909', Springer-Verlag, Berlin, pp. 196–213.

Bondyfalat D., Mourrain B. and Papadopoulo T. [1999], An application of automatic theorem proving in computer vision, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 207–231.

Boy de la Tour T., Fèvre S. and Wang D. [1998], Clifford term rewriting for geometric reasoning in 3d, *in* 'ADG'98, LNAI 1669', Beijing, pp. 130–155.

Brauer R. [1948], 'A note on Hilbert's nullstellensatz', *Bull. Amer. Math. Soc.* **54**, 894–896.

Brüderlin B. [1986], Constructing three-dimensional geometric objects defined by constraints, *in* 'Proc. of Workshop on Interactive 3D Graphics', ACM Press, New York, pp. 111–129.

Buchberger B. [1985], Gröbner bases: An algorithmic method in polynomial ideal theory, *in* N. Bose, ed., 'Multidimensional Systems Theory', D. Reidel Publ., pp. 184–232.

Buchberger B., Collins G. and Kutzler B. [1995], 'Algebraic methods for geometric reasoning', *Ann. Rev. Comput. Sci.* **19**, 85–119.

Cartan E. [1946], *Les Systèmes Différentiels Extérieurs et leurs Applications Géometriques*, Hermann, Paris.

Chou S. [1984], Proving elementary geometry theorems using Wu's algorithm, *in* W. Bledsoe and D. Loveland, eds, 'Automated Theorem Proving: After 25 years', AMS Contemporary Mathematics Series, pp. 243–286.

Chou S. [1987], 'A method for mechanical derivation of formulae in elementary geometry', *J. of Automated Reasoning* **3**, 291–299.

Chou S. [1988], *Mechanical Geometry Theorem Proving*, D. Reidel Publishing Company, Dordrecht, Netherlands.

Chou S. [1990], Automated reasoning in geometry using the CS and GB methods, *in* 'Proc. of ISSAC'90, (Tokyo)', ACM Press, New York, pp. 255–260.

Chou S. and Gao X. [1989], A collection of 120 computer solved geometry problems in mechanical formula derivation, Technical Report TR-89-22, CS Department, The Univ. of Texas at Austin.

Chou S. and Gao X. [1990*a*], Mechanical formula derivation in elementary geometries, *in* 'Proc. of ISSAC'90 (Tokyo)', ACM Press, New York, pp. 265–270.

Chou S. and Gao X. [1990*b*], Ritt-Wu's decomposition algorithm and geometry theorem proving, *in* 'Proc. of CADE-10', Springer-Verlag, Berlin, pp. 207–220.

Chou S. and Gao X. [1991], Geometry theorems proved mechanically using Wu's method, part on differential geometry, Technical Report MM Preprints, NO. 6, MMRC, Academia Sinica.

Chou S. and Gao X. [1992], Proving geometry statements of constructive type, *in* 'Proc. of CADE-11, LNCS 607', Springer-Verlag, Berlin, pp. 20–34.

Chou S. and Gao X. [1993*a*], 'Automated reasoning in differential geometry and mechanics: Part i. an improved version of Ritt-Wu's decomposition algorithm', *J. of Automated Reasoning* **10**, 161–172.

Chou S. and Gao X. [1993*b*], 'Automated reasoning in differential geometry and mechanics: Part ii. mechanical theorem proving', *J. of Automated Reasoning* **10**, 173–189.

Chou S. and Gao X. [1993*c*], 'Automated reasoning in differential geometry and mechanics: Part iii. mechanical formula derivation', *IFIP Tran. on Automated Reasoning* pp. 1–12.

Chou S. and Gao X. [1993*d*], 'Automated reasoning in differential geometry and mechanics: Part iv. Bertrand curves', *J. of Sys. Sci. and Math. Sci.* **6**, 186–192.

Chou S. and Gao X. [1993*e*], Mechanical theorem proving in Riemannian geometry using Wu's method, *in* W. Wu and M. Cheng, eds, 'Computer Mathematics', World Scientific, Singapore, pp. 136–158.

Chou S., Gao X. and Arnon D. [1992], 'On the mechanical proof of geometry theorems involving inequalities', *Advances in Computing Research* **6**, 139–181.

Chou S., Gao X. and Mcphee N. [1989], A combination of Ritt-Wu's method and Collins' method, Technical Report TR-89-28, CS Department, The Univ. of Texas at Austin.

Chou S., Gao X. and Zhang J. [1993*a*], Automated production of traditional proofs for constructive geometry theorems, *in* 'Proc. of Eighth IEEE Symposium on Logic in Computer Science', IEEE Computer Society Press, pp. 48–56.

Chou S., Gao X. and Zhang J. [1993*b*], Mechanical geometry theorem proving by vector calculation, *in* 'Proc. of ISSAC'93 (Kiev)', ACM Press, New York, pp. 284–291.

Chou S., Gao X. and Zhang J. [1994], *Machine Proofs in Geometry - Automated Production of Readable Proofs for Geometry Theorems*, World Scientific, Singapore.

Chou S., Gao X. and Zhang J. [1995], 'Automated production of traditional proofs in solid geometry', *J. of Automated Reasoning* **14**, 257–291.

Chou S., Gao X. and Zhang J. [1996*a*], 'Automated generation of readable proofs with geometric invariants part i: Multiple and shortest proof generation', *J. of Automated Reasoning* **17**, 325–347.

Chou S., Gao X. and Zhang J. [1996*b*], 'Automated generation of readable proofs with geometric invariants part ii: Theorem proving with full angles', *J. of Automated Reasoning* **17**, 349–370.

Chou S., Gao X. and Zhang J. [1996*c*], 'A deductive database approach to automated geometry theorem proving and discovering', *accepted by Journal of Automated Reasoning* .

Chou S. and Ko H. [1986], On mechanical theorem proving in Minkowskian plane geometry, *in* 'Proc. of IEEE Symp. of Logic in Computer Science', IEEE Computer Society Press, pp. 187–192.

Chou S. and Schelter W. F. [1986], 'Proving geometry theorems with rewrite rules', *J. of Automated Reasoning* **2**, 253–273.

Chou S. and Yang G. [1989], 'On the algebraic formulation of certain geometry statements and mechanical geometry theorem proving', *Algorithmica* **4**, 237–262.

Coelho H. and Perceira L. M. [1986], 'Automated reasoning in geometry theorem proving with Prolog', *J. of Automated Reasoning* **2**, 329–390.

Collins G. [1975], Quantifier elimination for real closed fields by cylindrical algebraic decomposition, *in* 'LNCS 33', Springer-Verlag, Berlin, pp. 134–183.

Collins G. and Hong H. [1991], 'Partial CAD construction in quantifier elimination', *J. of Symbolic Computation* pp. 299–328.

Conti P. and Traverso C. [1995], A case of automatic theorem proving in Euclidean geometry, *in* 'LNCS 948', Springer-Verlag, Berlin, pp. 183–193.

Davis P. [1995], 'The rise, fall and possible transfiguration of triangle geometry', *The American Mathematical Monthly* **102**, 204–214.

Davis P. and Cerutti E. [1969], 'Formac meets Pappus', *The American Mathematical Monthly* **76**, 895–905.

Dolzmann A. [1998], Solving geometric problems with real quantifier elimination, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 14–29.

Dolzmann A., Sturm T. and Weispfenning V. [1996], A new approach for automatic theorem proving in real geometry, Technical Report TR MIP-9611, to appear in JAR, Universität Passau.

(ed.) H. C. [1987], *Computer Aided Geometric Reasoning*, Proc. INRIA Workshop, Rocquencourt, France.

Fearnley-Sander D. [1998], Plane Euclidean reasoning, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 86–110.

Fearnley-Sander D. and Stokes T. [1998], Area in Grassmann geometry, *in* D. Wang, ed., 'LNAI 1360', Springer-Verlag, Berlin, pp. 141–170.

Ferro G. and Gallo G. [1990], 'A procedure to prove statements in differential geometry', *J. Automated Reasoning* **6**, 203–209.

Ferro G. and Gallo G. [1994], 'An extension of a procedure to prove statements in differential geometry', *J. Automated Reasoning* **12**, 351–358.

Ferro G., Gallo G. and Gennaro R. [1998], Probabilistic verification of elementary geometry statements, *in* D. Wang, ed., 'LNAI 1360', Berlin, pp. 87–101.

Fèvre S. [1998], Integration of reasoning and algebraic calculus in geometry, *in* D. Wang, ed., 'ADG'96, LNAI 1360', Springer-Verlag, Berlin, pp. 218–234.

Fèvre S. and Wang D. [1997], Proving geometric theorems using Clifford algebra and rewrite rules, *in* D. Wang, ed., 'LNAI 1421', Springer-Verlag, Berlin, pp. 17–32.

Fèvre S. and Wang D. [1998], Combining algebraic computing and term-rewriting for geometry theorem proving, *in* 'Proc. AISC'98', Plattsburgh, USA, pp. 145–156.

Fleuriot J. and Paulson L. [1998], Proving Newton's propositio Kepleriana using geometry and nonstandard analysis in Isabelle, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 47–66.

Gallaire H., Minker J. and Nicola J. M. [1984], 'Logic and databases: A deductive approach', *ACM Computing Surveys* **16**(2), 153–185.

Gao X. [1990], 'Transcendental functions and mechanical theorem proving in elementary geometries', *J. of Automated Reasoning* **6**, 403–417.

Gao X. and Cheng H. [1998], On the solution classification of the "P3P" problem, *in* Z. Li, ed., 'Proc. of ASCM'98', LanZhou Univ. Press, LanZhou, pp. 185–200.

Gao X. and Chou S. [1998*a*], 'Solving geometric constraint systems, i. a global propagation approach', *Computer Aideded Design* **30**(1), 47–54.

Gao X. and Chou S. [1998*b*], 'Solving geometric constraint systems, ii. a symbolic computational approach', *Computer Aideded Design* **30**(2), 115–122.

Gao X. and Wang D. [1995], 'On the automatic derivation of a set of geometric formulae', *J. of Geometry* **53**, 79–88.

Gao X., Zhang J. and Chou S. [1998], *Geometry Expert*, Nine Chapters Pub., TaiPei, Taiwan (in Chinese).

Gao X., Zhu C. C. and Huang Y. [1998*a*], Building dynamic mathematical models with geometry expert, ii. linkages, *in* Z. Li, ed., 'Proc. of ASCM'98', LanZhou Univ. Press, LanZhou, pp. 15–22.

Gao X., Zhu C. and Huang Y. [1998*b*], Building dynamic mathematical models with geometry expert, i. geometric transformations, functions and plane curves, *in* W. Yang, ed., 'Proc. of ATCM'98, (Tsukuba Japan)', Springer-Verlag, Singapore, pp. 216–224.

Gelernter H. [1959], Realization of a geometry theorem machine, *in* 'Proc. Int. Conf. in Info. Process', Paris, pp. 273–282.

Gelernter H., Hanson J. and Loveland D. [1960], Empirical explorations of the geometry-theorem proving machine, *in* 'Proc. West. Joint Computer Conf.', pp. 143–147.

Gianni P., Trager B. and Zacharias G. [1988], 'Gröbner bases and primary decomposition of polynomial ideals', *J. of Symbolic Computation* **6**, 149–169.

Gilmore P. [1970], 'An examination of the geometry theorem machine', *Artificial Intelligence* **1**, 171–187.

Guergueb A., Mainguené J. and Roy M. [1998], Examples of automatic theorem proving in real geometry, *in* 'Proc. ISSAC'94 (Oxford)', ACM Press, New York, pp. 20–24.

Havel T. [1991], 'Some examples of the use of distances as coordinates for Euclidean geometry', *J. of Symbolic Computation* **11**, 579–93.

Havel T. [1998], Computational synthetic geometry with Clifford algebraic, *in* D. Wang, ed., 'Automated Deduction in Geometry', Springer-Verlag, Berlin, pp. 102–114.

Hilbert D. [1971], *Foundations of Geometry*, Open Court Publishing Company, La Salla, Illinois.

Hoffmann C. [1995], Geometric constraint solving in R^2 and R^3, *in* 'Computing in Euclidean Geometry', World Scientific, Singapore, pp. 266–298.

Hong H., Wang D. and Winkler F. [1995], 'Short description of existing provers', *Ann. Math. Artif. Intell* **13**, 195–202.

Hong J. [1986], 'Can geometry be proved by an example?', *Scientia Sinica* **29**, 824–834.

Huang Y. Z. and Wu W. D. [1992], Kinematic solution of a Stewart platform, *in* 'Proc. IWMM'92', Inter. Academic Publishers, Beijing, pp. 181–188.

Kalkbrener M. [1995], 'A generalized Euclidean algorithm for geometry theorem proving', *Ann. of Math. and AI* **13**, 73–96.

Kapur D. [1986], Geometry theorem proving using Hilbert's nullstellensatz, *in* 'Proc. of SYMSAC'86, (Toronto)', ACM Press, New York, pp. 202–208.

Kapur D. [1988], 'A refutational approach to geometry theorem proving', *Artificial Intelligence* **37**, 61–93.

Kapur D. [1997], Automated geometric reasoning: Dixon resultants, Gröbner bases and characteristic sets, *in* 'Automated Deduction in Geometry', Springer-Verlag, Berlin, pp. 1–36.

Kapur D. and Mundy J. [1988], 'Wu's method and its application to perspective viewing', *Artificial Intelligence* **37**, 15–36.

Kapur D. and Saxena T. [1995], Automated geometry theorem proving using the Dixon resultants, *in* 'Invited Talk, IMACS-95', Albuquerque, USA.

Kapur D., Saxena T. and Yang L. [1994], Algebraic and geometric reasoning with Dixon resultants, *in* 'Proc. of ISSAC'94, (Oxford)', ACM Press, New York, pp. 99–107.

Kapur D. and Wan H. K. [1990], Refutational proofs of geometry theorems via characteristic set computation, *in* 'Proc. of ISSAC'90, (Tokyo)', ACM Press, New York, pp. 277–284.

Ko H. [1988], 'Geometry theorem proving by decomposition of quasi-algebraic sets: An application of the Ritt-Wu principle', *Artificial Intelligence* **37**, 95–122.

Ko H. and Hussain M. A. [1985], Alge-prover - an algebraic geometry theorem proving software, Technical Report Technical Report, 85CRD139, General Electric Company.

KOEDINGER K. R. AND ANDERSON J. R. [1990], 'Abstract planning and perceptual chunks: Elements of expertise in geometry', *Cognitive Science* **14**, 511–550.

KRAMER G. [1992], *Solving Geometric Constraint Systems*, MIT Press.

KUTZLER B. [1989], Careful algebraic translations of geometry theorems, *in* 'Proc. of ISSACC'89', ACM Press, New York, pp. 254–263.

KUTZLER B. AND STIFTER S. [1986], Automated geometry theorem proving using Buchberger's algorithm, *in* 'Proc. of SYMSAC'86, (Toronto)', ACM Press, New York, pp. 209–214.

LI C. AND ZHANG J. [1998], Readable machine solving in geometry and icai software msg, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 67–85.

LI H. [1995*a*], Automated reasoning with differential forms, *in* 'Proc. ASCM'95 (Beijing)', Scientist Press, Tokyo, pp. 29–32.

LI H. [1998*a*], Some applications of Clifford algebra to geometries, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 156–179.

LI H. [1998*b*], 'Vectorial equations solving for mechanical geometry theorem proving', *J. Automated Reasoning (accepted)* .

LI H. AND CHENG M. [1996], Proving geometric theorems with Clifford algebraic method, Technical Report MM Preprints, NO. 14, MMRC, Academia Sinica.

LI H. AND CHENG M. [1998], 'Proving geometric theorems with Clifford in differential geometries', *J. Automated Reasoning* **21**, 1–21.

LI H. AND SHI H. [1997], 'On erdos' ten-point problem', *Acta Mathematica Sinica, New Series* **13(2)**, 221–230.

LI Z. [1995*b*], 'Mechanical theorem-proving of the local theory of the surfaces', *Ann. of Math. and AI* **13**, 25–46.

LIN D. AND LIU Z. [1992], Some results on theorem proving in finite geometry, *in* 'Proc. IWMM'92', Inter. Academic Pub., Beijing, pp. 222–235.

MCCHAREN J., OVERBEEK R. AND WOS L. [1976], 'Problems and experiments for and with automated theorem-proving programs', *IEEE Trans. on Computers* **C-25**, 773–782.

MCPHEE N., CHOU S. AND GAO X. [1994], A combination of Ritt-Wu's method and Collins' method, *in* 'Proc. CADE-12', Springer-Verlag, Berlin, pp. 401–415.

MITRINOVIC D., PECARIC J. AND VOLENEC V. [1989], *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers.

NEVINS A. [1976], 'Plane geometry theorem proving using forward chaining', *Artificial Intelligence* **6**, 1–23.

OWEN J. [1991], Algebraic solution for geometry from dimensional constraints, *in* 'Proc. ACM Symp. Found. of Solid Modeling, (Austin TX)', ACM Press, New York, pp. 397–407.

QUAIFE A. [1989], 'Automated development of Tarski geometry', *J. of Automated Reasoning* **5**, 97–118.

REGE A. [1995], A complete and practical algorithm for geometric proving, *in* 'Proc. 11th Ann. Symp. Comp. Geom.', Vancouver, pp. 277–286.

REITER R. [1977], 'A semantically guided deductive system for automatic theorem proving', *IEEE Tras. on Computers* **C-25**(4), 328–334.

RICHTER-GEBERT J. [1995], 'Mechanical theorem proving in projective geometry', *Ann. of Math. and AI* **13**, 139–172.

RICHTER-GEBERT J. AND KORTENKAMP U. H. [1999], *Cinderella*, Springer-Verlag, Berlin.

RITT J. [1938], *Differential Equation from Algebraic Standpoint*, Vol. 14, AMS Colloquium Publications, New York.

RITT J. [1950], *Differential Algebra*, AMS Colloquium Publications, New York.

ROBINSON A. [1954], Proving a theorem (as done by man, logician, or machine), *in* J. Siekmann and G. Wrightson, eds, 'Automation of Reasoning', Springer-Verlag, Berlin, pp. 74–78.

SEIDENBERG A. [1954], 'A new decision method for elementary algebra', *Annals of Math.* **60**, 365–371.

Seidenberg A. [1955], 'An elimination theory for differential algebra', *University of California Pub. in Math., New Series* **3(2)**, 31–66.

Stifter S. [1993], Geometry theorem proving in vector spaces by means of Gröbner bases, *in* 'Proc. of ISSAC'93, (Kiev)', ACM Press, New York, pp. 301–310.

Stokes T. [1990], On the Algebraic and Algorithmic Properties of Some Generalized Algebras, PhD thesis, University of Tasmania.

Tarski A. [1951], A decision method for elementary algebra and geometry, Technical Report Report R-109, The Rand Corporation, Santa Monica. second revised ed.

Thomas J. M. [1954], *Differential Systems*, Amer. Math. Soc.

Wang D. [1988], 'Proving-by-examples method and inclusion of varieties', *Kexue Tongbao* **33**, 1121–1123.

Wang D. [1989], 'A new theorem discovered by computer prover', *J. of Geometry* **36**, 173–182.

Wang D. [1991], Reasoning about geometric problems using an elimination method, Technical Report Medlar 24-month deliverables, DOC, Imperial College, Univ. of London.

Wang D. [1992], Mechanical solution of a group of space geometry problems, *in* 'Proc. of IWMM'92', Int. Academic Pub., Beijing, China, pp. 236–243.

Wang D. [1993], Polynomial Equations Solving and Mechanical Geometric Theorem Proving, PhD thesis, Inst.of Sys. Sci., Academia Sinica, Beijing.

Wang D. [1995*a*], 'Elimination procedures for mechanical theorem proving in geometry', *Ann. of Math. and AI* **13**, 1–24.

Wang D. [1995*b*], 'A method for proving theorems in differential geometry and mechanics', *J. Univ. Comput. Sci.* **1**, 658–675.

Wang D. [1995*c*], Reasoning about geometric problems using an elimination method, *in* J. Pfalzgraf and D. Wang, eds, 'Automated practical reasoning: Algebraic approaches', Springer-Verlag, Berlin.

Wang D. [1996*a*], Algebraic factoring and geometry theorem proving, *in* 'LNAI 814', Springer-Verlag, Berlin.

Wang D. [1996*b*], Geometry machines: From AI to SMC, *in* 'Proc. AISMC-3, LNCS 1138', Springer-Verlag, Berlin, pp. 213–239.

Wang D. [1996*c*], Geother: A geometry theorem prover, *in* 'Proc. of CADE-13, (New Brunswick)', Springer-Verlag, Berlin.

Wang D. [1998*a*], Clifford algebraic calculus for geometric reasoning with application to computer vision, *in* D. Wang, ed., 'ADG'96, LNAI 1360', Springer-Verlag, Berlin.

Wang D. [1998*b*], Gröbner bases applied to geometric theorem proving and discovering, *in* B. Buchberger and F. Winkler, eds, 'Gröbner bases and applications', Cambridge Univ. Press, Cambridge.

Wang D. and Gao X. [1987], Geometry theorems proved mechanically using Wu's method, part on elementary geometries, Technical Report MM Research Preprint, No. 2, MMRC, Academia Sinica.

Wang D. and Hu S. [1987], 'Mechanical proving system for constructible theorems in elementary geometry', *J. Sys. Sci. and Math. Scis.* **7**, 163–172.

Wang D. and Zhi L. [1998], Algebraic factorization applied to geometric problems, *in* Z. Li, ed., 'Proc. ASCM'98', LanZhou Univ. Press, LanZhou, China, pp. 23–36.

Weispfenning V. [1994], Computational geometry problems in REDLOG, *in* D. Wang, ed., 'Automated Deduction in Geometry', Springer-Verlag, Berlin, pp. 386–400.

White N. L. and Mcmillan T. [1988], Cayley factorization, *in* 'Proc. of ISSAC'88', ACM Press, New York, pp. 4–8.

Winkler F. [1990], 'Gröbner bases in geometry theorem proving and simplest degeneracy conditions', *Math. Pannonica* **1**, 15–32.

Winkler F. [1992], 'Automated theorem proving in nonlinear geometry', *Issues in Robotics and Nonlinear Geometry* **6**, 183–197.

WU T. [1995], 'On a collision problem', *Acta Math. Scientia* **15**, 32–38.

WU W. [1978], 'On the decision problem and the mechanization of theorem proving in elementary geometry', *J. Sys. Sci. and Math. Scis.* **21**, 157–179.

WU W. [1979], 'Mechanical theorem proving in elementary differential geometry', *Scientia Sinica* pp. 94–102.

WU W. [1982*a*], Mechanical theorem proving in elementary geometry and differential geometry, *in* 'Proc. 1980 Beijing Symposium on Differential Geometry and Differential Equations', Vol. 2, Science Press, Beijing, pp. 125–138.

WU W. [1982*b*], 'Toward mechanization of geometry — some comments on Hilbert's Grundlagen der Geometrie', *Acta Math. Sci.* pp. 125–138.

WU W. [1984*a*], 'Basic principles of mechanical theorem proving in geometries', *J. Sys. Sci. and Math. Scis.* **4(3)**, 207–235.

WU W. [1984*b*], *Basic Principles of Mechanical Theorem Proving in Geometries (in Chinese)*, Science Press, Beijing.

WU W. [1984*c*], Some recent advances in mechanical theorem-proving of geometries, *in* 'Automated Theorem Proving: After 25 years', AMS, Contemporary Mathematics, pp. 235–242.

WU W. [1986*a*], 'A mechanization method of geometry and its applications i: Distances, areas, and volumes', *J. Sys. Sci. and Math. Scis.* **6**, 204–216.

WU W. [1986*b*], 'On reducibility problem in mechanical theorem proving of elementary geometries', *Chinese Quart. J. Math.* pp. 1–20.

WU W. [1987*a*], A constructive theory of differential algebraic geometry, *in* 'Lecture Notes in Math.', Vol. 1255, Springer-Verlag, Berlin, pp. 173–189.

WU W. [1987*b*], Mechanical derivation of Newton's gravitational laws from Kepler's laws, Technical Report MM Research Preprints No.1, MMRC, Academia Sinica.

WU W. [1987*c*], 'A mechanization method of geometry and its applications ii: Curve pairs of Bertrand type', *Kexue Tongbao* **32(9)**, 535–538.

WU W. [1989*a*], 'A mechanization method of geometry and its applications iv. some theorems in planar kinematics', *J. Sys. Scis and Math. Sci.* **2**, 97–109.

WU W. [1989*b*], A mechanization method of geometry and its applications vi. solving inverse kinematics equations of PUMA-type robotics, Technical Report MM Research Preprints No.4, MMRC, Academia Sinica.

WU W. [1989*c*], 'On the foundation of algebraic differential geometry', *Sys. Sci. and Math. Sci.* **2**, 289–312.

WU W. [1990], 'On a projection theorem of quasi-varieties in elimination theory', *Chinese Ann. of Math.* **11B**, 220–226.

WU W. [1992*a*], On a finiteness theorem about optimization problems, Technical Report MM Research Preprints No.8, MMRC, Academia Sinica.

WU W. [1992*b*], 'A report on mechanical geometry theorem proving', *Progress in Natural Sciences* **2**, 1–17.

WU W. [1993], On surface-fitting problem in CAGD, Technical Report MM Research Preprints No.10, MMRC, Academia Sinica.

YANG H., ZHANG S. AND FENG G. [1998], A Clifford algebraic method for geometric reasoning, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 111–129.

YANG L. [1998], Practical automated reasoning on inequalities, *in* 'Proc. of ATCM, (Tsukuba, Japan)', Springer-Verlag, Berlin, pp. 24–25.

YANG L., FU H. AND ZHENG Z. [1997], A practical symbolic algorithm for inverse kinematics of 6R manipulators with simple geometry, *in* 'Proc. of CADE-14, (Townsville, Australia)', Springer-Verlag, Berlin, pp. 73–88.

YANG L., GAO X., CHOU S. AND ZHANG J. [1998], Automated proving and discovering of theorems in non-Euclidean geometries, *in* D. Wang, ed., 'Automated Deduction in Geometry', Springer-Verlag, Berlin, pp. 171–188.

Yang L., Hou X. and Xia B. [1998], Automated discovering and proving for geometric inequalities, *in* 'ADG'98, LNAI 1669', Springer-Verlag, Berlin, pp. 30–46.

Yang L., Hou X. and Zeng Z. [1996], 'A complete discrimination system for polynomials', *Sciences in China* **E 39**, 628–646.

Yang L., Zhang J. and Hou X. [1992], A criterion of dependency between algebraic equations and its applications, *in* 'Proc. IWMM'92', Inter. Academic Publishers, Beijing, pp. 110–134.

Yang L., Zhang J. and Hou X. [1996], *Non-linear Algebraic Equations and Automated Theorem Proving*, ShangHai Science and Education Pub., ShangHai.

Zhang J. [1982], *How to Solve Geometry Problems Using Areas*, Shanghai Educational Publishing Inc., ShangHai.

Zhang J., Chou S. and Gao X. [1995], 'Automated production of traditional proofs for theorems in Euclidean geometry', *Annals of Math. and AI* **13**, 109–137.

Zhang J., Yang L. and Deng M. [1990], 'The parallel numerical method of mechanical theorem proving', *Theoretical Computer Science* **74**, 253–271.

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