Polynomial General Solution for First Order ODEs with Constant Coefficients *

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Abstract

We give a necessary and sufficient condition for an ODE to have polynomial type general solutions. For a first order ODEs of degree n and with constant coefficients, we give an algorithm of complexity $O(n^5)$ to decide whether it has a polynomial general solution and to compute one if it exists. Experiments show that this algorithm is quite effective in solving ODEs with high degrees and a large number of terms.

1 Introduction

Trying to find elementary function solutions for differential equations may be traced back to the work of Liouville. As a consequence, such solutions of differential equations are called *Liouvillian solutions*. In a pioneering paper [16], Risch gave an algorithm for finding Liouvillian solutions for the simplest differential equation $y' = f(x)$, that is, to find elementary function solutions to integration $\int f(x)\mathrm{d}x$. In [13], Kovacic presented a method for solving second order linear homogeneous differential equations. In [19], Singer proposed a method to find Liouvillian solutions of general homogeneous linear differential equations. Many other interesting results on finding Liouvillian solutions of linear ODEs are given in [1, 3, 2, 4, 5, 6, 8, 10, 12, 15, 18, 20]. In [14], Li and Schwarz gave a method to find rational solutions for a class of partial differential equations.

All these results are limited to linear cases. There seems no general methods to find Liouvillian solutions of nonlinear differential equations. In [7], Cano proposed an algorithm to find polynomial solutions to equations of the form $y' = R(x, y)$ where R is a rational function in x and y. In this paper, we try to find polynomial solutions to non-linear differential equations. Instead of finding arbitrary polynomial solutions, we will find the general solutions for ODEs of polynomial type. For example, the general solution for $(\frac{\mathrm{d}y}{\mathrm{d}x})^2 - 4y = 0$ is: $y = (x + c)^2$, where c is an arbitrary constant. Three main results are given in this paper. Firstly, we give a sufficient and necessary condition for an ODE to have polynomial general solutions. We also prove that

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the definition of general solutions of ODEs due to Ritt is equivalent to the definition in the usual sense, if the general solutions are in polynomial form. Secondly, we give a detailed analysis of the structure of first order ODEs with constant coefficients which have polynomial general solutions. Based on these results, an algorithm can be easily obtained. Thirdly, by introducing a novel method to evaluate the coefficients of polynomial general solutions of a first order ODE with constant coefficients, we get an algorithm to find polynomial general solutions of first order ODEs with constant coefficients with polynomial complexity. Our experiments show that this algorithm is quite effective in solving ODEs with high degree and terms.

The paper is organized as follows. In section 2, we give a brief introduction to the characteristic set method. In section 3, a criterion for an ODE to have polynomial general solutions is given. In section 4, we give the degree bound of solutions of first order ODEs with constant coefficients and an algorithm. In section 5, we analyse the structure of first order ODEs which have polynomial general solutions. In section 6, we present a polynomial-time algorithm to find polynomial general solutions of first order ODEs and some examples.

2 A brief introduction to the characteristic set method

In this section, we introduce some concepts and notations on the characteristic set method. More details can be found in [11, 17, 22].

Let $\mathbf{K}$ be a field and $z_1, z_2, \cdots, z_n$ indeterminants. Considering a polynomial P in $\mathbf{K}[z_1, \cdots, z_n]$, we can write it as the following form:

$$P = \sum a_{\alpha_1\alpha_2\cdots\alpha_k} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$$

where $a_{\alpha_1\alpha_2\cdots\alpha_k} \neq 0$ are in the field $\mathbf{K}$ and there exists an α_k which is not equal to zero. k is called the *class* of P, denoted by $\mathrm{cls}(P)$. $\deg(P, z_i)$ denotes the degree of P with respect to z_i when we regard P as a polynomial in z_i. The maximum of $\alpha_1 + \cdots + \alpha_k$ is called the *total degree* of P, denoted by $\mathrm{tdeg}(P)$. For a polynomial P of class $p > 0$, we may write P as the following form:

$$P = a_d z_p^d + a_{d-1} z_p^{d-1} + \cdots + a_0$$

where a_i are polynomials in $z_1, \cdots, z_{p-1}$ for $i = 0, \cdots, d$ and $a_d \neq 0$. a_d is called the *initial* of P and $S = \frac{\partial P}{\partial z_p}$ is called the *separant* of P. Let Q be another polynomial whose class is q. P is said to be *reduced* with respect to Q if $q > 0$ and $\deg(P, z_q) < \deg(Q, z_q)$. Let $R = \mathrm{prem}(P, Q)$ be the pseudo-remainder of P with respect to Q. We have the following *remainder formula* for R:

$$I^s P = BQ + R$$

where R is reduced with respect to Q and I is the initial of Q. A polynomial set $\mathcal{A}$ is called an *ascending chain* if its polynomials are all non-zero and can be arranged in the following sequence:

$$\mathcal{A}: \quad A_1, A_2, \cdots, A_r$$

satisfying the following conditions:

1. $\mathrm{cls}(A_1) < \mathrm{cls}(A_2) < \cdots < \mathrm{cls}(A_r)$
2. A_j is reduced with respect to A_i for all $i < j$.

A polynomial P is said to be *reduced* with respect to an ascending chain $\mathcal{A}$ if P is reduced with respect to every polynomial in $\mathcal{A}$. An ascending chain $\mathcal{A}$ is called *irreducible* if for any polynomials A_1, A_2 reduced with respect to $\mathcal{A}$, we have $\mathrm{prem}(A_1 A_2, \mathcal{A}) \neq 0$. Let $\mathcal{A}$ be an irreducible ascending chain. Let

$$\mathbf{SAT}(\mathcal{A}) = \{P \in \mathbf{K}[z_1, \cdots, z_n] \mid \exists J \ s.t. JP \in \mathrm{Ideal}(\mathcal{A})\}$$

where J is a product of the initials of the polynomials in $\mathcal{A}$. Then $\mathbf{SAT}(\mathcal{A})$ is a prime ideal and $\dim_{\mathbf{K}}(\mathbf{SAT}(\mathcal{A}))$ is equal to $n - r$ where r is the number of polynomials in $\mathcal{A}$ ([17]).

We will need the following Wu-Ritt's zero decomposition theorem ([17, 22]).

Theorem 2.1 *For any polynomial set PS, there are a finite number of irreducible ascending chains $\mathcal{A}_k$ such that*

$$\mathrm{Zero}(PS) = \cup_k \mathrm{Zero}(\mathbf{SAT}(\mathcal{A}_k))$$

Let Σ be a non-trivial prime ideal in $\mathbf{K}[z_1, \cdots, z_n]$ and $\mathbf{L}$ an extension field of $\mathbf{K}$. For a prime ideal Σ, a zero $\eta \in \mathbf{L}^n$ of Σ is called *generic zero* of Σ if for any polynomial $P \in \mathbf{K}[z_1, \cdots, z_n]$, $P(\eta) = 0$ implies that $P \in \Sigma$. It is well known that an ideal Σ is prime iff it has a generic zero. Let a generic zero of Σ be $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$. We call the transcendental degree of $\mathbf{K}(\eta_1, \eta_2, \cdots, \eta_n)$ over $\mathbf{K}$ *dimension* of η (denoted by $\dim_{\mathbf{K}}(\eta)$) which equals to $\dim_{\mathbf{K}}(\Sigma)$

3 Polynomial general solution to ODEs

In the following, let $\mathbf{K}$ be the differential field of meromorhpic functions in $\mathcal{C}$ with differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$. Let y be an indeterminate over $\mathbf{K}$ and we denote by y_i the ith derivative of y. We use $\mathbf{K}\{y\}$ to denote the ring of differential polynomials over differential field $\mathbf{K}$, which consists of the polynomials in the y_i with coefficients in $\mathbf{K}$. Let Σ be a system of differential polynomials in $\mathbf{K}\{y\}$. A *zero* of Σ is an element in an extension field of $\mathbf{K}$, which vanishes every differential polynomials in Σ. The totality of zeros in $\mathbf{K}$ is called the *restrict manifold* of Σ. (see [11, 17])

Let $P(y) \in \mathbf{K}\{y\}/\mathbf{K}$. We denote $\mathrm{ord}(P(y))$ the highest derivative of y in $P(y)$, called the *order* of $P(y)$. Let $\mathrm{ord}(P(y))$ be o. We can also regard $P(y)$ as an algebraic polynomial in y, y_1, $\cdots$, y_o with the coefficients in $\mathbf{K}$, then the *initial, separant*, $\deg(P(y), y_i)$ and $\mathrm{tdeg}(P(y))$ are defined as algebraic case. The definition of *generic zero* is also the same as it in algebraic case.

Let $P(y)$ be a differential polynomial of order o. A differential polynomial $Q(y)$ is said to be *reduced* with respect to $P(y)$ if $\mathrm{ord}(Q(y)) < o$ or $\mathrm{ord}(Q(y)) = o$ and $\deg(Q(y), y_o) < \deg(P(y), y_o)$. For two differential polynomials $P(y)$ and $Q(y)$, let $R = \mathrm{prem}(P(y), Q(y))$ be the differential pseudo-remainder of $P(y)$ with respect to $Q(y)$. We have the following *differential remainder formula* for $R(y)$ (see [11, 17])

$$J(y)P(y) = \sum_i B_i(y) Q^{(i)}(y) + R(y)$$

where $J(y)$ is a product of certain powers of the initial and separant of $Q(y)$, $Q^{(i)}(y)$ is the ith derivative of $Q(y)$ and $B_i(y)$ are differential polynomials. For a differential polynomial $P(y)$ with order o, we say $P(y)$ is *irreducible* if $P(y)$ is irreducible when $P(y)$ is treated as a polynomial in $\mathbf{K}[y, y_1, \cdots, y_o]$.

Let $P(y) \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial and

$$\Sigma_P = \{A \in \mathbf{K}\{y\} | SA \equiv 0 \, mod \, \{P(y)\}\} \tag{1}$$

Then Σ_P is a prime differential ideal and a differential polynomial $Q(y)$ belongs to Σ_P if and only if $\mathrm{prem}(Q(y), P(y)) = 0$. Furthermore, Ritt proved the following basic result [17].

Theorem 3.1 *Let $F(y)$ be an irreducible differential polynomial not in $\mathbf{K}$. Then the restrict manifold of Σ_F is not empty and a differential polynomial which vanishes over the restrict manifold of Σ_F is contained in Σ_F.*

We call a generic zero of Σ_F a *general solution* of $F(y) = 0$. By a *polynomial general solution* of $F(y) = 0$, we mean a general solution of $F(y) = 0$ in the following form

$$\hat{y} = \sum_{i=0}^{n} a_i x^i, \; (a_n \neq 0) \tag{2}$$

where a_i are constants which may be algebraically dependent over $\mathcal{C}$. In the literature in general, a *general solution* of $F(y) = 0$ is defined as a family of solutions with o independent parameters in a loose sense where $o = \mathrm{ord}(F(y))$. The definition given by Ritt is more precise. From Theorem 3.3, we can see that if $F(y) = 0$ has a polynomial general solution, then its polynomial general solution has o independent parameters.

Since the solutions of $y_{n+1} = 0$ are some polynomials with degrees not greater than n. Suppose that $F(y) = 0$ has a polynomial general solution with degree n. The investigate of the relation between y_{n+1} and $F(y)$ will lead to a sufficient and necessary condition for $F(y) = 0$ to have polynomial general solutions.

Theorem 3.2 *Let $F(y)$ be an irreducible differential polynomial. Then $F(y) = 0$ has polynomial general solutions iff there is a non-negative integer n such that* $\mathrm{prem}(y_{n+1}, F(y)) = 0$.

Proof: ($\Longrightarrow$) Suppose that $F(y) = 0$ has a polynomial general solution $\hat{y}$ with degree n. Since $y_{n+1}(\hat{y}) = 0$, $y_{n+1} \in \Sigma_F$ which means that $\mathrm{prem}(y_{n+1}, F(y)) = 0$ by Theorem 3.1.

($\Longleftarrow$) Assume that there exists an n such that $\mathrm{prem}(y_{n+1}, F(y)) = 0$ and n is the least. If $n = -1$, then $F(y) = y$. It is obvious. Now we suppose that $n \geq 0$. From Theorem 3.1, $y_{n+1} \in \Sigma_F$. Hence, all the elements in the restrict manifold of Σ_F must have the form: $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$. In particularly, the generic zero $\hat{y}$ has the form: $\hat{y} = \sum_{i=0}^{n} a_i x^i$. If $a_n = 0$, then $y_n(\hat{y}) = 0$ which implies that $y_n \in \Sigma_F$. Hence $\mathrm{prem}(y_n, F(y)) = 0$, a contradiction. The proof is complete. ∎

Theorem 3.3 *Let $F(y)$ be an irreducible differential polynomial with order o. If $F(y) = 0$ has a polynomial general solution of form (2), then the a_i depend on o independent parameters.*

Proof : Let $\hat{y}$ be a generic zero of Σ_F of the form (2). Let $a = (a_0, a_1, \cdots, a_n)$. Suppose that $\dim_{\mathcal{C}}(a) = d$. Then we need to prove that $d = o$. Let

$$\begin{array}{rcl} T : \mathbf{K}^{n+1} & \longrightarrow & \mathbf{K}^{n+1} \\ (z_0, z_1, \cdots, z_n) & \longrightarrow & (y, y_1, \cdots, y_n) \end{array}$$

where $y = \sum_0^n z_i x^i$ and $y_k = \sum_k^n i(i-1)\cdots(i-k+1) z_i x^{i-k}$. Note that if $z_i \in \mathcal{C}$, y_k is the kth derivative of y. Then T is an invertible linear transformation over $\mathbf{K}$. We have

$$d = \dim_{\mathcal{C}}(a) = \dim_{\mathbf{K}}(a) = \dim_{\mathbf{K}}(T(a))$$

Now we consider $\dim_{\mathbf{K}}(T(a))$. Let $F_{(i)}(y)$ be the ith derivative of $F(y)$ wrt x for $i = 0, \cdots, n-o$ and $S(y)$ be the separant of $F(y)$. Now we regard $F_{(i)}(y)$ and $S(y)$ as algebraic polynomials in $\mathbf{K}[y, y_1, \cdots, y_n]$ and denote them by $F_{(i)}(y, y_1, \cdots, y_n)$ and $S(y, y_1, \cdots, y_n)$ correspondingly. Then we have $F_{(i)}(T(a)) = 0$ and $S(T(a)) \neq 0$. Let $T(a) = (\eta_0, \eta_1, \cdots, \eta_n)$. Since the initials of $F_{(i)}(y, y_1, \cdots, y_n)(i > 0)$ are $S(y, y_1, \cdots, y_n)$ and y_{o+i} is linear in $F_{(i)}(y, y_1, \cdots, y_n)$, we have,

$$\eta_{o+i} = \frac{(F_{(i)} - S y_{o+i})(T(a))}{S(T(a))}$$

Hence we have $d \leq o$. In what follows, we will prove $\eta_0, \eta_1, \cdots, \eta_{o-1}$ are algebraically independent over $\mathbf{K}$. Otherwise, there is a polynomial $G(y, y_1, \cdots, y_{o-1})$ in $\mathbf{K}[y, y_1, \cdots, y_{o-1}]$ such that $G(\eta_0, \cdots, \eta_{o-1}) = 0$. If we regard $G(y, y_1, \cdots, y_{o-1})$ as a differential polynomial $G(y)$, $G(\hat{y}) = G(\eta_0, \cdots, \eta_{o-1}) = 0$ (note that $\eta_0 = \hat{y}$) which implies that $G(y) \in \Sigma_F$. In the other word, $\mathrm{prem}(G(y), F(y)) = 0$, a contradiction. Hence $d = o$. ∎

Remark 3.4 *Thereom 3.3 shows that Ritt's definition of general solutions is equivalent to the definition in usual sense if the general solutions are in polynomial form.*

Let

$$\mathbf{ID}(a) = \{P \in \mathcal{C}[z_0, \cdots, z_n] | P(a) = 0\}$$

where $a = (a_0, a_1, \cdots, a_n)$ is as in the proof of Theorem 3.3. Then $\mathbf{ID}(a)$ is a prime ideal. Let $\mathcal{A}$ be an irreducible ascending set of $\mathbf{ID}(a)$. Then $\mathcal{A}$ can be computed explicitly with the characteristic set method [17, 22]. Let $z = \sum_{i=0}^{n} z_i x^i$. Substituting y by z to $F(y)$, let $\mathbf{PS}$ be the set of the coefficients of the powers of x in $F(z)$ which is a set of polynomials in z_i. From Theorem 3.1, we get the following decomposition of the zero set of $\mathbf{PS}$:

$$\mathrm{Zero}(\mathbf{PS}) = \cup_i \mathrm{Zero}(\mathbf{SAT}(\mathcal{A}_i))$$

where $\mathcal{A}_i$ are irreducible ascending chains and any $\mathbf{SAT}(\mathcal{A}_i)$ do not include each other. Then we have the following theorem:

Theorem 3.5 *Suppose that $F(y) = 0$ has polynomial general solutions of the form (2). Then there is a unique $\mathcal{A}_{i_0}$ in the above decomposition such that $\mathbf{SAT}(\mathcal{A}_{i_0}) = \mathbf{ID}(a)$. Furthermore, all $\mathbf{SAT}(\mathcal{A}_i)$ but $\mathbf{SAT}(\mathcal{A}_{i_0})$ satisfy $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_i)) < \dim_{\mathcal{C}}(\mathbf{ID}(a))$.*

Proof : Let $\mathrm{ord}(F(y)) = o$. Then $\dim_{\mathcal{C}}(\mathbf{ID}(a)) = o$. Firstly, we show that for every i, $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_i)) \leq o$. Let $\xi_i = (\xi_{i,0}, \xi_{i,1}, \cdots, \xi_{i,n})$ be a generic zero of $\mathbf{SAT}(\mathcal{A}_i)$ and $\gamma_i = \sum_{k=0}^{n} \xi_{i,k} x^k$. Let

$$\Sigma_{\gamma_i} = \{P(y) \in \mathbf{K}\{y\} \,|\, P(\gamma_i) = 0\}$$

Then Σ_{γ_i} are differential prime ideals and γ_i are their generic zeros. From (P32, [17]), we know that there exist differential polynomials $G_i(y) \in \mathbf{K}\{y\}$ such that Σ_{γ_i} are general solutions of $G_i(y)$. That is said, $\Sigma_{\gamma_i} = \Sigma_{G_i}$ (Σ_{G_i} as in (1)). From Theorem 3.3, we have $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_i)) = \mathrm{ord}(G_i(y))$. Since $\xi_i \in Zero(\mathbf{PS})$, $F(\gamma_i) = 0$ which implies that $F(y) \in \Sigma_{\gamma_i}$. Hence $\mathrm{ord}(G_i(y)) \leq o$, that is, $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_i)) \leq o$. Since $a \in \mathrm{Zero}(\mathbf{PS})$, there is an $\mathcal{A}_{i_0}$ such that $a \in \mathrm{Zero}(\mathbf{SAT}(\mathcal{A}_{i_0}))$. We will show that $\mathbf{SAT}(\mathcal{A}_{i_0}) = \mathbf{ID}(a)$. Since a is a generic zero of $\mathbf{ID}(a)$, $\mathbf{SAT}(\mathcal{A}_{i_0}) \subseteq \mathbf{ID}(a)$. Hence $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_{i_0})) = \dim_{\mathcal{C}}(\mathbf{ID}(a))$ so that $\mathbf{SAT}(\mathcal{A}_{i_0}) = \mathbf{ID}(a)$. If there is another $\mathbf{SAT}(\mathcal{A}_k)$ such that $\dim_{\mathcal{C}}(\mathbf{SAT}(\mathcal{A}_k)) = o$, then $\mathrm{ord}(G_k(y)) = \mathrm{ord}(G_{i_0}(y)) = o$ which implies that $F(y) = \theta G_k(y) = \lambda G_{i_0}(y)$ where λ, $\theta \in \mathbf{K}$ and all are not zero. Hence

$\Sigma_{\gamma_{i_0}} = \Sigma_{\gamma_k} = \Sigma_F$. Let $P(z_0, \cdots, z_n) \in \mathbf{SAT}(\mathcal{A}_k)$. Then $P(T^{-1}((y, y_1, \cdots, y_n))) \in \mathbf{K}\{y\}$ where T is as in Theorem 3.3. Then

$$P(T^{-1}(\gamma_k)) = P(\xi_i) = 0 \Longleftrightarrow P(T^{-1}((y, y_1, \cdots, y_n))) \in \Sigma_{\gamma_k} = \Sigma_{\gamma_{i_0}}$$
$$\Longleftrightarrow P(T^{-1}(\gamma_{i_0})) = P(\xi_{i_0}) = 0 \Longleftrightarrow P(z_0, \cdots, z_n) \in \mathbf{SAT}(\mathcal{A}_{i_0})$$

Hence $\mathbf{SAT}(\mathcal{A}_k) = \mathbf{SAT}(\mathcal{A}_{i_0})$. Since $\mathbf{SAT}(\mathcal{A}_i)$ do not include each other, the proof is complete. ∎

Remark 3.6 *The coefficients of $F(y)$ in above sections and this section are in $\mathbf{K}$ and the order of $F(y)$ is arbitrary. In the following sections, we will always assume $F(y)$ has constant coefficients and order one.*

4 An algorithm for first order ODEs

In this section and following sections, if there is no other statement, $F(y)$ will always be a non-zero first order irreducible differential polynomial with coefficients in $\mathcal{C}$.

Theorem 4.1 *Let $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ be a solution of $F(y) = 0$, where $\bar{a}_i \in \mathcal{C}$, $n > 0$ and $\bar{a}_n \neq 0$. Then for an arbitrary constant c,*

$$\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i \tag{3}$$

is a polynomial general solution for $F(y) = 0$.

Proof : It is easy to show that $\hat{y}$ is still a zero of Σ_F. For any $G(y) \in \mathbf{K}\{y\}$ satisfying $G(\hat{y}) = 0$, let $R(y) = \mathrm{prem}(G(y), F(y))$. Then $R(\hat{y}) = 0$. Suppose that $R(y) \neq 0$. Since $F(y)$ is irreducible and $\deg(R(y), y_1) < \deg(F(y), y_1)$, there are two differential polynomials $P(y), Q(y) \in \mathbf{K}\{y\}$ such that $P(y)F(y) + Q(y)R(y) \in \mathbf{K}[y]$ and $P(y)F(y) + Q(y)R(y) \neq 0$. Thus $(PF + QR)(\hat{y}) = 0$. Because c is an arbitrary constant which is transcendental over $\mathbf{K}$, we have $P(y)F(y) + Q(y)R(y) = 0$, a contradiction. Hence $R(y) = 0$ which means that $G(y) \in \Sigma_F$. So $\hat{y}$ is a generic zero of Σ_F. The proof is complete. ∎

The above theorem reduces the problem of finding polynomial general solutions to the problem of finding a polynomial solution. In what below, we will show how to find such a solution.

Lemma 4.2 *Suppose that $deg(F(y), y_1) = m > 0$. If $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ ($\bar{a}_n \neq 0$) is a solution of $F(y) = 0$ where $\bar{a}_i \in \mathcal{C}$, then $n \leq m$.*

Proof : Assume that $F(y) = \sum_{i=0}^{l} c_{\alpha_i\beta_i} y^{\alpha_i} y_1^{\beta_i}$, where $c_{\alpha_i\beta_i} \neq 0$ and $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ if $i \neq j$. Substituting y in $F(y)$ by $\bar{y}$, we get a polynomial $F(\bar{y})$ in x. Assume that $n > m > 0$. Then $n \geq 2$. We consider the highest degree of x in $F(\bar{y})$ which is the largest number in $\{n\alpha_i + (n-1)\beta_i \quad \text{for } i = 0, \cdots, l\}$. If $\bar{y}$ is a solution of $F(y) = 0$, all the coefficients of $F(\bar{y})$ are zero. Hence the number of the terms $y^{\alpha_i} y_1^{\beta_i}$ such that $n\alpha_i + (n-1)\beta_i$ is the largest are at least two. Without lost of generality, we suppose that two of them are $n\alpha_1 + (n-1)\beta_1$ and $n\alpha_2 + (n-1)\beta_2$. Then we have $n(\alpha_1 - \alpha_2) = (n-1)(\beta_2 - \beta_1)$. Assume that $\beta_2 \geq \beta_1$. Since

$(n,\, n-1) = 1$, we have $n|(\beta_2 - \beta_1)$. But $0 \leq \beta_2 - \beta_1 \leq m < n$, which implies that $\beta_1 = \beta_2$. Hence $\alpha_1 = \alpha_2$. This contradicts $(\alpha_1,\, \beta_1) \neq (\alpha_2,\, \beta_2)$. Hence $n \leq m$. ▮

From Lemma 4.2 and Theorem 3.2, we have an algorithm to find a polynomial solution of $F(y) = 0$ if it exists.

Algorithm 4.3 *The input is $F(y)$. The output is a polynomial general solution of $F(y) = 0$ if it exists.*

1. Let $n = \deg(F(y), y_1)$.
2. Compute $R(y) = \mathrm{prem}(y_{n+1}, F(y))$. If $R(y) \neq 0$, then the algorithm terminates by Theorem 5.1, else goto step 3.
3. Let $z = a_n x^n + \cdots + a_1 x + a_0$ where a_i are indeterminants. Substitute y in $F(y)$ by z. Let **PS** be the set of the coefficients of $F(z)$ as a polynomial in x. Solve **PS**. If $Zero(\mathbf{PS}) \neq \emptyset$, we will get a polynomial general solution of $F(y) = 0$. If $Zero(\mathbf{PS}) = \emptyset$, then $F(y)$ has no polynomial general solutions.

It is known that general methods of equation solving are exponential algorithms. Therefore, the above algorithm might be ineffective.

In the next section, we will analyse the structure of the first order ODEs with constant coefficients which have polynomial solutions. After doing so, we can obtain an polynomial-time algorithm.

5 The structure of first order ODEs

If we have obtained a polynomial solution $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ of $F(y) = 0$, we regard x, y, y_1 as the independent indeterminants and eliminate x in the polynomial set $\{\sum_{i=0}^{n} \bar{a}_i x^i - y,\, \sum_{i=1}^{n} i\bar{a}_i x^{i-1} - y_1\}$. Then we will obtain a new differential polynomial $R(y)$. Theorem 5.2 below will give the relation between $R(y)$ and $F(y)$.

Lemma 5.1 *Let $f_1(y) = \sum_{i=0}^{n} \bar{a}_i x^i - y, \quad f_2(y) = \sum_{i=1}^{n} i\bar{a}_i x^{i-1} - y_1 \quad (n \geq 1,\, \bar{a}_n \neq 0,\, \bar{a}_i \in \mathcal{C})$. If $n \geq 2$, let $R(y)$ be the Sylvester-resultant of $f_1(y)$ and $f_2(y)$ with respect to x and if $n = 1$, let $R(y) = f_2(y)$. Then $R(y)$ is an irreducible polynomial in $\mathcal{C}[y, y_1]$ and has the form*

$$R(y) = (-1)^n \bar{a}_n^{n-1} y_1^n + (-1)^{n-1} n^n \bar{a}_n^n y^{n-1} + G(y,\, y_1) \tag{4}$$

where $\mathrm{tdeg}(G) \leq n-1$ and G does not contain the term y^{n-1}.

Proof : When $n = 1$, it is clear. Assume that $n \geq 2$. We know that $R(y)$ is the following determinant which has 2n-1 columns and rows.

$$\begin{vmatrix}
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 - y & & & \\
 & \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 - y & & \\
 & & \cdots & \cdots & \cdots & & & & \\
 & & & \bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 - y \\
n\bar{a}_n & (n-1)\bar{a}_{n-1} & (n-2)\bar{a}_{n-2} & \cdots & \bar{a}_1 - y_1 & & & & \\
 & n\bar{a}_n & (n-1)\bar{a}_{n-1} & \cdots & \bar{a}_2 & \bar{a}_1 - y_1 & & & \\
 & & \cdots & \cdots & \cdots & & & & \\
 & & & n\bar{a}_n & (n-1)\bar{a}_{n-1} & (n-2)\bar{a}_{n-2} & \cdots & \bar{a}_2 & \bar{a}_1 - y_1
\end{vmatrix}$$

Regard $\bar{a}_i$ in the above determinant as indeterminants. Let $R = \sum c_{\alpha_i\beta_i} y^{\alpha_i} y_1^{\beta_i}$, where $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ if $i \neq j$ and $c_{\alpha_i\beta_i}$ are non-zero polynomials in $\bar{a}_0, \cdots, \bar{a}_n$. We define a weight

$$w : \mathcal{C}[x, y, y_1, \bar{a}_i] \to \mathcal{Z}$$

which satisfies $w(st) = w(s) + w(t)$, $w(x) = 1$, $w(\bar{a}_i) = n - i$, $w(y) = n$, $w(y_1) = n - 1$ and $w(k) = 0$ for $k \in \mathcal{C}$. Then f_1 and f_2 are isobaric polynomials with the weight n and $n-1$. From [9], we know that the resultant of two homogeneous polynomials is still homogeneous. By the same way, we can show that the resultant of two isobaric polynomials with the weight n and $n-1$ is still an isobaric polynomial with the weight $n(n-1)$. Hence $R(y)$ is an isobaric polynomial with the weight $n(n-1)$. We have $w(y^{\alpha_i} y_1^{\beta_i}) = n\alpha_i + (n-1)\beta_i \leq n(n-1)$, which implies $\alpha_i \leq n-1$ and $\beta_i \leq n$. If $\alpha_i > 0$ then we have $\alpha_i + \beta_i < n$, because $n\alpha_i + (n-1)\beta_i = \alpha_i + (n-1)(\alpha_i + \beta_i)$. Since $w(y_1^n) = w(y^{n-1}) = n(n-1)$, the coefficients of y_1^n and y^{n-1} in $R(y)$ must be polynomials in $\bar{a}_n$. By the computation of the above determinant, the coefficients of y_1^n and y^{n-1} in $R(y)$ are $(-1)^n \bar{a}_n^{n-1}$ and $(-1)^{n-1} n^n \bar{a}_n^n$. Then the form of $R(y)$ is as (4). In the following, we take $\bar{a}_i$ as complex numbers. If $R(y)$ is reducible, we assume that $R(y) = F_1(y)F_2(y)$, where $0 < \mathrm{tdeg}(F_1(y)), \mathrm{tdeg}(F_2(y)) < n$. Since $R(y) = P(y)f_1(y) + Q(y)f_2(y)$, where $P(y)$, $Q(y)$ are two differential polynomials, we have $R(\bar{y}) \equiv 0$ which implies that $F_1(\bar{y}) = 0$ or $F_2(\bar{y}) = 0$. But we know that it is impossible by Lemma 4.2, because $\deg(F_1(y), y_1)$, $\deg(F_2(y), y_1) < n$, a contradiction. ∎

Theorem 5.2 *Use the same notations as in Lemma 5.1. If $\bar{y} = \sum_{i=0}^n \bar{a}_i x^i$ is a polynomial solution of $F(y) = 0$, then $R(y)|F(y)$. Since $F(y)$ is irreducible, $F(y) = \lambda R(y)$, where $\lambda \in \mathcal{C}$ and $\lambda \neq 0$.*

Proof : From Lemma 4.2 and Lemma 5.1, we know $\deg(F(y), y_1) \geq n = \deg(R(y), y_1)$. Let $T(y) = \mathrm{prem}(F(y), R(y))$. Then we have the remainder formula $J(y)^k F(y) = Q(y)R(y) + T(y)$, where $J(y)$ is the initial of $R(y)$ and $Q(y)$, $T(y) \in \mathcal{C}[y, y_1]$ and $\deg(T(y), y_1) < \deg(R(y), y_1)$. Since $F(\bar{y}) = 0$ and $R(\bar{y}) = 0$, we have $T(\bar{y}) = 0$. By Lemma 4.2, $T(y) = 0$. That is $J(y)^k F(y) = Q(y)R(y)$ which implies that $R(y)|F(y)$ because $R(y)$ is irreducible. Since $F(y)$ is irreducible, it is clear that $F(y) = \lambda R(y)$ where $\lambda \in \mathcal{C}$ and $\lambda \neq 0$. ∎

From Lemma 5.1 and Theorem 5.2, if $F(y)$ has polynomial solutions $\bar{y} = \sum_{i=0}^n \bar{a}_i x^i$, it must be the following form

$$F(y) = a y_1^n + b y^{n-1} + G(y, y_1) \tag{5}$$

where $a, b \in \mathcal{C}$ are not zero, $\mathrm{tdeg}(G) \leq n-1$ and G does not contain the term y^{n-1}.

As a consequence of Theorem 5.2 and Lemma 5.1, we have

Corollary 5.3 *Let $F(y)$ be of the form (5) and have a polynomial solution of the form (3). Then*

$$\bar{a}_n = -\frac{b}{n^n a}. \tag{6}$$

Lemma 5.4 *Let $F(y)$ be of the form (5) and have a polynomial general solution of the form (3). Then we may construct a new general solution of the following form for $F(y) = 0$*

$$\hat{y} = \bar{a}_n (x+c)^n + \sum_{i=0}^{n-2} \tilde{a}_i (x+c)^i. \tag{7}$$

In other words, we may assume that $\bar{a}_{n-1} = 0$ in the general solution of $F(y) = 0$.

Proof : It is clear that

$$\hat{y} = \bar{a}_n(x + c - \frac{\bar{a}_{n-1}}{n\bar{a}_n})^n + \sum_{i=0}^{n-2} \tilde{a}_i(x + c - \frac{\bar{a}_{n-1}}{n\bar{a}_n})^i$$

Since $c - \frac{\bar{a}_{n-1}}{n\bar{a}_n}$ is still an arbitrary constant, replacing $c - \frac{\bar{a}_{n-1}}{n\bar{a}_n}$ by c in the above equation, we get the form (7) and it is still a general solution of $F(y) = 0$. ∎

The following theorem will tell us that the value of $\bar{a}_k$ only depends on the values of $\bar{a}_i$ for $i \geq k$ if $F(y) = 0$ has polynomial general solutions.

Theorem 5.5 *Let $F(y)$ be of the form (5) and $z = (-b/n^n a)x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ where a_i are indeterminants. Substituting y by z in $F(y)$, the coefficients of $x^{(n-1)^2+i-1}$ in $F(z)$ are of the following form*

$$(\frac{-b}{n^n a})^{n-2}(n-1-i)ba_i + h_i(a_{n-1}, \cdots, a_{i+1}) \quad \text{for } i = n-2, \cdots, 0 \tag{8}$$

where $h_i(a_{n-1}, \cdots, a_{i+1})$ are polynomials in $a_{n-1}, \cdots, a_{i+1}$.

Proof : Let $\mathcal{C}_i$ be the coefficient of $x^{(n-1)^2+i-1}$ in $F(z)$ for $i = 0, \cdots, n-2$ where

$$F(z) = az_1^n + bz^{n-1} + G(z, z_1)$$

As in the proof of Lemma 5.1, we define a weight w. Then $z = (-b/n^n a)x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is an isobaric polynomial with the weight n. Hence $z^{\alpha_j} z_1^{\beta_j}$ is still an isobaric polynomial with the weight $n\alpha_j + (n-1)\beta_j$. Now we consider $\mathcal{C}_i$. By computation, we know that the highest weight of the terms in $F(z)$ is $n(n-1)$. Hence the highest weight in $\mathcal{C}_i$ is not greater than $n - i$. So a_k can not appear in $\mathcal{C}_i$ for $k \leq i - 1$ and if a_i appears in $\mathcal{C}_i$, then it must be linear and its coefficient must be constant. In the coefficients of $x^{(n-1)^2+i-1}$ in $az_1^n + bz^{n-1}$, the term in which a_i appears are $(\frac{-b}{n^n a})^{n-1} i n^n a a_i + (\frac{-b}{n^n a})^{n-2}(n-1)ba_i$. In the coefficients of $x^{(n-1)^2+i-1}$ in $G(z, z_1)$, since the weight of each term is less than $n - i$ (for $\mathrm{tdeg}(G) < n$), a_i can not appear. Therefore $\mathcal{C}_i$ has the form (8). ∎

6 A polynomial-time algorithm and experiment results

From the results of section 5, we have the following algorithm:

Algorithm 6.1 *The input is $F(y)$. The output is a polynomial general solution of $F(y) = 0$ if it exists.*

1. If $F(y)$ can be written as the form (5), then goto step 2. Otherwise, by Theorem 5.2, $F(y) = 0$ has no polynomial general solutions and the algorithm terminates.

2. Let $F(y)$ be of degree n in y_1. Let $\bar{a}_n = -\frac{b}{n^n a}$, $\bar{a}_{n-1} = 0$, $\bar{a}_i = -\frac{h_i(\bar{a}_{n-1}, \cdots, \bar{a}_{i+1})}{(-b/n^n a)^{n-2}(n-1-i)b}$, $i = n-2, \cdots, 0$, where h_i are from Lemma 5.5. We have $\bar{a}_i \in \mathcal{C}$.

3. Let $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$. If $F(\bar{y}) \equiv 0$ then $\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x+c)^i$ is a polynomial general solution of $F(y) = 0$. Otherwise, $F(y) = 0$ has no polynomial general solutions.

The correctness of Step 3 is due to the following facts. By Corollary 5.3, Lemmas 5.4 and 5.5, if $F(y) = 0$ has polynomial general solutions, then $\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x+c)^i$ must be such a solution. By Theorem 4.1, to check wether $\hat{y}$ is a polynomial general solution we need only to check whether $\bar{y}$ is a polynomial solution of $F(y) = 0$.

Now we give some examples.

Example 6.2 *Consider the differential polynomial:*

$$F_1(y) = {y_1}^4 - 8\,{y_1}^3 + (6 + 24\,y)\,{y_1}^2 + 257 + 528\,y^2 - 256\,y^3 - 552\,y$$

1. *$F_1(y)$ can be written as the form:*

$$F_1(y) = {y_1}^4 - 256\,y^3 + G(y, y_1)$$

where

$$G(y, y_1) = -8\,{y_1}^3 + (6 + 24\,y)\,{y_1}^2 + 257 + 528\,y^2 - 552\,y$$

$tdeg(G) \leq 3$

2. *If $F_1(y) = 0$ has a polynomial general solution, then its degree is four and the coefficient of x^4 must be $a_4 = \frac{256}{4^4} = 1$.*

3. *Let $z = x^4 + a_2x^2 + a_1x + a_0$. Replacing y by z in $F_1(y)$, we compute the coefficients of x^8, x^9, x^{10}, which are*

$$\begin{aligned} &768\,a_2 + 528 - 384\,{a_2}^2 - 768\,a_0 \\ &-512 - 512\,a_1 \\ &384 - 256\,a_2 \end{aligned}$$

Then we have $a_0 = \frac{17}{16}, a_1 = -1, a_2 = \frac{3}{2}$

4. *Let $\bar{y} = x^4 + \frac{3}{2}x^2 - x + \frac{17}{16}$. Substituting y by $\bar{y}$ in $F_1(y)$, $F_1(y)$ becomes zero. Hence a polynomial general solution of $F_1(y) = 0$ is*

$$\bar{y} = (x+c)^4 + \frac{3}{2}(x+c)^2 - (x+c) + \frac{17}{16}$$

Example 6.3 *Consider differential polynomial*

$$F_2(y) = y_1^5 - 16y^4 + y_1^3 + y^2 - y_1 y$$

Similar as Example 6.2, the coefficient of x^5 must be $a_5 = \frac{16}{5^5} = \frac{16}{3125}$. Compute the coefficients of x^{15}, x^{16}, x^{17}, x^{18}, which are

$$\begin{aligned} &-\frac{147456}{48828125}a_3a_2 - \frac{262144}{30517578125}a_0, \\ &-\frac{49152}{48828125}a_3^2 - \frac{196608}{30517578125}a_1, \\ &-\frac{131072}{30517578125}a_2, \quad -\frac{65536}{30517578125}a_3 \end{aligned}$$

Hence we have $a_3 = a_2 = a_1 = a_0 = 0$ and $\bar{y} = \frac{16}{3125}x^5$. Substituting $\bar{y}$ to $F_2(y)$, we have

$$F_2(\bar{y}) = -\frac{256}{1953125}x^9 + \frac{256}{9765625}x^{10} + \frac{4096}{244140625}x^{12} \neq 0$$

Hence $F_2(y) = 0$ has no polynomial general solutions.

In Step 2 of Algorithm 6.1, we need to compute $F(z)$ where $z = -\frac{b}{n^n a}x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and a_i are indeterminants. The complexity of it is very high. In order to give a polynomial-time algorithm, we should modify Step 2. From Theorem 5.5, to compute the value of $\bar{a}_k$, we need only compute $h_k(\bar{a}_{n-1}, \cdots, \bar{a}_{k+1})$. In other words, we need only compute $F(\bar{z})$ where $\bar{z} = \sum_{k+1}^{n} \bar{a}_i x^i$. Moreover, we need only compute the coefficient of $x^{(n-1)^2+k-1}$ in $F(\bar{z})$. To compute $F(\bar{z})$, we need to compute multiplication of two univariate polynomials which can be computed by the classical method or Karatsuba method ([21]).

Algorithm 6.4 *The inputs are $F(y)$ as the form (5) and $\bar{z} = \bar{a}_n x^n + \cdots + \bar{a}_0$ where $\bar{a}_i \in \mathcal{C}$. The output is the coefficient of $x^{(n-1)^2+k-1}$ in $F(\bar{z})$ for some k.*

1. Compute $\bar{z}^n$ and $\bar{z}_1^{n-1}$ where $\bar{z}_1$ is the derivative of $\bar{z}$ wrt x. We compute $\bar{z}^n$ step by step. That is, we compute $\bar{z}^2$ firstly, then compute multiplication of $\bar{z}^2$ and $\bar{z}$, and so on. Note that, after we computed $\bar{z}^n$, we have also obtained $\bar{z}^i$ for $i < n$. For $\bar{z}_1^{n-1}$, we compute it by the same way.
2. Write $F(y)$ as the form: $F(y) = \sum_{i=0}^{n}(d_{0,i} + d_{1,i}y + \cdots + d_{n-i,i}y^{n-i})y_1^i$ where $d_{i,j} \in \mathcal{C}$.
3. For i from 0 to n, compute $p_i = d_{0,i} + d_{1,i}\bar{z} + \cdots + d_{n-i,i}\bar{z}^{n-i}$
4. result:=0
   For j from 0 to n
   For i from 0 to $(n-1)^2 + k - 1$
   result:=result+coeff$(p_j, x, i)*$coeff$(y_1^j, x, (n-1)^2 + k - 1 - i)$
   where coeff(p, x, k) means the coefficient of x^k in p.
5. return(result)

In Algorithm 6.4, the complexity of Step 1 is $O(n^4)$. The complexity of Step 2 is $O(n^2)$. The complexity of Step 3 is $O(n^4)$. The complexity of Step 4 is $O(n^3)$. Hence the complexity of Algorithm 6.4 is $O(n^4)$. Here, we only consider the complexity on multiplications.

Algorithm 6.5 *The inputs are $F(y)$ as the form (5) and $z = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where a_i are interminates. The output is $\bar{a}_i$ for $i = n, \cdots, 0$ in Step 2 of Algorithm 6.1.*

1. Let $\bar{a}_n = -\frac{b}{n^n a}$, $\bar{a}_{n-1} = 0$.
2. Let $i = n - 2$.
   while $i \geq 0$ do
   (a) $\bar{y} := \bar{a}_n x^n + \cdots + \bar{a}_{i+1}x^{i+1}$.
   (b) $C_i :=$the coefficient of $x^{(n-1)^2+i-1}$ in $F(\bar{y})$ by Algorithm 6.4.
   (c) $\bar{a}_i := -\frac{C_i}{(-b/n^n a)^{n-2}(n-1-i)b}$.

(d) $i := i - 1$.

It is easy to know that the complexity of Algorithm 6.5 is $O(n^5)$. In Step 3 of Algorithm 6.1, we verify whether $\bar{y}$ is a polynomial solution of $F(y) = 0$. If $\bar{y}$ is not a polynomial solution of $F(y) = 0$, that is said $F(\bar{y}) \neq 0$, then $F(\bar{y})$ will be a polynomial in x with degree not greater than $n(n-1)$. Hence, if $F(\bar{y}) \neq 0$, then $F(\bar{y}) = 0$ as an equation in x has $n(n-1)$ roots at most. So we can verify it by numerical computation. If $F(\bar{y})(k) = 0$ for $k = -\frac{n(n-1)}{2}, -\frac{n(n-1)}{2} + 1, \cdots, \frac{n(n-1)}{2}$, then $F(\bar{y}) = 0$. Otherwise, $F(\bar{y}) \neq 0$.

Algorithm 6.6 *The inputs are $F(y)$ as the form (5) and $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ as in Algorithm 6.1. The output is "Yes" or "No" where "Yes" means $\bar{y}$ is a solution of $F(y) = 0$ and "No" means $\bar{y}$ is not a solution of $F(y) = 0$.*

1. Write $F(y)$ as the form: $F(y) = \sum_{j=0}^{n} (d_{0,j} + d_{1,j} y + \cdots + d_{n-j,j} y^{n-j}) y_1^j$ where $d_{i,j} \in \mathcal{C}$.
2. For k from $-n(n-1)/2$ to $n(n-1)/2$
   (a) Compute k^i for $i = 1 \cdots n$ by the same way as Step 1 of Algorithm 6.4.
   (b) Substitute x^i by k^i in $\bar{y}$ and $\bar{y}_1$. Then we get the values of $\bar{y}$ and $\bar{y}_1$ at k denoted by $\bar{y}(k)$ and $\bar{y}_1(k)$. Compute $\bar{y}_1(k)^i$ and $\bar{y}(k)^i$ for $i = 1 \cdots n$.
   (c) result:=0
   For j from 0 to n, compute $p_j(k) = d_{0,j} + d_{1,j}\bar{y}(k) + \cdots + d_{n-j,j}\bar{y}(k)^j$
   result:=result+$p_j(k)\bar{y}_1(k)^j$
   (d) If result$\neq 0$ then return(No)
3. return(Yes)

It is easy to see that the complexity of Algorithm 6.6 is $O(n^3)$. So we have the following theorem.

Theorem 6.7 *We can decide whether $F(y) = 0$ has a polynomial general solution and compute one if it exists with $O(n^5)$ multiplications.*

Table 1 shows the statistic results of running our algorithm for ten differential equations. Differential polynomials in our experiments are given in the Appendix of this paper. We only give the total degrees and terms of these differential polynomials here. In the table, F_i and G_i denote differential polynomials. Here, the coefficients of F_i and G_j are integers. The coefficients of F_i are less than 10^6 but that of G_j may be very large. The unit of running time is seconds. The column of "solution" means whether they have a polynomial general solution or not. The program is written in Maple. The running time is collected on a computer with Pentium 4, 2.66GHzCPU and and 256M memory.

7 Conclusion

We give a polynomial-time algorithm to compute a polynomial general solution of first order ODEs with constant coefficients. Our experiments show that the algorithm can be used to solve very large ODEs.

It is interesting to see whether the result can be extend to the case when the coefficients of first order ODEs are not constant or the case of high order ODEs with constant coefficients.

	tdegree	term	time(s)	solution		tdegree	term	time(s)	solution
G_6	6	17	0.077	Y	F_6	6	22	0.063	N
G_7	7	22	0.125	Y	F_7	7	28	0.266	N
G_8	8	30	0.312	Y	F_8	8	37	1.141	N
G_9	9	38	1.468	Y	F_9	9	45	3.500	N
G_{10}	10	47	8.108	Y	F_{10}	10	55	10.656	N
G_{11}	11	57	16.062	Y	F_{11}	11	60	31.345	N
G_{12}	12	68	34.250	Y	F_{12}	12	74	70.438	N
G_{13}	13	80	78.203	Y	F_{13}	13	80	148.984	N
G_{14}	14	93	178.469	Y	F_{14}	14	90	273.065	N
G_{15}	15	106	306.250	Y	F_{15}	15	110	434.514	N

Table 1: Statistics on Solving Differential Equations

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8 Appendix

$F_6 = 334y_1^6 + 279967y^5 + 194478 - 19838y^2y_1 + 113804yy_1^4 - 381141y^2y_1^2 - 219431y_1 + 209023y + 167749y^3y_1^2 - 382588yy_1^3 - 72424y^2y_1^3 + 72865y^3y_1 + 220632y^4y_1 + 213575yy_1 + 279767yy_1^2 - 276548y^3 - 148190y_1^2 + 90647y^2 + 37783y_1^4 + 46451y_1^3 - 248636y^4 - 385166y_1^5$

$F_7 = 34579 + 25778y^2y_1 + 80625yy_1^4 + 78679y^2y_1^2 + 43313y_1^6 + 46139y_1 - 73345y^5 - 50398y + 218y_1^7 + 96281y^6 + 6542y^2y_1^4 + 36183y^3y_1^3 - 119686y^5y_1 - 48030y^4y_1^2 + 69685yy_1^5 + 93551y^3y_1^2 + 74603yy_1^3 + 45035y^2y_1^3 + 32219y^3y_1 + 21648y^4y_1 - 111353yy_1 + 36169yy_1^2 + 51768y^3 + 4297y_1^2 - 147755y_1^4 - 33025y_1^3 + 112279y^4 + 89315y_1^5$

$F_8 = 68521 - 52460y^2y_1 + 184098yy_1^4 - 74708y^2y_1^2 + 117y_1^8 + 301474y^7 - 216839y^2y_1^5 + 431395y^6y_1 - 222129y^5y_1^2 + 190990y^3y_1^4 + 28550yy_1^6 - 253369y^4y_1^3 - 130938y_1^6 - 29609y_1 - 21177y^5 + 127956y + 239821y_1^7 - 37748y^6 - 266235y^2y_1^4 - 199885y^3y_1^3 - 465274y^5y_1 - 88894y^4y_1^2 - 105279yy_1^5 + 27277y^3y_1^2 - 111371yy_1^3 - 47233y^2y_1^3 - 51283y^3y_1 - 213137y^4y_1 - 79729yy_1 - 70345yy_1^2 - 255233y^3 - 64054y_1^2 - 53393y^2 + 16305y_1^4 + 123710y_1^3 + 91518y^4 - 164633y_1^5$

$F_9 = 897 - 171475y^2y_1 + 66023yy_1^4 - 168590y^2y_1^2 - 15692y_1^8 - 212470y^7 + 18496y^2y_1^5 - 90533y^6y_1 - 458442y^5y_1^2 - 3000y^3y_1^4 + 168380yy_1^6 - 26402y^4y_1^3 - 206060y_1^6 + 194497y_1 + 116603y^5 - 122110y + 77511y_1^7 + 62578y^6 + 301315y^2y_1^4 - 109153y^3y_1^3 - 221726y^5y_1 + 34509y^4y_1^2 - 241196yy_1^5 + 54585y^3y_1^2 + 330628y^3y_1^5 + 66y_1^9 + 210958y^8 - 185552y^7y_1 + 404735y^2y_1^6 + 17586yy_1^7 + 464890y^6y_1^2 + 41859y^5y_1^3 - 91674y^4y_1^4 + 25318yy_1^3 - 28988y^2y_1^3 - 320359y^3y_1 - 125446y^4y_1 + 41842yy_1^2 - 186212y^3 - 65069y_1^2 - 29310y^2 + 40296y_1^4 - 93934y_1^3 - 29771y^4 - 80489y_1^5$

$F_{10} = 165y_1^{10} - 113025y_1^9 + (-65389 + 46498y)y_1^8 + (123249y^2 + 914y - 176941)y_1^7 + (136118 - 73765y + 74818y^3 + 130026y^2)y_1^6 + (64371 + 117228y - 155272y^2 + 340028y^4)y_1^5 + (-16102y^2 + 22985y^5 - 113723y^3 + 31597 - 8059y - 123590y^4)y_1^4 + (-11444y^2 - 35359 - 4236y^4 + 26633y^6 - 116171y^5 + 560y + 46163y^3)y_1^3 + (32057y - 74344 - 52794y^7 - 81935y^2 - 94960y^5 + 184612y^6 - 59057y^3 + 125386y^4)y_1^2 + (8323 + 14585y^2 + 102546y^6 - 104936y^4 + 9467y + 74815y^7 + 267618y^5 + 114803y^3 - 289906y^8)y_1 + 82996 - 57968y - 227361y^5 + 42500y^7 - 85907y^2 - 16447y^6 - 137894y^8 - 30298y^4 - 69955y^3 + 633215y^9$

$F_{11} = 181y_1^{11} + 3840y_1^{10} + (-94780 + 12878y)y_1^9 + (22493y + 11288)y_1^8 + (-35340y - 72422y^3 + 4368 + 9846y^2)y_1^7 + (-19922y^2 - 71119 + 100156y^3)y_1^6 + (-93819y^2 + 146671y^4 + 22174y^3 - 72274y^5 - 138109y + 77309)y_1^5 + (107128y^4 - 24814y^2 - 28027y + 138939y^5 + 68990y^3 + 55960y^6)y_1^4 + (75561 + 149929y^6 - 8166y^7 + 57489y + 24849y^3 + 112586y^4 + 33916y^5 - 7177y^2)y_1^3 + (36767y^7 - 83358y^6 + 30821 + 58285y^3 - 50458y^8 - 196760y^2)y_1^2 + (128186y^5 - 16846 - 18936y + 125736y^7 + 45710y^2 + 235415y^9 + 40485y^4 + 112299y^6 + 88202y^3 - 6533y^8)y_1 + 9868 - 90092y^7 + 71113y^2 + 77724y^6 + 116126y^8 + 355681y^{10} + 47172y^4 + 17818y^3 - 14449y^9 + 23335y^5 + 28742y$

$F_{12} = 76y_1^{12} + 33722y_1^{11} + (84791y + 114149)y_1^{10} + (89932y - 162740 - 49006y^2)y_1^9 + (-26588 - 214315y^3$
$+ 16457y^2 - 137569y)y_1^8 + (69935y^4 + 87663y^2 + 20764 - 181741y^3 + 61867y)y_1^7 + (-36780 - 62219y^5 - 139248y^4$
$- 29929y + 52041y^3)y_1^6 + (-186473 + 71854y^5 + 72238y^2 - 30765y + 118595y^3 + 146395y^6 + 97372y^4)y_1^5 + (-31080y^3$
$+ 19498y^5 - 175150y^6 - 50050y^4 - 188845y^7 - 153372y - 6591 + 65853y^2)y_1^4 + (214886y^6 + 28473y^3 - 11026 + 49527y$
$+ 107562y^8 - 78969y^4 + 53763y^7 - 157844y^5)y_1^3 + (-56973y^5 + 69805y^2 - 44284y^8 - 33937y^4 + 50907y^3 + 79689y^7$
$- 108231y - 237536y^9 - 14906 + 25434y^6)y_1^2 + (112707y^7 - 399652y^{10} - 32365y^9 + 166580y^4 + 150190y^5 + 206297y^3$
$+ 7389y^6 - 17873y^2 + 21677y^8)y_1 + 53290 + 19052y^6 - 98408y - 25953y^5 + 256939y^7 + 242773y^{11} + 127722y^8$
$+ 93910y^3 + 222482y^9 + 105244y^{10} - 125372y^4$

$F_{13} = 135y_1^{13} + 47352y_1^{12} + (39554y + 114974)y_1^{11} + (-26453y - 108472y^2)y_1^{10} + (-45881y + 3448y^3 + 161186$
$- 78531y^2 + 88457y^4)y_1^8 + (-83844 + 90607y^5 + 131365y^4 + 101407y^2 + 34645y^3 - 139975y)y_1^7 + (116213y + 49054y^3$
$- 47811y^5 - 35338y^2 + 120450y^4 + 45941)y_1^6 + (70947 - 113751y^2 + 82734y^7 + 104310y + 51842y^4 + 46985y^6)y_1^5$
$+ (-161375y^5 + 25599y^7 + 47165y^2 + 29838y^3 + 260685y^8 + 86300y^4 + 89863y^6 - 7438 + 54365y)y_1^4 + (-78286y^7$
$+ 299955y^8 - 30142y^4 + 21679y^6 - 12137y^5 + 20605y^3 + 119671y^2 + 5052 + 275301y^9 - 45018y)y_1^3 + (110220y^5$
$- 43563y^6 + 50465y^8 - 163044y - 14511y^7 - 104411y^4 + 38802y^3 - 153933y^{10} + 74317y^9 - 33522y^2)y_1^2 + (75894y^4$
$+ 19612y^7 + 64802y^3 - 308216y^{11} + 219785y^{10} - 75054y^9 - 84388y^8 + 107556y - 73832y^6 + 118842y^2)y_1 + 28303$
$+ 21829y^6 + 86381y^8 - 257554y^{12} - 75232y^9 + 31873y^{11} - 6535y^4 - 23695y^5 - 14463y^7 - 442420y^{10} + 5739y^3$
$- 13745y^2$

$F_{14} = 47051 + 356y_1^{14} + 181442y^{12} + 23911y_1^{13} - 49639y^{13} + (-19535 - 149735y)y_1^{12} + (-32591 + 11835y^2$
$- 137727y)y_1^{10} + (-55189y^3 + 95505y^2 + 88591 - 23143y^4)y_1^9 + (-268746 - 13262y^2 + 36597y^3 - 94570y^5)y_1^8$
$- 322849y^2y_1^{11} + (58130y^6 - 60834y^3 + 173440 + 71192y^5 + 29352y)y_1^7 + (66865y^6 - 25802y^2 + 26422y^7 + 22248$
$+ 77469y^4 + 96906y^3 + 13758y^5 - 1587y)y_1^6 + (84650y^4 + 274940y^8 + 53913y^5 + 30005y^2 + 20384y - 151580y^7$
$+ 74588y^6 - 48558)y_1^5 + (-37977y^2 - 26154y^6 + 70806y^7 + 209515y^4 - 42854y^8 - 98749y^5 - 78896 - 75562y^3$
$- 31196y^9)y_1^4 + (76236 + 5397y^9 + 21870y^3 - 19996y + 6261y^2 + 47483y^{10} + 465y^5 + 39112y^8 + 146292y^6)y_1^3$
$+ (-182775y^9 + 74844y^4 + 53406y^{11} - 13323 + 26335y + 13293y^7 + 77477y^8 - 8099y^{10} - 25121y^3 - 61377y^5)y_1^2$
$+ (84612y^8 - 326734y^{12} + 163086y^9 + 4332y^7 - 115290y^4 - 43952y^{11} - 86323 + 202105y^6 - 97418y^{10} - 58002y^5$
$+ 15187y^3)y_1 + 85262y + 6192y^5 - 47024y^7 + 56564y^2 - 135289y^6 - 76342y^8 - 37712y^{11} + 93617y^4 - 41180y^3$
$- 181057y^9 + 28294y^{10}$

$F_{15} = -36816 - 60506y^3y_1^{10} - 148408yy_1 - 72370y^3y_1 + 96066yy_1^7 + 75950y^5y_1^6 - 135970yy_1^9 + 151078y^9y_1$
$- 55982y_1^7 + 83916y_1^9 + 171344y^4y_1^{10} + 46677y^3y_1^7 - 148125y^{10}y_1 - 233944yy_1^2 - 89697y^4y_1^3 - 30393y^3y_1^4$
$+ 74483y^2y_1^3 + 96626y^4y_1^2 + 97843y^7y_1 - 65185y_1^8 - 67716y_1^5 + 115526y_1^2 - 40601y^5y_1 - 91459y_1^6 - 33266y^5y_1^3$
$+ 79906y^5y_1^2 + 132103y^3y_1^8 + 465yy_1^4 + 17963y^2y_1^9 + 326yy_1^6 - 19920y^6y_1^5 + 142508y^6y_1 + 93012y^{11}y_1$
$- 53259y^8y_1^2 + 156381y^6y_1^4 + 86181y^2y_1 + 50879y^5y_1^5 + 64251y^4y_1^4 - 158941y^4y_1 - 7502y^3y_1^5 + 69757y^6y_1^2$
$+ 79041y^2y_1^{11} - 92201yy_1^{13} + 1811y^7y_1^5 + 8039y + 99967y_1 + 98128y^5 - 47895y^7 + 28509y^2 + 97742y^6 + 22883y^8$
$+ 154374y_1^{11} + 46882y_1^{12} - 114968y^{11} - 27662y^3 - 89980y^9 + 190735y^{10} + 94659y_1^3 - 68247y_1^4 + 105849y_1^{10}$
$+ 53268y^2y_1^5 + 18113y^2y_1^2 + 19294yy_1^5 + 19646yy_1^3 + 163996y^3y_1^3 - 9643y^8y_1^3 + 25209y^2y_1^7 - 136485y^7y_1^2$
$- 85493y^9y_1^2 - 54862y^4y_1^7 + 42306y^7y_1^4 + 83210y^2y_1^8 + 24478y^6y_1^3 + 56630y^5y_1^4 + 134851y^3y_1^6 - 33406y^4y_1^5$
$- 18545yy_1^8 - 108243y^{10}y_1^2 - 178390y^5y_1^7 + 125488y^8y_1^4 - 22599y^2y_1^{12} - 48811y^{12} - 141750y_1^{13} - 110379y^{12}y_1^2$
$- 74141y^3y_1^{11} - 71653y^4y_1^8 - 76398y^9y_1^5 + 46557y^5y_1^9 + 83305y^6y_1^8 - 300131y^9y_1^3 + 23870yy_1^{11} + 100912y^3y_1^9$
$- 162450y^7y_1^7 - 7384y_1^{14} + 33389y^{13} + 35253y^5y_1^8 - 145295y^7y_1^6 + 63153y^8y_1^5 - 88236y^6y_1^7 - 89364y^{11}y_1^2$
$+ 97372y^{12}y_1 + 68222y^6y_1^6 - 91325y^{10}y_1^3 + 131y_1^{15} + 145908y^{14} + 29104y^{11}y_1^3 - 11487y^{10}y_1^4 - 220024yy_1^{12}$
$+ 142894y^8y_1^6 - 341817y^{13}y_1$

$G_6 = -2239270123370508y^3y_1^2 - 733411666857995 22yy_1^3 - 348625916485227762y^2y_1^2 - 263548225827275057 64yy_1^2$
$- 27745457202367122742187604y - 1972143506316435278 07y^4 + 101621504799y_1^6 - 753859215536440896y^5$
$- 10703429963329336175328 96y^2 - 2057312626841398225038 0y^3 + 85030590682641y_1^5 - 462685425924899921661y_1^2$
$+ 7192539116156349 48y_1^3 + 20536740304090707y_1^4 - 138423274168410yy_1^4 - 90930477113712y^2y_1^3$
$- 28654017172999726865137 3524$

$G_7 = -678223072849y^6 + 29902831344442y^3y_1^2 + 8066721334y^3y_1^3 - 2567711542105192yy_1^3 - 1697982202yy_1^5$
$+ 8334405125940453y^2y_1^2 + 5897278090871023 78yy_1^2 + 1742252379y^2y_1^4 - 58714498185y^4y_1^2 - 39371277985848853484874y -$
$307303476489896 95y^4 + 583372188y_1^6 + 372111226698414y^5 - 1187254524283627954343y^2 - 1433205926188662 3712y^3 -$
$117649y_1^7 - 861043913576y_1^5 + 127333307950440516 33y_1^2 - 86753350653756045y_1^3 + 342142329427591y_1^4 + 5177695276740yy_1^4 -$
$16895153027179y^2y_1^3 - 47322705541136069861 2017$

$G_8 = 1902647099961771686675 28y^6 - 1469716239890230031664y^3y_1^2 - 130100372330945885424y^3y_1^3$
$+ 2135968555168228418 2368yy_1^3 + 635926239840095078 4yy_1^5 + 52694015782570975922904y^2y_1^2 - 40424508565134796 8y^2y_1^5 +$
$28738267971613360128y^4y_1^3 - 120372471054589132 8y^3y_1^4 - 29935320675415496758 2816yy_1^2 + 531176064955664556 24y^2y_1^4 -$
$1159897918917794331600y^4y_1^2 + 88600645401588006912y^5y_1^2 - 27513669434654592yy_1^6 - 284239492317637579935751 68 +$
$20104890609108772136359296y - 874170892507510121714208y^4 + 188446772808576072y_1^6 - 740887900495546107497472y^5 -$
$577483026442475082918393 6y^2 + 46443072751616096100066 24y^3 - 121165747909451065589 76y^7 - 575155251049320y_1^7 +$
$10030613004288y_1^8 - 27023008529215963800y_1^5 + 47574441340753413083702 4y_1^2 - 41979673480845703249728y_1^3$
$+ 155126538274387929717 6y_1^4 - 572043928477107148992yy_1^4 - 281478220366933171094 4y^2y_1^3$

$$
\begin{aligned}
G_9 = {} & 132420130032638282yy_1^7 + 1620480967429675732393666871\,9y^6 + 62843564622384142329614638\,0y^3y_1^2 \\
& + 1145295513375833522781988y^3y_1^3 + 2065000648445524941788302090\,0yy_1^3 - 2124169198354956357844\,4yy_1^5 \\
& - 10664230057889262699565292711\,0y^2y_1^2 - 148718084598451026211\,4y^2y_1^5 - 710365592430599892428\,24y^4y_1^3 \\
& - 249326238863621801168\,98y^3y_1^4 - 5643707609509233251023697305728yy_1^2 - 1357961334735693032y^3y_1^5 \\
& + 906939132053407653y^2y_1^6 - 1653884733037075179376349y^2y_1^4 + 271951074951408735992546\,93y^4y_1^2 \\
& - 678825585970804288872y^5y_1^3 + 1479242172860816017505\,56y^5y_1^2 - 14729983611494959411\,6yy_1^6 \\
& - 126031946325134888844y^4y_1^4 - 5588985403810644585\,03y^6y_1^2 + 242933187499533663647818406229180287\,6y \\
& + 12504785359533749257532025487928\,2y^4 + 390933567944588482199\,14y_1^6 + 180720713428267505734680618107\,0y^5 \\
& +151839141251814937126979739462952043y^2+550129665799323678888067038697402\,8y^3+834827111983319779676724\,10y^7- \\
& 564121528822815599\,27y_1^7 - 11688200277601y_1^9 + 194714675484358952176227y^8 + 42366494451615266y_1^8 \\
& - 1877365546884114900277077\,2y_1^5 - 2097597652144998556622278514909y_1^2 - 6916513610019210348925788751\,47y_1^3 \\
& + 4968018124896899041277366324y_1^4 - 43929195814266907214699544yy_1^4 + 53710852031387282866609880\,7y^2y_1^3 \\
& + 170529794906946238919854529226917743\,12
\end{aligned}
$$

$$
\begin{aligned}
G_{10} = {} & -448735886564127397151314549\,76y_1^{10} + 2832178749426013680072678434474226y_1^9 \\
& + (-5449073968798636595747944951505481976\,0 - 1132582117692273462323199315623210\,4y)y_1^8 \\
& + (520063140119015608703916472794207608251180 + 1569984100391791485109466655042112\,80y^2 \\
& + 225681679920155771172845673190012195\,16y)y_1^7 + (2097277279593757290141553430907263229056880y \\
& - 285359202857741182137454869760273035216847325\,8 - 597439859398700846535310329521120050264y^2 \\
& - 235107175459194840894709971630177305\,6y^3)y_1^6 + (-217064981458913480572531063044117984612671516\,36y \\
& + 4216113579664773477467246637056540455277\,2y^3 + 12919167531499406149459271798722444288y^4 \\
& + 951253098203505303585667158102519153036840457068\,4 + 28032892823513628198421231737659755353266\,34y^2)y_1^5 \\
& + (-2972063128390433603477585598097290758809\,18y^4 - 1257707106545678847519120132180570657461395066\,66y^2 \\
& - 1372737374185969523011324279415969600\,00y^5 + 817170272401934418965226780776651766053728978762\,40y \\
& - 1982187832015229988859457015239103604924986564139023\,6 \\
& - 52966219143949248315439071603313148723529722\,4y^3)y_1^4 \\
& + (393187691525649783335505437775147041144503994\,8y^4 + 7287730054852172049212581511352320000\,0y^6 \\
& - 10366676751810424128146919398902311847228924642700992\,4y \\
& + 26882776569544482043764398640367572134277756478442285672 \\
& + 914820958503798286311284369936943650828786811051526y^2 + 477999298443104158711822183549873474671770\,8y^5 \\
& + 3226746022414430123123223451571146122695473400056y^3)y_1^3 \\
& + (-322233116304275725186683367485077067701567356\,68y^5 \\
& + 2609712354498310903345750037388800000\,00y^7 - 2603229786880974749778677682026541636237128004536652682\,8y \\
& - 159292750222398614219464440059041127493769962152\,98y^4 \\
& - 223416778034098966270445029692371620621790976714685942221\,90 \\
& - 19369091855276976864409099358396266844494633544075064\,24y^2 - 197476517379607751541253203285054385284628\,46y^6 \\
& - 72409691502076338179634147257627525509882396283073\,20y^3)y_1^2 - 684770962896858408052906002933760000000000\,0y^9 \\
& + 14974521702758080861859291205449474993603932382668412609743290\,0y \\
& + 34103007667578130973552391563683496693581562221272922\,0y^5 \\
& + 146587809860044337777970398318126062112947792314\,4y^7 \\
& + 2098084411367035899975161353165361917958077707929975463968870943\,8 \\
& + 20627342157577987774513209892060375696646054469762675644139\,46y^2 \\
& + 88540889049009897573915873472651902461548251982177\,2y^6 \\
& + 897654251296761961089962991447183096112233414y^8 \\
& + 1362790300025853279976649467021399509440517083633334405042\,0y^3 \\
& + 814370687277355119377568244122233281275857147427370261\,06y^4
\end{aligned}
$$

$$
\begin{aligned}
G_{11} = {} & -579980646927311009285616\,49y_1^{11} - 800526721785128662263257367734\,75y_1^{10} \\
& + (-435416341841926386580041984883131274\,9 \\
& - 703381386888366491236906327843230y)y_1^9 + (-109210268220110650388889194987664452507249 \\
& - 5172858164758424736587999854863068y^2 - 68378520866612614493910729835988437500y)y_1^8 \\
& + (-245997750956362302619028563844584190154079\,6y - 18459788336479366060421866558337863146437026\,58 \\
& - 3258423870813499344489092811142179\,6y^3 - 6988102456665879295926830843578546101\,37y^2)y_1^7 \\
& + (-205291788185677052462994278378789181y^4 - 29094081335810932702759500741081509455825406y^2 \\
& - 414704686980963810787431526673719803893\,6y^3 - 2460657425450125671888458206830907116523178973259\,7 \\
& - 503089553056106758712933777695493749847755909\,54y)y_1^6 + (-90368813878948155551810143597985579\,0y^5 \\
& - 711765863631668862487242801407254141746814236098y^2 \\
& - 738437848493602258443176389113361077322374152397290y \\
& - 309695991391526182170952318080337402968\,02y^4 - 26649576429489778151675259719439406429950781\,0y^3 \\
& - 261146558416468152838716245689813869234070765252883845)y_1^5 \\
& + (-1094812694534033077313005700971635214229474241551371\,1y^2 \\
& - 8174955490082529337787806600243964583763687543565881438y \\
& - 614884384028373341747279162502284761286572217675\,0y^3 \\
& - 22096847731621423116478146458267905593431434034466003540\,48 \\
& - 10806049507108583898731271713094443281782650\,37y^4 \\
& + 80424728664575341086148612305653761227578y^5 - 6186609329963948462901329532389171166y^6)y_1^4 \\
& + (-98470220388804678499165415312950277220257747602821941759y^2 \\
& - 341004794330388833768568342573003073\,46y^7 \\
& - 7370748416632547800962508412797357750168709485185545\,4y^3 \\
& - 133600050146788332728401570719860007086441015390980266026320\,92 \\
& + 633724643014555054024165284473870188811798y^6 \\
& - 59076803330454341381656566909740532370085130208295607461996y
\end{aligned}
$$

$$
\begin{aligned}
&- 20802182352580648739584112986318976904120931045402y^4\\
&+ 830306104997681547746853471118569019851913008y^5)y_1^3\\
&+ (-271700440371782276838671603947999720067 6478y^7 - 986607912590569833484147068228409209 33y^8\\
&- 26461309518403646041454636699255556394787 4377876y^5\\
&- 4211722313967860491162499472837907948269593913808088207 20192y^2\\
&+ 6291901849286541987648353907512210029437386344y^6\\
&- 384545387354608160356812939205212135633189758611750134225 02859185\\
&- 4011689556551983838901639102931596451838611807350256420 46y^3\\
&- 15602782873566088844755745150289343805750670070775 4040y^4\\
&- 2052874774282382003782581651969479765111439081586995422779588 10y)y_1^2\\
&+ 21324606335713681818529186587695413829194655293415 6382y^6\\
&+ 23375774263167509141801964731510506121210981 27820y^8\\
&+ 17067604003667555094886935330944258355449718488338221879225826 6192326061\\
&+ 40842244526524306608467397889708029179261226517814287438 09976y^4\\
&- 6238416823092362727385191895443614072203y^{10}\\
&+ 63146860360348796368267086453578412181840217529630029062351576 80y^3\\
&+ 6310698653844265722701506445660193993826021 16y^9\\
&+ 2584582494884663928353792590222770931933275342793 628y^7\\
&+ 41847533595910927289006765595182603336535366918223067748347571 32805y^2\\
&+ 1340137538035593048286997569050298064919790293938101044908874338 485554y\\
&+ 1137407860786045608859758846695460959882801919571937572 98y^5
\end{aligned}
$$

$$
\begin{aligned}
G_{12} = {}&-7730993719707444524137094407y_1^{12} + 14743938098117819258708473061392 61y_1^{11}\\
&- 6238416823092362727385191895443614072203y^{10} + (-97765246895174545021407832306022824271\\
&- 6238416823092362727385191895443614072203y^{10} + 14685133877122994081514958879574180y)y_1^{10}\\
&+ (-2122624177119468539388449887809560971106y + 2703018213565254169448138842233 3905y^2\\
&+ 2539328131203837556692524382931625245616074)y_1^9 + (8963948776496327623712143214134136281784 4530y\\
&- 149851101745852841563072698065851833983 58y^2 - 25355913890345643131410283099456965234211508421\\
&- 593510514896769635243785516456415 16y^3)y_1^8 + (-33325137747734712682960045738215384 97y^4\\
&- 1394743288626618241029065239927183611962546158910y - 38856724561357007898336110158353414423380340148989\\
&- 81269561021414121735964117313121737008144y^3 + 9936115186571470939359617519393924744185716 28y^2)y_1^7\\
&+ (1985852665253110262222968824894119122y^5 - 305965351627675732101254853575366844390948y^4\\
&+ 6045646514258305242240024896407659960894033580529162y - 2147320759291548102517222967089481634616649225 9388y^2\\
&+ 1898586251891459632901323787098350933770511952053422131\\
&+ 653178375824906484597779750825547193094524552 8y^3)y_1^6\\
&+ (147563331319042527425637649179762970093685318 42y^4\\
&+ 132745317882359878246066504943317258878695287086854405y^2\\
&+ 3711569859522851122282739474608749079469074386594335022 4y\\
&- 18802109920679204484102005136084522894330289316460 6y^3\\
&- 585253979238075791889895628675719615488y^6 - 882661928680882733810290561501767578671 1736y^5\\
&- 13949873997532648713194252987162423937622263469260812570)y_1^5\\
&+ (-80443522271716899021379311378537468056663887122883434566835502\\
&+ 130934577370002818131298338948393886331699429724y^5\\
&- 75667430136266454925902111714137642203005099665 1869y^4 - 663592907765834830691664067275336578801961 24y^6\\
&+ 1485448572860514796039169471572951389915978380530626868y^3 - 710325793453487823202447025303240124748 8y^7\\
&- 1580477155572346925718918898091306310637491251192018166087 14y\\
&+ 32029321346425886563521564192195283768279037641106801130 5y^2)y_1^4\\
&+ (140813271867822919209292680082346603276062418y^7\\
&+ 17050899725771202661136385286168327222928522101103149477831 7y^2\\
&- 157284582680221443882214676824337532408154553950338021160489340y - 18905337442259392733102186378642854969344y^8\\
&+ 138267383482523689800094524914946741237604226095275534242208730984\\
&+ 35182035145800528710135630850484160903369407342 52y^6 + 409499591068945517647641293871786749754374731453724 0y^5\\
&+ 146246290746756235859345234173446823872078921284248773 24y^4\\
&+ 5720634417127169806375568374706317477548001866526657160260y^3)y_1^3\\
&+ (-107945764336259942542005289033899084751352371713734854729351700421 34y\\
&- 178207390612227592643300509000452215213260 8y^9 - 99044179664257616186016483619782084692483163230887473 6y^5\\
&- 49269817232030287509882075296000817422210846040511736325858y^4\\
&- 147555080356944583825846705545500627266974044852709610857465928307103 9\\
&+ 8854886838805095846906348690706073977804188784658y^7 - 695536145603228312167468190059784878264790 34y^8\\
&- 36053482568296148986522742455981971705258402599009369729129886026y^2\\
&+ 188209447368401848672272111431385363869288391662882 8y^6\\
&- 62499691632979880125764042459772683660134048001302186775005312y^3)y_1^2\\
&- 60549836206778416856585613775143575741525390168247563027 84y^7\\
&- 17370174813769865015018981687253142620672039627101084451537872112463509990379\\
&- 6849776045642392322700028019005996478112694543428016117198395y^6\\
&- 395976916008838467359383517256252902718504369534707772 2y^8\\
&- 2620447825568398086656200512811430688316660491298669853042465498158262y^3\\
&- 1690245669048371540642123199747215183102979785123800y^9 - 2364309858342351836139377701847058104293785 6y^{11}\\
&- 481082356773702612971914100998824045834935013814082225428785529194 0y^4\\
&- 63907524808772612863627564577341009460874354019135311986949185 94y^5\\
&- 33371802888669526493597713725427577919958207146 7y^{10}
\end{aligned}
$$

$$
\begin{aligned}
&- 1907998166247394972336149985745650317117090808761591061218848239361048268852y \\
&- 92967676797537834315631930213847755212350886697670721196473013905755250 4y^2
\end{aligned}
$$

$$
\begin{aligned}
G_{13} = &-7443086762947012794655647955104870295276301787136y^{12} - 5559903222673117407569247043993 6y_1^{13} \\
&+ 1139235522893187935464250508458231112109632273552695244766314676383351501373495947 60y \\
&- 767351731782655272538155190867076469621182583545523299729498555972359012y^5 \\
&+ 129532516878800900381872385113882039412029902282115274979051291600y^7 \\
&- 302826148506616081262359766382555179004092675437763255909483098135465343013230058y^2 \\
&- 281411836664322896897047148039862192575607610477809139912943415224052y^6 \\
&+ 4562764512773413237510243476587801909706042736495351151553126 70y^8 \\
&- 1220369547022884065482240866705203403 8y_1^{12} \\
&- 756257773935699303554681821027136037539268307879165 44y^{11} \\
&- 30070060329309139462828731821980472812799979460307373347840 4y^9 \\
&+ 770940279127265306861482326987630145302756451854933001715605077030403109344y^4 \\
&+ 786461731108832953571592639354878794652549277252199335882211487759717626051 20y^3 \\
&+ 32314037293222413321416850653387459569378057400919053198 2y^{10} \\
&- 1333517979621589096928547735768159257515043756285597193257974575838312496428633715545 6 \\
&+ (-1219231391416503592663874317307509505080480256y^8 \\
&- 8609280663823849356220847064682242146229728445058483831093 62y^4 \\
&+ 350374997694352673386222355096722846885611838352641 80y^6 \\
&+ 5740643583964482458718108967656118000135268349555097636y^5 \\
&- 2830760540760228406324215806781968942465716691492149220860129405172564y \\
&+ 1736080037005850034012132459947202779961017722923 2y^7 \\
&- 1008344022561201061610008374898024810267866474196826677472382044y^3 \\
&+ 4544127488223932364240790649942732875024034211711143690348975958106y^2 \\
&+ 48616337997393340727457241476884733297004258935914079019961924549460332 6)y_1^4 \\
&+ (683819386610614616406012585428783613118001666394 24y^9 \\
&+ 2649414128258087960442667280688908732791662011422335441768331132 8y^5 \\
&- 376258081065180784342024231718866664167318295147551649323319156415695776652 48y \\
&- 5711510207802912307978848402611360617024877055905886605682246773886 8y^4 \\
&+ 596916590020925025997778427004764358588337647065375485045085438245436361707208 2 \\
&+ 448929713882249170114839560774019456512143127868065575788y^7 \\
&- 254106123453535536703118638727626209470096503358504206 0y^8 \\
&+ 717355373389137352932640682621572031663142512728400443677459736003282611 34y^2 \\
&+ 634952533627588185279844799195443195467469824336078992092456 8y^6 \\
&- 71680434721058056722559719571269178316080455680y^{10} \\
&- 19133369421813227226880728015866571136712468868879273531693855722568124y^3)y_1^2 \\
&+ (39502756276642233194677461892942331434176606306191591 2y^7 \\
&- 20363551381973369777214536775279843220127106772054 4y^8 - 21743283267132336167906693059543581598671790 08y^9 \\
&+ 150137489175184181909884085962565089367139848460211368440294419163001858 64y \\
&- 8823074969571283281486066417463631412733637145845050562 76y^6 \\
&- 1711624058797739003978555830732575329005330827615438711402528y^5 \\
&+ 1227225795275961494998933465251493389617035396855242584967872180 4y^4 \\
&- 268584441523960747594494668519627896737467500004148391240976758323533 68y^2 \\
&+ 7910271828835666451914625485253687541230177779181732013020691176016y^3 \\
&- 2514843853805161170810586173587624186397407005010132963098118445463636332390)y_1^3 \\
&+ (170423080329800619782882184675626660264459805010696620 40y^4 \\
&+ 56418399814534849185558708982334257611179625733005363560424y^3 \\
&+ 852487062475807084286212789558870377922764 8y^7 + 8911744759490022332818966556463641502887987367944 8y^5 \\
&+ 33015506295901561358807389926878714530395479514127217754650437194 0y \\
&- 659451787919725265856939696151775547864449295803134221552636704410 46 \\
&- 41318577109844814764390822059970277876404156074213082522722784 2y^2 \\
&- 2650955580042345962371581743225281942977597232640y^6)y_1^5 \\
&+ (157487901555928416300909639425493168640987849 70y^4 \\
&+ 88159670506506511945564903953509443867807520524616y^3 + 13435272008937444718718954257135136244551 68y^5 \\
&- 9295132766079853100334349521121403351083745309743212487616 34 \\
&- 54750822116276973948689786588628698629487483239197574 2y^2 \\
&+ 13271204472758813740865405605744599606728778767552732601 16y)y_1^7 \\
&+ (-2596180273423874549847658820637482435927980765333626359036982 8y \\
&- 294202606796771373985952795655157988091269 12y^6 - 9495119349655175452267071981490086849455658232 4y^5 \\
&+ 8549293301116350378866802074551741303724954896452679305952652762 \\
&+ 2042359963600082645958503121961408536785266353445394304988 2y^2 \\
&- 5048983492477999676369202573384570319188415052064y^4 \\
&- 23467448861741444321720064817460979287985465135101694 28y^3)y_1^6 \\
&+ (-22667560896664850849321206875055324320092595168534 34 \\
&+ 63679638158087798911967180428389806462526907754 4y \\
&+ 37950782574450534667863045514504351655 68y^3 + 16342465759769983998857504504460378619299590y^2)y_1^9 \\
&+ (637548705634832914910358061049404446793536896492042766 62 \\
&- 4016938971219540673470111943761125536630186809993824 4y \\
&- 14065303009489387286294397759980518097893630 44y^3 + 74439728210142827888619968937895455388723805210 00y^2 \\
&- 32033354197702791042399712031264720218940 8y^4)y_1^8 + (15743436394331352347037995683971018993 6y
\end{aligned}
$$

$$+ 5909351138145686861266351474310881331633358)y_1^{11} + (-72305490068481896914309271900453202762108884y$$
$$- 30679533045563252472939617796401291838840y^2 + 248443862142048974062610293715814887526294920054)y_1^{10}$$

$$G_{14} = (-3626154718352545985958428434656392106230490029048151601690887882777$$
$$+ 96196264287721331317235846370744477422843271671813876339y^4$$
$$- 1631377636228334750783875071331608464489208911886646y^5$$
$$+ 4390015514290537715913945444707347000551606584086706236041573y^2$$
$$+ 12126159731628918345711594472118768025312913464807106111218y^3$$
$$- 3723293475561049484834405042781332760124959474805008253714945646464y)y_1^8$$
$$+ (35829166641605542378700561137236104365335874932963954774823100000011772y$$
$$- 368546793087861944937888280079728519383221440426734967229188y^4$$
$$+ 857681322603207530625537510435239175853280306918414223006y^5$$
$$+ 178562947494963148178213457246815548114527613494994488y^6$$
$$- 10720908960622289148481759019915275975327141684425388698961443288798404 9$$
$$+ 5316458606365101816026609186844067485831859428817318473286206929992y^2$$
$$- 5354098376232374462460332293028775892978716815559638465746037522y^3)y_1^7$$
$$+ (-2030452527610357039438993800827826598654217068 8y$$
$$- 755370635151282069914171604395036339746733702005 90)y_1^{12}$$
$$+ (30777576309723838721469107576150672151980704867254 34y$$
$$- 833409445890248246402399961988823618898781478747y^2$$
$$+ 8109701912427026097952834127545742780863507825128119 04)y_1^{11}$$
$$+ (-522458886204860491928165092079725602078795215345181 2y^2$$
$$- 91338753701165974460999025657513370185312455916209842666329$$
$$- 444614177199651991384487999523312496676629327623407069 84y$$
$$+ 101258838943176749942540618561216215892922837525 10y^3)y_1^{10}$$
$$+ (-113779036654587568230735582021778565426176095438831469 2y^3$$
$$- 8727787881657457297879420306480125947177455933283629590055081 5$$
$$+ 71688628173002169404865192949859280820125827327 05y^4$$
$$+ 263346848872559220880186787192233056116522732970625797687088 8y$$
$$+ 10135144936780074488023365409603099854236581838244208697 02y^2)y_1^9$$
$$+ (-383676148869173257738011692509797335124322480580814476568376448254312191720902y^5$$
$$- 42101319200926578827018404773453934883200590429762568388165900776105530511y^6$$
$$- 3264958112294860310129046419809056415478418202262990474426955319284542507733122608080 7554y$$
$$- 1809553302504283674036686850098037032063532025454411873417238y^{10}$$
$$- 3014290003192936732889477425303343066823673384555301898252471151252290900960221674878140058$$
$$+ 1329654197805510682721347495676907610442935099395863662624942954 2y^9$$
$$- 151689633419558404725143719201311807821583192295180951020904456900805419709555891016339y^2$$
$$- 343589459692178642291853498524817253723538419779844742144y^{11}$$
$$- 383504826617449900344324140255168272128946159439988329670517140875061228524483363334y^3$$
$$+ 11889325399584243178305847165223120085997425332523172207301919766602829 6y^7$$
$$+ 762534278549480193836200713065582341105984142533621162619336937191 73y^8$$
$$- 543140336304016677279311018472718056564433683139268525553848675336049212748324150y^4)y_1^2$$
$$+ (-139835169457840667026202785927864643009576741715726012y^7$$
$$- 295898901948050308171732828876493652789369798209216997644799145493395237320 4$$
$$- 1881219023751448516499663684421877735302677366862874272004050y^5$$
$$- 523302155903253168103013604654274014453843612026159867235254963159497162 6y$$
$$- 220779731991571671575231247521731699481391125050356715744 8y^6$$
$$+ 41034800732732789728679495717589653134706231539604441243940384221548 19y^2$$
$$+ 2349632763424388724522102758011011166524986466003584719728356101y^4$$
$$+ 69649598279755847605607376911597557731678315926762585090491165963 16y^3)y_1^6$$
$$+ (7472446364763737034332490659860300463831979883702214272y^9$$
$$- 5604094825830001034899033200376333892390105434779748150301199256 93y^6$$
$$- 1934254379740093191959391948815515910012231663315198035436830110224730454030y^3$$
$$- 111916835155252414290521085414356709245293028711645776896104227911631892487440712957$$
$$- 55245348004399384675271983517362577225085159189242475318956980084305045887668756 2y$$
$$- 17143929977445049861786550303212555214892698210612624507647033985940 42y^5$$
$$- 1175821943424403507844285317222796945380565106414203930966809588012889524399158y^2$$
$$- 2501249744966899230970323167199143000827838356263467848778360327588819282y^4$$
$$+ 12440295634504700043613792441956340269770309243154476343845 2y^8$$
$$+ 763433274655946091046759710578172304965197287753065897878640 18y^7)y_1^4$$
$$+ (4374842824643603481837715390257508805869120469415550340208872344351195247 4y^5$$
$$+ 468849031365739998538884046199499294817597154504624974544957421150565924471823747 1y^2$$
$$+ 327696812456147317751802075546102987725627285996815609833106546488783441815388356640y$$
$$+ 1732353481210450625487955914814752220785320496988545033939210336686330128629658 4y^3$$
$$- 981819509998554135286429532186187451799444697043701946731430419387746837111991314002 05$$
$$- 24185941864000275332453309653095149304133153059696502803835 88y^9$$
$$- 684138676621424750595536216793270715357937263498987844962807286708 0y^7$$
$$+ 195179513496634442262789835594147598344709611599551575253774120857322 96y^6$$
$$+ 3676237557947893914942081895695812797647361261176723579941583968136843448171 4y^4$$
$$- 87649594849027869707754997560840290018600173988850115942822415 60y^8$$
$$- 1654938293090097627069224616969099234126276831513191669 76y^{10})y_1^3$$

$+ (-2552945119009539037676587638067152474897784342603426436291067191865 5y^4$

$+ 894115151055872373985852558870077934215706252245591712132161649 44y^5$

$- 17050254289310384957736717501084930629641333766108556 80y^8$

$- 773622078028842418203109886676237334758195158948422713365988179608821604 5648954$

$- 390435845690603841185479840069172622871645824949681521227697900218564 64y^3$

$- 813601104345278203360051280219705659587511112170388181656660513357551791 84y$

$+ 570657695996971010261955038732394154723041813729410615139097 20y^6$

$+ 280421402268173696693131115880499786846645135024015966573535270028957933 02y^2$

$+ 139100284392359278117962866886337035291446539147008620636 8y^7)y_1^5$

$- 601294182047104525844632957484511341271224629207428172332827604490427607664780432484344005441009 60$

$- 24508519547995298487969667054590364893439y_1^{14}$

$+ 94560216438901981548322466813729852654282865 2y_1^{13}$

$- 2053059995950985095425663877955950728003479204955956516117462617y^{12}$

$- 3483213718856050011073060141711656412640178754924893367336 96y^{13}$

$- 943423672460782941725877165438066027467088696401712317029244304409425985622668244139677206815504y$

$+ 0$

$- 761787459548036588637819068186072768066164340081561349437269368220524183091951969294 50y^5$

$- 7497545170960097160327050809392104167656223078315462948641972205453780528182112y^7$

$- 63167074686989608657878513026199956515253095793213797146285627040027413379758114666321166264 65y^2$

$- 58243099846938261179076113889791552579467544070887862476864181180048515740924738669y^6$

$+ 281026670478474349155949217360201013083382623538466274418314363215761816140 51y^8$

$- 31634248925551616426178803748948013309819615662213274698589487503 44y^{11}$

$- 542525321382873288059342422193025074102686469602083414269120471314382436598241230936219 02y^4$

$- 2368395533432563752616883260192083856136727447577323434494978780315558016609308427862592256 4y^3$

$+ 24156156493878272826885966626955833203533024552269819890468250680561815570y^9$

$+ 5165893538396468223429948576443617236866500623476226317138561279572674y^{10}$

$G_{15} = 1626011754096644428761022477075176735284520538349344950526449397617783477200yy_1^7$

$+ 111230337187800655132023318434929879216003112384814271327427343 0y^5y_1^7$

$- 2307028575491304550986699899359712434709010533742287779288549603 60y^7y_1^5$

$+ 649892959375332813933621713593892417797172300651475673560763098462002553607 90y^7y_1^2$

$- 5566929522830383615230826068654934122904176695079344627971305568547818651871415y^4y_1^4$

$- 68798114779425193533997246209137861570729984429767821600414154486844740yy_1^8$

$- 417188373435239796200741195635374677259355608436496271639491474993159496576309709751506 05y_1^4$

$+ 469450510667303465055088820586180976819477881149082273847613472328810760314740113719 0y_1^5$

$+ 98921979885631479216023855050946950923135289527644250169182814446304008991005284 0y_1^6$

$+ 10735524228859355160726276125437015990760049555348087218255180y^3y_1^9$

$- 1301213352258002864387634169692369362426955104093418683323548161473788166 0y^7y_1^3$

$+ 490958574048862499715172164118566056523326246716659038299524381644310290y^5y_1^5$

$- 2027939075861473598002371910311819610207064807708649819604299416907259991638084 0yy_1^6$

$- 2555092126415126185400072481282449874989316250301981148 10y^3y_1^{10}$

$- 800911192436732045253192319812805150483088693752710426051481915465130 85y^4y_1^6$

$+ 917016393770080245374685831018919467650758583613307449112089210731347377790y^6y_1^3$

$- 4623265499106243589404742338174317929588206542308616478215747250368921472624 85y_1^7$

$+ 86186501315951332577431596976433711066149739422584058707108894431235207470y_1^8$

$- 1006414002894815354978076349044082318727352477386795775479328347888792 5y_1^9$

$+ 149423489554106170006846340401883822613657934796423773666539289429177399748032 20y^5y_1^3$

$- 66797482199727254685554288433597724295955936323376997881229446 15y^{10}y_1^3$

$- 181906782462728539133830821961259644851661377272460181920301596704787009690638508151158985299 0yy_1^2$

$- 15108711998573983972668493958565401173495427661782462518401912 0y^4y_1^8$

$- 453795915654514515462417546627296638256922144486381061408817 20y^8y_1^5$

$- 32242478891839351622003274398754414230597744061779779028967642546 0y^3y_1^8$

$- 411922491342131083701357231341281167446612886204325673343133745 00y^{11}y_1^2$

$+ 1390523120275979234344390230905636868867512320241193626493192560515y^8y_1^4$

$+ 15168181771938677139479916520476886276127275813693550469635952431898993584098322044 0yy_1^5$

$+ 17762226401446525194264045581510251791014448367295839713447642506639655811313859340y^4y_1^3$

$- 2239873138233803288018234103815057500626898611363653501400834506519489140795004886906 5y^4y_1^2$

$+ 33764119186901153695361507037009891535959793199743851960 65y^4y_1^9$

$+ 124533782611240020023181814680869211361613580721734918685642605929421066604831376987896739365y_1^3$

$- 39684077379169495632937207892811007128285406811790848896273509346542878153 20y^5y_1^4$

$- 6776839137785969408147361270671909639363058587107631613308544897122902357860290506807 10yy_1^4$

$- 1621418301689455371275454480716404927160758980810741706653248622588688 5y^8y_1^3$

$+ 3885932172845960358626342791647557497322023494888026017563627399651 5y^{10}y_1^2$

$- 14746239385604341789417200662865749014814820841980576171831425597353424490080121112253287430479 5y_1^2$

$+ 127213975591076599140491119537778220488860892979985586072972039924994354748005535323 80y^3y_1^3$

$- 9177591540395193766922912858827590454919964495764648840466630117436031 5y^6y_1^4$

$+ 1391641191606994158215406699186441906432264028712828017310383 00yy_1^{10}$

$- 308586176242100737840902679752419757298433495209659884203924140263 40y^5y_1^6$

$- 14671548369539108385794481913813337884320754214517183860583346039657329050008558612572650y^3y_1^2$

$- 964850159838941617809431446331920534279851917829896438313062804920979644 50y^3y_1^6$

$+ 43175749287783503322348874831855209594557434782082107728427743176 15y^4y_1^7$

$- 767969482926794114814889147987548944821874599334550619424 0yy_1^{11}$

$+ 8407229909744360885261566000352093446019584827839088079466830y^7y_1^6$
$+ 8189298218987898213423858170783270209375327396734900668137730046 75y_1^{10}$
$- 48042159120598533786544221864234708549847668110346087770730765y_1^{11}$
$+ 20195150823621892272854154697599766257730522582930319851 35y_1^{12}$
$+ 125318151956807260345912152530456236458995079142375984498861668022575237954652 90y^6y_1^2$
$+ 88919330410549227676328336929094751217442749604071476770551647990737257883 0y^4y_1^5$
$+ 244045368763935228764119163214221577620093171191441626082802 7865y^6y_1^6$
$- 583009097202580072843653624553978210253784980847262 80y_1^{13}$
$+ 10331306098190267928049185608970265540542472079 10y_1^{14}$
$+ 23614881430638034916141531626901235644450864633938630614974703991292 30y^7y_1^4$
$+ 919142341494350202005264267177172970276189866988526607627248683932218244443885453821801558136482580y$
$+ 2074489800903719650290851364370916453344080358340340141190627269 30yy_1^9$
$- 18454804983918841462488887298301023500745956072168567412338752428003808518559548080y^5y_1^2$
$- 904834610296200246769404788519200294779799914105377486 5y^2y_1^{11}$
$- 18902455823612473383631466677981746203775041594 50yy_1^{13}$
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$- 1415029166916629884011775428141626540225221484917440665656211699057 80y^2y_1^8$
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$+ 44333977084700492407044965168228114623762979539826430033373723274932711 50y^{11}$
$+ 35628458472256356015112900106851735289603765620536142254738813686018 35y^{12}$
$+ 103237724911944772524505995128188125507716412994661808843754419961 0y^{13}$
$+ 8126303971562043208018613285174848628531036980303387375952818953998376885711711691872777300233334 1165$