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Automated Reasoning in Differential Geometry and Mechanics Using Characteristic Method[1]

III. Mechanical Formula Derivation

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Abstract In this paper, methods of mechanical deriving of formulas from a set of differential polynomial equations and a set of differential polynomial inequations are presented. The methods have been used successfully to many problems in space curve theory and mechanics. In particular, a mechanical derivation of Newton's gravitational law from Kepler's laws has been given without knowing Newton's laws in advance. We also give a partial method to derive algebraic relations from a set of differential polynomial equations.

1. Introduction

In the third paper of this series, we will present methods for mechanical formula derivation in differential geometry and mechanics using Ritt–Wu's decomposition algorithm for differential polynomials described in the first paper of this series [1]. For a closely related topic, mechanical theorem proving in differential geometry, see [8, 2, 5].

In [3], we give a precise formulation and complete methods of mechanical formula derivation in elementary geometry. In this paper, the formulation and the method based on Ritt–Wu's decomposition algorithm have been extended to the differential polynomial case with several modifications. The extended method has been used to solve quite a few examples in differential geometry and mechanics successfully. But for some problems the method fails to give the desired results. For example, Newton's gravitational law can not be derived from Kepler's laws by the method. Wu gave a derivation of the Newton's law from Kepler's laws in [7], but the desired relations do not occur in the final ascending chain obtained by Ritt–Wu's well ordering algorithm, instead they occur in the middle steps of the process. In our point of view, this is not an automated process. More human interactions must be involved: one has to check the differential polynomials (ab. d-pols) produced in each step carefully, and the occurrence of the desired formula cannot be guaranteed.

The problem is that some relatively simple relations (i.e., with lower orders) among certain variables are what we actually seek. Therefore, we further reformulate the problem and present a new method which can be used to find such simple relations mechanically. The method is

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used to plane mechanics and differential geometry and are quite successful for certain problems in mechanics. In particular, we give an automated derivation of Newton's square inverse gravitational law from Kepler's laws in the sense that the formula representing Newton's law occurs as the first element of the ascending chain of the non-degenerate component. The derivation procedure is carried out without human assistant. About ten problems in mechanics are solved in this way [4].

Unlike in the elementary geometry, in the case of d-pols, the relations we find are generally some differential equations. But generally, what we really want is the solutions of these differential equations. So, to make the algorithm more complete, we need a method of solving multivariate algebraic differential equations. In this paper, we give a partial method to decide whether the general solution of a multivariate algebraic differential equation is a polynomial equation with constant coefficients. We also present a partial method to decide whether some variables in a geometric problem satisfy a polynomial equation with constant coefficients.

We assume the reader is familiar with Ritt–Wu's zero decomposition theorem for d-pols a detailed description of which can be found in [1].

The paper is organized as follows. In Section 2, we present an algorithm of mechanical formula derivation. In Section 3, a method of finding algebraic relations from a geometric problem is given. In Section 4, we shall show the defect of the derivation algorithm when we try to derive Newton's laws from Kepler's laws and a solution to the problem in this concrete case. In Section 5, we give a refined method based on the idea of Section 4.

2. Methods of Mechanical Formula Derivation (1)

2.1. Formulation of the Problem

Let K be a computable field with characteristic zero (in practice, $K = \mathbf{Q}(t)$) and $x_1, ..., x_n$ be indeterminates. We use $K\{x_1, ..., x_n\}$, or $K\{x\}$ to denote the d-pol ring of the indeterminates x with coefficients in K. For convenience, we often rename the x as two groups of indeterminates $u_1, ..., u_q$ and $y_1, ..., y_p$ where $q + p = n$.

Definition 2.1. Let D be an ideal in $K\{u_1, ..., u_q, y_1, ..., y_p\}$. Then $\{u_1, ..., u_q\}$ is called a parameter set of D if (1) the u are algebraically independent with respect to (ab. wrpt) D, i.e., $D \cap K\{u\} = \{0\}$; and (2) each y_i is algebraic dependent on the u, i.e., $D \cap K\{u, y_i\} \neq \{0\}$.

We shall consider geometric problems, after adopting an appropriate coordinate system, whose geometric configurations can be expressed by several d-pol equations

$$h_1(u_1, ..., u_q, y_1, ..., y_p) = 0 \wedge \cdots \wedge h_s(u_1, ..., u_q, y_1, ..., y_p) = 0$$

together with some d-pol inequations

$$d_1(u_1, ..., u_q, y_1, ..., y_p) \neq 0 \wedge \cdots \wedge d_l(u_1, ..., u_q, y_1, ..., y_p) \neq 0.$$

Let $HS = \{h_1, ..., h_s\}$ and $DS = \{d_1, ..., d_l\}$, then such a *geometric problem* can be represented by $S = (HS, DS)$. For the above geometric problem, let

$$HD = Ideal\{h_1, ..., h_s, z_1 d_1 - 1, ..., z_l d_l - 1\} \tag{2.2}$$

where $z_1, ..., z_l$ are distinct new variables. We say the u are a *parameter set of S* if they are a parameter set of the ideal HD. The y are called *dependent variables*. The geometric meaning of the parameter set may be explained below. In S, the parameters u_i can generally take any value and the dependent variables y_j can be determined as some functions of the u by the geometric hypotheses.

Before formulating our problem, we introduce a new order among d-pols. For d-pols f and g, we define an order $f \prec g$ inductively as follows: (1) if f is in K and g is not, $f < g$; (2) otherwise, $f \prec g$ if either f is of lower rank than that of g or f and g have the same rank and $int(f) \prec int(g)$. It is easy to see that the order $\prec$ is well-ordered, i.e. there exists no strictly decreasing d-pol sequence under the order $\prec$. The maximal monomial of a d-pol P under the order $\prec$ is called the *leading term* of P.

Definition 2.3. Let $S = (HS, DS)$ be a geometric problem with parameters $u_1, ..., u_q$, and let HD be defined as (2.2). For a dependent variable y_{i_0}, the *relation set* among the u and y_{i_0} is a set of d-pol equations $r_1(u, y_{i_0}) = 0$, ...,$r_k(u, y_{i_0}) = 0$, all containing y_{i_0}, such that: (1) all $r_i(u, y_{i_0})$ are monic and irreducible; (2) there is a non-zero d-pol U containing the u only (*We will call such a d-pol a u–pol*) such that $U \cdot r_1(u, y_{i_0}) \cdots r_k(u, y_{i_0})$ is in $radical(HD)$; (3) the d-pol $\prod_{i=1}^{k} r_i$ is minimal under the order $\prec$ to satisfy (2).

Theorem 2.4. Let the notations and conditions be the same as Definition 2.3, then

$$(2.4.1) \quad \forall u, y[(h_1 = 0 \wedge ... \wedge h_s = 0 \wedge d_1 \neq 0 \wedge ... \wedge d_l \neq 0 \wedge U \neq 0) \rightarrow (r_1 = 0 \vee \cdots \vee r_k = 0)]$$

where U is the u–pol in (2) of Definition 2.3.

Proof. From (2) of Definition 2.3, we have

$$\forall xuz[(h_1 = 0 \wedge ... \wedge h_s = 0 \wedge d_1 z_1 - 1 = 0 \wedge \cdots \wedge d_l z_l - 1 = 0 \wedge U \neq 0) \rightarrow (r_1 \cdots r_k = 0)].$$

Since $\exists z_i(d_i z_i - 1 = 0)$ is equivalent to $d_i \neq 0$, (2.4.1) is equivalent to the above formula. ∎

Theorem 2.5. Let the notations be the same as Definition 2.3. Then the relation set for a geometric problem $S = (HS, DS)$ exists and is unique.

Proof. Let M be the d-pols in HD which involve the u and y_{i_0}. Since the u are a parameter set of HD, M is not empty. For a d-pol P in M, let

$$P = U \cdot r_1^{s_1}(u, y_{i_0}) \cdots r_k^{s_k}(u, y_{i_0})$$

where U is a u–pol, $deg(r_i, y_{i_0}) \geq 1$ and $s_i \geq 1$ for all $i = 1, ..., k$, and the r_i are distinct monic irreducible d-pols. We have $k > 0$. Let $R(P) = \{r_1(u, y_{i_0}), ..., r_k(u, y_{i_0})\}$, then $R(P)$ satisfies conditions (1)–(2) in 2.3 and all d-pol sets satisfying (1) and (2) of Definition 2.3 must be the form $R(P)$ for some d-pol P in M. Since $\prec$ is well ordering, there is a minimal d-pol in $\{\prod_{h \in R(P)} h \ : \ P \in M\}$. Let a minimal d-pol be $\prod_{h \in R(P)} h$ for some P in M, then $R(P)$ is a relation set.

To prove the uniqueness, let $R_1 = \{r_1, ..., r_t\}$ and $R_2 = \{f_1, ..., f_s\}$ be two relation sets of S. Since $\prec$ is a total order among the monomials, $\prod_{1 \leq i \leq t} r_i$ and $\prod_{1 \leq i \leq s} f_i$ must have the same leading term. By (2) of Definition 2.3, there are u-pols U_1 and U_2 such that $P_1 = U_1 \cdot \prod_{1 \leq i \leq t} r_i$ and $P_2 = U_2 \cdot \prod_{1 \leq i \leq s} f_i$ are in $radical(HD)$. Let $P = U_2 P_1 - U_1 P_2$. If $P \neq 0$, the P must involve the y and $R(P)$ also satisfies (1) and (2) of Definition 2.3. Since r_i and f_i have the same leading terms, $\prod_{f \in R(P)} f$ must be less than $\prod_{g \in R_1} g$ under the order $\prec$. This contradicts to (3) of Definition 2.3. Thus we have $P = 0$, i.e. $U_1 P_2 = U_2 P_1$. Note that U_1 and U_2 are u-pols, then $\prod_{1 \leq i \leq t} r_i = \prod_{1 \leq i \leq s} f_i$. By (1), we have $R_1 = R_2$. ∎

2.2. A Method of Finding the Relation Set

As an example, let us consider the relation set between the u and y_1 for a geometric problem (HS, DS). According to Ritt–Wu's decomposition algorithm we have the following decomposition in the variable ordering $u < y_1 < y_2 < \cdots < y_p$:

$$\text{(2.6)} \qquad \text{E-Zero}(HS/DS) = \cup_{i=1}^{a}\text{E-Zero}(PD(ASC_i^*)/DS) \bigcup \cup_{j=1}^{b}\text{E-Zero}(PD(ASC_j)/DS),$$

where all ascending chains ASC_i^* and ASC_j are irreducible such that all ASC_i^* do not contain any u-pol and each ASC_j contains at least one u-pol.

Theorem 2.7. Let the notations be the same as in the previous paragraph and HD is defined as (2.2). Then

(1) The u are algebraically independent wrpt HD iff $a > 0$.

(2) The variables y_k are algebraically dependent on the u iff each y_k appears as a leading variable in each ASC_i^*.

(3) Assume that the u are a parameter set of HD. Let $r_i(u, y_1)$ $(i = 1, ..., a)$ be the first d-pol in ASC_i^*. Then a minimal subset R of $R' = \{r_1(u, y_1), ..., r_a(u, y_1)\}$ which satisfies

$$\text{(2.7.1)} \qquad \cup_{I\in R}\text{E-Zero}(PD(I)) = \cup_{J\in R'}\text{E-Zero}(PD(J))$$

is the relation set of (HS, DS).

Proof. First we state the following repeatedly used fact: For a d-pol P in the u and y,

$$\text{(2.7.1)} \qquad \text{E-Zero}(HD) \subset \text{E-Zero}(P) \iff \text{E-Zero}(HS/DS) \subset \text{E-Zero}(P)$$

(1) Suppose $a = 0$, then according to decomposition (2.6), there is a u–pol U such that E-Zero$(HS/DS) \subset$ E-Zero(U). Thus E-Zero$(HD) \subset$ E-Zero(U). Therefore, U is in Radical(HD); hence for some k, U^k is in HD. The u are algebraically dependent. Now suppose that the u are algebraically dependent, i.e., HD contains a u–pol U. Then E-Zero$(HD) \subset$ E-Zero(U), which is equivalent to E-Zero$(HS/DS) \subset$ E-Zero(U). Since E-Zero(U) does not contain each E-Zero$(PD(ASC_i^*)/DS)$, a must be zero.

(2) Each y_i appears as a leading variable in each ASC_j^* iff E-Zero(HS/DS) is of zero dimension over $K(u)$ which is equivalent to that for each y_i there is a d-pol P_i of y_i and the u such that E-Zero$(HS/DS) \subset$ E-Zero(P_i). By (2.7.1), this is equivalent to that each y_i is algebraically dependent on the u.

(3) From decomposition (2.6), there is a u–pol U such that E-Zero$(HS/DS) \subset$ E-Zero$(U \cdot r_1 \cdots r_k)$. Thus E-Zero$(HD) \subset$ E-Zero$(U \cdot r_1 \cdots r_k)$. By Hilbert Nullstellensatz in differential case [6], $U \cdot r_1 \cdots r_k$ is in Radical(HD), i.e., R satisfies (1) and (2) of Definition 2.3. The proof of the minimal property of R is straight forward. ∎

To the best of our knowledge, there is no method to delete the redundant components in (2.7.1). Thus (3) of Theorem 2.7 is useless in practice. In practical case, we have.

Definition 2.8. For a statement (HS, DS), assume that the u are a parameter set of HD and RS be the relation set among y_1 and the u. A set of d-pols $RS' = \{r_1(u, y_1), ..., r_k(u, y_1)\}$ is called a *weak relation set* of (HS, DS) if $RS \subset RS'$, and

$$\cup_{I\in RS}\text{E-Zero}(PD(I)) = \cup_{J\in RS'}\text{E-Zero}(PD(J)).$$

Algorithm 2.9. For a geometric problem $S = (HS, DS)$ where HS and DS are two finite sets in $K\{u, y\}$, the algorithm decides whether the u are a parameter set of S, and if it is, finds a weak relation set among y_1 and $u_1, .., u_q$.

Step 1. Use Ritt–Wu's decomposition algorithm to obtain a decomposition as (2.6).

Step 2. If $a = 0$, then give the answer: "the parameters u are not algebraically independent."

Step 3. Suppose $a > 0$. If each y_j appears as a leading variable in all ASC_i^*, goto step 4. Otherwise give the answer: "the u are not a parameter set of S."

Step 4. By (3) of Theorem 2.7, the first d-pols of ASC_i^*, $i = 1, \cdots, a$, consist of a weak relation set for (HS, DS). The U in (2) of Definition 2.3 can be obtained as follows: $U = \prod_{i=1}^{b} P_i$ where $P_i \in ASC_i \cap K\{u\}$ $(i = 1, ..., b)$. ▮

Remark In the implementation, we do not need to compute the degenerate part

$$\cup_{i=1}^{b}\text{E-Zero}(PD(ASC_i)/DS)$$

explicitly. During the decomposition process, whenever a u–pol appears in a d-pol set, we can delete that d-pol set, and add that u–pol as a factor of the d-pol U in Definition 2.3. This leads to a speedup of the process.

2.3. Two Examples

Example 2.10. Compute the curvature of a straight line.

Let the curve be $C = (x, y, z)$, then the problem can be represented as (HS, DS) ([2]), where $DS = \emptyset$, $HS = \{h_1, ..., h_5\}$ and

$$
\begin{aligned}
&h_1 = z'^2 + y'^2 + x'^2 - 1 && C \text{ with its arc as parameter}\\
&h_2 = z''^2 + y''^2 + x''^2 - k^2 && k = |C''|\\
&h_3 = y'z'' - y''z' && \\
&h_4 = x'z'' - x''z' && (\text{fix-line } C)\\
&h_5 = x'y'' - x''y'. &&
\end{aligned}
$$

There are no parameters for this problem. Using Ritt–Wu's decomposition algorithm under the variable order $k < x < y < z$, we have

$$\text{E-Zero}(HS) = \cup_{1\le i\le 3}\text{E-Zero}(PD(ASC_i)) \quad \text{where}$$

$$
\begin{array}{lllll}
ASC_1 = & k & x' & y' + 1 & z'\\
ASC_2 = & k & x'' & y'' & z'^2 + y'^2 + x'^2 - 1\\
ASC_3 = & k & x' + 1 & y' & z'
\end{array}
$$

The relation set is $\{k\}$, i.e., for a straight line $k = 0$.

Example 2.11. (Bertrand Curves) A pair of space curves having their principal normals in common are said to be associate Bertrand curves. Find the relation between the curvature and the torsion of a Bertrand curve.

Given two space curves C_1 and C_2 in a one to one correspondence, following [8], let us attach *moving triads* $(C_1, e_{11}, e_{12}, e_{13})$ and $(C_2, e_{21}, e_{22}, e_{23})$ to C_1 and C_2 at the corresponding points of C_1 and C_2 respectively. We denote the arcs, curvature and torsions of C_1 and C_2 by s_1, k_{11}, t_1 and s_2, k_2, t_2 respectively. Then all the quantities introduced above can be looked as functions of s_1. Let $r = \frac{ds_2}{ds_1}$. We have $C_2 = C_1 + a_2E_{12}$ and

$$
\begin{aligned}
e_{21} &= u_{11}e_{11} + u_{13}e_{13}\\
e_{22} &= e_{12}\\
e_{23} &= -u_{13}e_{11} + u_{11}e_{13}
\end{aligned}
$$

where $h_1 = u_{13}^2 + u_{11}^2 - 1 = 0$. Using Frenet formulas for both curves, the problem can be represented as (HS, DS).

$HS = \{h_1, ..., h_{10}\}$ where
$h_2 = ru_{13} - t_1 a_2$
$h_3 = a_2'$
$h_4 = ru_{11} + k_1 a_2 - 1$
$h_5 = u_{13}'$
$h_6 = rk_2 + t_1 u_{13} - k_1 u_{11}$
$h_7 = ru_{11} t_2 - ru_{13} k_2 - t_1$
$h_8 = ru_{13} t_2 + ru_{11} k_2 - k_1$
$h_9 = u_{11}'$
$h_{10} = rt_2 - k_1 u_{13} - t_1 u_{11}$

$DS = \{k_1, t_1, k_2, a_2\}$

By Algorithm 2.9, we get a relation set between k_1 and t_1: $\{k_1' t_1'' - k_1'' t_1'\}$.

3. Methods of Finding Algebraic Relations

As we mentioned in Section 1, Algorithm 2.9 usually gives a relation set consisting of d-pol equations of $y_1, u_1, ..., u_q$. We further want to know the general form of the solution of these differential equations. For example, in Example 2.11, Algorithm 2.9 gives a relation $k_1' t_1'' - t_1' k_1'' = 0$ between k_1 and t_1. Now we want to know if there exists any algebraic relation between k_1 and t_1. In this section, we will answer this question.

Problem 3.1. If $x_1, ..., x_n$ satisfy a d-pol equation: $P(x_1, ..., x_n) = 0$, decide whether the x satisfy a relation $R(a_1, ..., a_m, x_1, ..., x_n) = 0$ where R is a polynomial with coefficients in Q and $a_1, ..., a_m$ are constants.

Lemma 3.2. Variables $x_1, ..., x_n$ satisfy a polynomial equation of degree d with constants coefficients iff

$$LD_d = ld(1, x_1, ..., x_n, x_2^2, ..., x_n^2, ..., x_1^d, ..., x_n^d) = 0$$

Proof. This is a consequence of Lemma II.3.1 (i.e., Lemma 3.1. in Part II [2]). ∎

Here we only have a partial solution to Problem 3.1: if such a relation exists, our method can find it in a finite number of steps, but if such relation does not exit, our method will not stop.

Algorithm 3.3. A partial solution to Problem 3.1.

Step 1. For $i = 1$ to infinite, decide whether $prem(LD_i, P) = 0$.

Step 2. Let i_0 be the first number such that $prem(LD_{i_0}, P) = 0$, then by Lemma 3.2, the x satisfy a polynomial equation of degree i_0 with constant coefficients.

Step 3. By lemma II.3.1, we can go further to decide which monomials effectively occur in the polynomial relation (for an example, see Example 3.7.). ∎

Example 3.4. (Continuation of Example 2.11.) Algorithm 2.9 gives a relation $h_1 = k_1' t_1'' - t_1' k_1'' = 0$ between k_1 and t_1. Following Algorithm 3.3, $LD_1 = ld(1, k_1, t_1) = k_1' t_1'' - t_1' k_1'' = h_1$, then k_1 and t_1 satisfy a linear relation $ak_1 + bt_1 + c = 0$ for arbitrary constants a, b and c.

We may combine the ideas used in Algorithm 2.9 and Algorithm 3.3 and ask whether certain variables in a geometric problem (HS, DS) satisfying polynomial equations with constant coefficients. Precisely, we propose

Problem 3.5. Let (HS, DS) be a geometric problem where $HS = \{h_1, ..., h_s\}$, $DS = \{d_1, ..., d_l\}$ and the h_i and d_i are d-pols in $K\{x_1, ..., x_n\}$. We ask whether there is a polynomial $P \in Q[a_1, ..., a_m, x_1, ..., x_n]$ such that

$$\forall a, x[(h_1 = 0 \wedge ... \wedge h_s = 0 \wedge d_1 \neq 0 \wedge ... \wedge d_l \neq 0 \wedge a'_1 = 0 \wedge ... \wedge a'_m = 0) \Rightarrow P = 0]$$

Algorithm 3.6. A partial solution for Problem 3.5.

Step 1. By Ritt-Wu's decomposition algorithm (Here the variable order is not important), we have

$$\text{E-Zero}(HS/DS) = \cup_{i=1}^{t} \text{E-Zero}(PD(ASC_i)/DS)$$

Step 2. For $k = 1$ to infinite, let k_0 be the first number such that $prem(LD_{k_0}, ASC_i) = 0$ for $i = 1, ..., t$. By Lemma 3.2, the x satisfy a polynomial equation of degree k_0 with constant coefficients.

Step 3. By lemma II.3.1, we can go further to decide which monomials effectively occur in the polynomial equation. ▮

Remark. If the relations of a geometric problem cannot expressed as polynomial equations, then Algorithm 3.6 will not stop and we have to use Algorithm 2.9. But for some geometric problems, e.g., Example 3.7., which have no relations according to Definition 2.3, we can find polynomial relations using Algorithm 3.6.

Example 3.7. If a particle moves in a plane under a central force which is proportional to the radius drawn from the particle to the force center, find the orbit of the particle.

Let the the coordinates of the particle be $(x(t), y(t))$ where t is the time. We assume the force center is the origin point $(0, 0)$. Let a be the magnitude of the acceleration of the particle and r be the length of the radius vector drawn from the particle to the force center. The problem can be represented as (HS, DS) where

$HS = \{h_1, h_2, h_3, h_4\}$
$h_1 = r^2 - x^2 - y^2 = 0$
$h_2 = a^2 - x''^2 - y''^2 = 0$
$h_3 = x''y - y''x = 0$ — the force is toward the origin,
$h_4 = ld(a, r) = a'r - r'a$ — a is proportional to r.
$DS = \{d_1\}$
$d_1 = ld(1, x, y) = x'y'' - y'x'' \neq 0$ — The orbit is not a straight line.

By Ritt-Wu's decomposition algorithm, under the variable order $x < y < r < a$ we have

$$\text{E-Zero}(HS/DS) = \cup_{1 \leq i \leq 2} \text{E-Zero}(PD(ASC_i)/DS)$$

where $ASC_1 = \{xx''' - x'x'', xy'' - x''y, r^2 - y^2 - x^2, a^2 - y''^2 - x''^2\}$ and $ASC_2 = \{x'', y'', r^2 - y^2 - x^2, a\}$. The least k such that prem$(LD_k, ASC_1) =$ prem$(LD_k, ASC_2) = 0$ is 2. Thus the orbit is a conics. We can further check that $prem(ld(1, x^2, xy, y^2), ASC_i) = 0, i = 1, 2$ and by deleting any terms from $\{1, x^2, xy, y^2\}$ the conics will not vanish on E-Zero(HS/DS). Thus the orbit is a conics with $(0, 0)$ as its center, i.e. $ax^2 + bxy + cy^2 = 1$ for arbitrary constants a, b, and c.

4. Mechanical Derivation of Newton's Law from Kepler's Laws

Suppose we have known Kepler's laws which were obtained by experimental observation. We want to know whether these laws imply certain relations between the acceleration a and the radius r. Use the same coordinate system and differential equations for Kepler's laws as in [2]. The geometric problem can be expressed as (HS, DS) where

$HS = \{h_1, h_2, k_1, k_2\}$
$h_1 = r^2 - x^2 - y^2 = 0$
$h_2 = a^2 - x''^2 - y''^2 = 0$
$k_1 = ld(1, x, r) = x''r' - r''x' = 0$ **K1**
$k_2 = x''y - y''x = 0$ **K2**

$DS = \{d_1, d_2\}$
$d_1 = ld(1, x, y) \neq 0$ The orbit is not a straight line.
$d_2 = a \neq 0$ The force is not zero.

We want to find the relations between a and r. By Algorithm 2.9, using Ritt–Wu's decomposition algorithm under the following order $r < a < x < y$, we know that E-Zero(HS/DS) = E-Zero$(PD(ASC_1)/DS)$ where ASC_1 is

$$\begin{array}{l} r^2r'r'''' + (-r^2r'' + 6rr'^2)r''' + 6r'^3r'' \\ r'a - rr''' - 3r'r'' \\ (r'^2a - rr''^2 - r'^2r'')x^2 + r^3r''^2 \\ y^2 + x^2 - r^2 \end{array} \tag{4.1}$$

From (4.1), we know that there is no relation between a and r according to Definition 2.3, because neither r nor a is a parameter of the statement.

The relation we want to find is $ld(1, ar^2) = r(a'r + 2r'a) = 0$ in which the highest order for a and r is one. But in (4.1), the highest order for r is four. We want to find a relation between r and a with lower orders. This suggests that if only a, a', r and r' are allowed during the decomposition, we may hopefully get the relation we want.

Note that r'' occurs in k_1. We eliminate it by taking the pseudo remainder of k_1 wrpt h_1. The remainder is

$$k_{11} = (x'y^3 + x^2x'y)y'' + x^2x'y'^2 + (-x''y^3 + (-x^2x'' - 2xx'^2)y)y' + x'^3y^2$$

which involves x and y alone.

Now the problem can be understood as follows: $k_{11} = 0$ and $k_2 = 0$ determine the motion completely by giving the differential equations for x and y, and we want to find the relations between two functions of x and y: $r^2 = x^2 + y^2$ and $a^2 = x''^2 + y''^2$ under the condition of $k_{11} = 0$ and $k_2 = 0$. To do this, our first step is to determine the relation among x and y defined by $k_{11} = 0$ and $k_2 = 0$. By Ritt–Wu's decomposition algorithm, E-Zero$(\{k_{11}, k_2\}/DS)$ = E-Zero$(PD(ASC_2)/DS)$ where ASC_2 is

$$\begin{array}{l} 3x^2x'x''x'''' - 4x^2x'x'''^2 + (-3x^2x''^2 + 2xx'^2x'')x''' + 2x'^3x''^2 \\ 3x''^2y^2 + x^2x'x''' + 2xx'^2x'' \end{array} \tag{4.2}$$

Since we want to find relations among a, a', r and r', we have to represent r' and a' as functions of x and y. This can be done by taking the pseudo remainders of h_1' and h_2' wrpt ASC_2. Taking the pseudo remainders of h_1, h_1', h_2, and h_2' wrpt ASC_2, and deleting some factors which are contradict to DS, we have

$$
\begin{array}{l}
x^2x'x''' + (-3x^2+3r^2)x''^2 + 2xx'^2x'' \\
x^2x'x'''^2 + (-3x^2x''^2 + xx'^2x'')x''' + (3xx' - 9rr')x''^3 - 2x'^3x''^2 \\
xx'x''' - 3xx''^2 + 2x'^2x'' + 3a^2x \\
2x^2x'x'''^2 + (-6x^2x''^2 + 2xx'^2x'')x''' + 6xx'x''^3 - 4x'^3x''^2 + 9aa'x^2x''
\end{array}
\tag{4.3}
$$

We want to keep the order of r and a less than one which can be done by treat the d-pols in (4.2) and (4.3) as ordinary polynomials in the variables $x, x', x'', x''', a, a', r, r'$. Applying Algorithms in [3] (or Algorithm 2.9 in the sense of ordinary polynomials) to find the relation among a, a', r and r' defined by statement $((4.3), DS)$. Under the variable order: $r < r' < a < a' < x < x' < x'' < x'''$, we have

$$
\text{E-Zero}((4.3)/DS) = \text{E-Zero}(PD(ASC_3)/DS) \cup \text{E-Zero}(PD(ASC_4)/DS)
$$

where

$$
\begin{array}{ll}
ASC_3 = & ASC_4 = \\
ra' + 2r'a & ra' + 2r'a \\
rx'^2 - r'xx' - ax^2 + ar^2 & rx'^2 - r'xx' + ax^2 - r^2a \\
rx'' - ax & rx'' + ax \\
x^2x'x''' + (-3x^2+3r^2)x''^2 + 2xx'^2x'' & x^2x'x''' + (-3x^2+3r^2)x''^2 + 2xx'^2x''
\end{array}
$$

The first equation of ASC_1 and ASC_2 actually means that a is reversely proportional to r^2. This fact can also be obtained by Algorithm 2.12. Hence we have derived **N1** from **K1** and **K2** mechanically. The crucial point is that we only allow r', a' occurring in the process and all higher orders are forbidden. We do this by treating the elements in (4.3) as ordinary polynomials of the new variables: $a, a', r, r'x, x', x''$, and x'''. In Section 5, we will give a general method based on this idea.

5. Methods of Mechanical Formula Derivation (2)

Based on the idea of Section 4, we propose the following problem.

Problem 5.1. Given two sets of d-pol equations

$$h_1(x_1, ..., x_n) = 0, ..., h_r(x_1, ..., x_n) = 0$$
$$q_1(w_1, x_1, ..., x_n) = 0, ..., q_s(w_s, x_1, ..., x_n) = 0$$

and a set of d-pol inequations

$d_1(x_1, ..., x_n) \neq 0, ..., d_t(x_1, ..., x_n) \neq 0$

Find the relations among $w_1, ..., w_s$ with the lowest order for each w_i under the above conditions. Precisely, let

$$ID = Ideal(h_1, ..., h_r, q_1, ..., q_s, z_1 d_1 - 1, ..., z_t d_t - 1) \tag{5.1.1}$$

where z_i are some new distinct variables. We want to find a set of d-pols $RS = \{r_1, ..., r_l\}$ of the w such that (1) the r_i are irreducible d-pols of the w; and (2) $r_1 ... r_l$ is in the $radical(ID)$; and (3) the r_i are the d-pols with least order in each w_j to satisfy (2).

Algorithm 5.2. A solution to Problem 5.1. For convenience, we assume that the order of w_i in q_i is 0. Let $VS = \{w_1, ..., w_s\}, HS = \{h_1, ..., h_r\}, QS = \{q_1, ..., q_s\}$, $DS = \{d_1, ..., d_t\}$.

Step 1. Use Algorithm 2.9 to find relations among w_s and $w_1, ..., w_{s-1}$ determined by $S = (HS \cup QS, DS)$. If $\{w_1, ..., w_{s-1}\}$ is algebraicly independent wrpt ID (defined as (5.1.1)) and w_s is dependent on $\{w_1, ..., w_{s-1}\}$, then we can find a weak relation set among the w and the algorithm terminates. If $w_1, ..., w_s$ are algebraic independent wrpt ID, then there are no relations among the w and the algorithm terminates. Otherwise $w_1, ..., w_{q-1}$ are not algebraic independent wrpt ID. Goto Step 2.

Step 2. By Ritt–Wu's decomposition algorithm, we have

$$\text{E-Zero}(HS/DS) = \cup_{i=1}^{t} \text{E-Zero}(ASC_i/DS \cup J_i)$$

where J_i are the initial and separant sets of ASC_i.

Step 3. For each component E-Zero$(ASC_{i_0}/DS \cup J_{i_0})$, let QS_1 be the set of the pseudo remainders of the d-pols in QS wrpt ASC_{i_0}.

Step 4. Treat the d-pols in $PS_1 = QS \cup QS_1 \cup ASC_{i_0}$ as ordinary polynomials in variables $w_1, ..., w_s, u_1, u_1', ..., x_p, x_p' ....$ Use the algorithms in [3] or Algorithm 2.9 in the sense of ordinary polynomials to find the relation set among the variables in VS determined by the geometric problem $(PS_1,\ DS \cup J_{i_0})$.

Step 5. If we can find a relation set among the variables in VS, the algorithm terminates. Otherwise goto Step 6.

Step 6. For $j = 1$ to infinite, $i = 1$ to s do: adding $w_{i,j}$ (the j-th derivation of w_i) to VS, adding $q_{i,j}$ (the j-th derivation of q_i) to QS, and repeating step 3, 4, and 5 for the new VS and QS. As there are actually some relations among $w_1, ..., w_s$ by step 1, the process must terminate at a finite number of steps. ∎

Note that the Step 1 of Algorithm 5.2 is actually Algorithm 2.9. Newton's law can be described from Kepler's automatically by using Algorithm 5.2.

Example 5.3. (The inverse of Example 3.7.) If the orbit of a particle described under a central attractive force is an ellipse having its center at the center of the force. Find the relation between the force and the radius drawn from the particle to the force center.

Use the same notations as example 3.7. Let the equation of the ellipse be $ax^2 + by^2 + c = 0$ for constants a, b, and c, or equivalently $h_5 = ld(1, x^2, y^2) = 0$. The problem here is to find the relation of a and r under the condition $h_5 = 0 \wedge h_3 = 0 \wedge d_1 = ld(1, x, y) \neq 0$, i.e., in the terms of Problem 5.1, $HS = \{h_3, h_5\}$, $QS = \{h_1, h_2\}$, and $DS = \{d_1\}$.

Using Theorem 2.7, we find that neither r nor a is a parameter, hence there are no relations between r and a according to Definition 2.3. We have to use Algorithm 5.2. First using Ritt–Wu's decomposition algorithm, we have E-Zero(HS/DS) = E-Zero$(PD(ASC_1)/DS)$ where $ASC_1 = \{x'''x - x''x', x'y' - yx''\}$. The pseudo remainders of h_1, h'_1, h_2, and h'_2 wrpt ASC_1 are

$$\begin{array}{l} y^2 + x^2 - r^2 \\ y^2x'' + x'^2x - r'r \\ y^2x''^2 + x''^2x^2 - x^2a^2 \\ y^2x''^3 + x''^2x'^2x - x'x^2 - a'a. \end{array} \tag{5.3.1}$$

Set an order for the variables: $a < a' < r < r' < x < x' < x'' < x''' < y < y' < y''$. In the sense of ordinary polynomials, we have

$$\text{E-Zero}(ASC_1 \cup (5.3.1) \cup \{h_3, h_5\}/DS) = \text{E-Zero}(PD(ASC_2)/DS) \cup \text{E-Zero}(PD(ASC_3)/DS)$$

where

$$\begin{array}{ll} ASC_2 = & ASC_3 = \\ r'a - ra' & r'a - ra' \\ x'^2 + xr - x'r'r^2 - x^3a + xr^2a & x'^2 + xr - x'r'r^2 + x^3a - xr^2a \\ x''r - xa & x''r + xa \\ x'''x - x''x' & x'''x - x''x' \\ y^2 + x^2 - r^2 & y^2 + x^2 - r^2 \\ y'x' - yx'' & y'x' - yx'' \\ xy'' - x''y & xy'' - x''y \end{array}$$

The first d-pol of both ascending chains is $ld(a, r) = a'r - r'a$, i.e., a is proportional to r by Algorithm 3.3.

For more examples from mechanics, see [4].

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