Complete Solution Classification for the Perspective-Three-Point Problem [1]

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Abstract

In this paper, we use two approaches to solve the Perspective-Three-Point (P3P) problem: the algebraic approach and the geometric approach. In the algebraic approach, we use Wu-Ritt's zero decomposition algorithm to give a complete triangular decomposition for the P3P equation system. This decomposition provides the first complete analytical solution to the $P3P$ problem. We also give a complete solution classification for the P3P equation system, i.e., we give explicit criteria for the P3P problem to have one, two, three, and four solutions. Combining the analytical solutions with the criteria, we provide an algorithm, $CASSC$, which may be used to find complete and robust numerical solutions to the P3P problem. In the geometric approach, we give some pure geometric criteria for the number of real physical solutions.

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Key Words: Perspective-Three-Point problem, pose determination, analytical solutions, solution classification, geometric criteria, Wu-Ritt's zero decomposition method.

1 Introduction

The Perspective-n-Point (PnP) problem is originated from camera calibration [1, 2, 3, 4]. Also known as pose estimation, it is to determine the position and orientation of the camera with respect to a scene object from n correspondent points. It concerns many important fields, such as computer animation [5], computer vision [3], automation, image analysis and automated cartography [2], photogrammetry [6], robotics [1] and model based machine vision system [7], etc. Fischler and Bolles [2] summarized the problem as follows:

> " Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (C_P), find the lengths of the line segments joining C_P to each of the control points."

The study of the PnP problem mainly consists of two aspects:

1. Design fast and stable algorithms that can be used to find all or some of the solutions of the PnP problem.
2. Give a classification for the solutions of the PnP problem, i.e., give the conditions under which the problem has one, two, three or four solutions.

There are many results for the first problem and the second problem is still open. The aim of this paper is to give a complete and effective solution to the above two problems for the P3P problem.

The P3P problem is the smallest subset of control points that yields a finite number of solutions. In 1981, Fischler and Bolles [2] presented the RANSAC algorithm. They have noticed that there are at most four possible solutions to the P3P equation system. Hung et al [8] presented an algorithm for computing the 3D coordinates of the perspective center relative to the camera frame. In 1991, Haralick *et al* [9] reviewed the major direct solutions up to 1991, including six algorithms given by Grunert(1841), Finsterwalder(1903), Merritt(1949), Fischler and Bolles(1981), Linnainmaa *et al*(1988) and Grafarrend *et al*(1989), respectively. They also give the analytical solution for the P3P problem with resultant computation. DeMenthon *et al* [10, 11] showed that by using approximations to the perspective, simpler computational solutions can be obtained. Quan and Lan [4] reduced the problem to a new quartic equation with Sylvester resultant and proposed a linear algebra algorithm to solve the PnP problem.

One of the important research directions on the $P3P$ problem is its multi-solution phenomenon. Fischler and Bolles [2] presented some examples of multi-solutions of the P3P problem. In 1986, Wolfe et al [12] pointed out that the six permutations of the three control points combined with four-solution possibility can produce 24 possible camera-triangle configurations consistent with a single perspective view [6, 7]. Yuan [6] gave a necessary condition for the existence of the solution for first time. In 1991, Wolfe *et al* [7] gave a geometric explanation to this multi-solution phenomenon in the image plane under the assumption of "canonical view."

In 1997, Su *et al* [5] applied Wu-Ritt's zero decomposition method to find the main solution branch and some non-degenerate branches for the P3P problem. But a complete decomposition was not given. In [13], they used the Sturm sequence to give some conditions to adjudicate the number of solutions. In 1998, Yang [14] gave partial solution classifications of the P3P problem under some non-degenerate conditions.

The P3P problem is the most basic case of the PnP problems. All other PnP ($n > 3$) problems include the P3P problem as a special case. Therefore, a complete study of this problem is desirable. This paper is an effort toward this goal. We use two approaches to solve the P3P problem: the algebraic approach and the geometric approach. In the algebraic approach, we apply Wu-Ritt's zero decomposition algorithm [15, 16, 17] to find a complete zero decomposition for the P3P equation system. This decomposition provides the first complete analytical solution to the P3P problem. Based on this decomposition, we give a complete solution classification to the P3P equation system for the first time, i.e., we give explicit criteria for the P3P problem to have one, two, three, or four solutions. The procedure of obtaining this classification consists of the most difficult part of this paper. With these criteria, we introduce the concept of stable and critical values for the input parametric values. If a set of values is a stable, then a small variation of these values will give the same number of solutions. Therefore, for a set of stable values, we may use the usual floating-point calculations to enhance the computation speed; and for a set of critical values, we may use high precision computation tools [18] to provide more robust computation.

Combining the analytical solutions with the criteria, we provide an algorithm, *CASSC* (Complete Analytical Solution with the assistance of Solution Classification), which can be used to find complete and robust numerical solutions to the P3P problem. Our experimental results support this assertion.

In the geometric approach, we consider the three perspective angles separately. Then the locus of the center of perspective point for each angle is a toroid and the center of perspective is the intersection of three toroids. In this way, we give some pure geometric criteria for the number of solutions of the P3P problem. One interesting result is "The P3P problem can have only one solution if the three perspective angles are obtuse." This kind of criteria is much simpler than the algebraic one and gives some insight into the multi-solution phenomenon. On the other hand, since the field of view of most cameras is much less than 90 degrees, this result does not have much practical value. In any case, to find the solutions we must use the algebraic computation approach.

The rest of the paper is organized as follows. In Section 2, we present the zero decomposition for the P3P equation system. In Section 3, we present the solution classification. In Section 4, we present the CASSC algorithm and the experimental results. In Section 5, we present the geometric approach. In Section 6, we present the conclusions.

2 Zero Structure for the P3P Equation System

2.1 Simplification of the Equation System

Let P be the Center of Perspective, and A, B, C the control points. Let $|PA| = X, |PB| = Y, |PC| = Z$, $\alpha = \angle BPC$, $\beta = \angle APC$, $\gamma = \angle APB$, $p = 2\cos\alpha, q = 2\cos\beta, r = 2\cos\gamma$, $|AB| = c', |BC| = a', |AC| = b'$. From triangles PBC, PAC, and PAB, we obtain the *P3P equation system*:

$$\begin{cases} Y^2 + Z^2 - YZp - a'^2 = 0 \\ Z^2 + X^2 - XZq - b'^2 = 0 \\ X^2 + Y^2 - XYr - c'^2 = 0. \end{cases} \tag{1}$$

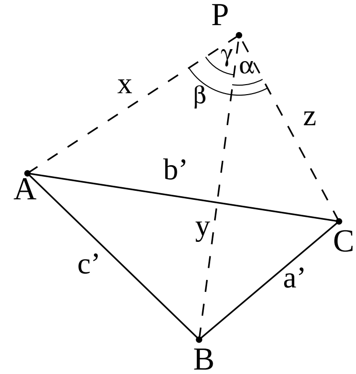


Figure 1. The P3P problem

A set of solutions for X, Y, Z is call a set of *physical solutions* if the following "reality conditions" are satisfied. These conditions are assumed through out the paper.

$$\begin{cases} X > 0, Y > 0, Z > 0, a' > 0, b' > 0, c' > 0, a' + b' > c', a' + c' > b', b' + c' > a' \\ 0 < \alpha, \beta, \gamma < \pi, 0 < \alpha + \beta + \gamma < 2\pi \\ \alpha + \beta > \gamma, \alpha + \gamma > \beta, \gamma + \beta > \alpha \\ I_0 = p^2 + q^2 + r^2 - pqr - 1 \neq 0 \text{ (Points } P, A, B, C \text{ are not co-planar [5]).} \end{cases} \quad (2)$$

To simplify the equation system, let $X = xZ, Y = yZ, |AB| = \sqrt{v}Z, |BC| = \sqrt{av}Z$, $|AC| = \sqrt{bv}Z$. Since $Z = |PC| \neq 0$, we obtain the following equivalent equation system:

$$\begin{cases} y^2 + 1 - yp - av = 0 \\ x^2 + 1 - xq - bv = 0 \\ x^2 + y^2 - xyr - v = 0. \end{cases} \quad (3)$$

Since $|r| < 2$, we have $v = x^2 + y^2 - xyr > 0$. Thus Z can be uniquely determined by $Z = |AB|/\sqrt{v}$. Eliminating v from (3), we have

$$\begin{cases} p_1 = (1-a)y^2 - ax^2 - py + arxy + 1 = 0 \\ p_2 = (1-b)x^2 - by^2 - qx + brxy + 1 = 0 \end{cases} \quad (ES)$$

which has the same number of physical solutions with (1). Now the P3P problem is reduced to finding the positive solutions of two quadratic equations. As a consequence, we obtain the following result: *the P3P problem has either an infinite number of solutions or at most four physical solutions.* This result was known before only for the "main part" of the P3P problem.

2.2 Zero Structure for the P3P Equation System

Wu-Ritt's zero decomposition method [15, 16, 17] is a general method to solve systems of algebraic equations. It may be used to represent the zero set of a polynomial equation system as the union of zero sets of equations in *triangular form*, that is, equation systems like

$$f_1(u, x_1) = 0, f_2(u, x_1, x_2) = 0, \ldots, f_p(u, x_1, \ldots, x_p) = 0$$

where the u could be considered as a set of parameters and the x are the variables to be determined. As shown in [15], solutions for an equation system in triangular form are well-determined. For instance, the solution of an equation system in triangular form can be easily reduced to the solution of univariate equations. For a polynomial set PS and a polynomial I, let $\text{Zero}(PS)$ be the set of solutions of the equation system $PS = 0$, and $\text{Zero}(PS/I) = \text{Zero}(PS) - \text{Zero}(I)$.

Among the "reality conditions" listed in (2), $I_0 \neq 0$ could be used to simplify the computation. Therefore, we consider $\text{Zero}(ES/I_0)$. Using Wu-Ritt's zero decomposition method [15], we decompose $\text{Zero}(ES/I_0)$ into ten *disjoint* components:

$$\text{Zero}(ES/I_0) = \bigcup_{i=1}^{10} \mathbf{C}_i. \tag{DE}$$

In the above formula, $\mathbf{C}_i = \text{Zero}(TS_i/T_i), i = 1, \cdots, 9$ and $C_{10} = \text{Zero}(TS_{10}/T_{10}) \cup \text{Zero}(TS_{11}/T_{11})$, where TS_i are polynomial equations in triangular form and T_i are polynomials. TS_i and T_i may be found in Appendix A.

Among the ten components, $\text{Zero}(TS_1/T_1)$ is called the *main component* of the P3P equation system, which is of the following form.

$$\begin{cases} f = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \\ g = b_0y - b_1 = 0. \end{cases} \tag{TS_1}$$

The coefficients a_i and b_j may be found in Appendix A. This component has been obtained in [2, 4, 5, 9]. All other components could be considered as *degenerate cases.* This is because, in these cases the parameters a, b, p, q, r must satisfy certain algebraic relations. In other words, we are considering special cases of the problem.

Comparing to the main component, the "degenerate" cases are less possible to occur. But, they are still important due to the following reasons. Solutions satisfying some degenerate conditions, such as $a+b-1 = 0$ (in TS_4) meaning that $\angle ACB$ is a right angle, may occur quite often if points A, B, C are from man-made structures like buildings, where many right angles exist. In the general case, the degenerate cases, such as TS_2, could be complicated and have no clear geometric meaning. Therefore, it is difficult to tell when it will occur.

The following table gives the maximal number of solutions for each component.

$\mathbf{C}_i, i =$	1	2	3	4	5	6	7	8	9	10
NO. of solutions	4	3	2	2	1	1	1	2	4	3

Table 1. The maximal number of solutions for each component

Since C_i and $C_j (i \neq j)$ are disjoint, to solve the equation numerically, for a set of specific values of the parameters p, q, r, a, b, either

$$\text{Zero}(PS/I_0) = \text{Zero}(TS_k/T_k)$$

if k satisfying $1 \leq k \leq 9$ and $T_k(p,q,r,a,b) \neq 0$, or

$$\text{Zero}(PS/I_0) = \text{Zero}(TS_{10}/T_{10}) \cup \text{Zero}(TS_{11}/T_{11})$$

if $T_{10}(p,q,r,a,b) = T_{11}(p,q,r,a,b) \neq 0$. Since the polynomials in TS_i are of degree ≤ 4, the solution of the P3P problem is reduced to the solution of equation systems in triangular form, and hence to the solution of univariate equations of degree ≤ 4.

From this decomposition, we have the following observations.

1. Since the solutions for each triangular set are well-determined, this decomposition provides a complete set of analytical solutions for the P3P problem.
2. From Table 1 and the analysis following the table, it is easy to see that there are at most four distinct solutions under the reality condition (2). Notice that this result was proved previously only for the main component.
3. From the experimental results in Section 4, we can see that the above decomposition provides a complete and robust way to find the solutions to the P3P problem.

3 Complete Solution Classification for the P3P Equation System

For polynomials $f(x)$ and $g(x)$, let V_f be the number of real solutions of $f(x)$, and $V_f(g > 0)$ the number of real solutions of $f(x)$ such that $g > 0$. If $f(x)$ has n real solutions, then let $C_f^{(n,j)}(g > 0)$ denote the conditions that make $f(x)$ having j real solutions such that $g > 0$. The following lemmas will be used in this section.

Lemma 1 *(***Descartes' Rule of Sign** *[19, 16]) Let $f = \sum_{i=k}^{n} a_i x^i, (a_n a_k \neq 0)$ be a polynomial with real numbers as coefficients. Then the number of positive roots of f is less than or congruent to the number of sign changes in the sequence of coefficients $a_n, \ldots, a_k($ mod $2)$.*

Lemma 2 *[16] Let $f(x)$, $g(x)$ be two polynomials, and f of degree n. Let*

$$r(T) = resultant(f, g - T, x) = c\prod_{i=1}^{n}(g(x_i) - T)$$

where $x_i, i = 1, \ldots, n$ are the roots of $f(x) = 0$. If all the solutions of $f(x) = 0$ are real, then $V_f(g > 0) = V_r(T > 0)$.

Let $f_i(x), g_i(x,y)$ be the first two polynomials in TS_i.

In components $TS_i, i = 5, 6, 7, 10$, $f_i(x)$ and $g_i(x,y)$ are linear in x and y respectively. In these cases, each component can have only one positive solution and it is trivial to give the conditions for them to have such solutions.

In components $TS_i, i = 3, 4, 8, 9, 11$, $f_i(x)$ and $g_i(x,y)$ are either linear or quadratic in x and y respectively. We will treat these cases in Section 3.1.

In component TS_2, $f_2(x)$ is a cubic equation and $g_2(x,y)$ is linear in y. We will treat this case in Section 3.2.

In component TS_1, $f_1(x)$ is a quartic equation and $g_1(x,y)$ is linear in y. We will treat this case in Section 3.3.

3.1 The Quadratic Cases

The quadratic cases may have three forms: (1) $f_i(x)$ is quadratic and $g_i(x,y)$ is linear in y; (2) $f_i(x)$ is linear and $g_i(x,y)$ is quadratic in y; (3) $f_i(x)$ and $g_i(x,y)$ are quadratic in x and y respectively. All of them can be treated similarly. We will take TS_9 as an illustrative example. The equation system is

$$\begin{cases} f_9 = (-1+a+b)x^2 + (-qa+q)x - 1 + a - b = 0, \\ g_9 = (-1+a+b)y^2 - 1 - a + qxa + b = 0, \\ p = 0, r = 0, x > 0, y > 0, a + b - 1 \neq 0. \end{cases} \tag{4}$$

The number of solutions for the above system is the same as the following *equation system*:

$$\begin{cases} f_9 = (-1+a+b)x^2 + (-qa+q)x - 1 + a - b = 0, \\ g = (-1+a+b)(1 + a - qxa - b) > 0, \\ p = 0, r = 0, a + b - 1 \neq 0, x > 0 \end{cases} \tag{5}$$

We first assume that $resultant(f_9, g, x) \neq 0$, $resultant(f_9, x, x) = a - b - 1 \neq 0$ and $resultant(g, x, x) = (a+b-1)(a-b+1) \neq 0$. Let

$$r_{11}(T) = resultant(f_9, g - T, x) = r_{110}T^2 + r_{111}T + r_{112},$$

$$r_{12}(T) = resultant(f_9, xg - T, x) = r_{120}T^2 + r_{121}T + r_{122}.$$

By computation, we obtain the *Sylvester-Habicht* sequences[21] of $(f_9, diff(f_9, x))$ which are denoted by D_{11}, D_{12} and D_{13} (the discriminant) respectively. So f_9 has two real solutions iff $D_{13} > 0$ and one real solution iff $D_{13} = 0$. Since $resultant(f_9, g, x) \neq 0$, $resultant(f_9, x, x) \neq 0$ and $resultant(g, x, x)) \neq 0$, we have

$$\begin{cases} V_{f_9} = V_{f_9}(x > 0) + V_{f_9}(x < 0), \\ V_{f_9}(x < 0) = V_{f_9}(x < 0, g > 0) + V_{f_9}(x < 0, g < 0), \\ V_{f_9}(g > 0) = V_{f_9}(x > 0, g > 0) + V_{f_9}(x < 0, g > 0), \\ V_{f_9}(xg > 0) = V_{f_9}(x > 0, g > 0) + V_{f_9}(x < 0, g < 0). \end{cases} \tag{6}$$

From the above equations, we obtain the following formula for the number of physical solutions:

$$V_{f_9}(x > 0, g > 0) = \frac{1}{2}(V_{f_9}(x > 0) + V_{f_9}(g > 0) + V_{f_9}(xg > 0) - V_{f_9}) \tag{7}$$

By Lemmas 1 and 2, we have the following results

$$\begin{aligned} V_{f_9}(x > 0) &= \text{the number of sign changes in coefficients of } f_9(x), \\ V_{f_9}(g > 0) &= \text{the number of sign changes in coefficients of } r_{11}(T), \\ V_{f_9}(xg > 0) &= \text{the number of sign changes in coefficients of } r_{12}(T). \end{aligned}$$

If $resultant(f_9, g, x) = 0$, (5) becomes:

$$\begin{cases} f_9 = -aq(a + b - 1)x - q^2a + q^2a^2 - 2b - a^2 + 1 + b^2 = 0, \\ g = (-1 + a + b)(1 + a - qxa - b) > 0, \\ p = 0, r = 0, \\ a + b - 1 \neq 0, x > 0 \end{cases} \tag{8}$$

which may have one positive solution and the condition for that is easy to obtain. For $resultant(f_9, x, x) = a - b - 1 = 0$ and $resultant(g, x, x) = (a + b - 1)(a - b + 1) = 0$, we can deal with them similarly.

Theorem 3 *We have the following necessary and sufficient conditions to adjudicate the number of physical solutions $V_{f_9}(x > 0, g > 0)$ of $Zero(TS_9/T_9)$.*

1. *(4) has two physical solutions iff one of the following statements holds;*

   1.1) $[\quad p = r = 0, a + b - 1 > 0, q > 0, D_{13} > 0, \Delta_1 < 0, \Delta_2 < 0, \Delta_3 < 0]$

   1.2) $[\quad p = r = 0, a + b - 1 < 0, q > 0, D_{13} > 0, \Delta_1 > 0, \Delta_2 < 0, \Delta_3 > 0]$

2. (4) has one physical solution iff one of the following statements holds;

2.1) $[\ p = r = 0, a = 1, q + b - 2 < 0]$

2.2) $[\ p = r = 0, a - b + 1 = 0, q \neq 0]$

2.3) $[\ p = r = 0, a - b - 1 = 0, q > 0, q^2 a < 4]$

2.4) $[\ p = r = 0, a + b - 1 > 0, D_{13} = 0, \Delta_1 < 0]$

2.5) $[\ p = r = 0, a + b - 1 < 0, D_{13} = 0, \Delta_1 > 0]$

2.6) $[\ p = r = 0, a + b - 1 > 0, \Delta_2 = 0, \Delta_1 < 0]$

2.7) $[\ p = r = 0, a + b - 1 < 0, \Delta_2 = 0, \Delta_1 > 0]$

2.8) $[\ p = r = q = 0, (a + b - 1)(a - b - 1) < 0, (a + b - 1)(a - b + 1) > 0]$

2.9) $[\ p = r = 0, q > 0, a - b - 1 > 0, D_{13} > 0, \Delta_1 < 0, \Delta_2 > 0, \Delta_3 \neq 0]$

2.10) $[\ p = r = 0, q > 0, a - b - 1 > 0, D_{13} > 0, \Delta_1 > 0, \Delta_2 < 0, \Delta_3 \neq 0]$

2.11) $[\ p = r = 0, q > 0, a + b - 1 > 0, D_{13} > 0, \Delta_1 < 0, \Delta_2 < 0, \Delta_3 \neq 0]$

2.12) $[\ p = r = 0, q > 0, a + b - 1 > 0, D_{13} > 0, \Delta_1 > 0, \Delta_2 < 0, \Delta_3 \neq 0]$

where

$$
\begin{aligned}
D_{13} &= q^2(a-1)^2 - 4(a+b-1)(a-b-1), \\
\Delta_1 &= q^2 a(a-1) - 2(a-b+1)(a+b-1), \\
\Delta_2 &= q^2 a - (a-b+1)^2, \\
\Delta_3 &= q^2 a(a-1)^2 - (a+b-1)(b - 3ab - 1 - 2a + 3a^2).
\end{aligned}
$$

Proof. We know that (4) has two physical solutions iff $V_{f_9}(x > 0, g > 0) = 2$. From (7), this is possible iff

$$V_{f_9}(x > 0) = 2, V_{f_9}(g > 0) = 2, V_{f_9}(xg > 0) = 2, V_{f_9} = 2.$$

The first part of the Theorem 3 follows directly from these conditions. For the second part, there are four cases. Suppose $resultant(f_9, g, x) \neq 0$, $resultant(f_9, x, x) \neq 0$ and $resultant(g, x, x) \neq 0$. In this case, (4) has one physical solutions iff $V_{f_9}(x > 0, g > 0) = 1$. From (7), (4) has one physical solution iff one of the following conditions holds:

$$
\begin{aligned}
&V_{f_9}(x > 0) = 2, V_{f_9}(g > 0) = 2, V_{f_9}(xg > 0) = 0, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 2, V_{f_9}(g > 0) = 1, V_{f_9}(xg > 0) = 1, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 2, V_{f_9}(g > 0) = 0, V_{f_9}(xg > 0) = 2, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 1, V_{f_9}(g > 0) = 2, V_{f_9}(xg > 0) = 1, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 1, V_{f_9}(g > 0) = 1, V_{f_9}(xg > 0) = 2, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 0, V_{f_9}(g > 0) = 2, V_{f_9}(xg > 0) = 2, V_{f_9} = 2, \\
&V_{f_9}(x > 0) = 1, V_{f_9}(g > 0) = 1, V_{f_9}(xg > 0) = 1, V_{f_9} = 1.
\end{aligned}
$$

Analysing these conditions will lead to some of the conditions in part two of Theorem 3. The other three cases are $resultant(f_9, g, x) = 0$, $resultant(f_9, x, x) = 0$ and $resultant(g, x, x)) = 0$ respectively. We can deal with them similarly. Combining the four cases will lead to the second part of Theorem 3.

3.2 The Cubic Case

For TS_2, we need to consider the *polynomial system:*

$$\begin{cases} f_2 = a_5 x^3 + a_6 x^2 + a_7 x + a_8, \\ g_2 = b_2 y - b_3 = 0, \\ x > 0, y > 0, a_5 \neq 0, b_2 \neq 0 \end{cases} \tag{9}$$

where the a_i and b_j could be found in Appendix A. The number of solutions for the above system is the same as the following *equation system*:

$$\begin{cases} f_2 = a_5 x^3 + a_6 x^2 + a_7 x + a_8, \\ g = b_2 b_3 > 0 \\ x > 0, a_5 \neq 0 \end{cases} \tag{10}$$

Let $Resultant(f_2, g, x) \neq 0$, and D_i the *Sylvester-Habicht* sequences of $(f_2, diff(f_2))$, $i = 5, \ldots, 8$ [21]. We know that $f(x)$ has three real solutions iff $D_8 > 0$ [20]. Let

$$r_7(T) = resultant(f_2, g - T, x) = r_{70} T^3 + r_{71} T^2 + r_{72} T + r_{73},$$

$$r_8(T) = resultant(f_2, xg - T, x) = r_{80} T^3 + r_{81} T^2 + r_{82} T + r_{83}.$$

If $D_8 > 0$, then by Lemmas 1 and 2, we can give the conditions for the equation system to have one, two or three positive solutions. If $D_8 < 0$, $f_2(x)$ has only one real solution and two complex solutions. By Descartes rule of sign, $V_{f_2}(x > 0) = 1$ iff the number of sign changes in coefficients of $f_2(x)$ is 1 or 3. Now, we consider the number of positive solutions of $r_7(T)$ and $r_8(T)$. Let x_1 be real, x_2, x_3 be complex. If $g(x_2)$ is real, then $g(x_3)$ is also real, and the signs of $g(x_2)$ and $g(x_3)$ are the same. So by the Descartes rule of sign, $V_{r_j}(T > 0) = 1$ iff the number of sign changes in coefficients of $r_j(x)$ is 1 or 3, $j = 7, 8$. If $D_8 = 0$, $D_7 \neq 0$, then there exists polynomial $Q_2(x) = q_{20} x + q_{21}$ such that $f_2(x) = c Q_2(x) H_3^2(x)$, where $H_3(x) = h_{30} x + h_{31}$ is the pseudo-remainder of f_2 with $diff(f_2)$ for variable x This case is easy to solve. If $D_8 = 0$, $D_7 = 0$, $D_6 \neq 0$, then $f_2(x) = c H_4^3(x)$, where $H_4(x) = 3a_5 x + a_6$.

From the above discussion, we obtain the following result.

Theorem 4 *For TS_2, we have the following results:*

1. *(9) has three physical solutions iff one of the following statements holds;*

$$[\quad D_8 > 0, C_{f_2}^{(3,3)}(x>0), C_{f_2}^{(3,3)}(g>0), C_{f_2}^{(3,3)}(xg>0)].$$

2. *(9) has two physical solutions iff one of the following statements holds;*

$$\begin{array}{ll}
2.1\ ) & [D_8 > 0, C_{f_2}^{(3,3)}(x>0), C_{f_2}^{(3,2)}(g>0), C_{f_2}^{(3,2)}(xg>0)] \\
2.2\ ) & [D_8 > 0, C_{f_2}^{(3,2)}(x>0), C_{f_2}^{(3,3)}(g>0), C_{f_2}^{(3,2)}(xg>0)] \\
2.3\ ) & [D_8 > 0, C_{f_2}^{(3,2)}(x>0), C_{f_2}^{(3,2)}(g>0), C_{f_2}^{(3,2)}(xg>0)] \\
2.4\ ) & [D_8 = 0, D_7 \neq 0, \frac{h_{31}}{h_{30}} < 0, g(\frac{-h_{31}}{h_{30}}) > 0, \frac{q_{21}}{q_{20}} < 0, g(\frac{-q_{21}}{q_{20}}) > 0]
\end{array}$$

3. *(9) PS has one physical solution iff one of the following statements holds;*

$$\begin{array}{ll}
3.1) & [\quad D_8 = 0, D_7 \neq 0, \frac{q_{21}}{q_{20}} < 0, g(\frac{-q_{21}}{q_{20}}) > 0] \\
3.2) & [\quad D_8 = 0, D_7 \neq 0, \frac{h_{31}}{h_{30}} < 0, g(\frac{-h_{31}}{h_{30}}) > 0] \\
3.3) & [\quad D_8 = 0, D_7 = 0, D_6 \neq 0, \frac{a_2}{a_1} < 0, g(\frac{-a_2}{3a_1}) > 0] \\
3.4) & [\quad D_8 < 0, C_{f_2}^{(1,1)}(x>0), C_{f_2}^{(1,1)}(g>0), C_{f_2}^{(1,1)}(xg>0)] \\
3.5) & [\quad D_8 > 0, C_{f_2}^{(3,3)}(x>0), C_{f_2}^{(3,1)}(g>0), C_{f_2}^{(3,1)}(xg>0)] \\
3.6) & [\quad D_8 > 0, C_{f_2}^{(3,2)}(x>0), C_{f_2}^{(3,2)}(g>0), C_{f_2}^{(3,1)}(xg>0)] \\
3.7) & [\quad D_8 > 0, C_{f_2}^{(3,2)}(x>0), C_{f_2}^{(3,1)}(g>0), C_{f_2}^{(3,2)}(xg>0)] \\
3.8) & [\quad D_8 > 0, C_{f_2}^{(3,1)}(x>0), C_{f_2}^{(3,3)}(g>0), C_{f_2}^{(3,1)}(xg>0)] \\
3.9) & [\quad D_8 > 0, C_{f_2}^{(3,1)}(x>0), C_{f_2}^{(3,2)}(g>0), C_{f_2}^{(3,2)}(xg>0)] \\
3.10) & [\quad D_8 > 0, C_{f_2}^{(3,1)}(x>0), C_{f_2}^{(3,1)}(g>0), C_{f_2}^{(3,3)}(xg>0)]
\end{array}$$

The explicit expressions for all $C_f^{(n,j)}$ may be found in Appendix B.

If $Resultant(f_2, g, x) = 0$, the problem becomes a quadratic case and can be treated similarly as in the preceding section.

3.3 The Quartic Case

For TS_1, we need to count the number of solutions for the following system.

$$\begin{cases} f_1 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \\ g_1 = b_0y - b_1 = 0, \\ x > 0, y > 0, a_0 \neq 0, b_0 \neq 0 \end{cases} \tag{11}$$

where a_i and b_i may be found in Appendix A. Since $b_0 \geq 0$ (see Appendix A), it is equivalent to count the number of positive solutions for the following system

$$\begin{cases} f_1 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \\ b_1 > 0 \\ x > 0, a_0 \neq 0, b_0 \neq 0. \end{cases} \quad (12)$$

We will first assume that $Resultant(f_1, b_1, x) \neq 0$, and $Resultant(f_1(x), x, x) \neq 0$. Let $f' = Diff(f_1, x)$, and D_i, A_i, B_i, C_i the *Sylvester-Habicht* sequences of (f_1, f'), (f_1, xf'), $(f_1(x), b_1f')$ and $(f_1(x), xb_1f')$ respectively, $i = 0, .., 4$ [21]. First, let us assume $D_4 \neq 0$. It is known that $V_{f_1} = 4$ iff $D_2 > 0$, $D_3 > 0$, $D_4 > 0$ which is denoted by $C_{f_1}^{(4)}$ [20]. Let

$$\begin{aligned} r_1(T) = resultant(f_1, b_1 - T, x) &= a_0^4 \prod_{i=1}^{4} (b_1(x_i) - T) \\ &= r_{10}T^4 + r_{11}T^3 + r_{12}T^2 + r_{13}T + r_{14} \\ r_2(T) = resultant(f_1, xb_1 - T, x) &= a_0^4 \prod_{i=1}^{4} (xb_1(x_i) - T) \\ &= r_{20}T^4 + r_{21}T^3 + r_{22}T^2 + r_{23}T + r_{24} \end{aligned}$$

where $f(x_i) = 0$, $i = 1, 2, 3, 4$.

By the Lemmas 1 and 2, we have the following results:

$$\begin{aligned} V_{f_1}(x > 0) &= \text{the number of sign changes in coefficients of } f(x) \\ V_{f_1}(b_1 > 0) &= \text{the number of sign changes in coefficients of } r_1(T) \\ V_{f_1}(xb_1 > 0) &= \text{the number of sign changes in coefficients of } r_2(T) \end{aligned}$$

By the complete discrimination method [22], $V_{f_1} = 2$ iff one of the following conditions holds.

$$\begin{cases} 1) \quad D_2 > 0,\ D_3 > 0,\ D_4 < 0 \\ 2) \quad D_2 \neq 0,\ D_3 < 0,\ D_4 < 0 \\ 3) \quad D_2 \leq 0,\ D_3 > 0,\ D_4 > 0 \\ 4) \quad D_2 > 0,\ D_3 = 0,\ D_4 > 0 \end{cases} \quad (13)$$

We denote one of the conditions by $C_{f_1}^{(2)}$. It is clear that there are the following results:

$$\begin{aligned} V_{f_1}(x > 0) - V_{f_1}(x < 0)) &= \tfrac{1}{2}[sign(A_1) + sign(A_1A_2) + sign(A_2A_3) + sign(A_3A_4)]; \\ V_{f_1}(b_1 > 0) - V_{f_1}(b_1 < 0) &= \tfrac{1}{2}[sign(B_1) + sign(B_1B_2) + sign(B_2B_3) + sign(B_3B_4)]; \\ V_{f_1}(xb_1 > 0) - V_{f_1}(xb_1 < 0) &= \tfrac{1}{2}[sign(C_1) + sign(C_1C_2) + sign(C_2C_3) + sign(C_3C_4)]. \end{aligned}$$

Since $Resultant(f_1, b_1, x) \neq 0$, and $Resultant(f_1(x), x, x) \neq 0$, we have

$$V_{f_1}(x>0)=\tfrac{1}{2}[V_{f_1}+sign(A_1)+sign(A_1A_2)+sign(A_2A_3)+sign(A_3A_4)];$$
$$V_{f_1}(b_1>0)=\tfrac{1}{2}[V_{f_1}+sign(B_1)+sign(B_1B_2)+sign(B_2B_3)+sign(B_3B_4)];$$
$$V_{f_1}(xb_1>0)=\tfrac{1}{2}[V_{f_1}+sign(C_1)+sign(C_1C_2)+sign(C_2C_3)+sign(C_3C_4)].$$

For any two equations $f(x)$ and $g(x)$, if $Resultant(f(x),g(x),x)\neq 0$, then $V_{f(x)} = V_{f(x)}(g(x)>0)+V_{f(x)}(g(x)<0)$

Then, we have

$$V_{f_1}(x>0,b_1>0)=\frac{1}{2}(V_{f_1}(x>0)+V_{f_1}(b_1>0)+V_{f_1}(xb_1>0)-V_{f_1}).$$

So, for $a_0\neq 0$ and $D_4\neq 0$, we can solve the equation system completely. Note that D_4 is the discriminant for f_1. From the above discussion, we proved

Theorem 5 *For $a_0\neq 0$, $D_4\neq 0$, we have*

1. *(11) has four physical solutions iff $C_{f_1}^{(4,4)}(x>0)$, $C_{f_1}^{(4,4)}(b_1>0)$, $C_{f_1}^{(4,4)}(xb_1>0)$ and $C_{f_1}^{(4)}$ hold.*
2. *(11) has three physical solutions iff one of the following statements holds;*

   2.1) $[C_{f_1}^{(4,4)}(x>0),C_{f_1}^{(4,3)}(b_1>0),C_{f_1}^{(4,3)}(xb_1>0),C_{f_1}^{(4)}]$
   2.2) $[C_{f_1}^{(4,3)}(x>0),C_{f_1}^{(4,4)}(b_1>0),C_{f_1}^{(4,3)}(xb_1>0),C_{f_1}^{(4)}]$
   2.3) $[C_{f_1}^{(4,3)}(x>0),C_{f_1}^{(4,3)}(b_1>0),C_{f_1}^{(4,3)}(xb_1>0),C_{f_1}^{(4)}]$

3. *(11) has two physical solutions iff one of the following statements holds;*

   3.1) $[C_{f_1}^{(2,2)}(x>0),C_{f_1}^{(2,2)}(b_1>0),C_{f_1}^{(2,2)}(xb_1>0),C_{f_1}^{(2)}]$
   3.2) $[C_{f_1}^{(4,4)}(x>0),C_{f_1}^{(4,2)}(b_1>0),C_{f_1}^{(4,2)}(xb_1>0),C_{f_1}^{(4)}]$
   3.3) $[C_{f_1}^{(4,3)}(x>0),C_{f_1}^{(4,3)}(b_1>0),C_{f_1}^{(4,2)}(xb_1>0),C_{f_1}^{(4)}]$
   3.4) $[C_{f_1}^{(4,3)}(x>0),C_{f_1}^{(4,2)}(b_1>0),C_{f_1}^{(4,3)}(xb_1>0),C_{f_1}^{(4)}]$
   3.5) $[C_{f_1}^{(4,2)}(x>0),C_{f_1}^{(4,4)}(b_1>0),C_{f_1}^{(4,2)}(xb_1>0),C_{f_1}^{(4)}]$
   3.6) $[C_{f_1}^{(4,2)}(x>0),C_{f_1}^{(4,3)}(b_1>0),C_{f_1}^{(4,3)}(xb_1>0),C_{f_1}^{(4)}]$
   3.7) $[C_{f_1}^{(4,2)}(x>0),C_{f_1}^{(4,2)}(b_1>0),C_{f_1}^{(4,4)}(xb_1>0),C_{f_1}^{(4)}]$

4. *(11) has one physical solution iff one of the following statements holds;*

   4.1) $[C_{f_1}^{(2,2)}(x>0),C_{f_1}^{(2,1)}(b_1>0),C_{f_1}^{(2,1)}(xb_1>0),C_{f_1}^{(2)}]$
   4.2) $[C_{f_1}^{(2,1)}(x>0),C_{f_1}^{(2,2)}(b_1>0),C_{f_1}^{(2,1)}(xb_1>0),C_{f_1}^{(2)}]$

4.3) $[\ C_{f_1}^{(2,1)}(x>0), C_{f_1}^{(2,1)}(b_1>0), C_{f_1}^{(2,2)}(xb_1>0), C_{f_1}^{(2)}]$
4.4) $[\ C_{f_1}^{(4,4)}(x>0), C_{f_1}^{(4,1)}(b_1>0), C_{f_1}^{(4,1)}(xb_1>0), C_{f_1}^{(4)}]$
4.5) $[\ C_{f_1}^{(4,3)}(x>0), C_{f_1}^{(4,2)}(b_1>0), C_{f_1}^{(4,1)}(xb_1>0), C_{f_1}^{(4)}]$
4.6) $[\ C_{f_1}^{(4,3)}(x>0), C_{f_1}^{(4,1)}(b_1>0), C_{f_1}^{(4,2)}(xb_1>0), C_{f_1}^{(4)}]$
4.7) $[\ C_{f_1}^{(4,2)}(x>0), C_{f_1}^{(4,3)}(b_1>0), C_{f_1}^{(4,1)}(xb_1>0), C_{f_1}^{(4)}]$
4.8) $[\ C_{f_1}^{(4,2)}(x>0), C_{f_1}^{(4,2)}(b_1>0), C_{f_1}^{(4,2)}(xb_1>0), C_{f_1}^{(4)}]$
4.9) $[\ C_{f_1}^{(4,2)}(x>0), C_{f_1}^{(4,1)}(b_1>0), C_{f_1}^{(4,3)}(xb_1>0), C_{f_1}^{(4)}]$
4.10) $[\ C_{f_1}^{(4,1)}(x>0), C_{f_1}^{(4,4)}(b_1>0), C_{f_1}^{(4,1)}(xb_1>0), C_{f_1}^{(4)}]$
4.11) $[\ C_{f_1}^{(4,1)}(x>0), C_{f_1}^{(4,3)}(b_1>0), C_{f_1}^{(4,2)}(xb_1>0), C_{f_1}^{(4)}]$
4.12) $[\ C_{f_1}^{(4,1)}(x>0), C_{f_1}^{(4,2)}(b_1>0), C_{f_1}^{(4,3)}(xb_1>0), C_{f_1}^{(4)}]$
4.13) $[\ C_{f_1}^{(4,1)}(x>0), C_{f_1}^{(4,1)}(b_1>0), C_{f_1}^{(4,4)}(xb_1>0), C_{f_1}^{(4)}]$

The explicit expressions for all $C_f^{(n,j)}$ may be found in Appendix B.

If $D_4 = 0$ and $D_3 \neq 0$, then we know that there exists a polynomial $Q_1(x) = q_{10}x^2 + q_{11}x + q_{12}$ such that $f_1(x) = cQ_1(x)H_1^2(x)$, where $H_1(x) = D_3x + \bar{D}_3$. Let $D_{Q_1} = resultant(Q_1, diff(Q_1), x)$. If $D_{Q_1} > 0$, then $Q_1(x)$ has two different real solutions. Let

$$r_3(T) = resultant(Q_1, b_1 - T, x) = r_{30}T^2 + r_{31}T + r_{32},$$

$$r_4(T) = resultant(Q_1, xb_1 - T, x) = r_{40}T^2 + r_{41}T + r_{42}.$$

By Lemmas 1 and 2, we have:

$$V_{Q_1}(x>0) = \text{the number of sign changes in coefficients of } Q_1(x)$$
$$V_{Q_1}(b_1>0) = \text{the number of sign changes in coefficients of } r_3(T)$$
$$V_{Q_1}(xb_1>0) = \text{the number of sign changes in coefficients of } r_4(T).$$

Similarly, we have $V_{Q_1}(x>0, b_1>0) = \frac{1}{2}[V_{Q_1}(x>0) + V_{Q_1}(b_1>0) + V_{Q_1}(xb_1>0) - V_{Q_1}]$. If $D_{Q_1} = 0$, then $Q_1(x) = q_0(x + \frac{q_{11}}{2q_{10}})^2$.

For $D_4 = 0$ and $D_3 = 0$, $D_2 \neq 0$, let $H_1(x) = D_2x^2 + \bar{D}_2x + \bar{\bar{D}}_2$. Then $f_1(x) = cH_1^2(x)$. If $D_{H_1} > 0$, then $V_{H_1} = 2$. Let

$$r_5(T) = resultant(H_1, b_1 - T, x) = r_{50}T^2 + r_{51}T + r_{52},$$

$$r_6(T) = resultant(H_1, xb_1 - T, x) = r_{60}T^2 + r_{61}T + r_{62}.$$

Similar to the above discussion, we can solve $V_{H_1}(x > 0, b_1 > 0)$. If $D_4 = 0$, $D_3 = 0$, $D_2 = 0$, $D_1 \neq 0$, then $f_1(x) = cH_2^4$, where $H_2(x) = 4a_0x + a_1$.

From the above discussion, we can obtain the following theorem.

Theorem 6 *For $D_4 = 0$, we have*

1. *(11) has three physical solutions iff $[D_3 \neq 0, -\frac{\bar{D}_3}{D_3} > 0, b_1(-\frac{\bar{D}_3}{D_3}) > 0, D_{Q_1} > 0, C_{Q_1}^{(2,2)}(x > 0), C_{Q_1}^{(2,2)}(b_1 > 0), C_{Q_1}^{(2,2)}(xb_1 > 0)]$ holds;*
2. *(11) has two physical solutions iff one of the following statements holds;*

2.1) $[\quad D_3 \neq 0, D_{Q_1} = 0, \frac{q_{11}}{q_{10}} < 0, b_1(-\frac{q_{11}}{q_{10}}), \frac{\bar{D}_3}{D_3} < 0, b_1(-\frac{\bar{D}_3}{D_3})]$

2.2) $[\quad D_3 \neq 0, D_{Q_1} > 0, C_{Q_1}^{(2,2)}(x > 0), C_{Q_1}^{(2,2)}(b_1 > 0), C_{Q_1}^{(2,2)}(xb_1 > 0)]$

2.3) $[\quad D_3 = 0, D_2 \neq 0, D_{H_1} > 0, C_{H_1}^{(2,2)}(x > 0), C_{H_1}^{(2,2)}(b_1 > 0), C_{H_2}^{(2,2)}(xb_1 > 0)]$

2.4) $[\quad D_3 \neq 0, D_{Q_1} > 0, \frac{\bar{D}_3}{D_3} < 0, b_1(-\frac{\bar{D}_3}{D_3}) > 0, C_{Q_1}^{(2,2)}(x > 0), C_{Q_1}^{(2,1)}(b_1 > 0), C_{Q_1}^{(2,1)}(xb_1 > 0)]$

2.5) $[\quad D_3 \neq 0, D_{Q_1} > 0, \frac{\bar{D}_3}{D_3} < 0, b_1(-\frac{\bar{D}_3}{D_3}) > 0, C_{Q_1}^{(2,1)}(x > 0), C_{Q_1}^{(2,1)}(b_1 > 0), C_{Q_1}^{(2,1)}(xb_1 > 0)]$

2.6) $[\quad D_3 \neq 0, D_{Q_1} > 0, \frac{\bar{D}_3}{D_3} < 0, b_1(-\frac{\bar{D}_3}{D_3}) > 0, C_{Q_1}^{(2,1)}(x > 0), C_{Q_1}^{(2,1)}(b_1 > 0), C_{Q_1}^{(2,2)}(xb_1 > 0)]$

3. *(11) has one physical solution iff one of the following statements holds;*

3.1) $[\quad D_3 \neq 0, \frac{\bar{D}_3}{D_3} < 0, b_1(\frac{-\bar{D}_3}{D_3}) > 0]$

3.2) $[\quad D_3 \neq 0, D_{Q_1} = 0, -\frac{q_{11}}{q_{10}} > 0, b_1(\frac{-q_{11}}{q_{10}}) > 0]$

3.3) $[\quad D_3 = 0, D_2 = 0, D_1 \neq 0, \frac{a_1}{a_0} < 0, b_1(\frac{-a_1}{4a_0}) > 0]$

3.4) $[\quad D_3 = 0, D_2 \neq 0, D_{H_2} = 0, \frac{\bar{D}_2}{D_2} < 0, b_1(\frac{-\bar{D}_2}{D_2}) > 0]$

3.5) $[\quad D_3 \neq 0, D_{Q_1} > 0, C_{Q_1}^{(2,2)}(x > 0), C_{Q_1}^{(2,1)}(b_1 > 0), C_{Q_1}^{(2,1)}(xb_1 > 0]$

3.6) $[\quad D_3 \neq 0, D_{Q_1} > 0, C_{Q_1}^{(2,1)}(x > 0), C_{Q_1}^{(2,2)}(b_1 > 0), C_{Q_1}^{(2,1)}(xb_1 > 0)]$

3.7) $[\quad D_3 \neq 0, D_{Q_1} > 0, C_{Q_1}^{(2,1)}(x > 0), C_{Q_1}^{(2,1)}(b_1 > 0), C_{Q_1}^{(2,2)}(xb_1 > 0)]$

3.8) $[\quad D_3 = 0, D_2 \neq 0, D_{H_1} > 0, C_{H_1}^{(2,2)}(x > 0), C_{H_1}^{(2,1)}(b_1 > 0), C_{H_2}^{(2,1)}(xb_1 > 0)]$

3.9) $[\quad D_3 = 0, D_2 \neq 0, D_{H_1} > 0, C_{H_1}^{(2,1)}(x > 0), C_{H_1}^{(2,2)}(b_1 > 0), C_{H_2}^{(2,1)}(xb_1 > 0)]$

3.10) $[\quad D_3 = 0, D_2 \neq 0, D_{H_1} > 0, C_{H_1}^{(2,1)}(x > 0), C_{H_1}^{(2,1)}(b_1 > 0), C_{H_2}^{(2,2)}(xb_1 > 0)]$

The explicit expressions for all $C_f^{(n,j)}$ may be found in Appendix B.

If $Resultant(f_1, b_1, x) = 0$, then the equation system becomes the following form

$$\begin{cases} f_{11} = a_{10}x^3 + a_{11}x^2 + a_{12}x + a_{13} = 0, \\ b_{11} = b_{10}x^4 + b_{11}x^3 + b_{12}x^2 + b_{13}x + b_{14} > 0, \\ x > 0. \end{cases} \tag{14}$$

which can be treated with the method in Section 3.2. We may solve the case for $Resultant(f_1, x, x) = 0$ and $Resultant(f_1, b_1, x) = 0$ similarly.

3.4 A Special Case of the P3P Problem

Let us assume that $a = b = 1$ and $r = q$. Since the formulas in this case are quite simple, we may have an intuitive idea about the distribution of the solutions. The P3P equation system becomes

$$\begin{cases} f = -x^2 - py + qxy + 1 \\ g = -y^2 - qx + pxy + 1. \end{cases} \tag{15}$$

Using Wu-Ritt's method, this equation system has the following two components:

$$\begin{cases} f_1 = x^2 - qx + p - 1 \\ f_2 = y - 1 \end{cases} \tag{16}$$

$$\begin{cases} g_1 = (-1 + q^2)x^2 + (-q - qp)x + 1 + p \\ g_2 = y - qx - 1 \end{cases} \tag{17}$$

It is clear that the number of positive solutions of (16) is determined by $f_1(x) = 0$. Notice that $f_1(x)$ is a quadratic equation in x, we have the following results.

- Equation system (16) has one positive solution iff,

$$\begin{cases} q > 0 \\ \frac{1}{2}p = \frac{4+q^2}{4} \text{ or } 1 \end{cases} \quad \text{or} \quad p < 1.$$

- Equation system (16) has two positive solutions iff,

$$q > 0 \text{ and } \frac{4+q^2}{4} > p > 1.$$

Now we will discuss (17). Let $g = qx + 1$.

$$\begin{cases} R_1(t) = resultant(g_1, g - t, x) = (q^2 - 1)t^2 + (q^2 - q^2p - 2)t + (q^2 - 1) \\ R_2(t) = resultant(g_1, xg - t, x) = R_{20}t^2 + R_{21}t + R_{22} \end{cases} \tag{18}$$

Here, $R_{20} = (q-1)^2(q+1)^2$, $R_{21} = -q(1+p)(q^2p - 2q^2 + 3)$, $R_{22} = (q-1)(q+1)(1+p)$. Denote the discriminant of g_1 by Δ. We have

$$\Delta = (1+p)(q^2p - 3q^2 + 4).$$

By Descartes rules of sign, (17) has two positive solutions iff

$$q > 1 \text{ and } p > 3 - \tfrac{4}{q^2}.$$

Otherwise ES_2 has no positive solution.

We still need to consider the reality conditions (2): $0 < \alpha, \beta < \pi$, $0 < \alpha + 2\beta < 2\pi$, and $2\beta > \alpha$, which can be reduced to

$$-2 < p < 2, \text{-}2 < q < 2, \mathrm{q}^2 - 2 < p.$$

Combining the above conditions, we have the following classification for the P3P problem.

1. Point P has four solutions, iff

$$2 > q > 1, \quad \tfrac{4+q^2}{4} > p > 1, \quad \text{ and } p > 3 - \tfrac{4}{q^2}.$$

2. Point P has three solutions, iff

$$\begin{cases} 1 < q < \sqrt{2} \\ 3 - \frac{4}{q^2} < p \leq 1 \end{cases} \text{or} \begin{cases} 1 < q < 2 \\ p = \frac{4+q^2}{4}. \end{cases}$$

3. Point P has two solutions, iff

$$\begin{cases} 0 < q \leq 1 \\ 1 < p < \frac{4+q^2}{4} \end{cases} \text{or} \begin{cases} 1 < q < 2 \\ \frac{4+q^2}{4} < p < 2 \end{cases} \text{or} \begin{cases} \sqrt{2} < q < 2 \\ 1 < p \leq 3 - \frac{4}{q^2} \text{ and } p > q^2 - 2. \end{cases}$$

4. Point P has one solution, iff

$$\begin{cases} -\sqrt{3} < q < 1 \; or \; \sqrt{2} \leq q < \sqrt{3} \\ q^2 - 2 < p < 1 \end{cases} \text{or} \begin{cases} 1 < q < \sqrt{2} \\ q^2 - 2 < p \leq 3 - \frac{4}{q^2} \end{cases}$$

$$\text{or} \begin{cases} 0 < q \leq 1 \text{ or } \sqrt{2} \leq q < \sqrt{3} \\ p = 1 \end{cases} \text{or} \begin{cases} 0 < q \leq 1 \\ p = \frac{4+q^2}{4}. \end{cases}$$

Figure 2 is the solution distribution diagram for this special case. L_1 is $p = \frac{4+q^2}{4}$, L_2 is $p = q^2 - 2$, and L_3 is $p = 3 - \frac{4}{q^2}$. The following table shows where the solutions come from for each region.

Solution number	1	2a	2b	2c	3	4
Equation systems	(16)	(17)	(17)	(16)	(16) and (17)	(16) and (17)

Table 2. Number of solutions and the component providing the solutions

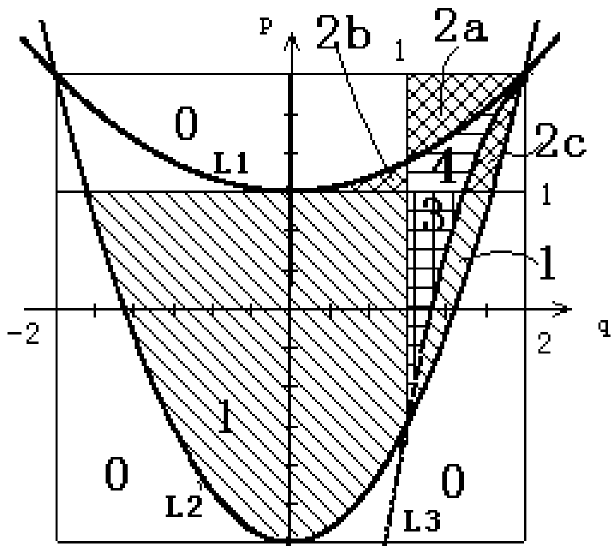


Figure 2. Solution distribution for case $a = b = 1, q = r$

Here are some general observations from this diagram. The P3P most probably will have one solution. The probabilities to have two, three and four solutions decrease in order. The P3P problem tends to have more solutions when the three perspective angles are small. The most complicated case occurs when the three perspective angles are almost equal to the correspondent inner angles of triangle ABC.

4 Experimental Results on the Algebraic Method

Two sets of formulas are given in Sections 2 and 3 respectively. In this section, we will show how to combine them to obtain more robust numerical solutions.

In Section 2, we give a set of complete analytical solutions to the P3P problem. Previous methods only give the main component, which, although covering most of the cases, may not provide solutions in many cases. With these formulas, numerical solutions to the P3P problems could be found in all cases. Solution of the P3P problem is reduced to the computation of Zero(TS_i/T_i). The main component Zero(TS_1/T_1) has been studied in previous work [9, 4], in which we need to solve a quartic equation. To solve cases $\mathrm{Zero}(TS_i.T_i), i > 1$, we need to solve equations with degrees ≤ 3. To solve linear and quadratic equations is trivial. For the cubic and quartic equations, we may write their roots as formulas with radicals. Because these formulas involve $\sqrt{-1}$, it is generally difficult to distinguish real roots from complex roots. In Section 3, we also give explicit formulas to determine the number of real positive roots. Combining these formulas with the solutions in radical form gives us an efficient and stable method to solve these equations.

Based on the criteria obtained in Section 3, we introduce the concept of stable and critical values for the parameters. For a condition $\mathbf{C}$ in these criteria, let $\mathcal{F}(\mathbf{C})$ be the set of ϕ such that one of the following formulas: $\phi > 0$, $\phi < 0$, $\phi \neq 0$ occurs in $\mathbf{C}$. For instance, let $\mathbf{C}$ be condition 1.1 in Theorem 3. Then

$$\mathcal{F}(\mathbf{C}) = \{a + b - 1, q, D_{13}, \Delta_1, \Delta_2, \Delta_3\}.$$

A set of values for the parameters a, b, p, q, r is said to be *stable* for condition $\mathbf{C}$ if for each $\phi \in \mathcal{F}(\mathbf{C})$, $|\phi(a, b, p, q, r)| > \delta$, where $\delta > 0$ is a small number, say 1% of the value range of the parameters. Otherwise, it is *critical*. Basically speaking, the critical values are those which will approximately vanish the expressions in $\mathcal{F}(\mathbf{C})$. Therefore, the probability for the occurrence of the critical values is near zero.

If a set of values is stable, then a small variation of these values will give the same number of solutions. On the contrary, for a set of critical values, a small variation may lead to changes in the number of solutions. Therefore, for a set of stable values, we may use usual floating-point calculations to enhance the computation speed; and for a set of critical values, we may use high precision computation tools provided by symbolic computation software or by other special tools, like the exact geometric computation method[18].

From the above discussion, we propose the following the CASSC algorithm.

CASSC Algorithm.

Input: A set of values for parameters a, b, p, q, r.

Output: The physical solutions.

S1 In this step, we will decide which of the ten components in (DE) will provide the solution. Let ES_i be the set of polynomials in TS_i involving the parameters a, b, p, q, r only. Then for a set of parameter values a, b, p, q, r, the solutions will be provided by C_k if $ES_k(a, b, p, q, r) = 0$ and $t_k = |T_k(a, b, p, q, r)| \neq 0$. In practice, we use the criteria: $|P(a, b, p, q, r)| < 10^{-4}m$ for each $P \in ES_k$ and $t_k > 10^{-2}m$, where

$m = \max(a, b, p, q, r)$. We will find the smallest k such that the above conditions are satisfied. If no such k, then there exists no solution. Otherwise, goto S2.

S2 Determine the number of physical solutions with the criteria given in Section 3. Let N be the number and ϕ the criterion used to determine this number. If $N = 0$, the algorithm terminates. Otherwise goto S3.

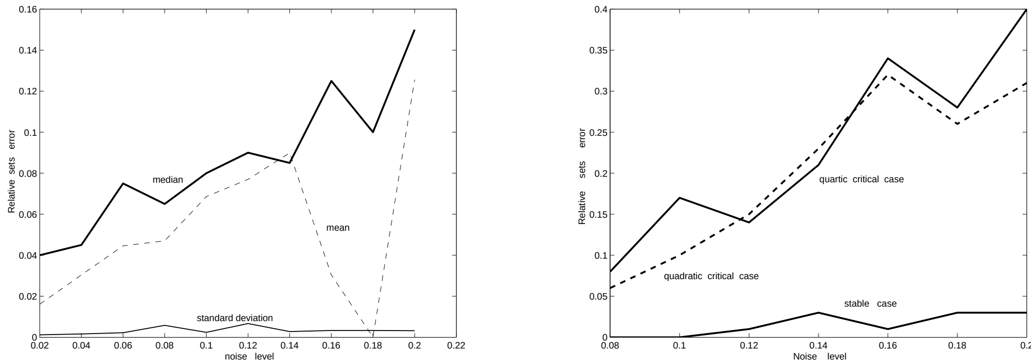
S3 In this step, we will decide the digits of precisions used in the computation. With the criterion ϕ obtained in step S2, we may determine whether a, b, p, q, r are stable values. If they are stable, let $M = 16$ (usual floating-point number); otherwise, let $M = 40$.

S4 Find all the solutions of $TS_k = 0$ using high precision numbers with M digits.

S5 If the number of solutions obtained in S4 is the same as N, then these are the solutions. Otherwise, we need to select N "right" solutions from them. We first replace a complex number $u + vi$ in the solutions with u if v is very small, say $|v| < 10^{-4}m$. Then, we select the N largest positive solutions. According to Step S2, this is possible.

The following experiments are done with Maple. The first experiment is to show the stability of the criteria in Section 3. These formulas use arithmetic operations (+, -, *) only and are of moderate size. The computation will be robust. Also, from the above analysis, only for critical values of the parameters, the computation will be un-stable. We also know that the probability for the occurrence of the critical values is near zero. This observation gives another assurance that the computation is stable. The following experimental results support this statement.

The parameters a, b, p, q, and r are randomly generated within some ranges by a random number generator in Maple 7. We take a, $b \in (\frac{1}{10}\ 10)$, and p, q, $r \in (-2\ 2)$. One hundred trials are carried out and 100 sets of parameters are generated for each trial. For each set of parametric values, two results are computed: one with the original parametric values; the other with the parametric values perturbed by random noises in certain level. In trial i, let n_i be the number of the parametric values such that the two results are the same and let $\frac{\|n_i - n\|}{n}$ (here $n = 100$) be the relative sets error. Figure 3 (a) gives the median, mean, and standard deviation of the relative errors w.r.t. varying noises. We observe that the algorithm yields very graceful degradation with increasing noises and are, therefore, very stable.



(a) (b)

Figure 3. Experimental results

The second experiment concerns the stability of the whole CASSC algorithm. We choose three sets of parametric values $S_1 = \{a = 0.6, b = 0.5, p = r = 0, q = 0\}$, $S_2 = \{a = 1, b = 1, p = 1.2, r = 1.2, q = 1.3\}$, and $S_3 = \{a = 1.35, b = 1.65, p = 1.2, r = 1.0, q = 1.67\}$. We choose $n = 100$ random numbers as noises nearing each of the seven numbers: $e_1 = 0.08, e_2 = 0.1, e_3 = 0.12, e_4 = 0.14, e_5 = 0.16, e_6 = 0.18, e_7 = 0.2$ and compute the solutions for S_1, S_2 and S_3 with our algorithm. Let n_i be the number of parametric values near e_i which give different number of solutions with the original parametric values, and $\frac{\|n_i\|}{n}$ the relative sets error. The experimental results are illustrated in Figure 3 (b). It is easy to check that S_3 is stable and S_1, S_2 are critical. The experimental results strongly support the fact that our definition of stable and critical values are meaningful. From Figure 3 (b), we see that even for a set of stable values (S_3), the computation may be unstable. This is caused by the high noise level. The computation is stable for noises less that 10% of the value range of the parameters, which is quite reasonable.

We test our algorithm with a larger set of samples. For a set of solutions obtained with the algorithm, we substitute them into (ES) and check whether the substituted values are zero or not. We take 100 sets of parameters randomly. The maximal substituted value into (ES) is $0.3 * 10^{-10}$ for the equation systems, which is satisfactory.

We also tested the speed for the CASSC algorithm, which should be fast for the following reasons. Steps S1, S2, S3 only involve the evaluation of rational expressions of moderate size. Step S4 is to solve univariate equations of degrees at most four. Step S5 is computationally trivial. We test our algorithm with one hundred randomly chosen samples. The average running time for steps S1, S3 and S5 is almost zero; the average running time for step S2 is 0.011 second; the average running time for step S4 is 0.013 second. The data is collected with Maple V on a PC with a 2G CPU. Step S4, which is to solve the quartic equations, is the most time consuming step. Note that this step is needed in most previous approaches to solving the P3P problem. The implementation is based on the interpreter language of Maple for symbolic computation, which is known to be much slower than implementations with C languages for the tasks mentioned above.

As a conclusion, the CASSC algorithm is quite fast. There is no problem to provide realtime solutions to the P3P problem.

5 The Geometric Approach

Let us consider the three conditions $\alpha = \angle BPC$, $\beta = \angle APC$, and $\gamma = \angle APB$ separately. The set of all P satisfying condition $\angle APB = \gamma$ is part of a toroid (part-toroid) S'_{AB}. Similarly, we can define S'_{AC} and S'_{BC}. Because the three part-toroids are symmetric with the plane ABC, we need only consider what happens on one side of plane ABC. Let S_{AB} denote the half of S'_{AB} which is on one side of plane ABC. We can similarly define S_{AC} and S_{BC}.

We divide the problem into two steps: first, we determine the intersection curve C_A of surfaces S_{AB} and S_{AC}; then, we determine the intersection of C_A with S_{BC}. We have solved the first step completely. For the second step, we have some partial results.

5.1 Determine $C_A = S_{AB} \cap S_{AC}$.

Let $\widehat{AB}_i$ ($\widehat{AB}_e$) denote the intersection of S_{AB} and plane ABC which is on the same (opposite) side of AB with point C. Since the axes of symmetry for part-toroids S'_{AB} and S'_{AC} meet in point A and point A is also on the part-toroids, from the shape of the part-toroid each branch of C_A must pass through plane ABC. That is, C_A must meet with plane ABC. Curve C_A intersects with plane ABC in at most four points: $J = \widehat{AB}_e \cap \widehat{AC}_e, H = \widehat{AB}_e \cap \widehat{AC}_i, K = \widehat{AB}_i \cap \widehat{AC}_e$, and $I = \widehat{AB}_i \cap \widehat{AC}_i$. Please note that in some cases (e.g., in Figure 7) point A may not be on C_A. In this case, point A is on the intersection curve of the parts of the toroids that are excluded by us.

From now on, we also use A, B, C to denote the angles of $\angle A, \angle B, \angle C$. We first give the existence conditions for points J, H, K and I.

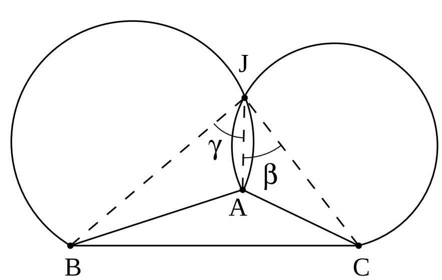


Figure 4. Existence conditions for point J

- Point J exists if $\beta+\gamma<A$ (Figure 4). In Figure 4, $\angle BJA=\gamma$, $\angle CJA=\beta$, and $\angle BJC=\beta+\gamma$. If $\beta+\gamma$ is large enough, $\widehat{AB_e}$ and $\widehat{AC_e}$ will have no intersection point. If $\beta+\gamma=A$, $\widehat{AB_e}$ is tangent to $\widehat{AC_e}$ at point A. If $\beta+\gamma<A$, the intersection of $\widehat{AB_e}$ and $\widehat{AC_e}$ will exist.

- Point H. There are two cases. If $B<\beta$, then point B is outside of S_{AC} and H_1 exists if $B<\beta$ and $\beta+A<\gamma$. In Figure 5, $\angle BH_1A=\gamma$, $\angle CH_1A=\beta$. To ensure the existence of H_1, γ must be greater than $\beta+A$. If $\gamma=\beta+A$, $\widehat{AC_i}$ is tangent to $\widehat{AB_e}$ at point A. If $\beta<B$, then point B is inside S_{AC} and H_2 exists if $\beta<B$ and $\gamma<\beta+A$.

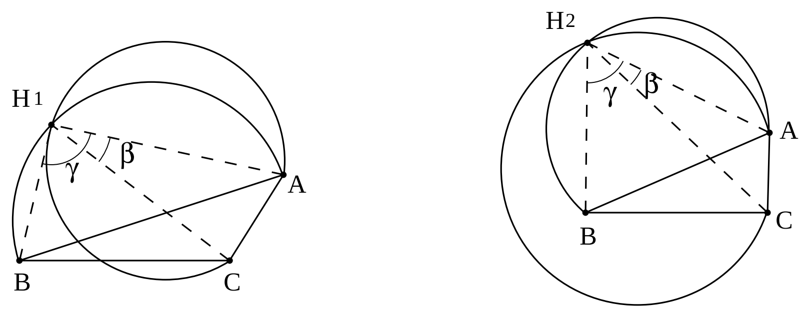


Figure 5. Two cases for point H

- Point K. There are two cases. K_1 exists if $C<\gamma$ and $\gamma+A<\beta$. K_2 exists if $\gamma<C$ and $\beta<\gamma+A$.

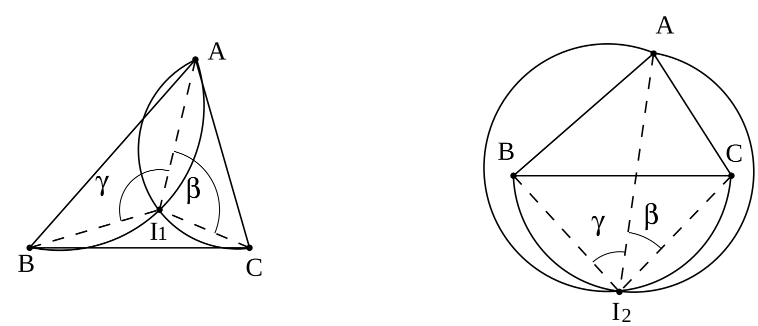


Figure 6. Existence conditions for point I

- Point I. There are two cases: I_1 exists if $B<\beta, C<\gamma$, and $\beta+\gamma+A<2\pi$ (Figure 6). I_2 exists if $\beta<B$ and $\gamma<C$ (Figure 6).

We will now give a classification of C_A by counting the intersections of C_A with plane ABC. Suppose that S,U,V are points. We use $E(S)$ ($\overline{E(S)}$) to denote the existence (nonexistence) condition of point S. Notation $S\ \setminus (U,V)$ means that if S exists then U

and V will not exist. Notation $S, U \Rightarrow V$ means that if S and U exist then V exists. From the results in the preceding sections, we have the following conclusion.

$$\begin{cases} J \setminus (K_1, H_1) \\ H_1 \setminus (J, K_1, H_2, I_2), H_2 \setminus (H_1, I_1) \\ K_1 \setminus (J, H_1, K_2, I_2), K_2 \setminus (K_1, I_1) \\ I_1 \setminus (I_2, K_2, H_2), I_2 \setminus (I_1, K_1, H_1) \\ K_2, H_2 \Rightarrow I_2, J, I_2 \Rightarrow H_2, K_2. \end{cases} \tag{19}$$

- C_A intersects plane ABC in four points. From the above analysis, the four points must come from $H_1, H_2, K_1, K_2, I_1, I_2$, and J. Since $H_1 \setminus (H_2)$, $K_1 \setminus (K_2)$ and $I_1 \setminus (I_2)$, the fourth point must be J. From $J \setminus (K_1, H_1)$, K_2 and H_2 must exist. Finally from $K_2, H_2 \Rightarrow I_2$ we get the fourth point I_2. So the four points are J, H_2, K_2, I_2. Then the condition of this case should be $E(J) \cap E(H_2) \cap E(K_2) \cap E(I_2) \cap \overline{E(H_1)} \cap \overline{E(K_1)} \cap \overline{E(I_1)}$, which is equivalent to $E(J) \cap E(I_2)$ by (19). That is,

$$\beta + \gamma < A, \beta < B, \text{ and } \gamma < C.$$

In this case, C_A consists of two spatial curves: one is from point J to I_2 and the other is from H_2 to K_2. Figure 7 shows the case in the ABC plane and the spatial case. Note that in this case, point A is not on the curves.

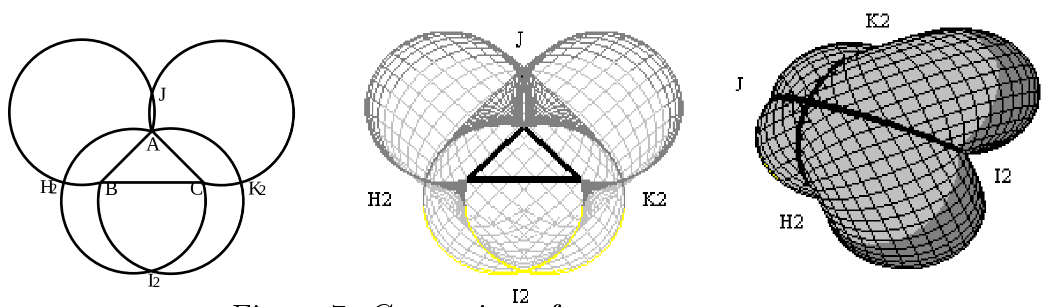


Figure 7. C_A consists of two curves

- C_A intersects plane ABC in three points. From $J \setminus (K_1, H_1)$, we know that if J exists, at lease one of H_2 and K_2 should exist. Actually only one of H_2 and K_2 can exist. Otherwise from $K_2, H_2 \Rightarrow I_2$ we know that there will be four points! Then we know that either H_2 or K_2 exists. From $H_2 \setminus (H_1, I_1)$ and $K_2 \setminus (K_1, I_1)$, we know that I_2 must exist. Since $J, I_2 \Rightarrow H_2, K_2$, point J must not exist. Since $H_1 \setminus (H_2), K_1 \setminus (K_2)$ and $I_1 \setminus (I_2)$, if we assume that H_1 exists, from $H_1 \setminus (J, K_1, H_2, I_2)$ we know that the other two points are K_2 and I_1. This contradicts to $K_2 \setminus (K_1, I_1)$. Thus H_2 must exist. From $H_2 \setminus (H_1, I_1)$ we know the other two points are K_2 and I_2. The condition of this case should be $E(H_2) \cap E(K_2) \cap E(I_2) \cap \overline{E(J)} \cap \overline{E(H_1)} \cap \overline{E(K_1)} \cap \overline{E(I_1)}$. Using (19) we can simplify this condition to $E(H_2) \cap E(K_2) \cap \overline{E(J)}$. That is,

$$|\beta - \gamma| < A < \beta + \gamma, B < \beta, \text{ and } \gamma < C.$$

In this case, C_A consists of two spatial curves: one is from A to I_2 and the other is from H_2 to K_2. Since the detailed analysis is the same, we will omit it below.

- C_A intersects plane ABC in two points. There are five sub-cases.

Case 1 The intersections are J, K_2 (J, H_2) if

$$\begin{cases} \beta+\gamma<A \\ B<\beta \\ \gamma<C \end{cases} \qquad \left(\begin{cases} \beta+\gamma<A \\ \beta<B \\ C<\gamma \end{cases}\right).$$

In this case, C_A consists of one spatial curve from J to K_2 (J to H_2).

Case 2 The intersections are J, I_1 if

$$\beta+\gamma<A, B<\beta, \text{ and } C<\gamma.$$

In this case, C_A consists of one spatial curve from J to I_1.

Case 3 The intersections are H_2, K_1 (H_1, K_2) if

$$\begin{cases} \gamma+A<\beta \\ \beta<B \\ C<\gamma \end{cases} \qquad \left(\begin{cases} \beta+A<\gamma \\ B<\beta \\ \gamma<C \end{cases}\right).$$

In this case, C_A consists of one spatial curve from H_2 to K_1 (or H_1 to K_2).

Case 4 The intersections are K_1, I_1 (H_1, I_1) if

$$\begin{cases} \gamma+A<\beta \\ B<\beta \\ C<\gamma \\ \beta+\gamma+A<2\pi \end{cases} \qquad \left(\begin{cases} \beta+A<\gamma \\ B<\beta \\ C<\gamma \\ \beta+\gamma+A<2\pi \end{cases}\right).$$

In this case, C_A consists of one spatial curve from K_1 to I_1 (H_1 to I_1).

Case 5 The intersections are K_2, I_2 (H_2, I_2) if

$$\begin{cases} \beta+A<\gamma \\ \beta<B \\ \gamma<C \end{cases} \qquad \left(\begin{cases} \gamma+A<\beta \\ \beta<B \\ \gamma<C \end{cases}\right).$$

In this case, C_A consists of one spatial curve from K_2 to I_2 (H_2 to I_2).

- C_A intersects plane ABC in one point. We need to consider two sub-cases.

  Case 1 The intersection is H_2 (K_2) if

  $$\begin{cases} |\beta-\gamma| < A < \beta+\gamma \\ \beta < B \\ C < \gamma \end{cases} \qquad \left(\begin{cases} |\beta-\gamma| < A < \beta+\gamma \\ B < \beta \\ \gamma < C \end{cases} \right).$$

  In this case, C_A consists of one spatial curve from A to H_2 (A to K_2).

  Case 2 The intersection is I_1 if

  $$|\beta-\gamma| < A < \beta+\gamma, B < \beta, C < \gamma, \text{ and } \beta+\gamma+A < 2\pi.$$

  In this case, C_A consists of one spatial curve from A to I_1.

5.2 Determine $C_A \cap S_{BC}$.

Determine the intersection of C_A and S_{BC} is much more difficult than determine C_A. We will discuss the reason in Section 6. Here, we will report some partial results.

Lemma 7 *The P3P problem has one or three solutions if C_A consists of one spatial curve and the two intersection points of plane ABC and C_A are not in the same side of S_{BC}.*

Proof: Since C_A is a continuous spatial curve and the two intersection points of plane ABC and C_A are not in the same side of S_{BC}, C_A must intersect S_{BC} for odd times. In addition, the maximum number of solutions is four, hence the problem has a unique solution or three solutions.

Lemma 8 *If β, γ ($\alpha, \beta; \gamma, \alpha$) are obtuse angles and $\alpha > A$ ($\beta > B$; $\gamma > C$), then the P3P problem can only have one or three solutions.*

Proof: See Figure 8. We have $\angle BI_1A = \gamma > \frac{\pi}{2}$, $\angle CI_1A = \beta > \frac{\pi}{2}$. Point I_1 is on the same side of BC with point A. According to the "reality condition", we know that $\alpha+\beta+\gamma < 2\pi$, which implies that point I_1 is inside S_{BC}. Condition $\alpha > A$ means that point A is in the outside of S_{BC}. Thus the result follows from Lemma 7.

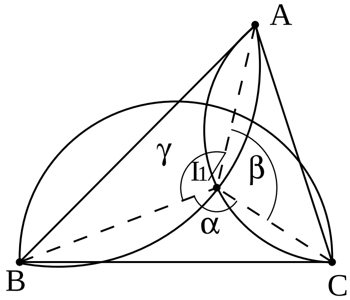


Figure 8. A unique solution exists

Theorem 9 *Under the reality conditions (2), if β, α, and γ are obtuse, then the P3P problem can only have one solution. Furthermore if $A < \alpha, B < \beta, C < \gamma$, then the P3P problem has a unique solution.*

Proof: From Lemma 8, we know that the problem will have one or three solutions since β, α, and γ are obtuse and at least one of A, B, C is acute. Since the three angles are all obtuse, the three part-toroids and their intersection curves are concave. This implies that they can only have one intersection point. If $A < \alpha, B < \beta, C < \gamma$, from Section 5.1, point I_1 must exist. Similar to Lemma 8, points A and I_1 must be in different sides of S_{BC}. Similar to the proof of Lemma 7, a solution must exist.

6 Conclusion

In this paper, we give a complete and robust algorithm CASSC to find the numerical solutions for the P3P problem. This algorithm is based on two sets of formulas obtained by us. The first is a set of complete analytical solutions to the P3P problem. The second is a set of formulas to determine the number of real positive solutions to the P3P problem.

We also give partial geometric criteria for the number of solutions of the P3P problem. This kind of results, like Theorem 9, involves linear inequalities only, and hence is simpler and more intuitive than the algebraic approach. To find a complete geometric classification for the P3P problem is a still challenging problem. There might be two difficulties in doing so. The complete results reported in Section 5.1 are based on geometric intuition coming from a dynamic geometry software: *Geometry Expert* [23]. Using *Geometry Expert*, we can see clearly how $\widehat{AB}_i$ and $\widehat{AB}_e$ change when changing the six free parameters continuously. But for the 3D case, there is still no adequate software to get an intuitive idea of how C_A looks like. Also, it is doubtful that the complete classification of the P3P problem can be expressed with linear inequalities only.

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Appendix A The triangular sets in the zero decomposition for the P3P problem

$$\begin{cases} a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4, \\ b_0y - b_1. \end{cases} \qquad (TS_1)$$

$$\begin{cases} a_5x^3 + a_6x^2 + a_7x + a_8, \\ b_2y - b_3, \\ a^2 + (-2 + 2b - br^2)a - 2b + b^2 + 1. \end{cases} \qquad (TS_2)$$

$$\begin{cases} (r^2p^2 - 4pqr + r^2q^2)x^2 + (4p^2q - p^2r^2q)x - 4p^2 + r^2p^2, \\ b_4y - b_5, \\ (-4p^2 + 4pqr + r^2p^2 + r^2q^2 - r^3pq - 4q^2)a + r^2p^2 - 4pqr + 4q^2, \\ (-4p^2 + 4pqr + r^2p^2 + r^2q^2 - r^3pq - 4q^2)b + r^2q^2 + 4p^2 - 4pqr. \end{cases} \qquad (TS_3)$$

$$\begin{cases} (p^2b + q^2b - p^2)x^2 + (-4bq + p^2q)x + 4b - p^2, \\ py + qx - 2, \\ a + b - 1, \\ r. \end{cases} \qquad (TS_4)$$

$$\begin{cases} qx - 1, \\ py - 1, \\ (p^2 + q^2)a - q^2, \\ (p^2 + q^2)b - p^2, \\ r. \end{cases} \qquad (TS_5)$$

$$\begin{cases} qx - 1, \\ py - 1, \\ (p^4 - 2p^2q^2 + q^4)a - p^2q^2 - q^4, \\ (p^4 - 2p^2q^2 + q^4)b - p^2q^2 - p^4, \\ (p^2 + q^2)r - 4pq. \end{cases} \qquad (TS_6)$$

$$\begin{cases} (p^2q - 2pr)x - p^2 + r^2, \\ py - 1, \\ (4r^2 + p^2q^2 + p^4 - r^4 - p^3qr + pr^3q - 4qpr)b \\ +2pr^3q - 2p^2r^2 + 2p^3qr - p^2q^2r^2 - p^4 - r^4. \end{cases} \qquad (TS_7)$$

$$
\left\{
\begin{array}{l}
(2pr^3q - 2p^2r^2 + 4br^2 + p^2q^2b + p^4b - r^4b + 2p^3qr - p^2q^2r^2 - p^4 - r^4 - \\
p^3qrb + r^3pbq - 4qbpr)x^2 + (-q^2r^3pb + 2pq^2br + 2p^2r^2q + p^2q^3r^2 + r^4q + qr^4b \\
+p^4q - 2bpr^3 + 3qr^2bp^2 - 4r^2bq + 8rpb - 2rbp^3 - 2p^3q^2r - 4bp^2q - 2r^3pq^2)x \\
-p^2q^2r^2 + 2pr^3q + 2p^3qr - p^4 - r^4 - 2p^2r^2 - 4qbpr + q^2br^2 + 4bp^2, \\
(-qpr + p^2 + r^2)y + pqx - 2rx - 2p + qr.
\end{array}
\right. \quad (TS_8)
$$

$$
\left\{
\begin{array}{l}
(-1 + a + b)x^2 + (-qa + q)x - 1 + a - b, \\
(-1 + a + b)y^2 - 1 - a + qxa + b, \\
p, \\
r.
\end{array}
\right. \quad (TS_9)
$$

$$
\left\{
\begin{array}{l}
(2pr - p^2q)x - r^2 + p^2, \\
py - 1, \\
(-pqr^3 + r^4 + rp^3q - 4r^2 - p^2q^2 + 4rpq - p^4)a + p^2q^2 - 4rpq + 4r^2, \\
(-pqr^3 + r^4 + rp^3q - 4r^2 - p^2q^2 + 4rpq - p^4)b+ \\
p^4 + r^4 + 2r^2p^2 + p^2r^2q^2 - 2rp^3q - 2pqr^3.
\end{array}
\right. \quad (TS_{10})
$$

$$
\left\{
\begin{array}{l}
rx - p, \\
(-p^2r^2 + r^3qp - r^4)y^2 + (p^3r^2 - p^2r^3q + r^4p)y \\
(-pqr^3 + r^4 + rp^3q - 4r^2 - p^2q^2 + 4rpq - p^4)a + p^2q^2 - 4rpq + 4r^2, \\
(-pqr^3 + r^4 + rp^3q - 4r^2 - p^2q^2 + 4rpq - p^4)b+ \\
p^4 + r^4 + 2r^2p^2 + p^2r^2q^2 - 2rp^3q - 2pqr^3.
\end{array}
\right. \quad (TS_{11})
$$

$$(20)$$

Here,

$$
\begin{aligned}
a_0 &= -2b + b^2 + a^2 + 1 - br^2a + 2ba - 2a \\
a_1 &= -2bqa - 2a^2q + br^2qa - 2q + 2bq + 4aq + pbr + brpa - b^2rp \\
a_2 &= q^2 + b^2r^2 - bp^2 - qpbr + b^2p^2 - br^2a + 2 - 2b^2 - abrpq + 2a^2 - 4a - 2q^2a + q^2a^2) \\
a_3 &= -b^2rp + brpa - 2a^2q + qp^2b + 2bqa + 4aq + pbr - 2bq - 2q \\
a_4 &= 1 - 2a + 2b + b^2 - bp^2 + a^2 - 2ba, \\
a_5 &= (apr + 2qa - rpb + 2bq - 2q - ar^2q + pr, \\
a_6 &= (-2q^2 + r^2 - 4 + r^2q^2 - pqr)a + br^2 - p^2 - bq^2 + bp^2 + 2q^2 + 4 - pqr - 4b, \\
a_7 &= (6q + pr - 2r^2q)a + pr - 6q - rpb + 2bq + qp^2, \\
a_8 &= 4 - 4a - p^2 + ar^2,
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= b(p^2a - p^2 + bp^2 + pqr - qarp + ar^2 - r^2 - br^2)^2, \\
b_1 &= ((1-a-b)x^2 + (qa-q)x + 1 - a + b)((a^2r^3 + 2br^3a - br^5a - 2ar^3 + r^3 + b^2r^3 - 2r^3b)x^3 + \\
&\quad (pr^2 + pa^2r^2 - 2br^3qa + 2r^3bq - 2r^3q - 2par^2 - 2pr^2b + r^4pb + 4ar^3q + bqar^5 - 2r^3a^2q \\
&\quad +2r^2pba + b^2r^2p - r^4pb^2)x^2 + (r^3q^2 + r^5b^2 + rp^2b^2 - 4ar^3 - 2ar^3q^2 + r^3q^2a^2 + \\
&\quad 2a^2r^3 - 2b^2r^3 - 2p^2br + 4par^2q + 2ap^2rb - 2ar^2qbp - 2p^2ar + rp^2 - br^5a + 2pr^2bq + \\
&\quad rp^2a^2 - 2pqr^2 + 2r^3 - 2r^2pa^2q - r^4qbp)x + 4ar^3q + pr^2q^2 + 2p^3ba - 4par^2 + \\
&\quad -2r^3bq - 2p^2qr - 2b^2r^2p + r^4pb + 2pa^2r^2 - 2r^3a^2q - 2p^3a + p^3a^2 + 2pr^2 + p^3 + 2br^3qa \\
&\quad +2qp^2br + 4qarp^2 - 2par^2q^2 - 2p^2a^2rq + pa^2r^2q^2 - 2r^3q - 2p^3b + p^3b^2 - 2p^2brqa), \\
b_2 &= b(-4ar^3 + 4r^3 + ar^5 - 2p^3q + 4rp^2 - 6pqr^2 - 4rp^2b - 4p^2ar + 6par^2q + \\
&\quad 2p^2rq^2 + 2p^2ar^3 + 2p^3bq + 2p^3qa + p^4ar + p^2ar^3q^2 - 2p^2rq^2a - p^2rbq^2 - \\
&\quad 2p^3ar^2q - 2par^4q + 2pr^2bq), \\
b_3 &= ((-1+a+b)x^2 + (-qa+q)x - 1 + a - b)((-par^3 + ar^4q - 2ar^2q - 2r^2bq + 2r^2q - pr^3 + \\
&\quad r^3bp)x^2 + (-r^2p^2a + 2r^3paq + 4ar^2 - r^4q^2a - ar^4 - 2qarp + 2ar^2q^2 - r^4b + r^2bq^2 + r^3pq \\
&\quad +2pqr + 4r^2b - 2qpbr - 2r^2q^2 - 4r^2)x - p^3ar + 2ar^4q - par^3q^2 + 2p^2ar^2q - 2par^3 + 2prq^2a \\
&\quad -2p^2aq - 6ar^2q + 4apr - pr^3 + 4pbr + prbq^2 - 2r^2bq - 2prq^2 + 2qp^2 - 2bqp^2 - 4pr + 6r^2q), \\
b_4 &= r^2p^2(rq^2 + p^2r - 4pq)(p^2 - pqr + r^2 + q^2 - 4), \\
b_5 &= r^2q((rp^2 + rq^2 - r^2pq)x + pr^2 - 4p)((rp^2 + rq^2 - 4pq)x + q^2p - qp^2r + p^3), \\
I_1 &= a_0, I_2 = b_0, I_3 = a_5, I_4 = b_2, I_5 = r, I_6 = rp - 4pq + rq^2, I_7 = p, I_8 = (p^2+q^2)b - p^2. \\
T_1 &= I_0I_1I_2, T_2 = I_0I_2I_3I_4, T_3 = I_0I_2I_4I_5I_6I_7, T_4 = I_0I_2I_4I_6I_7I_8, \\
T_5 &= I_0I_2I_4I_6I_7, T_6 = I_0I_2I_4I_5I_7, T_7 = I_0I_2I_3, T_8 = T_7, T_9 = I_0I_1, T_{10} = I_0I_1, T_{11} = I_0I_1.
\end{aligned}
$$

Appendix B. The explicit formulas for the conditions in the theorems in Section 3 can be found in http://www.mmrc.iss.ac.cn/˜ xgao/paper/appendix.ps.