Automated Production of Traditional Proofs for Theorems in Euclidean Geometry †

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Abstract

We present a method which can produce traditional proofs for a class of geometry statements whose hypotheses can be described constructively and whose conclusions can be represented by polynomial equations of three kinds of geometry quantities: ratios of line segments, areas of triangles, and Pythagoras differences of triangles. This class covers a large portion of the geometry theorems about straight lines and circles. The method involves the elimination of the constructed points from the conclusion using a few basic geometry propositions. Our program based on the method can produce short and readable proofs of many hard geometry theorems such as Pappus' theorem, Simson's theorem, the Butterfly theorem, Pascal's theorem, etc. Currently, it has produced proofs of 400 nontrivial theorems *entirely* automatically. The proofs produced by our program are generally short and readable. This method seems to be the first to produce traditional proofs for hard geometry theorems efficiently.

Keywords. Machine proof, automated geometry theorem proving, Euclidean traditional proof, area method, Pythagoras difference, constructive geometry statements.

1 Introduction

Geometry theorem proving on computers began in earnest in the 50s with the work of Gelernter, J. R. Hanson, and D. W. Loveland [8]. This work and most of the subsequent work [10, 12, 7] was *synthetic*, i.e., researchers focused on the automation of the traditional proof method. The main problem of this approach was controlling the search space, or equivalently, guiding the program toward the right deductions. Despite the initial success, this approach did not make much progress in proving many difficult theorems.

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On the other hand, earlier in the 1930s, A. Tarski, introduced a quantifier elimination method based on the *algebraic approach* [13] to prove theorems in *elementary geometry.* A breakthrough in the use of algebraic method came with the work of Wen-Tsün Wu, who introduced an algebraic method which, for the first time, was used to prove hundreds of geometry theorems automatically [14]. Since Wu's work, highly successful algebraic methods for automated proving geometry theorems have been developed. Computer programs based on these methods have been used to prove many non–trivial geometry theorems [6, 9, 11, 15]. Especially the program developed at the University of Texas has proved about 600 theorems from Euclidean and non–Euclidean geometries [1]. Many hard theorems whose traditional proofs need an enormous amount of human intelligence, such as Feuerbach's theorem, Morley's trisector theorem, etc., can be proved by computer programs based on algebraic methods within seconds.

Algebraic methods, which are very different from the traditional proof methods used by geometers since Euclid, generally can only tell whether a statement is true or not. If one wishes to look at the proofs produced by the machine, he/she will find tedious and formidable computations of polynomials. The polynomials involved in the proofs can contain *hundreds* of terms with more than a dozen variables. Because of this, producing short, readable proofs remains a prominent challenge.

In [18], by combining ideas from both the algebraic approach and the synthetic approach, we present a method that can produce short and readable proofs for more than 100 theorems about line intersections. The success of the method is based on the extensive study of the traditional area method [16, 17]. In [4], we extend the area method to the volume method which is very successful in automated theorem proving in solid geometry.

This paper, which is the full version of the extended abstract [2], is a further extension of the area method in plane geometry to a wider class of constructive geometry statements involving perpendicularity and circles. The concept of Pythagoras differences of triangles is introduced as the key tool in dealing with perpendiculars and circles. Most of the geometry theorems of equality type in geometry textbooks are in this class. Among the 512 theorems in [1], about 420 are in this class. The method is complete and the complexity of the algorithm is given.

Our program[1] implements this method and can produce traditional proofs of many hard geometry theorems such as Simson's theorem, the Butterfly theorem, Pascal's theorem, the Pascal conic theorems, etc. Currently, it has produced proofs for 400 nontrivial theorems *entirely* automatically [5]. The program is very efficient. Most of the 400 theorems are proved within a few seconds. The most important feature of our work is that the proofs produced by the program are generally short: the formulas in the proofs usually contain *several terms*, and hence readable by people. This is the main theme of our research. To achieve this, we need to use proper geometry quantities and to find proper ways of doing elimination. For the detailed statistics of the lengths and the timings of the proofs for the 400 examples, see Section 6. This method seems to be the first that can produce readable proofs for hard geometry theorems efficiently.

Based on the performance of our prover, we believe that the area method may have potential use in geometry education, since the proofs produced according to the method are generally short and in a shape that a student of mathematics could learn to design with pencil and paper.

In Section 2, we present the basic propositions which are the deductive basis of the method. In Section 3, we define the constructive statements. In Section 4, we present the method. In

[1]b The prover is available via ftp at emcity.cs.twsu.edu: pub/geometry.

Section 5, we present some techniques of producing short proofs. In Section 6, we give the experiment results and comparisons.

2 Basic Geometry Quantities and Propositions

We use three basic *geometry quantities*: the ratio of parallel line segments, the signed area, and the Pythagoras difference. The basic propositions, which formally describe the properties of these quantities, are the deductive basis of the area method. The validity of these propositions are taken for granted in this paper. However their proofs can be found in the appendix of the technical report form of [2].

We use capital English letters to denote points in the Euclidean plane. Let $\mathbf{R}$ be the field of the real numbers. The following proposition formally defines the ratio of line segments.

Proposition 2.1 For four collinear points P, Q, A, and B such that $A \neq B$, $\frac{\overline{PQ}}{\overline{AB}}$, *the ratio of the directed segments*, is an element in $\mathbf{R}$ and satisfies

1. $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.
2. $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.
3. $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.
4. For $r \in \mathbf{R}$, there exists a unique point P which is collinear with A and B and satisfies $\frac{\overline{AP}}{\overline{AB}} = r$.

Let $r = \frac{\overline{PQ}}{\overline{AB}}$. We sometimes also write $\overline{PQ} = r\overline{AB}$. A point P on line AB is determined uniquely by $\frac{\overline{AP}}{\overline{AB}}$ or $\frac{\overline{PB}}{\overline{AB}}$. We thus call

$$x_P = \frac{\overline{AP}}{\overline{AB}}, \quad y_P = \frac{\overline{PB}}{\overline{AB}}$$

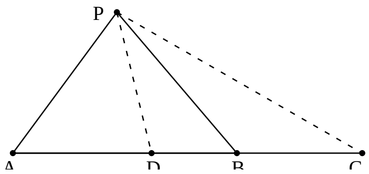
the *position ratio* or *position coordinates* of the point P with respect to AB. It is clear that $x_P + y_P = 1$.

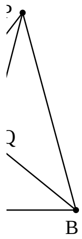
2.1 Propositions about Signed Areas

We denote by S_{ABC} the *signed area* of the triangle ABC.

Proposition 2.2 For any points A, B, C, and D, we have

1. $S_{ABC} = S_{CAB} = S_{BCA} = -S_{ACB} = -S_{BAC} = -S_{CBA}$.
2. $S_{ABC} = 0$ iff A, B, and C are collinear.
3. $S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}$.





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Proposition 2.3 If points C and D are on line AB and P is any point not on line AB (Figure 1), then

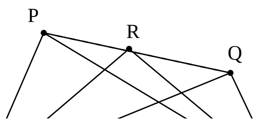
$$\frac{S_{PCD}}{S_{PAB}} = \frac{\overline{CD}}{\overline{AB}}.$$

Proposition 2.4 (The Co-side Theorem) Let M be the intersection of two lines AB and PQ and $Q \neq M$ (Figure 2). Then

$$\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}; \quad \frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}}; \quad \frac{\overline{QM}}{\overline{PQ}} = \frac{S_{QAB}}{S_{PAQB}}.$$

Proposition 2.5 Let R be a point on line PQ. Then for any two points A and B

$$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}} S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}} S_{PAB}.$$



Definition 2.6 We use the notation $AB \parallel PQ$ to denote the fact that A, B, P, and Q satisfy one of the following conditions: (1) $A = B$ or $P = Q$; (2) A, B, P and Q are on the same line; or (3) line AB and line PQ do not have a common point.

Proposition 2.7 $PQ \parallel AB$ iff $S_{PAB} = S_{QAB}$, i.e., iff $S_{PAQB} = 0$.

A *parallelogram* is a quadrilateral $ABCD$ such that $AB \parallel CD$, $BC \parallel AD$, and no three vertices of it are on the same line. Let $ABCD$ be a parallelogram and P, Q be two points on CD. We define the *ratio of two parallel line segments* as follows

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}}.$$

In our machine proofs, *auxiliary parallelograms* are often added automatically and the following two propositions are used frequently.

Proposition 2.8 Let $ABCD$ be a parallelogram. Then for two points P and Q, we have

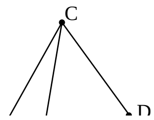
$$S_{APQ} + S_{CPQ} = S_{BPQ} + S_{DPQ} \text{ or } S_{PAQB} = S_{PDQC}.$$

Proposition 2.9 Let $ABCD$ be a parallelogram and P be any point. Then

$$S_{PAB} = S_{PDC} - S_{ADC} = S_{PDAC}.$$

We use a simple example to illustrate how to use these propositions to prove theorems. The following proof is essentially the same as the proof produced by our prover.

Example 2.10 (Ceva's Theorem) Let $\triangle ABC$ be a triangle and P be any point in the plane. Let $D = AP \cap CB$, $E = BP \cap AC$, and $F = CP \cap AB$ (Figure 4). Show that



$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Proof. Our aim is to eliminate the constructed points F, E and D from the left hand side of the conclusion. Using the co-side theorem three times, we can eliminate E, F, and D

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{S_{APC}}{S_{BCP}} \cdot \frac{S_{BPA}}{S_{CAP}} \cdot \frac{S_{CPB}}{S_{ABP}} = 1. \quad \blacksquare$$

2.2 Propositions about Pythagoras Differences

For three points A, B, and C, the *Pythagoras difference* P_{ABC} is defined to be

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$

It is easy to check that

1. $P_{AAB} = 0$; $P_{ABC} = P_{CBA}$.
2. $P_{ABA} = 2\overline{AB}^2$.
3. If A, B, C are collinear, $P_{ABC} = 2\overline{BA} \cdot \overline{BC}$.

For a quadrilateral $ABCD$, we define

$$P_{ABCD} = P_{ABD} - P_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

Then we have $P_{ABCD} = -P_{ADCB} = P_{BADC} = -P_{BCDA} = P_{CDAB} = -P_{CBAD} = P_{DCBA} = -P_{DABC}$.

Definition 2.11 For four points A, B, C, and D, the notation $AB \perp CD$ implies that one of the following conditions is true: $A = B$, or $C = D$, or the line AB is perpendicular to line CD.

Proposition 2.12 (Pythagorean Theorem) $AB \perp BC$ iff $P_{ABC} = 0$.

Proposition 2.13 $AB \perp CD$ iff $P_{ACD} = P_{BCD}$ or $P_{ACBD} = 0$.

The above generalized Pythagorean proposition is one of the most useful tools in our mechanical theorem proving method.

Proposition 2.14 Let D be the foot of the perpendicular drawn from point P to a line AB (Figure 5). Then we have

$$\frac{\overline{AD}}{\overline{DB}} = \frac{P_{PAB}}{P_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$

Proposition 2.15 Let AB and PQ be two non-perpendicular lines and Y be the intersection of line PQ and the line passing through A and perpendicular to AB (Figure 6). Then

$$\frac{\overline{PY}}{\overline{QY}} = \frac{P_{PAB}}{P_{QAB}}, \frac{\overline{PY}}{\overline{PQ}} = \frac{P_{PAB}}{P_{PAQB}}, \frac{\overline{QY}}{\overline{PQ}} = \frac{P_{QAB}}{P_{PAQB}}.$$

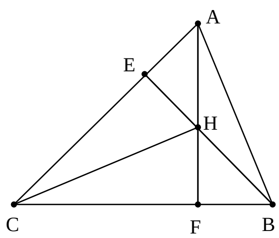
Proposition 2.16 Let R be a point on line PQ with position ratios $r_1 = \frac{\overline{PR}}{\overline{PQ}}, r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ with respect to PQ. Then for points A, B, we have

$$\begin{aligned} P_{RAB} &= r_1 P_{QAB} + r_2 P_{PAB} \\ P_{ARB} &= r_1 P_{AQB} + r_2 P_{APB} - r_1 r_2 P_{PQP}. \end{aligned}$$

Proposition 2.17 Let $ABCD$ be a parallelogram. Then for any points P and Q, we have

$$\begin{aligned} P_{APQ} + P_{CPQ} &= P_{BPQ} + P_{DPQ} \quad \text{or} \quad P_{APBQ} = P_{DPCQ} \\ P_{PAQ} + P_{PCQ} &= P_{PBQ} + P_{PDQ} + 2P_{BAD} \end{aligned}$$

Example 2.18 (The Orthocenter Theorem) Show that the three altitudes of a triangle are concurrent.



Proof. Let the two altitudes AF and BE of triangle ABC meet in H. We only need to prove $CH \perp AB$, i.e., $P_{ACH} = P_{BCH}$. Since $BH \perp AC$ and $AH \perp BC$, by Proposition 2.13, $P_{ACH} = P_{ACB} = P_{BCA} = P_{BCH}$. ∎

Remark 2.19 The 14 propositions are not independent. Actually all the propositions can be derived from Propositions 2.1, 2.2, 2.3, and 2.13. We use all of them as basic propositions with the intention of producing short proofs.

3 The Constructive Geometry Statements

3.1 Constructive Geometry Statements

In Section 2, we have introduced three *geometric quantities*: the area of a triangle or a quadrilateral, the Pythagoras difference of a triangle or a quadrilateral, and the ratio of parallel line segments.

Points are the basic geometry objects, from which we can introduce two other basic *geometric objects*: lines and circles. A *straight line* can be given in one of the following four forms

(LINE U V) is the line passing through two points U and V.

(PLINE W U V) is the line passing through point W and parallel to (LINE U V).

(TLINE W U V) is the line passing through point W and perpendicular to (LINE U V).

(BLINE U V) is the perpendicular-bisector of UV.

To make sure that the four kinds of lines are well defined, we need to assume $U \neq V$ which is called the *nondegenerate condition* (ndg) of the corresponding line.

A *circle* with point O as its center and passing through point U is denoted by (CIR O U).

A *construction* is one of the following ways of introducing new points. For each construction, we also give its ndg condition and the degree of freedom for the constructed point.

C1 (POINT[S] $Y_1, \cdots, Y_l$). Take arbitrary points $Y_1, \cdots, Y_l$ in the plane. Each Y_i has two degrees of freedom.

C2 (ON Y ln). Take a point Y on a line ln. The ndg condition of C2 is the ndg condition of the line ln. Point Y has one degree of freedom.

C3 (ON Y (CIR O P)). Take a point Y on a circle (CIR O P). The ndg condition is $O \neq P$. Point Y has one degree of freedom.

C4 (INTER Y $ln1$ $ln2$). Point Y is the intersection of line $ln1$ and line $ln2$. Point Y is a fixed point. The ndg condition is $ln1 \nparallel ln2$. More precisely, we have

1. If $ln1$ is (LINE U V) or (PLINE W U V) and $ln2$ is (LINE P Q) or (PLINE R P Q), then the ndg condition is $UV \nparallel PQ$.
2. If $ln1$ is (LINE U V) or (PLINE W U V) and $ln2$ is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $\neg(UV \perp PQ)$.
3. If $ln1$ is (BLINE U V) or (TLINE W U V) and $ln2$ is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $UV \nparallel PQ$.

C5 (INTER Y ln (CIR O P)). Point Y is the intersection of line ln and circle (CIR O P) other than point P. Line ln could be (LINE P U), (PLINE P U V), and (TLINE P U V). The ndg conditions are $O \neq P$, $Y \neq P$, and line ln is not degenerate. Point Y is a fixed point.

C6 (INTER Y (CIR O_1 P) (CIR O_2 P)). Point Y is the intersection of the circle (CIR O_1 P) and the circle (CIR O_2 P) other than point P. The ndg condition is that O_1, O_2, and P are not collinear. Point Y is a fixed point.

C7 (PRATIO Y W U V r). Take a point Y on the line (PLINE W U V) such that $\overline{WY} = r\overline{UV}$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity then Y is a fixed point; if r is a variable then Y has one degree of freedom. The ndg condition is $U \neq V$. If r is a rational expression in geometry quantities then we will further assume that the denominator of r could not be zero.

C8 (TRATIO Y U V r). Take a point Y on line (TLINE U U V) such that $r = \frac{4S_{UVY}}{P_{UVU}} (= \frac{\overline{UY}}{\overline{UV}})$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity then Y is a fixed point; if r is a variable then Y has one degree of freedom. The ndg condition is the same as that of C7.

Since there are four kinds of lines, constructions C2, C4, and C5 have 4, 10, and 3 possible forms respectively. Thus, totally we have 22 different forms of constructions.

Definition 3.1 Now class **C**, the class of *constructive geometry statements*, can be defined as follows. A statement in class **C** is a list $S = (C_1, C_2, \ldots, C_k, G)$ where C_i, $i = 1, \ldots, k$, are constructions such that each C_i introduces a new point from the points introduced before; and $G = (E_1, E_2)$ where E_1 and E_2 are polynomials in geometric quantities of the points introduced by the C_i and $E_1 = E_2$ is the conclusion of the statement.

Let $S = (C_1, C_2, \ldots, C_k, (E_1, E_2))$ be a statement in **C**. The *ndg condition* of S is the set of ndg conditions of the C_i plus the condition that the denominators of the length ratios in E_1 and E_2 are not equal to zero.

Example 3.2 (Ceva's Theorem) Continue from Example 2.10. The constructive description for Ceva's theorem is as follows.

((c POINTS A B C P)
(c INTER D (c LINE B C) (c LINE P A))
(c INTER E (c LINE A C) (c LINE P B))
(c INTER F (c LINE A B) (c LINE P C))
($\frac{\overline{AF}}{\overline{FB}} \frac{\overline{BD}}{\overline{DC}} \frac{\overline{CE}}{\overline{EA}} = 1$))

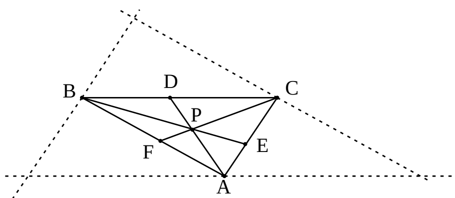


Figure 8

The ndg conditions for Ceva's theorem are

$$BC \not\parallel AP; AC \not\parallel BP; AB \not\parallel CP; F \neq B; D \neq C; E \neq A,$$

i.e., point P can not be on the three sides of $\triangle ABC$ and the three dotted lines in Figure 8. You may wonder that the condition "A, B, and C not collinear" is not in the ndg conditions. Indeed, when A, B, and C are three different (this comes from the ndg condition) points on the same line, the Ceva's theorem is still true (now $F = C, D = A$, and $E = B$) and the proofs based on the area method is still valid in this case. The ndg conditions produced according to our method guarantee that we can produce a proof for the statement. Certainly, we can avoid this seemingly unpleasant fact by introduce a new construction: TRIANGLE which introduces three non-collinear points. But theoretically, this is not necessary.

The 22 constructions are not independent to each other. We now introduce a minimal set of constructions which are equivalent to all the 22 constructions but much few in number.

A *minimal set of constructions* consists of C1, C7, C8 and the following two constructions.

C41 (INTER Y (LINE U V) (LINE P Q)).

C42 (FOOT Y P U V), or equivalently (INTER Y (LINE U V) (TLINE P U V))). The ndg condition is $U \neq V$.

We first show how to represent the four kinds of lines by one kind: (LINE U V).

For ln = (PLINE W U V), we first introduce a new point N by (PRATIO N W U V 1). Then ln = (LINE W N).

For ln = (TLINE W U V), we have two cases: if W, U, V are collinear, ln = (LINE N W) where N is introduced by (TRATIO N W U 1); otherwise ln = (LINE N W) where N is given by (FOOT N W U V).

(BLINE U V) can be written as (LINE N M) where N and M are introduced as follows (MIDPOINT M U V) (i.e., (PRATIO M U U V 1/2)), (TRATIO N M U 1).

Since now there is only one kind of line, to represent all the 22 constructions by the constructions in the minimal set we only need to consider the following cases.

- (ON Y (LINE U V)) is equivalent to (PRATIO Y U U V r) where r is an *indeterminate*.
- (INTER Y (LINE U V) (CIR O U)) is equivalent to two constructions: (FOOT N O U V), (PRATIO Y N N U -1).
- C6 can be reduced to (FOOT N P O_1 O_2) and (PRATIO Y N N P -1).
- For C3, i.e., to take an arbitrary point Y on a circle (CIR O P), we first take an arbitrary point Q. Then Y is introduced by (INTER Y (LINE P Q) (CIR O P)).

3.2 The Predicate Form

The constructive description of geometry statements can be transformed into the commonly used predicate form. We introduce five predicates.

1. Point $POINT(P)$: P is a point in the plane.
2. Collinear $COLL(P_1, P_2, P_3)$: points P_1, P_2, P_3 are on the same line. It is equivalent to $S_{P_1P_2P_3} = 0$.
3. Parallel $PARA(P_1, P_2, P_3, P_4)$: $P_1P_2 \parallel P_3P_4$. It is equivalent to $S_{P_1P_3P_2P_4} = 0$.
4. Perpendicular ($PERP$ P_1, P_2, P_3, P_4): $P_1P_2 \perp P_3P_4$. It is equivalent to $P_{P_1P_3P_2P_4} = 0$.

5. Congruence ($CONG\ P_1, P_2, P_3, P_4$): Segment P_1P_2 is congruent to P_3P_4. It is equivalent to $P_{P_1P_2P_1} = P_{P_3P_4P_3}$.

To transform constructions into predicate forms, we only need to consider the minimal set of constructions introduced in the preceding subsection.

C41 (INTER Y (LINE $U\ V$) (LINE $P\ Q$)) is equivalent to (COLL $Y\ U\ V$), (COLL $Y\ P\ Q$), and $\neg$(PARA $U\ V\ P\ Q$).

C42 (FOOT $Y\ P\ U\ V$) is equivalent to (COLL $Y\ U\ V$), (PERP $Y\ P\ U\ V$), and $U \neq V$.

C7 (PRATIO $Y\ W\ U\ V\ r$) is equivalent to (PARA $Y\ W\ U\ V$), $\frac{\overline{WY}}{\overline{UV}} = r$, and $U \neq V$.

C8 (TRATIO $Y\ U\ V\ r$) is equivalent to (PERP $Y\ U\ U\ V$), $r = \frac{4S_{UVY}}{P_{UVU}}$, and $U \neq V$.

Now a constructive statement $S = (C_1, \cdots, C_k, (E, F))$ can be transformed into the following predicate form

$$\forall P_i[(P(C_1) \wedge \cdots \wedge P(C_k)) \Rightarrow (E = F)]$$

where $P(C_i)$ is the predicate form for C_i and P_i is the point introduced by C_i.

We now discuss what geometry properties can be the conclusion of a geometry statement in **C**, i.e., what geometry properties can be represented by polynomial equations of geometry quantities. To see that, let us give an algebraic interpretation for the area and Pythagoras difference. Let A, B, C, and D be four points in the Euclidean plane. Then S_{ABCD} and P_{ABCD} are propositional to the exterior and inner product of the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ of the quadrilateral ABC:

$$S_{ABCD} = \frac{1}{2}[\overrightarrow{AC}, \overrightarrow{BD}], \quad P_{ABCD} = 2\langle\overrightarrow{AC}, \overrightarrow{DB}\rangle.$$

So any geometry property that can be represented by an equation of the inner and exterior products can be the conclusion of a geometry statement. As examples, we show how to represent several often used geometry properties by the geometry quantities.

(COLLINEAR $A\ B\ C$). Points A, B, and C are collinear iff $S_{ABC} = 0$.

(PARALLEL $A\ B\ C\ D$). AB is parallel to CD iff $S_{ACD} = S_{BCD}$.

(PERPENDICULAR $A\ B\ C\ D$). AB is perpendicular to CD iff $P_{ACD} = P_{BCD}$.

(MIDPOINT $O\ A\ B$). O is the midpoint of AB iff $\frac{\overline{AO}}{\overline{OB}} = 1$.

(EQDISTANCE $A\ B\ C\ D$). AB has the same length as CD iff $P_{ABA} = P_{CDC}$.

(HARMONIC $A\ B\ C\ D$). A, B and C, D are harmonic points iff $\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$.

(COCIRCLE $A\ B\ C\ D$). Points A, B, C, and D are co-circle iff $S_{CAD}P_{CBD} = P_{CAD}P_{CBD}$.

Example 3.3 (Ceva's Theorem) Continue from Example 3.2. The predicate form for Ceva's theorem is

$$\forall A, B, C, P, E, F, D(HYP \Rightarrow CONC)$$

where

$$\begin{aligned}
HYP = \quad & (COLL\ D\ B\ C) \wedge (COLL\ D\ A\ O) \wedge \neg(PARA\ B\ C\ A\ O) \wedge \\
& (COLL\ E\ A\ C) \wedge (COLL\ E\ B\ O) \wedge \neg(PARA\ A\ C\ B\ O) \wedge \\
& (COLL\ F\ A\ B) \wedge (COLL\ F\ C\ D) \wedge \neg(PARA\ A\ B\ C\ O) \wedge \\
& B \neq F \wedge D \neq C \wedge A \neq E \\
CONC = \quad & (\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1).
\end{aligned}$$

4 The Algorithm

Before presenting the method, let us re-examine the proof of Ceva's theorem. By describing Ceva's theorem constructively, we can introduce an order among the points naturally: A, B, C, P, D, E, and F, i.e., the order according to which the points are introduced. The proof is actually to eliminate the points from the conclusions according to the reverse order: F, E, D, P, C, B, and A. We thus have the proof:

$$\begin{array}{ll}
\frac{\overline{AF}}{\overline{FB}} = -\frac{S_{ACP}}{S_{BCP}} & \text{Eliminate point } F. \\
\frac{\overline{CE}}{\overline{EA}} = \frac{S_{BCP}}{S_{ABP}} & \text{Eliminate point } E. \\
\frac{\overline{BD}}{\overline{DC}} = -\frac{S_{ABP}}{S_{ACP}} & \text{Eliminate point } D.
\end{array}$$

Then

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{S_{ACP}S_{BCP}S_{ABP}}{S_{BCP}S_{ACP}S_{ABP}} = 1.$$

Thus the key step of the method is to *eliminate points from geometry quantities*. We will show how this is done in the next subsection.

4.1 The Elimination Procedures

As mentioned in Section 3, we only need to consider the minimal set of constructions: C1, C7, C8, C41, C42. We will discuss C1 in Section 4.2. Thus we need to eliminate points introduced by four constructions from three kinds of geometry quantities.

Let $G(Y)$ be one of the following geometry quantities: $S_{ABY}, S_{ABCY}, P_{ABY}$, or P_{ABCY} for distinct points A, B, C, and Y. For three collinear points Y, U, and V, by Propositions 2.5 and 2.16 we have

$$(I) \qquad G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U).$$

We call $G(Y)$ a *linear geometry quantity* for variable Y. Elimination procedures for all linear geometry quantities are similar for constructions C7, C41, and C42.

Lemma 4.1 Let $G(Y)$ be a linear geometry quantity and point Y be introduced by construction (PRATIO $Y\ W\ U\ V\ r$). Then we have $G(Y) = G(W) + r(G(V) - G(U))$.

Proof. Take a point S such that $\overline{WS} = \overline{UV}$. By (I)

$$G(Y) = \frac{\overline{WY}}{\overline{WS}}G(S) + \frac{\overline{YS}}{\overline{WS}}G(W) = rG(S) + (1-r)G(W).$$

By Propositions 2.8 and 2.17, $G(S) = G(W) + G(V) - G(U)$. Substituting this into the above equation, we obtain the result. Notice that we need the ndg condition $U \neq V$. ∎

Lemma 4.2 Let $G(Y)$ be a linear geometry quantity and Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then

$$G(Y) = \frac{S_{UPQ}G(V) - S_{VPQ}G(U)}{S_{UPVQ}}.$$

Proof. By the co-side theorem, $\frac{\overline{UY}}{\overline{UV}} = \frac{S_{UPQ}}{S_{UPVQ}}, \frac{\overline{YV}}{\overline{UV}} = -\frac{S_{VPQ}}{S_{UPVQ}}$. Substituting these into (I), we prove the result. ∎

Lemma 4.3 Let $G(Y)$ be a linear geometry quantity and Y be introduced by (FOOT Y P U V). Then

$$G(Y) = \frac{P_{PUV}G(V) + P_{PVU}G(U)}{2\overline{UV}^2}.$$

Proof. By Proposition 2.14, $\frac{\overline{UY}}{\overline{UV}} = \frac{P_{PUV}}{P_{UVU}}, \frac{\overline{YV}}{\overline{UV}} = \frac{P_{PVU}}{P_{UVU}}$. Substituting these into (I), we prove the result, ∎

Let $G(Y) = P_{AYB}$. By Proposition 2.16, for three collinear points Y, U, and V

$$(II) \qquad G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U) - \frac{\overline{UY}}{\overline{UV}} \cdot \frac{\overline{YV}}{\overline{UV}}P_{UVU}.$$

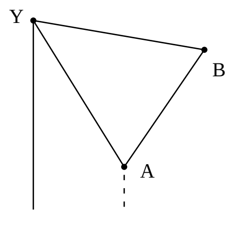
Since we have obtained the position ratios $\frac{\overline{UY}}{\overline{UV}}, \frac{\overline{YV}}{\overline{UV}}$ for Y when it is introduced by $C7$, $C41$, $C42$ in the above three lemmas, we can substitute them into (II) to eliminate point Y from $G(Y)$. Notice that in the case of construction $C7$, we need to use the second formula of Proposition 2.17. The result is as follows.

Lemma 4.4 Let Y be introduced by (PRATIO Y W U V r). Then we have

$$P_{AYB} = P_{AWB} + r(P_{AVB} - P_{AUB} + P_{WUV}) - r(1-r)P_{UVU}.$$

Construction C8 needs special treatment.

Lemma 4.5 Let Y be introduced by (TRATIO Y P Q r). Then we have $S_{ABY} = S_{ABP} - \frac{r}{4}P_{PAQB}$.

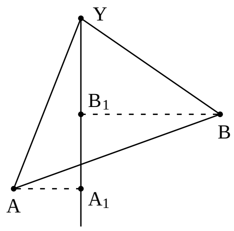


Proof. Let A_1 be the orthogonal projection from A to PQ. Then by Propositions 2.7 and 2.14

$$\frac{S_{PAY}}{S_{PQY}} = \frac{S_{PA_1Y}}{S_{PQY}} = \frac{\overline{PA_1}}{\overline{PQ}} = \frac{P_{A_1PQ}}{P_{QPQ}} = \frac{P_{APQ}}{P_{QPQ}}.$$

Thus $S_{PAY} = \frac{P_{APQ}}{P_{QPQ}} S_{PQY} = \frac{r}{4} P_{APQ}$. Similarly, $S_{PBY} = \frac{P_{BPQ}}{P_{QPQ}} S_{PQY} = \frac{r}{4} P_{BPQ}$. Now $S_{ABY} = S_{ABP} + S_{PBY} - S_{PAY} = S_{ABP} - \frac{r}{4} P_{PAQB}$. ∎

Lemma 4.6 Let Y be introduced by (TRATIO Y P Q r). Then we have $P_{ABY} = P_{ABP} - 4rS_{PAQB}$.



Proof. Let the orthogonal projections from A and B to PY be A_1 and B_1. Then

$$\frac{P_{BPAY}}{P_{YPY}} = \frac{P_{B_1PA_1Y}}{P_{YPY}} = \frac{\overline{A_1B_1}}{\overline{PY}} = \frac{S_{PA_1QB_1}}{S_{PQY}} = \frac{S_{PAQB}}{S_{PQY}}.$$

Since $PY \perp PQ$, $S^2_{PQY} = \frac{1}{4}\overline{PQ}^2 \cdot \overline{PY}^2$. Then $P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY}$. Therefore $P_{ABY} = P_{ABP} - P_{BPAY} = P_{ABP} - 4rS_{PAQB}$. ∎

Lemma 4.7 Let Y be introduced by (TRATIO Y P Q r). Then we have

$$P_{AYB} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ}).$$

Proof. By Lemma 4.6,

$$P_{APY} = 4rS_{APQ}, P_{BPY} = 4rS_{BPQ}.$$

Then

$$P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY} = r^2 P_{PQP}.$$

Then $P_{AYB} = P_{APB} - P_{APY} - P_{BPY} + P_{YPY} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ})$. ∎

Now we consider how to eliminate points from the ratio of lengths.

Lemma 4.8 Let point Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{AUV}}{S_{CUDV}} & \text{if } A \text{ is not on } UV \\ \frac{S_{APQ}}{S_{CPDQ}} & \text{otherwise} \end{cases}$$

Proof. If A is not on UV, let S be a point such that $\overline{AS} = \overline{UV}$. By Propositions 2.4 and 2.8. $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{S_{AUV}}{S_{AUSV}} = \frac{S_{AUV}}{S_{CUDV}}$. ∎

Lemma 4.9 Let Y be introduced by (FOOT Y P U V). We assume $D \neq U$; otherwise interchange U and V.

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{P_{PCAD}}{P_{CDC}} & \text{if } A \in UV. \\ \frac{S_{AUV}}{S_{CUDV}} & \text{if } A \notin UV. \end{cases}$$

Proof. If $A \in UV$, let T be a point such that $\overline{AT} = \overline{CD}$. By Propositions 2.14 and 2.17

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AT}} = \frac{P_{PAT}}{P_{ATA}} = \frac{P_{PCAD}}{P_{CDC}}.$$

The second equation is a direct consequence of the co-side theorem. ∎

Lemma 4.10 Let point Y be introduced by construction (PRATIO $Y\ R\ P\ Q\ r$). Then we have

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in RY. \\ \frac{S_{APRQ}}{S_{CPDQ}} & \text{if } A \notin RY. \end{cases}$$

Proof. The first case is obvious. For the second case, take points T and S such that $\frac{\overline{RT}}{\overline{PQ}} = 1$ and $\frac{\overline{AS}}{\overline{CD}} = 1$. By the co-side theorem,

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{S_{ART}}{S_{ARST}} = \frac{S_{APRQ}}{S_{CPDQ}}. \quad \blacksquare$$

Lemma 4.11 Let Y be introduced by (TRATIO $Y\ P\ Q\ r$).

$$G = \frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{P_{APQ}}{P_{CPDQ}} & \text{if } A \notin PY. \\ \frac{S_{APQ}-\frac{r}{4}P_{PQP}}{S_{CPDQ}} & \text{if } A \in PY. \end{cases}$$

Proof. The first case is a direct consequence of Proposition 2.15. If $A \in PY$, then $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AP}}{\overline{CD}} - \frac{\overline{YP}}{\overline{CD}}$. By the co-side theorem,

$$\frac{\overline{AP}}{\overline{CD}} = \frac{S_{APQ}}{S_{CPDQ}}; \frac{\overline{YP}}{\overline{CD}} = \frac{S_{YPQ}}{S_{CPDQ}} = \frac{rP_{PQP}}{4S_{CPDQ}}.$$

Now the second result follows immediately. $\blacksquare$

4.2 Free Points and the Algorithm

For a geometry statement $S = (C_1, C_2, \ldots, C_k, (E, F))$, after eliminating all the nonfree points introduced by C_i from E and F using the lemmas in the preceding subsection, we obtain two rational expressions E' and F' in indeterminates, areas and Pythagoras differences of *free points*. These geometric quantities are generally not independent, e.g. for any four points A, B, C, D we have

$$S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}.$$

We thus need to reduce E' and F' to expressions in independent variables. To do that, we need the concept of area coordinates.

Definition 4.12 Let A, O, U, and V be four points such that O, U, and V are not collinear. The area coordinates of A with respect to OUV are

$$x_A = \frac{S_{OUA}}{S_{OUV}}, \quad y_A = \frac{S_{OAV}}{S_{OUV}}, \quad z_A = \frac{S_{AUV}}{S_{OUV}}.$$

It is clear that $x_A + y_A + z_A = 1$. Since x_A, y_A, and z_A are not independent, we also call x_A, y_A the area coordinates of Q with respect to OUV.

It is clear that the points in the plane are in a one to one correspondence with their area coordinates. To represent E and F as expressions in independent variables, we first introduce three new points O, U, V such that $UO \perp OV$. We will reduce E and F to expressions in the area coordinates of the free points with respect to to OUV.

Lemma 4.13 For any free points A, B, C, we have

1. $S_{ABC} = \frac{(S_{OVB}-S_{OVC})S_{OUA}+(S_{OVC}-S_{OVA})S_{OUB}+(S_{OVA}-S_{OVB})S_{OUC}}{S_{OUV}}$.
2. $P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AB}^2$.
3. $\overline{AB}^2 = \frac{\overline{OU}^2(S_{OVA}-S_{OVB})^2}{S_{OUV}^2} + \frac{\overline{OV}^2(S_{OUA}-S_{OUB})^2}{S_{OUV}^2}$.
4. $S_{OUV}^2 = \frac{\overline{OU}^2 \cdot \overline{OV}^2}{4}$.

Proof. For the proof of 1, see Case 15 of Algorithm ELIM in [18]. Case 2 is the definition of Pythagoras difference. For case 3, we introduce a new point M by construction (INTER M (PLINE A O U) (PLINE B O V)). Then by Proposition 2.12, $\overline{AB}^2 = \overline{AM}^2 + \overline{BM}^2$. By the second case of Lemma 4.10, $\frac{\overline{AM}}{\overline{OU}} = \frac{S_{AOBV}}{S_{OOUV}} = \frac{S_{AOV}-S_{BOV}}{S_{OUV}}$; $\frac{\overline{BM}}{\overline{OV}} = \frac{S_{AOU}-S_{BOU}}{S_{OUV}}$. We have proved 3. Case 4 is another basic fact taken for granted.

Using Lemma 4.13, E and F can be written as expressions in $\overline{OU}, \overline{OV}$, and the area coordinates of the free points. Since the area coordinates of free points are independent, $E = F$ iff E and F are literally the same.

Algorithm 4.14 (AREA)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a statement in **C**.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \cdots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the ndg conditions of C_i are satisfied. The ndg condition of a construction has three forms: $A \neq B$, $PQ \not\parallel UV$, or $PQ \not\perp UV$. For the first case, we check whether $P_{ABA} = 2\overline{AB}^2 = 0$. For the second case, we check whether $S_{PUV} = S_{QUV}$. For the third case, we check whether $P_{PUV} = P_{QUV}$. If a ndg condition of a geometry statement is not satisfied, the statement is *trivially true.* The algorithm terminates.

S3. Let $G_1, \cdots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \cdots, s$ do S4.

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using the lemmas in this section and replace G_j by H_j in E and F to obtain the new E and F.

S5. Now E and F are rational expressions in independent variables. Hence if $E = F$, S is true. Otherwise S is false.

Proof of the correctness. Only the last step needs explanation. If $E = F$, the statement is obviously true. Note that the ndg conditions ensure that the denominators of all the expressions occurring in the proof do not vanish.

Otherwise, since the geometric quantities in E and F are all free parameters, i.e., in the geometric configuration of S they can take arbitrary values. Since $E \neq F$, we can take some concrete values for these quantities such that when replacing these quantities by the corresponding values in E and F, we obtain two different numbers. In other words, we obtain a counter example for S. ∎

For the complexity of the algorithm, let n be the number of the non-free points in a statement which is described using constructions C1–C8. By the analysis in Section 3, we will use at most $5n$ constructions in the minimal set to represent the hypotheses (we need five minimal constructions to represent construction (INTER A (BLINE U V) (BLINE P Q))). Then we will use at most $5n$ minimal constructions to describe the statement. Notice that each lemma will replace a geometric quantity by a rational expression with degree less than or equal to three. Then if the conclusion of the geometry statement is of degree d, the output of our algorithm is at most degree $3^{5n}d$. In the last step, we need to represent the area and Pythagoras difference by area coordinates. In the worst case, a geometry quantity (Pythagoras difference) will be replaced by an expression of degree five. Thus the degree of the final polynomial is at most $5d3^{5n}$.

Example 4.15 Continue from Example 3.3. The *machine produced proof* (in Latex form) for Ceva's theorem is as follows. In the proof, $a \overset{P}{=} b$ means that b is the result obtained by eliminating point P from a; $a \overset{simplify}{=} b$ means that b is obtained by canceling some common factors from the denominator and numerator of a; "eliminants" are the results obtained by eliminating points from separate geometry quantities.

The machine proof

$$-\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}\cdot\frac{\overline{AF}}{\overline{BF}}$$

$$\overset{F}{=} \frac{-(-S_{ACP})}{-S_{BCP}}\cdot\frac{\overline{CE}}{\overline{AE}}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{E}{=} \frac{-S_{BCP}\cdot S_{ACP}}{S_{BCP}\cdot(-S_{ABP})}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{simplify}{=} \frac{S_{ACP}}{S_{ABP}}\cdot\frac{\overline{BD}}{\overline{CD}}$$

$$\overset{D}{=} \frac{S_{ABP}\cdot S_{ACP}}{S_{ABP}\cdot S_{ACP}} \overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AF}}{\overline{BF}} \overset{F}{=} \frac{S_{ACP}}{S_{BCP}}$$

$$\frac{\overline{CE}}{\overline{AE}} \overset{E}{=} \frac{S_{BCP}}{-S_{ABP}}$$

$$\frac{\overline{BD}}{\overline{CD}} \overset{D}{=} \frac{S_{ABP}}{S_{ACP}}$$

We use a sequence of consecutive equations to represent a machine proof. It is very easy to rewrite a proof in consecutive equations as the usual form. For instance, the proof of Ceva's theorem on page 11 can be obtained from the above machine proof easily.

5 Producing Short and Readable Proofs

We have presented a complete method for proving geometry statements in class **C** by considering a minimal set of constructions. But if only using those five constructions, we have to introduce many auxiliary points in the description of geometry statements. More points usually mean longer proofs. In this section we will introduce more constructions and more elimination techniques which will enable us to obtain shorter proofs.

5.1 Refined Elimination Techniques

Each lemma in Subsection 4.1 only gives the elimination result in the general case. In some special cases, the results are much more simple. For the construction FOOT, we have.

Proposition 5.1 Let point Y be introduced by construction (FOOT $Y\ P\ U\ V$). Then

$$S_{ABY} = \begin{cases} S_{ABU} & \text{if } AB \parallel UV; \\ S_{ABP} & \text{if } AB \perp UV; \\ \frac{S_{UBV}P_{PUAV}}{P_{UVU}} & \text{if } U, V, \text{ and } A \text{ are collinear;} \\ \frac{S_{AUV}S_{PUBV}}{S_{UVU}} & \text{if } U, V, \text{ and } B \text{ are collinear.} \end{cases}$$

$$P_{ABY} = \begin{cases} P_{ABP} & \text{if } AB \parallel UV; \\ P_{ABU} & \text{if } AB \perp UV; \\ \frac{P_{ABU}P_{PBU}}{P_{UBU}} & \text{if } U, V, \text{ and } B \text{ are collinear.} \end{cases}$$

$$P_{AYB} = \begin{cases} \frac{16S_{PUV}^2}{P_{UVU}} & \text{if } A = B = P; \\ \frac{P_{PUV}^2}{P_{UVU}} & \text{if } A = B = U; \\ \frac{P_{PVU}^2}{P_{UVU}} & \text{if } A = B = V; \\ \frac{-P_{PVU}P_{PUV}}{P_{UVU}} & \text{if } A = U, B = V. \end{cases}$$

The proof is omitted. We use this kind of refined elimination techniques for all constructions.

Example 5.2 Continue from Example 2.18. The following machine proof of the orthocenter theorem uses the above proposition.

Constructive description
((c POINTS $A\ B\ C$)
(c FOOT $E\ B\ A\ C$)
(c FOOT $F\ A\ B\ C$)
(c INTER H (c LINE $A\ F$) (c LINE $B\ E$))
(c PERPENDICULAR $A\ B\ C\ H$))

The machine proof

$$\frac{P_{ACH}}{P_{BCH}} \overset{H}{=} \frac{P_{ACB}}{P_{ACB}} \overset{simplify}{=} 1$$

The eliminants

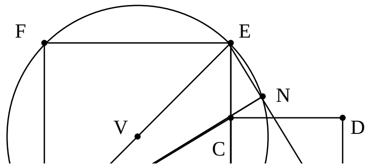
$P_{BCH} \overset{H}{=} P_{ACB}$
$P_{ACH} \overset{H}{=} P_{ACB}$

Since $BC \perp AH$ and $AC \perp BH$, by Proposition 5.1 $P_{BCH} = P_{BCA}$ and $P_{ACH} = P_{ACB}$.

Example 5.3 [2] Let M be a point on line AB. Two squares $AMCD$ and $BMEF$ are drawn on the same side of AB. Let U and V be the center of the squares $AMCD$ and $BMEF$. Line BC and circle VB meet in N. Show that A, E, and N are collinear.

[2]b This is a problem from the 1959 International Mathematical Olympiad.

Constructive description
((c POINTS A B)
(c ON M (c LINE A B))
(c TRATIO C M A 1) (c TRATIO E M B −1)
(c MIDPOINT V E B)
(c INTER N (c LINE B C) (c CIR V B))
(c INTER T (c LINE B C) (c LINE A E))
$(\frac{\overline{BN}}{\overline{CN}} = \frac{\overline{BT}}{\overline{CT}})$)
The ndg conditions:
$A \neq B$, $M \neq A$, $M \neq B$, $B \neq E$,
$B \neq C$, $V \neq B$, $BC \not\parallel AE$, $C \neq N$, and $C \neq T$.



The machine proof

$$(\frac{\overline{BN}}{\overline{CN}})/(\frac{\overline{BT}}{\overline{CT}})$$
$$\stackrel{T}{=} \frac{-S_{ACE}}{-S_{ABE}} \cdot \frac{\overline{BN}}{\overline{CN}}$$
$$\stackrel{N}{=} \frac{P_{CBV} \cdot S_{ACE}}{S_{ABE} \cdot (P_{CBV} - \frac{1}{2} P_{BCB})}$$
$$\stackrel{V}{=} \frac{(\frac{1}{2} P_{CBE}) \cdot S_{ACE}}{S_{ABE} \cdot (\frac{1}{2} P_{CBE} - \frac{1}{2} P_{BCB})}$$
$$\stackrel{E}{=} \frac{(P_{MBC} + 4 S_{BMC}) \cdot (\frac{1}{4} P_{MABC} - S_{AMC})}{(-\frac{1}{4} P_{ABM}) \cdot (P_{MBC} - P_{BCB} + 4 S_{BMC})}$$
$$\stackrel{C}{=} \frac{-(P_{BMB} - P_{AMB}) \cdot (P_{BMB} + P_{AMA} - P_{ABM})}{P_{ABM} \cdot (-P_{AMB} - P_{AMA})}$$
$$\stackrel{M}{=} \frac{(-P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}} + P_{ABA}) \cdot (2 P_{ABA} \cdot (\frac{\overline{AM}}{\overline{AB}})^2 - P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}})}{(-P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}} + P_{ABA}) \cdot (2 P_{ABA} \cdot (\frac{\overline{AM}}{\overline{AB}})^2 - P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}})}$$
$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BT}}{\overline{CT}} \stackrel{T}{=} \frac{S_{ABE}}{S_{ACE}}$$
$$\frac{\overline{BN}}{\overline{CN}} \stackrel{N}{=} \frac{P_{CBV}}{(\frac{1}{2}) \cdot (2 P_{CBV} - P_{BCB})}$$
$$P_{CBV} \stackrel{V}{=} \frac{1}{2}(P_{CBE})$$
$$S_{ABE} \stackrel{E}{=} -\frac{1}{4}(P_{ABM})$$
$$S_{ACE} \stackrel{E}{=} \frac{1}{4}(P_{MABC} - 4 S_{AMC})$$
$$P_{CBE} \stackrel{E}{=} P_{MBC} + 4 S_{BMC}$$
$$P_{BCB} \stackrel{C}{=} P_{BMB} + P_{AMA}$$
$$S_{AMC} \stackrel{C}{=} -\frac{1}{4}(P_{AMA})$$
$$P_{MABC} \stackrel{C}{=} P_{BMB} - P_{ABM}$$
$$S_{BMC} \stackrel{C}{=} -\frac{1}{4}(P_{AMB})$$
$$P_{MBC} \stackrel{C}{=} P_{BMB}$$
$$P_{ABM} \stackrel{M}{=} -((\frac{\overline{AM}}{\overline{AB}} - 1) \cdot P_{ABA})$$
$$P_{AMA} \stackrel{M}{=} P_{ABA} \cdot (\frac{\overline{AM}}{\overline{AB}})^2$$
$$P_{AMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}} - 1) \cdot P_{ABA} \cdot \frac{\overline{AM}}{\overline{AB}}$$
$$P_{BMB} \stackrel{M}{=} (\frac{\overline{AM}}{\overline{AB}} - 1)^2 \cdot P_{ABA}$$

In this example, we use several refined elimination techniques, such as $P_{MBC} \stackrel{C}{=} P_{BMB}$, $P_{AMA} \stackrel{M}{=} P_{ABA} \cdot (\frac{\overline{AM}}{\overline{AB}})^2$. The geometric meaning for them are very clear.

5.2 Co-Circle Points

We first introduce a new construction.

C9 (CIRCLE $A_1 \cdots A_s$), $(s \geq 3)$. Points $A_1 \cdots A_s$ are on the same circle. There is no ndg condition for this construction. The degree of freedom of all the points is $s + 3$.

Let $A_1 \cdots A_s$ be points on a circle with center O. We choose a point, say A_1, as the reference point. Then point A_i is uniquely determined by the oriented angle $\frac{\angle A_1 O A_i}{2}$ (we assume that all angles have values from $-\pi$ to π). We thus can also talk about oriented chords.

Lemma 5.4 Let A, B, C, D be points on a circle with center O and diameter δ, and A the reference point. We denote $\angle B$ to be $\frac{\angle AOB}{2}$. Then

$S_{BCD} = \frac{\overline{BC} \cdot \overline{CD} \cdot \overline{BD}}{2\delta}$.

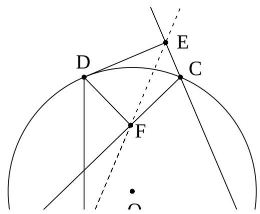
$P_{BCD} = 2\overline{BC} \cdot \overline{DC} \cos(\angle D - \angle B)$.

$\overline{BC} = \delta \sin(\angle C - \angle B)$.

This lemma can be proved using the sine and cosine laws. In this lemma, we actually use something more than the basic propositions in Section 2. Some simple properties of trigonometric functions are used. These properties can be developed using the basic propositions in Section 2 alone. See [2] for more details.

Using Lemma 5.4, an expression of areas and Pythagoras differences of points on a circle can be reduced to an expression of the diameter δ of the circle and trigonometric functions of independent angles. Two such expressions have the same value iff when substituting, for each angle α, $(\sin\alpha)^2$ by $1-(\cos\alpha)^2$ the resulting expression should be the same. We thus have a complete method for this construction. The reader may have noticed that this construction can only be the first construction in the description of the statement. Otherwise, in the next step, we do not know how to eliminate those trigonometric functions introduced in this step.

Example 5.5 (Simson's Theorem) Let D be a point on the circumscribed circle of triangle ABC. From D three perpendiculars are drawn to the three sides BC, AC, and AB of triangle ABC. Let E, F, and G be the three feet respectively. Show that E, F and G are collinear.



Here is the input to our program.

((b CIRCLE $A\ B\ C\ D$)
(FOOT $E\ D\ B\ C$)
(FOOT $F\ D\ A\ C$)
(FOOT $G\ D\ A\ B$)
(INTER G_1 (b LINE $E\ F$) (b LINE $A\ B$))
($\frac{\overline{AG}}{\overline{BG}} = \frac{\overline{AG_1}}{\overline{BG_1}}$))

The ndg conditions: $B \neq C$, $A \neq C$, $A \neq B$, $EF \not\parallel AB$, $B \neq G$, $B \neq G_1$.

Here is the machine proof. The last step of the proof ($\stackrel{co-cir}{=}$) uses Lemma 5.4 to eliminate the co-circle points.

The machine proof

$(\frac{\overline{AG}}{\overline{BG}})/(\frac{\overline{AG_1}}{\overline{BG_1}})$

$\stackrel{G_1}{=} \frac{S_{BEF}}{S_{AEF}} \cdot \frac{\overline{AG}}{\overline{BG}}$

$\stackrel{G}{=} \frac{P_{BAD} \cdot S_{BEF}}{S_{AEF} \cdot (-P_{ABD})}$

$\stackrel{F}{=} \frac{-P_{BAD} \cdot P_{ACD} \cdot S_{ABE} \cdot P_{ACA}}{(-P_{CAD} \cdot S_{ACE}) \cdot P_{ABD} \cdot P_{ACA}}$

$\stackrel{simplify}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot S_{ABE}}{P_{CAD} \cdot S_{ACE} \cdot P_{ABD}}$

The eliminants

$\frac{\overline{AG_1}}{\overline{BG_1}} \stackrel{G_1}{=} \frac{S_{AEF}}{S_{BEF}}$

$\frac{\overline{AG}}{\overline{BG}} \stackrel{G}{=} \frac{P_{BAD}}{-P_{ABD}}$

$S_{AEF} \stackrel{F}{=} \frac{-P_{CAD} \cdot S_{ACE}}{P_{ACA}}$

$S_{BEF} \stackrel{F}{=} \frac{P_{ACD} \cdot S_{ABE}}{P_{ACA}}$

$S_{ACE} \stackrel{E}{=} \frac{-P_{BCD} \cdot S_{ABC}}{P_{BCB}}$

$S_{ABE} \stackrel{E}{=} \frac{P_{CBD} \cdot S_{ABC}}{P_{BCB}}$

$P_{ABD} = -2(\widetilde{BD} \cdot \widetilde{AB} \cdot \cos(AD))$

$$\overset{E}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot P_{CBD} \cdot S_{ABC} \cdot P_{BCB}}{P_{CAD} \cdot (-P_{BCD} \cdot S_{ABC}) \cdot P_{ABD} \cdot P_{BCB}}$$

$$\overset{simplify}{=} \frac{P_{BAD} \cdot P_{ACD} \cdot P_{CBD}}{-P_{CAD} \cdot P_{BCD} \cdot P_{ABD}}$$

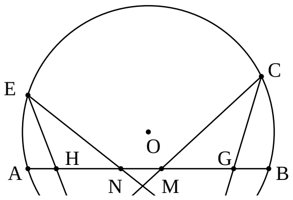
$$\overset{co-cir}{=} \frac{(2\widetilde{AD} \cdot \widetilde{AB} \cdot \cos(BD)) \cdot (-2\widetilde{CD} \cdot \widetilde{AC} \cdot \cos(AD)) \cdot (2\widetilde{BD} \cdot \widetilde{BC} \cdot \cos(CD))}{-(2\widetilde{AD} \cdot \widetilde{AC} \cdot \cos(CD)) \cdot (-2\widetilde{CD} \cdot \widetilde{BC} \cdot \cos(BD)) \cdot (-2\widetilde{BD} \cdot \widetilde{AB} \cdot \cos(AD))}$$

$$\overset{simplify}{=} 1$$

$$P_{BCD} = -2(\widetilde{CD} \cdot \widetilde{BC} \cdot \cos(BD))$$
$$P_{CAD} = 2(\widetilde{AD} \cdot \widetilde{AC} \cdot \cos(CD))$$
$$P_{CBD} = 2(\widetilde{BD} \cdot \widetilde{BC} \cdot \cos(CD))$$
$$P_{ACD} = -2(\widetilde{CD} \cdot \widetilde{AC} \cdot \cos(AD))$$
$$P_{BAD} = 2(\widetilde{AD} \cdot \widetilde{AB} \cdot \cos(BD))$$

Example 5.6 (The General Butterfly Theorem.) As in the figure, A, B, C, D, E, F are six points on a circle. $M = AB \cap CD; N = AB \cap EF; G = AB \cap CF; H = AB \cap DE$. Show that $\frac{\overline{MG}}{\overline{AG}} \frac{\overline{BH}}{\overline{NH}} \frac{\overline{AN}}{\overline{MB}} = 1$.

Constructive description
((b CIRCLE $A\ B\ C\ D\ E\ F$)
(b INTER M (b LINE $D\ C$) (b LINE $A\ B$))
(b INTER N (b LINE $E\ F$) (b LINE $A\ B$))
(b INTER G (b LINE $A\ B$) (b LINE $C\ F$))
(b INTER H (b LINE $D\ E$) (b LINE $A\ B$))
($\frac{\overline{MG}}{\overline{AG}} \frac{\overline{BH}}{\overline{NH}} = \frac{\overline{BM}}{\overline{AB}} \frac{\overline{BA}}{\overline{AN}}$))



The machine proof

$$\frac{\frac{\overline{MG}}{\overline{AG}} \cdot \frac{\overline{BH}}{\overline{NH}}}{-\frac{\overline{BM}}{\overline{AB}} \cdot \frac{\overline{AB}}{\overline{AN}}}$$

$$\overset{H}{=} \frac{S_{BDE}}{-\frac{\overline{BM}}{\overline{AB}} \cdot \frac{\overline{AB}}{\overline{AN}} \cdot S_{DEN}} \cdot \frac{\overline{MG}}{\overline{AG}}$$

$$\overset{G}{=} \frac{-S_{CFM} \cdot S_{BDE}}{\frac{\overline{BM}}{\overline{AB}} \cdot \frac{\overline{AB}}{\overline{AN}} \cdot S_{DEN} \cdot S_{ACF}}$$

$$\overset{N}{=} \frac{-S_{CFM} \cdot S_{BDE} \cdot (-S_{AEBF}) \cdot S_{AEF}}{\frac{\overline{BM}}{\overline{AB}} \cdot S_{AEBF} \cdot S_{DEF} \cdot S_{ABE} \cdot S_{ACF}}$$

$$\overset{simplify}{=} \frac{S_{CFM} \cdot S_{BDE} \cdot S_{AEF}}{\frac{\overline{BM}}{\overline{AB}} \cdot S_{DEF} \cdot S_{ABE} \cdot S_{ACF}}$$

$$\overset{M}{=} \frac{(-S_{CDF} \cdot S_{ABC}) \cdot S_{BDE} \cdot S_{AEF} \cdot (-S_{ACBD})}{(-S_{BCD}) \cdot S_{DEF} \cdot S_{ABE} \cdot S_{ACF} \cdot (-S_{ACBD})}$$

$$\overset{simplify}{=} \frac{S_{CDF} \cdot S_{ABC} \cdot S_{BDE} \cdot S_{AEF}}{S_{BCD} \cdot S_{DEF} \cdot S_{ABE} \cdot S_{ACF}}$$

$$\overset{co-cir}{=} \frac{(-\widetilde{DF} \cdot \widetilde{CF} \cdot \widetilde{CD}) \cdot (-\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}) \cdot (-\widetilde{DE} \cdot \widetilde{BE} \cdot \widetilde{BD}) \cdot (-\widetilde{EF} \cdot \widetilde{AF} \cdot \widetilde{AE}) \cdot ((2d))^4}{(-\widetilde{CD} \cdot \widetilde{BD} \cdot \widetilde{BC}) \cdot (-\widetilde{EF} \cdot \widetilde{DF} \cdot \widetilde{DE}) \cdot (-\widetilde{BE} \cdot \widetilde{AE} \cdot \widetilde{AB}) \cdot (-\widetilde{CF} \cdot \widetilde{AF} \cdot \widetilde{AC}) \cdot ((2d))^4}$$

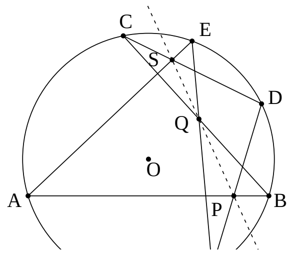
$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{BH}}{\overline{NH}} \overset{H}{=} \frac{S_{BDE}}{S_{DEN}}$$
$$\frac{\overline{MG}}{\overline{AG}} \overset{G}{=} \frac{S_{CFM}}{S_{ACF}}$$
$$S_{DEN} \overset{N}{=} \frac{S_{DEF} \cdot S_{ABE}}{-S_{AEBF}}$$
$$\frac{\overline{AB}}{\overline{AN}} \overset{N}{=} \frac{S_{AEBF}}{S_{AEF}}$$
$$\frac{\overline{BM}}{\overline{AB}} \overset{M}{=} \frac{S_{BCD}}{S_{ACBD}}$$
$$S_{CFM} \overset{M}{=} \frac{S_{CDF} \cdot S_{ABC}}{S_{ACBD}}$$
$$S_{ACF} = \frac{\widetilde{CF} \cdot \widetilde{AF} \cdot \widetilde{AC}}{(-2) \cdot d}$$
$$S_{ABE} = \frac{\widetilde{BE} \cdot \widetilde{AE} \cdot \widetilde{AB}}{(-2) \cdot d}$$
$$S_{DEF} = \frac{\widetilde{EF} \cdot \widetilde{DF} \cdot \widetilde{DE}}{(-2) \cdot d}$$
$$S_{BCD} = \frac{\widetilde{CD} \cdot \widetilde{BD} \cdot \widetilde{BC}}{(-2) \cdot d}$$
$$S_{AEF} = \frac{\widetilde{EF} \cdot \widetilde{AF} \cdot \widetilde{AE}}{(-2) \cdot d}$$
$$S_{BDE} = \frac{\widetilde{DE} \cdot \widetilde{BE} \cdot \widetilde{BD}}{(-2) \cdot d}$$
$$S_{ABC} = \frac{\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}}{(-2) \cdot d}$$
$$S_{CDF} = \frac{\widetilde{DF} \cdot \widetilde{CF} \cdot \widetilde{CD}}{(-2) \cdot d}$$

Example 5.7 (Pascal's Theorem on a Circle) Let A, B, C, D, E, and F be six points on a circle. Let $P = AB \cap DF$, $Q = BC \cap EF$, and $S = CD \cap EA$. Show that P, Q, and S are collinear.

Here is the input to the program.



((b CIRCLE $A\ B\ C\ D\ F\ E$)
(b INTER P (b LINE $D\ F$) (b LINE $A\ B$))
(b INTER Q (b LINE $F\ E$) (b LINE $B\ C$))
(b INTER S (b LINE $E\ A$) (b LINE $C\ D$))
(b INTER S_1 (b LINE $P\ Q$) (b LINE $C\ D$))
($\frac{\overline{CS}}{\overline{DS}} = \frac{\overline{CS_1}}{\overline{DS_1}}$))

The ndg conditions:
$DF \not\parallel AB$, $EF \not\parallel BC$, $AE \not\parallel CD$, $PQ \not\parallel CD$, $D \neq S$, $D \neq S_1$.

The machine proof

$$(\frac{\overline{CS}}{\overline{DS}})/(\frac{\overline{CS_1}}{\overline{DS_1}})$$

$$\stackrel{S_1}{=} \frac{S_{DPQ}}{S_{CPQ}} \cdot \frac{\overline{CS}}{\overline{DS}}$$

$$\stackrel{S}{=} \frac{S_{ACE} \cdot S_{DPQ}}{S_{CPQ} \cdot S_{ADE}}$$

$$\stackrel{Q}{=} \frac{S_{ACE} \cdot (-S_{DEP} \cdot S_{BCF}) \cdot S_{BFCE}}{(-S_{CFE} \cdot S_{BCP}) \cdot S_{ADE} \cdot (-S_{BFCE})}$$

$$\stackrel{simplify}{=} \frac{-S_{ACE} \cdot S_{DEP} \cdot S_{BCF}}{S_{CFE} \cdot S_{BCP} \cdot S_{ADE}}$$

$$\stackrel{P}{=} \frac{-S_{ACE} \cdot (-S_{DFE} \cdot S_{ABD}) \cdot S_{BCF} \cdot S_{ADBF}}{S_{CFE} \cdot (-S_{BDF} \cdot S_{ABC}) \cdot S_{ADE} \cdot (-S_{ADBF})}$$

$$\stackrel{simplify}{=} \frac{S_{ACE} \cdot S_{DFE} \cdot S_{ABD} \cdot S_{BCF}}{S_{CFE} \cdot S_{BDF} \cdot S_{ABC} \cdot S_{ADE}}$$

$$\stackrel{co-cir}{=} \frac{(-\widetilde{CE} \cdot \widetilde{AE} \cdot \widetilde{AC}) \cdot (-\widetilde{FE} \cdot \widetilde{DE} \cdot \widetilde{DF}) \cdot (-\widetilde{BD} \cdot \widetilde{AD} \cdot \widetilde{AB}) \cdot (-\widetilde{CF} \cdot \widetilde{BF} \cdot \widetilde{BC}) \cdot ((2d))^4}{(-\widetilde{FE} \cdot \widetilde{CE} \cdot \widetilde{CF}) \cdot (-\widetilde{DF} \cdot \widetilde{BF} \cdot \widetilde{BD}) \cdot (-\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}) \cdot (-\widetilde{DE} \cdot \widetilde{AE} \cdot \widetilde{AD}) \cdot ((2d))^4}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$\frac{\overline{CS_1}}{\overline{DS_1}} \stackrel{S_1}{=} \frac{S_{CPQ}}{S_{DPQ}}$$

$$\frac{\overline{CS}}{\overline{DS}} \stackrel{S}{=} \frac{S_{ACE}}{S_{ADE}}$$

$$S_{CPQ} \stackrel{Q}{=} \frac{-S_{CFE} \cdot S_{BCP}}{S_{BFCE}}$$

$$S_{DPQ} \stackrel{Q}{=} \frac{S_{DEP} \cdot S_{BCF}}{S_{BFCE}}$$

$$S_{BCP} \stackrel{P}{=} \frac{-S_{BDF} \cdot S_{ABC}}{S_{ADBF}}$$

$$S_{DEP} \stackrel{P}{=} \frac{S_{DFE} \cdot S_{ABD}}{S_{ADBF}}$$

$$S_{ADE} = \frac{\widetilde{DE} \cdot \widetilde{AE} \cdot \widetilde{AD}}{(-2) \cdot d}$$

$$S_{ABC} = \frac{\widetilde{BC} \cdot \widetilde{AC} \cdot \widetilde{AB}}{(-2) \cdot d}$$

$$S_{BDF} = \frac{\widetilde{DF} \cdot \widetilde{BF} \cdot \widetilde{BD}}{(-2) \cdot d}$$

$$S_{CFE} = \frac{\widetilde{FE} \cdot \widetilde{CE} \cdot \widetilde{CF}}{(-2) \cdot d}$$

$$S_{BCF} = \frac{\widetilde{CF} \cdot \widetilde{BF} \cdot \widetilde{BC}}{(-2) \cdot d}$$

$$S_{ABD} = \frac{\widetilde{BD} \cdot \widetilde{AD} \cdot \widetilde{AB}}{(-2) \cdot d}$$

$$S_{DFE} = \frac{\widetilde{FE} \cdot \widetilde{DE} \cdot \widetilde{DF}}{(-2) \cdot d}$$

$$S_{ACE} = \frac{\widetilde{CE} \cdot \widetilde{AE} \cdot \widetilde{AC}}{(-2) \cdot d}$$

5.3 Area Coordinates and Special Points of Triangles

We introduce a new construction.

C9 (ARATIO $A\ O\ U\ V\ r_O\ r_U\ r_V$). Take a point A such that

$$r_O = \frac{S_{AUV}}{S_{OUV}}, \quad r_U = \frac{S_{OAV}}{S_{OUV}}, \quad r_V = \frac{S_{OUA}}{S_{OUV}}$$

are the *area coordinates* of A with respect to OUV. The r_O, r_U, and r_V could be rational numbers, rational expressions in geometric quantities, and indeterminates. The ndg condition is that O, U, and V are not collinear. The degree of freedom for A is dependent on the number of indeterminates in $\{r_O, r_U, r_V\}$.

Lemma 5.8 Let $G(Y)$ be a linear geometry quantity and Y be introduced by (ARATIO Y O U V r_O r_U r_V). Then

$$G(Y) = r_O G(O) + r_U G(U) + r_V G(V).$$

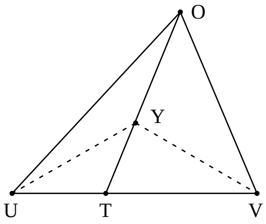


Figure 15

Proof. Without loss of generality, let OY intersect UV at T. If OY is parallel to UV, we may consider the intersection of UY and OV or the intersection of VY and OU since one of them must exist. By Proposition 2.5,

$$G(Y) = \frac{\overline{OY}}{\overline{OT}} G(T) + \frac{\overline{YT}}{\overline{OT}} G(O) = \frac{\overline{OY}}{\overline{OT}} \left(\frac{\overline{UT}}{\overline{UV}} G(V) + \frac{\overline{TV}}{\overline{UV}} G(U)\right) + \frac{\overline{YT}}{\overline{OT}} G(O).$$

By the co-side theorem, $\frac{\overline{YT}}{\overline{OT}} = r_O$; $\frac{\overline{OY}}{\overline{OT}} = \frac{S_{OUYV}}{S_{OUV}}$; $\frac{\overline{UT}}{\overline{UV}} = \frac{S_{OUY}}{S_{OUYV}}$; $\frac{\overline{TV}}{\overline{UV}} = \frac{S_{OYV}}{S_{OUYV}}$. Substituting these into the above formula, we obtain the desired result. ∎

Lemma 5.9 Let $G(Y) = P_{AYB}$ and Y be introduced by (ARATIO Y O U V r_O r_U r_V). Then

$$G(Y) = r_O G(O) + r_U G(U) + r_V G(V) - 2(r_O r_U \overline{OU}^2 + r_O r_V \overline{OV}^2 + r_U r_V \overline{UV}^2).$$

Proof. Continue from the proof of Lemma 5.8, By (II) on page 12

$$G(Y) = \frac{\overline{OY}}{\overline{OT}} G(T) + \frac{\overline{YT}}{\overline{OT}} G(O) - \frac{\overline{OY}}{\overline{OT}}\frac{\overline{YT}}{\overline{OT}} P_{OTO}$$

$$G(T) = \frac{\overline{UT}}{\overline{UV}} G(V) + \frac{\overline{TV}}{\overline{UV}} G(U) - \frac{\overline{UT}}{\overline{UV}}\frac{\overline{TV}}{\overline{UV}} P_{UVU}.$$

Substituting $G(T)$ into $G(Y)$, we have

$$G(Y) - r = -\frac{\overline{OY}}{\overline{OT}}\frac{\overline{UT}}{\overline{UV}}\frac{\overline{TV}}{\overline{UV}} P_{UVU} - \frac{\overline{OY}}{\overline{OT}}\frac{\overline{YT}}{\overline{OT}} P_{OTO} = -r_V \frac{\overline{TV}}{\overline{UV}} P_{UVU} - r_A \frac{\overline{OY}}{\overline{OT}} P_{OTO},$$

where $r = r_O G(O) + r_U G(U) + r_V G(V)$. By (II),

$$P_{OTO} = \frac{\overline{UT}}{\overline{UV}} P_{OVO} + \frac{\overline{TV}}{\overline{UV}} P_{OUO} - \frac{\overline{UT}}{\overline{UV}}\frac{\overline{TV}}{\overline{UV}} P_{UVU}.$$

Then

$$\begin{aligned} & G(Y) - r \\ = \; & -r_V \frac{\overline{TV}}{\overline{UV}} P_{UVU} - r_O \frac{\overline{OY}}{\overline{OT}}\frac{\overline{UT}}{\overline{UV}} P_{OVO} - r_O \frac{\overline{OY}}{\overline{OT}}\frac{\overline{TV}}{\overline{UV}} P_{OUO} + r_O \frac{\overline{OY}}{\overline{OT}}\frac{\overline{UT}}{\overline{UV}}\frac{\overline{TV}}{\overline{UV}} P_{UVU} \\ = \; & -r_O r_V P_{OVO} - r_O r_U P_{OUO} - r_U r_V \left(-\frac{S_{YUV}}{S_{OUYV}} + \frac{S_{OUV}}{S_{OUYV}}\right) P_{UVU} \\ = \; & -r_O r_V P_{OVO} - r_O r_U P_{OUO} - r_U r_V P_{UVU}. \; ∎ \end{aligned}$$

If Y is introduced by construction ARATIO and we need to eliminate Y from $G = \frac{\overline{AY}}{\overline{CD}}$. One of O, U, and V, say O, satisfies the condition that A, Y, and O are not collinear. Then $G = \frac{S_{OAY}}{S_{OCAD}}$. Now, we can use Lemma 5.8 to eliminate Y.

By using the construction ARATIO, we can treat the following often used constructions easily.

- (CENTROID $G\ A\ B\ C$). G is the centroid of triangle ABC. This is equivalent to

$$(\text{ARATIO } G\ A\ B\ C\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$$

  The ndg condition is that A, B, C are not collinear.

- (ORTHOCENTER $H\ A\ B\ C$). H is the orthocenter of the triangle ABC. This is equivalent to.

$$(\text{ARATIO } H\ A\ B\ C\ \frac{P_{ABC}P_{ACB}}{16S_{ABC}^2}\qquad \frac{P_{BAC}P_{BCA}}{16S_{ABC}^2}\qquad \frac{P_{CAB}P_{CBA}}{16S_{ABC}^2})$$

  The ndg condition is that A, B, C are not collinear.

- (CIRCUMCENTER $O\ A\ B\ C$). O is the circumcenter of triangle ABC. This is equivalent to

$$(\text{ARATIO } O\ A\ B\ C\ \frac{P_{BCB}P_{BAC}}{32S_{ABC}^2}\qquad \frac{P_{ACA}P_{ABC}}{32S_{ABC}^2}\qquad \frac{P_{ABA}P_{ACB}}{32S_{ABC}^2})$$

  The ndg condition is that A, B, C are not collinear.

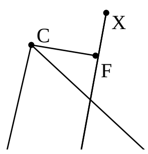
- (INCENTER $C\ I\ A\ B$) I is the center of the inscribed circle of triangle ABC. This construction is to construct point C from points I, A, and B. This is equivalent to

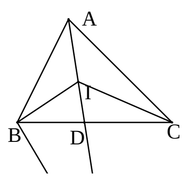
$$(\text{ARATIO } C\ I\ A\ B\ -\frac{2P_{IAB}P_{IBA}}{P_{AIB}P_{ABA}}\qquad \frac{P_{IAB}P_{IBI}}{P_{AIB}P_{ABA}}\qquad \frac{P_{IBA}P_{IAI}}{P_{AIB}P_{ABA}})$$

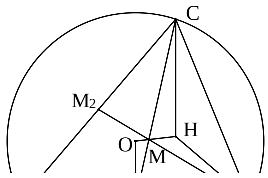
  The ndg conditions are $A \neq B$ and IA is not perpendicular to IB.

The reader may check these results by direct calculation or just treat them as basic propositions. The construction INCENTER needs some explanation. If three vertices of a triangle are given and we need to find the coordinates of the incenter, we generally have an equation of degree four. The reason is that we can not distinguish the incenter and the three excenters without using inequalities. What we do here is to reverse the problem: when an incenter or an excenter and two vertices of a triangle are given the third vertex is uniquely determined and can be constructed using the constructions in this paper.

Remark 5.10 In this section, we actually use the centroid theorem, the orthocenter theorem (Example 2.18), the circumcenter theorem, and the incenter theorem, in the proof of more complicated theorems. The four theorems themselves can be proved using the basic propositions.







Example 5.11 A line is drawn through the centroid of a triangle. Show that the sum of the distances of the line from the two vertices of the triangle situated on the same side of the line is equal to the distance of the line from the third vertex. (Figure 16)

Constructive description

((b POINTS $A\ B\ C\ X$)
(b CENTROID $G\ A\ B\ C$)
(b FOOT $D\ A\ G\ X$)
(b FOOT $E\ B\ G\ X$)
(b FOOT $F\ C\ G\ X$)
($\frac{\overline{EB}}{\overline{DA}} + \frac{\overline{FC}}{\overline{DA}} = -1$))

The machine proof

$$-\left(\frac{\overline{CF}}{\overline{AD}} + \frac{\overline{BE}}{\overline{AD}}\right)$$

$$\overset{F}{=} \frac{-\frac{\overline{BE}}{\overline{AD}} \cdot S_{AXG} - S_{CXG}}{-(-S_{AXG})}$$

$$\overset{E}{=} \frac{-(-S_{CXG} \cdot S_{AXG} - S_{BXG} \cdot S_{AXG})}{S_{AXG} \cdot (-S_{AXG})}$$

$$\overset{simplify}{=} \frac{-(S_{CXG} + S_{BXG})}{S_{AXG}}$$

$$\overset{G}{=} \frac{-(3S_{ACX} + 3S_{ABX}) \cdot (3)}{(-S_{ACX} - S_{ABX}) \cdot ((3))^2}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{CF}}{\overline{AD}} \overset{F}{=} \frac{S_{CXG}}{S_{AXG}}$$

$$\frac{\overline{BE}}{\overline{AD}} \overset{E}{=} \frac{S_{BXG}}{S_{AXG}}$$

$$S_{AXG} \overset{G}{=} -\frac{1}{3}(S_{ACX} + S_{ABX})$$

$$S_{BXG} \overset{G}{=} -\frac{1}{3}(S_{BCX} - S_{ABX})$$

$$S_{CXG} \overset{G}{=} \frac{1}{3}(S_{BCX} + S_{ACX})$$

The results $\frac{\overline{CF}}{\overline{AD}} \overset{F}{=} \frac{S_{CXG}}{S_{AXG}}$ and $\frac{\overline{BE}}{\overline{AD}} \overset{E}{=} \frac{S_{BXG}}{S_{AXG}}$ are obtained by refined elimination techniques.

Example 5.12 Two tritangent centers divide the bisector on which they are located, harmonically (Figure 17).

Constructive description

((b POINTS $B\ C\ I$)
(b INCENTER $A\ I\ C\ B$)
(b INTER D (b LINE $A\ I$) (b LINE $B\ C$))
(b INTER I_A (b LINE $A\ I$) (b TLINE $B\ B\ I$))
(b HARMONIC $A\ D\ I\ I_A$))

The machine proof

$$\left(-\frac{\overline{IA}}{\overline{ID}}\right) / \left(\frac{\overline{AI_A}}{\overline{DI_A}}\right)$$

$$\overset{I_A}{=} \frac{P_{IBD}}{P_{IBA}} \cdot -\frac{\overline{IA}}{\overline{ID}}$$

$$\overset{D}{=} \frac{(-S_{BICA}) \cdot P_{CBI} \cdot S_{BIA}}{P_{IBA} \cdot S_{BICA} \cdot S_{BCI}}$$

$$\overset{simplify}{=} \frac{P_{CBI} \cdot S_{BIA}}{P_{IBA} \cdot S_{BCI}}$$

$$\overset{A}{=} \frac{P_{CBI} \cdot (-P_{BIB} \cdot P_{BCI} \cdot S_{BCI}) \cdot P_{BIC} \cdot P_{BCB}}{(-P_{CBI} \cdot P_{BIB} \cdot P_{BCI}) \cdot S_{BCI} \cdot P_{BIC} \cdot P_{BCB}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AI_A}}{\overline{DI_A}} \overset{I_A}{=} \frac{P_{IBA}}{P_{IBD}}$$

$$P_{IBD} \overset{D}{=} \frac{P_{CBI} \cdot S_{BIA}}{S_{BICA}}$$

$$\frac{\overline{IA}}{\overline{ID}} \overset{D}{=} \frac{-S_{BICA}}{S_{BCI}}$$

$$P_{IBA} \overset{A}{=} \frac{-P_{CBI} \cdot P_{BIB} \cdot P_{BCI}}{P_{BIC} \cdot P_{BCB}}$$

$$S_{BIA} \overset{A}{=} \frac{-P_{BIB} \cdot P_{BCI} \cdot S_{BCI}}{P_{BIC} \cdot P_{BCB}}$$

The last tow eliminants are obtained by refined elimination techniques.

Example 5.13 (Euler's Theorem) The centroid of a triangle is on the segment determined by the circumcenter O and the orthocenter H of the same triangle and divides OH in the ratio of 1:2 (Figure 18).

Constructive description

((b POINTS $A\ B\ C$)
(b CIRCUMCENTER $O\ A\ B\ C$)
(b CENTROID $M\ A\ B\ C$)
(b LRATIO $H\ M\ O\ -2$)
(b PERPENDICULAR $A\ H\ B\ C$))

The machine proof

$$\frac{P_{ABC}}{P_{CBH}}$$

$$\overset{H}{=} \frac{P_{ABC}}{3P_{CBM} - 2P_{CBO}}$$

$$\overset{M}{=} \frac{P_{ABC} \cdot (3)}{-6P_{CBO} + 3P_{BCB} + 3P_{ABC}}$$

$$\overset{O}{=} \frac{-P_{ABC} \cdot (2)}{-2P_{ABC}}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$P_{CBH} \overset{H}{=} 3P_{CBM} - 2P_{CBO}$$

$$P_{CBM} \overset{M}{=} \frac{1}{3}(P_{BCB} + P_{ABC})$$

$$P_{CBO} \overset{O}{=} \frac{1}{2}(P_{BCB})$$

$p_{CBO} \overset{O}{=} \frac{1}{2}(p_{BCB})$ is obtained by refined elimination techniques.

6 Conclusion Remarks

We have implemented the algorithm using Common Lisp (AKCL) on a NeXT workstation. [5] is a collection of 400 geometry theorems proved by our prover and machine proofs of 100 theorems. The following tables contain some timing and proof length statistics about the examples in this paper and the 400 theorems. Maxterm means the number of terms of the maximal polynomial occurring in a proof.

Examples	2.10	2.18	5.3	5.5	5.7	5.6	5.11	5.12	5.13
Time (secs)	0.06	0.01	0.750	0.03	0.05	0.03	0.067	0.033	0.01
Maxterm	1	1	3	1	1	1	2	1	3

Table 1. The examples in this paper

The Length of the Proofs		The Proving Time	
Maxterm	No. of Theorems	Time (secs)	No. of Theorems
$m = 1$	61	$t \leq 0.1$	117
$m = 2$	53	$0.1 < t \leq 0.5$	120
$2 < m \leq 5$	136	$0.5 < t \leq 1$	58
$5 < m \leq 10$	57	$1 < t \leq 2$	47
$10 < m \leq 20$	45	$2 < t \leq 5$	36
$20 < m \leq 94$	48	$5 < t \leq 31$	22

Table 2. Statistics for the 400 theorems

From Table 2, we can see that our program is very fast and can produce short proofs for many difficult geometry theorems. If we set a standard that a short proof means the maxterm in the proof is less than or equal to 10. Then 76.7 percent (or 307) of the proofs of the 400 theorems produced by our prover are short and can be considered readable.

There are still many problems not solved or unsolved satisfactorily for this approach. Though a large portion of the geometry theorems in text book of high school or college geometry can be proved by our prover, there are still equational theorems which are not in class **C**, e.g., theorems which can not be described constructively. These will be our further research topics. From Table 2, the proofs produced by our prover for many theorems are still too long. Therefore we need more elimination techniques to obtain shorter proofs.

Comparing to the algebraic methods [14, 1, 9, 11], the area method uses geometry invariants like areas and Pythagoras differences as basic geometry quantities and the proofs produced by the area method are generally short and readable. On the other hand, the scope of the algebraic methods are larger than that of our current method.

Comparing to the synthetic approach, the area method is much more efficient and is complete for a class of geometry statements. Also the area method is of diagram independent. We choose the basic propositions according to the standard that most theorems can be deduced from them easily instead of the usual standard of independence and simplicity. Most of the synthetic approaches use the properties of congruent triangles as their basic propositions. The difficulty of using congruent triangles is that rarely are there congruent triangles in the diagram of a geometry statement and there exists no automated method of adding auxiliary lines to obtain congruent triangles. So it is difficult for the approach based on congruent triangles to be complete for a specific class of geometry statements.

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