Polynomial General Solution for First Order ODEs with Constant Coefficients *

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Abstract

We give a necessary and sufficient condition for an ODE to have polynomial type general solutions. For a first order ODEs of degree $n$ and with constant coefficients, we give an algorithm of complexity $O(n^5)$ to decide whether it has a polynomial general solution and to compute one if it exists. Experiments show that this algorithm is quite effective in solving ODEs with high degrees and a large number of terms.

1 Introduction

Trying to find elementary function solutions for differential equations may be traced back to the work of Liouville. As a consequence, such solutions of differential equations are called Liouvillian solutions. In a pioneering paper [16], Risch gave an algorithm for finding Liouvillian solutions for the simplest differential equation $y' = f(x)$, that is, to find elementary function solutions to integration $\int f(x)dx$. In [13], Kovacic presented a method for solving second order linear homogeneous differential equations. In [19], Singer proposed a method to find Liouvillian solutions of general homogeneous linear differential equations. Many other interesting results on finding Liouvillian solutions of linear ODEs are given in [1, 3, 2, 4, 5, 6, 8, 10, 12, 15, 18, 20]. In [14], Li and Schwarz gave a method to find rational solutions for a class of partial differential equations.

All these results are limited to linear cases. There seems no general methods to find Liouvillian solutions of nonlinear differential equations. In [7], Cano proposed an algorithm to find polynomial solutions to equations of the form $y' = R(x, y)$ where $R$ is a rational function in $x$ and $y$. In this paper, we try to find polynomial solutions to non-linear differential equations. Instead of finding arbitrary polynomial solutions, we will find the general solutions for ODEs of polynomial type. For example, the general solution for $(\frac{dy}{dx})^2 - 4y = 0$ is: $y = (x + c)^2$, where $c$ is an arbitrary constant. Three main results are given in this paper. Firstly, we give a sufficient and necessary condition for an ODE to have polynomial general solutions. We also prove that

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the definition of general solutions of ODEs due to Ritt is equivalent to the definition in the usual sense, if the general solutions are in polynomial form. Secondly, we give a detailed analysis of the structure of first order ODEs with constant coefficients which have polynomial general solutions. Based on these results, an algorithm can be easily obtained. Thirdly, by introducing a novel method to evaluate the coefficients of polynomial general solutions of a first order ODE with constant coefficients, we get an algorithm to find polynomial general solutions of first order ODEs with constant coefficients with polynomial complexity. Our experiments show that this algorithm is quite effective in solving ODEs with high degree and terms.

The paper is organized as follows. In section 2, we give a brief introduction to the characteristic set method. In section 3, a criterion for an ODE to have polynomial general solutions is given. In section 4, we give the degree bound of solutions of first order ODEs with constant coefficients and an algorithm. In section 5, we analyse the structure of first order ODEs which have polynomial general solutions. In section 6, we present a polynomial-time algorithm to find polynomial general solutions of first order ODEs and some examples.

2 A brief introduction to the characteristic set method

In this section, we introduce some concepts and notations on the characteristic set method. More details can be found in [11, 17, 22].

Let $\mathbf{K}$ be a field and $z_1, z_2, \cdots, z_n$ indeterminants. Considering a polynomial $P$ in $\mathbf{K}[z_1, \cdots, z_n]$, we can write it as the following form:

$$P = \sum a_{\alpha_1, \alpha_2, \cdots, \alpha_k} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$$

where $a_{\alpha_1, \alpha_2, \cdots, \alpha_k} \neq 0$ are in the field $\mathbf{K}$ and there exists an $\alpha_k$ which is not equal to zero. $k$ is called the class of $P$, denoted by $\text{cls}(P)$. $\text{deg}(P, z_i)$ denotes the degree of $P$ with respect to $z_i$ when we regard $P$ as a polynomial in $z_i$. The maximum of $\alpha_1 + \cdots + \alpha_k$ is called the total degree of $P$, denoted by $\text{tdeg}(P)$. For a polynomial $P$ of class $p > 0$, we may write $P$ as the following form:

$$P = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$$

where $a_i$ are polynomials in $z_1, \cdots, z_{p-1}$ for $i = 0, \cdots, d$ and $a_d \neq 0$. $a_d$ is called the initial of $P$ and $S = \frac{\partial P}{\partial z_p}$ is called the separant of $P$. Let $Q$ be another polynomial whose class is $q$. $P$ is said to be reduced with respect to $Q$ if $q > 0$ and $\text{deg}(P, z_q) < \text{deg}(Q, z_q)$. Let $R = \text{prem}(P, Q)$ be the pseudo-remainder of $P$ with respect to $Q$. We have the following remainder formula for $R$:

$$I^s P = BQ + R$$

where $R$ is reduced with respect to $Q$ and $I$ is the initial of $Q$. A polynomial set $\mathcal{A}$ is called an ascending chain if its polynomials are all non-zero and can be arranged in the following sequence:

$$\mathcal{A} : A_1, A_2, \cdots, A_r$$

satisfying the following conditions:

1. $\text{cls}(A_1) < \text{cls}(A_2) < \cdots < \text{cls}(A_r)$

2. $A_j$ is reduced with respect to $A_i$ for all $i < j$. 


A polynomial $P$ is said to be reduced with respect to an ascending chain $A$ if $P$ is reduced with respect to every polynomial in $A$. An ascending chain $A$ is called irreducible if for any polynomials $A_1, A_2$ reduced with respect to $A$, we have $\text{prem}(A_1, A_2, A) \neq 0$. Let $A$ be an irreducible ascending chain. Let

$$\text{SAT}(A) = \{ P \in K[z_1, \cdots, z_n] \mid \exists J \text{ s.t. } J P \in \text{Ideal}(A) \}$$

where $J$ is a product of the initials of the polynomials in $A$. Then $\text{SAT}(A)$ is a prime ideal and $\text{dim}_K(\text{SAT}(A))$ is equal to $n - r$ where $r$ is the number of polynomials in $A$ ([17]).

We will need the following Wu-Ritt’s zero decomposition theorem ([17, 22]).

**Theorem 2.1** For any polynomial set $PS$, there are a finite number of irreducible ascending chains $A_k$ such that

$$\text{Zero}(PS) = \bigcup_k \text{Zero}(\text{SAT}(A_k))$$

Let $\Sigma$ be a non-trivial prime ideal in $K[z_1, \cdots, z_n]$ and $L$ an extension field of $K$. For a prime ideal $\Sigma$, a zero $\eta \in L^n$ of $\Sigma$ is called generic zero of $\Sigma$ if for any polynomial $P \in K[z_1, \cdots, z_n]$, $P(\eta) = 0$ implies that $P \in \Sigma$. It is well known that an ideal $\Sigma$ is prime iff it has a generic zero. Let a generic zero of $\Sigma$ be $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$. We call the transcendental degree of $K(\eta_1, \eta_2, \cdots, \eta_n)$ over $K$ dimension of $\eta$ (denoted by $\text{dim}_K(\eta)$) which equals to $\text{dim}_K(\Sigma)$.

### 3 Polynomial general solution to ODEs

In the following, let $K$ be the differential field of meromorphic functions in $C$ with differential operator $\frac{d}{dz}$. Let $y$ be an indeterminate over $K$ and we denote by $y_i$ the $i$th derivative of $y$. We use $K\{ y \}$ to denote the ring of differential polynomials over differential field $K$, which consists of the polynomials in the $y_i$ with coefficients in $K$. Let $\Sigma$ be a system of differential polynomials in $K\{ y \}$. A zero of $\Sigma$ is an element in an extension field of $K$, which vanishes every differential polynomials in $\Sigma$. The totality of zeros in $K$ is called the restrict manifold of $\Sigma$. (see [11, 17])

Let $P(y) \in K\{ y \}/K$. We denote $\text{ord}(P(y))$ the highest derivative of $y$ in $P(y)$, called the order of $P(y)$. Let $\text{ord}(P(y)) = 0$. We can also regard $P(y)$ as an algebraic polynomial in $y, y_1, \cdots, y_o$ with the coefficients in $K$, then the initial, separant, $\text{deg}(P(y), y_i)$ and $\text{tdeg}(P(y))$ are defined as algebraic case. The definition of generic zero is also the same as it in algebraic case.

Let $P(y)$ be a differential polynomial of order $o$. A differential polynomial $Q(y)$ is said to be reduced with respect to $P(y)$ if $\text{ord}(Q(y)) < o$ or $\text{ord}(Q(y)) = o$ and $\text{deg}(Q(y), y_o) < \text{deg}(P(y), y_o)$. For two differential polynomials $P(y)$ and $Q(y)$, let $R = \text{prem}(P(y), Q(y))$ be the differential pseudo-remainder of $P(y)$ with respect to $Q(y)$. We have the following differential remainder formula for $R(y)$ (see [11, 17])

$$J(y)P(y) = \sum_i B_i(y)Q^{(i)}(y) + R(y)$$

where $J(y)$ is a product of certain powers of the initial and separant of $Q(y)$, $Q^{(i)}(y)$ is the $i$th derivative of $Q(y)$ and $B_i(y)$ are differential polynomials. For a differential polynomial $P(y)$ with order $o$, we say $P(y)$ is irreducible if $P(y)$ is irreducible when $P(y)$ is treated as a polynomial in $K[y, y_1, \cdots, y_o]$. 

3
Let \( P(y) \in K\{y\}/K \) be an irreducible differential polynomial and
\[
\Sigma_P = \{ A \in K\{y\} | SA \equiv 0 \mod \{P(y)\} \}
\] (1)
Then \( \Sigma_P \) is a prime differential ideal and a differential polynomial \( Q(y) \) belongs to \( \Sigma_P \) if and only if \( \text{prem}(Q(y), P(y)) = 0 \). Furthermore, Ritt proved the following basic result [17].

**Theorem 3.1** Let \( F(y) \) be an irreducible differential polynomial not in \( K \). Then the restrict manifold of \( \Sigma_F \) is not empty and a differential polynomial which vanishes over the restrict manifold of \( \Sigma_F \) is contained in \( \Sigma_F \).

We call a generic zero of \( \Sigma_F \) a general solution of \( F(y) = 0 \). By a polynomial general solution of \( F(y) = 0 \), we mean a general solution of \( F(y) = 0 \) in the following form
\[
\hat{y} = \sum_{i=0}^{n} a_i x^i, \quad (a_n \neq 0)
\] (2)
where \( a_i \) are constants which may be algebraically dependent over \( C \). In the literature in general, a general solution of \( F(y) = 0 \) is defined as a family of solutions with \( o \) independent parameters in a loose sense where \( o = \text{ord}(F(y)) \). The definition given by Ritt is more precise. From Theorem 3.3, we can see that if \( F(y) = 0 \) has a polynomial general solution, then its polynomial general solution has \( o \) independent parameters.

Since the solutions of \( y_{n+1} = 0 \) are some polynomials with degrees not greater than \( n \). Suppose that \( F(y) = 0 \) has a polynomial general solution with degree \( n \). The investigate of the relation between \( y_{n+1} \) and \( F(y) \) will lead to a sufficient and necessary condition for \( F(y) = 0 \) to have polynomial general solutions.

**Theorem 3.2** Let \( F(y) \) be an irreducible differential polynomial. Then \( F(y) = 0 \) has polynomial general solutions iff there is a non-negative integer \( n \) such that \( \text{prem}(y_{n+1}, F(y)) = 0 \).

**Proof:** \((\Longrightarrow)\) Suppose that \( F(y) = 0 \) has a polynomial general solution \( \hat{y} \) with degree \( n \). Since \( y_{n+1}(\hat{y}) = 0 \), \( y_{n+1} \in \Sigma_F \) which means that \( \text{prem}(y_{n+1}, F(y)) = 0 \) by Theorem 3.1.

\((\Longleftarrow)\) Assume that there exists an \( n \) such that \( \text{prem}(y_{n+1}, F(y)) = 0 \) and \( n \) is the least. If \( n = -1 \), then \( F(y) = y \). It is obvious. Now we suppose that \( n \geq 0 \). From Theorem 3.1, \( y_{n+1} \in \Sigma_F \). Hence, all the elements in the restrict manifold of \( \Sigma_F \) must have the form: \( \hat{y} = \sum_{i=0}^{n} a_i x^i \). In particular, the generic zero \( \hat{y} \) has the form: \( \hat{y} = \sum_{i=0}^{n} a_i x^i \). If \( a_n = 0 \), then \( y_n(\hat{y}) = 0 \) which implies that \( y_n \in \Sigma_F \). Hence \( \text{prem}(y_n, F(y)) = 0 \), a contradiction. The proof is complete.

**Theorem 3.3** Let \( F(y) \) be an irreducible differential polynomial with order \( o \). If \( F(y) = 0 \) has a polynomial general solution of form (2), then the \( a_i \) depend on \( o \) independent parameters.

**Proof:** Let \( \hat{y} \) be a generic zero of \( \Sigma_F \) of the form (2). Let \( a = (a_0, a_1, \ldots, a_n) \). Suppose that \( \dim_C(a) = d \). Then we need to prove that \( d = o \). Let
\[
T : K^{n+1} \rightarrow K^{n+1}
\]
\[
(z_0, z_1, \ldots, z_n) \rightarrow (y, y_1, \ldots, y_n)
\]
where \( y = \sum_{i=0}^{n} z_i x^i \) and \( y_k = \sum_{i=0}^{k} i(i - 1) \cdots (i - k + 1) z_i x^{i-k} \). Note that if \( z_i \in C \), \( y_k \) is the \( k \)th derivative of \( y \). Then \( T \) is an invertible linear transformation over \( K \). We have
\[
d = \dim_C(a) = \dim_K(a) = \dim_K(T(a))
\]
Now we consider \( \dim_{K}(T(a)) \). Let \( F_{(i)}(y) \) be the \( i \)th derivative of \( F(y) \) wrt \( x \) for \( i = 0, \cdots, n - o \) and \( S(y) \) be the separant of \( F(y) \). Now we regard \( F_{(i)}(y) \) and \( S(y) \) as algebraic polynomials in \( K[y, y_{1}, \cdots, y_{n}] \) and denote them by \( F_{(i)}(y, y_{1}, \cdots, y_{n}) \) and \( S(y, y_{1}, \cdots, y_{n}) \) correspondingly. Then we have \( F_{(i)}(T(a)) = 0 \) and \( S(T(a)) \neq 0 \). Let \( T(a) = (\eta_{0}, \eta_{1}, \cdots, \eta_{n}) \). Since the initials of \( F_{(i)}(y, y_{1}, \cdots, y_{n}) (i > 0) \) are \( S(y, y_{1}, \cdots, y_{n}) \) and \( y_{o+i} \) is linear in \( F_{(i)}(y, y_{1}, \cdots, y_{n}) \), we have,
\[
\eta_{o+i} = \frac{(F_{(i)} - Sy_{o+i})(T(a))}{S(T(a))}
\]
Hence we have \( d \leq o \). In what follows, we will prove \( \eta_{0}, \eta_{1}, \cdots, \eta_{o-1} \) are algebraically independent over \( K \). Otherwise, there is a polynomial \( G(y, y_{1}, \cdots, y_{o-1}) \) in \( K[y, y_{1}, \cdots, y_{o-1}] \) such that \( G(\eta_{0}, \cdots, \eta_{o-1}) = 0 \). If we regard \( G(y, y_{1}, \cdots, y_{o-1}) \) as a differential polynomial \( G(y) \), \( G(\hat{y}) = G(\eta_{0}, \cdots, \eta_{o-1}) = 0 \) (note that \( \eta_{0} = \hat{y} \)) which implies that \( G(y) \in \Sigma_{F} \). In the other word, \( \text{prem}(G(y), F(y)) = 0 \), a contradiction. Hence \( d = o \).

\textbf{Remark 3.4} Thereom 3.3 shows that Ritt’s definition of general solutions is equivalent to the definition in usual sense if the general solutions are in polynomial form.

Let
\[
\text{ID}(a) = \{ P \in C[z_{0}, \cdots, z_{n}] | P(a) = 0 \}
\]
where \( a = (a_{0}, a_{1}, \cdots, a_{n}) \) is as in the proof of Theorem 3.3. Then \( \text{ID}(a) \) is a prime ideal. Let \( \mathcal{A} \) be an irreducible ascending set of \( \text{ID}(a) \). Then \( \mathcal{A} \) can be computed explicitly with the characteristic set method [17, 22]. Let \( z = \sum_{i=0}^{n} z_{i}x^{i} \). Substituting \( y \) by \( z \) to \( F(y) \), let \( \mathcal{P} \mathcal{S} \) be the set of the coefficients of the powers of \( x \) in \( F(z) \) which is a set of polynomials in \( z_{i} \). From Theorem 3.1, we get the following decomposition of the zero set of \( \mathcal{P} \mathcal{S} \):
\[
\text{Zero}(\mathcal{P} \mathcal{S}) = \bigcup \text{Zero}(\text{SAT}(\mathcal{A}_{i}))
\]
where \( \mathcal{A}_{i} \) are irreducible ascending chains and any \( \text{SAT}(\mathcal{A}_{i}) \) do not include each other. Then we have the following theorem:

\textbf{Theorem 3.5} Suppose that \( F(y) = 0 \) has polynomial general solutions of the form (2). Then there is a unique \( \mathcal{A}_{i} \) in the above decomposition such that \( \text{SAT}(\mathcal{A}_{i}) = \text{ID}(a) \). Furthermore, all \( \text{SAT}(\mathcal{A}_{i}) \) but \( \text{SAT}(\mathcal{A}_{i_{0}}) \) satisfy \( \dim_{C} \text{SAT}(\mathcal{A}_{i}) < \dim_{C} \text{ID}(a) \).

\textbf{Proof:} Let \( \ord(F(y)) = o \). Then \( \dim_{C}(\text{ID}(a)) = o \). Firstly, we show that for every \( i \), \( \dim_{C}(\text{SAT}(\mathcal{A}_{i})) \leq o \). Let \( \xi_{i} = (\xi_{i,0}, \xi_{i,1}, \cdots, \xi_{i,n}) \) be a generic zero of \( \text{SAT}(\mathcal{A}_{i}) \) and \( \gamma_{i} = \sum_{k=0}^{n} \xi_{i,k}x^{k} \). Let
\[
\Sigma_{\gamma_{i}} = \{ P(y) \in K[y] | P(\gamma_{i}) = 0 \}
\]
Then \( \Sigma_{\gamma_{i}} \) are differential prime ideals and \( \gamma_{i} \) are their generic zeros. From (P32, [17]), we know that there exist differential polynomials \( \Gamma_{i}(y) \in K[y] \) such that \( \Sigma_{\gamma_{i}} \) are general solutions of \( \Gamma_{i}(y) \). That is said, \( \Sigma_{\gamma_{i}} = \Sigma_{\gamma_{i}}(\Sigma_{\Gamma_{i}}) \) as in (1)). From Theorem 3.3, we have \( \dim_{C}(\text{SAT}(\mathcal{A}_{i})) = \ord(\Gamma_{i}(y)) \). Since \( \xi_{i} \in \text{Zero}(\mathcal{P} \mathcal{S}) \), \( F(\gamma_{i}) = 0 \) which implies that \( F(y) \in \Sigma_{\gamma_{i}} \). Hence \( \ord(\Gamma_{i}(y)) \leq o \), that is, \( \dim_{C}(\text{SAT}(\mathcal{A}_{i})) \leq o \). Since \( a \in \text{Zero}(\mathcal{P} \mathcal{S}) \), there is an \( \mathcal{A}_{i_{0}} \) such that \( a \in \text{Zero}(\text{SAT}(\mathcal{A}_{i_{0}})) \). We will show that \( \text{SAT}(\mathcal{A}_{i_{0}}) = \text{ID}(a) \). Since \( a \) is a generic zero of \( \text{ID}(a), \text{SAT}(\mathcal{A}_{i_{0}}) \subseteq \text{ID}(a) \). Hence \( \dim_{C}(\text{SAT}(\mathcal{A}_{i_{0}})) = \dim_{C}(\text{ID}(a)) \) so that \( \text{SAT}(\mathcal{A}_{i_{0}}) = \text{ID}(a) \).

If there is another \( \text{SAT}(\mathcal{A}_{k}) \) such that \( \dim_{C}(\text{SAT}(\mathcal{A}_{k})) = o \), then \( \ord(G_{k}(y)) = \ord(G_{i_{0}}(y)) = o \) which implies that \( F(y) = \theta G_{k}(y) = \lambda G_{i_{0}}(y) \) where \( \lambda, \theta \in K \) and all are not zero. Hence
\[ \Sigma_{\gamma_0} = \Sigma_{\gamma_k} = \Sigma_F. \] Let \( P(z_0, \cdots, z_n) \in \text{SAT}(A_k) \). Then \( P(T^{-1}((y, y_1, \cdots, y_n))) \in K\{y\} \) where \( T \) is as in Theorem 3.3. Then
\[
\begin{align*}
P(T^{-1}(\gamma_k)) &= P(\xi_i) = 0 \iff P(T^{-1}((y, y_1, \cdots, y_n))) \in \Sigma_{\gamma_k} = \Sigma_{\gamma_0} \\
&\iff P(T^{-1}(\gamma_0)) = P(\xi_0) = 0 \iff P(z_0, \cdots, z_n) \in \text{SAT}(A_0)
\end{align*}
\]
Hence \( \text{SAT}(A_k) = \text{SAT}(A_0) \). Since \( \text{SAT}(A_i) \) do not include each other, the proof is complete. \( \blacksquare \)

**Remark 3.6** The coefficients of \( F(y) \) in above sections and this section are in \( K \) and the order of \( F(y) \) is arbitrary. In the following sections, we will always assume \( F(y) \) has constant coefficients and order one.

## 4 An algorithm for first order ODEs

In this section and following sections, if there is no other statement, \( F(y) \) will always be a non-zero first order irreducible differential polynomial with coefficients in \( C \).

**Theorem 4.1** Let \( \hat{y} = \sum_{i=0}^{n} \bar{a}_i x^i \) be a solution of \( F(y) = 0 \), where \( \bar{a}_i \in C \), \( n > 0 \) and \( \bar{a}_n \neq 0 \). Then for an arbitrary constant \( c \),
\[
\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i
\]
is a polynomial general solution for \( F(y) = 0 \).

**Proof :** It is easy to show that \( \hat{y} \) is still a zero of \( \Sigma_F \). For any \( G(y) \in K\{y\} \) satisfying \( G(\hat{y}) = 0 \), let \( R(y) = \text{prem}(G(y), F(y)) \). Then \( R(\hat{y}) = 0 \). Suppose that \( R(y) \neq 0 \). Since \( F(y) \) is irreducible and \( \deg(R(y), y_1) < \deg(F(y), y_1) \), there are two differential polynomials \( P(y), Q(y) \in K\{y\} \) such that \( P(y)F(y) + Q(y)R(y) \in K[y] \) and \( P(y)F(y) + Q(y)R(y) \neq 0 \). Thus \( (PF + QR)(\hat{y}) = 0 \). Because \( c \) is an arbitrary constant which is transcendental over \( K \), we have \( P(y)F(y) + Q(y)R(y) = 0 \), a contradiction. Hence \( R(y) = 0 \) which means that \( G(y) \in \Sigma_F \). So \( \hat{y} \) is a generic zero of \( \Sigma_F \). The proof is complete. \( \blacksquare \)

The above theorem reduces the problem of finding polynomial general solutions to the problem of finding a polynomial solution. In what below, we will show how to find such a solution.

**Lemma 4.2** Suppose that \( \deg(F(y), y_1) = m > 0 \). If \( \bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i \) \( (\bar{a}_n \neq 0) \) is a solution of \( F(y) = 0 \) where \( \bar{a}_i \in C \), then \( n \leq m \).

**Proof :** Assume that \( F(y) = \sum_{i=0}^{l} c_{\alpha_i, \beta_i} y^{\alpha_i} y_1^{\beta_i} \), where \( c_{\alpha_i, \beta_i} \neq 0 \) and \( (\alpha_i, \beta_i) \neq (\alpha_j, \beta_j) \) if \( i \neq j \). Substituting \( y \) in \( F(y) \) by \( \hat{y} \), we get a polynomial \( F(\hat{y}) \) in \( x \). Assume that \( n > m > 0 \).

Then \( n \geq 2 \). We consider the highest degree of \( x \) in \( F(\hat{y}) \) which is the largest number in \( \{ \alpha_i + (n - 1)\beta_i \} \) for \( i = 0, \cdots, l \). If \( \hat{y} \) is a solution of \( F(y) = 0 \), all the coefficients of \( F(\hat{y}) \) are zero. Hence the number of the terms \( y^{\alpha_i} y_1^{\beta_i} \) such that \( \alpha_i + (n - 1)\beta_i \) is the largest are at least two. Without lost of generality, we suppose that two of them are \( n\alpha_1 + (n - 1)\beta_1 \) and \( n\alpha_2 + (n - 1)\beta_2 \). Then we have \( n(\alpha_1 - \alpha_2) = (n - 1)(\beta_2 - \beta_1) \). Assume that \( \beta_2 \geq \beta_1 \). Since
(n, n − 1) = 1, we have n|(β2 − β1). But 0 ≤ β2 − β1 ≤ m < n, which implies that β1 = β2. Hence α1 = α2. This contradicts (α1, β1) ≠ (α2, β2). Hence n ≤ m.

From Lemma 4.2 and Theorem 3.2, we have an algorithm to find a polynomial solution of \( F(y) = 0 \) if it exists.

**Algorithm 4.3** The input is \( F(y) \). The output is a polynomial general solution of \( F(y) = 0 \) if it exists.

1. Let \( n = \deg(F(y), y_1) \).
2. Compute \( R(y) = \text{prem}(y_{n+1}, F(y)) \). If \( R(y) \neq 0 \), then the algorithm terminates by Theorem 5.1, else goto step 3.
3. Let \( z = a_n x^n + \cdots + a_1 x + a_0 \) where \( a_i \) are indeterminants. Substitute \( y \) in \( F(y) \) by \( z \). Let \( \text{PS} \) be the set of the coefficients of \( F(z) \) as a polynomial in \( x \). Solve \( \text{PS} \). If \( \text{Zero}(\text{PS}) \neq \emptyset \), we will get a polynomial general solution of \( F(y) = 0 \). If \( \text{Zero}(\text{PS}) = \emptyset \), then \( F(y) \) has no polynomial general solutions.

It is known that general methods of equation solving are exponential algorithms. Therefore, the above algorithm might be ineffective.

In the next section, we will analyse the structure of the first order ODEs with constant coefficients which have polynomial solutions. After doing so, we can obtain an polynomial-time algorithm.

## 5 The structure of first order ODEs

If we have obtained a polynomial solution \( \bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i \) of \( F(y) = 0 \), we regard \( x, y, y_1 \) as the independent indeterminants and eliminate \( x \) in the polynomial set \( \{ \sum_{i=0}^{n} \bar{a}_i x^i - y, \sum_{i=1}^{n} i \bar{a}_i x^{i-1} - y_1 \} \). Then we will obtain a new differential polynomial \( R(y) \). Theorem 5.2 below will give the relation between \( R(y) \) and \( F(y) \).

**Lemma 5.1** Let \( f_1(y) = \sum_{i=0}^{n} \bar{a}_i x^i - y, \quad f_2(y) = \sum_{i=1}^{n} i \bar{a}_i x^{i-1} - y_1 \) \((n \geq 1, \bar{a}_n \neq 0, \bar{a}_i \in \mathbb{C})\). If \( n \geq 2 \), let \( R(y) \) be the Sylvester-resultant of \( f_1(y) \) and \( f_2(y) \) with respect to \( x \) and if \( n = 1 \), let \( R(y) = f_2(y) \). Then \( R(y) \) is an irreducible polynomial in \( \mathbb{C}[y, y_1] \) and has the form

\[
R(y) = (-1)^n \bar{a}_n^{-1} y_1^n + (-1)^{n-1} n \bar{a}_n y^{n-1} + G(y, y_1)
\]

where \( \text{tdeg}(G) \leq n - 1 \) and \( G \) does not contain the term \( y^{n-1} \).

**Proof:** When \( n = 1 \), it is clear. Assume that \( n \geq 2 \). We know that \( R(y) \) is the following determinant which has 2n-1 columns and rows.

\[
\begin{vmatrix}
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 - y & \bar{a}_0 - y \\
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 - y \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 - y & \bar{a}_0 - y \\
n\bar{a}_n & (n-1)\bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 - y & \bar{a}_0 - y \\
(n-1)\bar{a}_n & (n-2)\bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 - y \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n\bar{a}_n & (n-1)\bar{a}_{n-1} & (n-2)\bar{a}_{n-2} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 - y \\
\end{vmatrix}
\]
Regard $\bar{a}_i$ in the above determinant as indeterminants. Let $R = \sum c_{\alpha_i, \beta_i} y^{\alpha_i} y_1^{\beta_i}$, where $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ if $i \neq j$ and $c_{\alpha_i, \beta_i}$ are non-zero polynomials in $\bar{a}_0, \ldots, \bar{a}_n$. We define a weight

$$w : \mathcal{C}[x, y, y_1, \bar{a}_i] \to \mathcal{Z}$$

which satisfies $w(st) = w(s) + w(t)$, $w(x) = 1$, $w(\bar{a}_i) = n - i$, $w(y) = n$, $w(y_1) = n - 1$ and $w(k) = 0$ for $k \in \mathcal{C}$. Then $f_1$ and $f_2$ are isobaric polynomials with the weight $n$ and $n - 1$. From [9], we know that the resultant of two homogeneous polynomials is still homogeneous. By the same way, we can show that the resultant of two isobaric polynomials with the weight $n$ and $n - 1$ is still an isobaric polynomial with the weight $n(n - 1)$. Hence $R(y)$ is an isobaric polynomial with the weight $n(n - 1)$. We have $w(y^{\alpha_i} y_1^{\beta_i}) = n\alpha_i + (n - 1)\beta_i \leq n(n - 1)$, which implies $\alpha_i \leq n - 1$ and $\beta_i \leq n$. If $\alpha_i > 0$ then we have $\alpha_i + \beta_i < n$, because $n\alpha_i + (n - 1)\beta_i = \alpha_i + (n - 1)(\alpha_i + \beta_i)$.

Since $w(y_1^n) = w(y^{n-1}) = n(n - 1)$, the coefficients of $y_1^n$ and $y^{n-1}$ in $R(y)$ must be polynomials in $\bar{a}_n$. By the computation of the above determinant, the coefficients of $y_1^n$ and $y^{n-1}$ in $R(y)$ are $(-1)^n\bar{a}_n^{-1}$ and $(-1)^{n-1}n\bar{a}_n^n$. Then the form of $R(y)$ is as (4). In the following, we take $\bar{a}_i$ as complex numbers. If $R(y)$ is reducible, we assume that $R(y) = F_1(y)F_2(y)$, where $0 < \text{tdeg}(F_1(y)), \text{tdeg}(F_2(y)) < n$. Since $R(y) = P(y)f_1(y) + Q(y)f_2(y)$, where $P(y), Q(y)$ are two differential polynomials, we have $R(\bar{y}) \equiv 0$ which implies that $F_1(\bar{y}) = 0$ or $F_2(\bar{y}) = 0$. But we know that it is impossible by Lemma 4.2, because $\text{deg}(F_1(y), y_1), \text{deg}(F_2(y), y_1) < n$, a contradiction.

**Theorem 5.2** Use the same notations as in Lemma 5.1. If $\bar{y} = \sum_{i=0}^{n} \hat{a}_ix^i$ is a polynomial solution of $F(y) = 0$, then $R(y)|F(y)$. Since $F(y)$ is irreducible, $F(y) = \lambda R(y)$, where $\lambda \in \mathcal{C}$ and $\lambda \neq 0$.

**Proof:** From Lemma 4.2 and Lemma 5.1, we know $\text{deg}(F(y), y_1) \geq n = \text{deg}(R(y), y_1)$. Let $T(y) = \text{prem}(F(y), y_1)$. Then we have the remainder formula $J(y)^k F(y) = Q(y)R(y) + T(y)$, where $J(y)$ is the initial of $R(y)$ and $Q(y), T(y) \in \mathcal{C}[y, y_1]$ and $\text{deg}(T(y), y_1) < \text{deg}(R(y), y_1)$.

Since $F(\bar{y}) = 0$ and $R(\bar{y}) = 0$, we have $T(\bar{y}) = 0$. By Lemma 4.2, $T(y) = 0$. That is $J(y)^k F(y) = Q(y)R(y)$ which implies that $R(y)|F(y)$ because $R(y)$ is irreducible. Since $F(y)$ is irreducible, it is clear that $F(y) = \lambda R(y)$ where $\lambda \in \mathcal{C}$ and $\lambda \neq 0$.

From Lemma 5.1 and Theorem 5.2, if $F(y)$ has polynomial solutions $\bar{y} = \sum_{i=0}^{n} \hat{a}_ix^i$, it must be the following form

$$F(y) = ay_1^n + by_1^{n-1} + G(y, y_1)$$

where $a, b \in \mathcal{C}$ are not zero, $\text{tdeg}(G) \leq n - 1$ and $G$ does not contain the term $y_1^{n-1}$.

As a consequence of Theorem 5.2 and Lemma 5.1, we have

**Corollary 5.3** Let $F(y)$ be of the form (5) and have a polynomial solution of the form (3). Then

$$\hat{a}_n = -\frac{b}{n^na}$$

**Lemma 5.4** Let $F(y)$ be of the form (5) and have a polynomial general solution of the form (3). Then we may construct a new general solution of the following form for $F(y) = 0$

$$\hat{y} = \hat{a}_n(x + c)^n + \sum_{i=0}^{n-2} \hat{a}_i(x + c)^i.$$
In other words, we may assume that $\bar{a}_{n-1} = 0$ in the general solution of $F(y) = 0$.

**Proof:** It is clear that
\[
\hat{y} = \bar{a}_n(x + c - \frac{\bar{a}_{n-1}}{n\bar{a}_n})^n + \sum_{i=0}^{n-2} \bar{a}_i(x + c - \frac{\bar{a}_{n-1}}{n\bar{a}_n})^i.
\]
Since $c - \frac{\bar{a}_{n-1}}{n\bar{a}_n}$ is still an arbitrary constant, replacing $c - \frac{\bar{a}_{n-1}}{n\bar{a}_n}$ by $c$ in the above equation, we get the form (7) and it is still a general solution of $F(y) = 0$.

The following theorem will tell us that the value of $\bar{a}_k$ only depends on the values of $\bar{a}_i$ for $i \geq k$ if $F(y) = 0$ has polynomial general solutions.

**Theorem 5.5** Let $F(y)$ be of the form (5) and $z = (\frac{-b}{n^n}a)x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ where $a_i$ are indeterminants. Substituting $y$ by $z$ in $F(y)$, the coefficients of $x^{(n-1)^2+i-1}$ in $F(z)$ are of the following form
\[
\left(\frac{-b}{n^n}a\right)^{n-i}ba_i + h_i(a_{n-1}, \ldots, a_{i+1}) \text{ for } i = n-2, \ldots, 0
\]
where $h_i(a_{n-1}, \ldots, a_{i+1})$ are polynomials in $a_{n-1}, \ldots, a_{i+1}$.

**Proof:** Let $C_i$ be the coefficient of $x^{(n-1)^2+i-1}$ in $F(z)$ for $i = 0, \ldots, n-2$ where
\[
F(z) = az^n + bz^{n-1} + G(z, z_1)
\]
As in the proof of Lemma 5.1, we define a weight $w$. Then $z = (\frac{-b}{n^n}a)x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is an isobaric polynomial with the weight $n$. Hence $z^{\alpha_j}z_1^{\beta_j}$ is still an isobaric polynomial with the weight $n\alpha_j + (n-1)\beta_j$. Now we consider $C_i$. By computation, we know that the highest weight of the terms in $F(z)$ is $n(n-1)$. Hence the highest weight in $C_i$ is not greater than $n-i$. So $a_k$ can not appear in $C_i$ for $k \leq i-1$ and if $a_i$ appears in $C_i$, then it must be linear and its coefficient must be constant. In the coefficients of $x^{(n-1)^2+i-1}$ in $az^n + bz^{n-1}$, the term in which $a_i$ appears are $\left(\frac{-b}{n^n}a\right)^{n-i}ia_i + \left(\frac{-b}{n^n}a\right)^{n-2}(n-1)a_i$. In the coefficients of $x^{(n-1)^2+i-1}$ in $G(z, z_1)$, since the weight of each term is less than $n-i$ (for $\text{deg}(G) < n$), $a_i$ can not appear. Therefore $C_i$ has the form (8).

\section{A polynomial-time algorithm and experiment results}

From the results of section 5, we have the following algorithm:

**Algorithm 6.1** The input is $F(y)$. The output is a polynomial general solution of $F(y) = 0$ if it exists.

1. If $F(y)$ can be written as the form (5), then goto step 2. Otherwise, by Theorem 5.2, $F(y) = 0$ has no polynomial general solutions and the algorithm terminates.

2. Let $F(y)$ be of degree $n$ in $y_1$. Let $\bar{a}_n = \frac{b}{n^n}, \bar{a}_{n-1} = 0,$
\[
\bar{a}_i = \frac{h_i(\bar{a}_{n-1}, \ldots, \bar{a}_{i+1})}{(-b/n^n)a^{n-2}(n-1)} \text{ for } i = n-2, \ldots, 0,
\]
where $h_i$ are from Lemma 5.5. We have $\bar{a}_i \in C$. 


3. Let \( \bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i \). If \( F(\bar{y}) \equiv 0 \) then \( \hat{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i \) is a polynomial general solution of \( F(y) = 0 \). Otherwise, \( F(y) = 0 \) has no polynomial general solutions.

The correctness of Step 3 is due to the following facts. By Corollary 5.3, Lemmas 5.4 and 5.5, if \( F(y) = 0 \) has polynomial general solutions, then \( \bar{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i \) must be such a solution. By Theorem 4.1, to check whether \( \hat{y} \) is a polynomial general solution we need only to check whether \( \bar{y} \) is a polynomial solution of \( F(y) = 0 \).

Now we give some examples.

Example 6.2 Consider the differential polynomial:

\[
F_1(y) = y_1^4 - 8 y_1^3 + (6 + 24 y) y_1^2 + 257 + 528 y^2 - 256 y^3 - 552 y
\]

1. \( F_1(y) \) can be written as the form:

\[
F_1(y) = y_1^4 - 256 y^3 + G(y, y_1)
\]

where

\[
G(y, y_1) = -8 y_1^3 + (6 + 24 y) y_1^2 + 257 + 528 y^2 - 552 y
\]

\( \text{tdeg}(G) \leq 3 \)

2. If \( F_1(y) = 0 \) has a polynomial general solution, then its degree is four and the coefficient of \( x^4 \) must be \( a_4 = \frac{256}{256} = 1 \).

3. Let \( z = x^4 + a_2 x^2 + a_1 x + a_0 \). Replacing \( y \) by \( z \) in \( F_1(y) \), we compute the coefficients of \( x^8, x^9, x^{10}, \) which are

\[
768 a_2 + 528 - 384 a_2^2 - 768 a_0 \\
-512 - 512 a_1 \\
384 - 256 a_2
\]

Then we have \( a_0 = \frac{17}{16}, a_1 = -1, a_2 = \frac{3}{2} \)

4. Let \( \bar{y} = x^4 + \frac{3}{2} x^2 - x + \frac{17}{16} \). Substituting \( y \) by \( \bar{y} \) in \( F_1(y) \), \( F_1(y) \) becomes zero. Hence a polynomial general solution of \( F_1(y) = 0 \) is

\[
\bar{y} = (x + c)^4 + \frac{3}{2} (x + c)^2 - (x + c) + \frac{17}{16}
\]

Example 6.3 Consider differential polynomial

\[
F_2(y) = y_1^5 - 16 y_1^4 + y_1^3 + y^2 - y_1 y
\]

Similar as Example 6.2, the coefficient of \( x^5 \) must be \( a_5 = \frac{16}{32} = \frac{16}{3125} \). Compute the coefficients of \( x^{15}, x^{16}, x^{17}, x^{18}, \) which are

\[
-\frac{147456}{48828125} a_3 a_2 - \frac{262144}{30517578125} a_0, \\
-\frac{49152}{48828125} a_7 - \frac{196608}{30517578125} a_1, \\
-\frac{131072}{30517578125} a_2, -\frac{65536}{30517578125} a_3
\]

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Hence we have $a_3 = a_2 = a_1 = a_0 = 0$ and $\bar{y} = \frac{16}{3125} x^5$. Substituting $\bar{y}$ to $F_2(y)$, we have

$$F_2(\bar{y}) = -\frac{256}{1953125} x^9 + \frac{256}{9765625} x^{10} + \frac{4096}{244140625} x^{12} \neq 0$$

Hence $F_2(y) = 0$ has no polynomial general solutions.

In Step 2 of Algorithm 6.1, we need to compute $F(z)$ where $z = -\frac{b}{n^a} x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $a_i$ are indeterminants. The complexity of it is very high. In order to give a polynomial-time algorithm, we should modify Step 2. From Theorem 5.5, to compute the value of $\bar{y}$ and $\bar{a}$.

Algorithm 6.4 The inputs are $F(y)$ as the form (5) and $\bar{z} = \bar{a}_n x^n + \cdots + \bar{a}_0$ where $\bar{a}_i \in C$. The output is the coefficient of $x^{(n-1)^2 + k - 1}$ in $F(\bar{z})$ for some $k$.

1. Compute $\bar{z}^n$ and $\bar{z}^{n-1}$ where $\bar{z}_1$ is the derivative of $\bar{z}$ wrt $x$. We compute $\bar{z}^n$ step by step. That is, we compute $\bar{z}^2$ firstly, then compute multiplication of $\bar{z}^2$ and $\bar{z}$, and so on. Note that, after we computed $\bar{z}^n$, we have also obtained $\bar{z}^i$ for $i < n$. For $\bar{z}^{n-1}$, we compute it by the same way.

2. Write $F(y)$ as the form: $F(y) = \sum_{i=0}^{n} (d_{0,i} + d_{1,i} y + \cdots + d_{n-i,i} y^{n-i}) y_i^j$ where $d_{i,j} \in C$.

3. For $i$ from 0 to $n$, compute $p_i = d_{0,i} + d_{1,i} \bar{z} + \cdots + d_{n-i,i} \bar{z}^{n-i}$

4. result:=0
   For $j$ from 0 to $n$
      For $i$ from 0 to $(n - 1)^2 + k - 1$
          result:=result+coef($p_j$, $x$, $i$)*coef($y_i^j$, $x$, $(n - 1)^2 + k - 1 - i$)
          where coef($p$, $x$, $k$) means the coefficient of $x^k$ in $p$.

5. return(result)

In Algorithm 6.4, the complexity of Step 1 is $O(n^4)$. The complexity of Step 2 is $O(n^2)$. The complexity of Step 3 is $O(n^4)$. The complexity of Step 4 is $O(n^3)$. Hence the complexity of Algorithm 6.4 is $O(n^4)$. Here, we only consider the complexity on multiplications.

Algorithm 6.5 The inputs are $F(y)$ as the form (5) and $\bar{z} = \bar{a}_n x^n + \bar{a}_{n-1} x^{n-1} + \cdots + \bar{a}_0$ where $\bar{a}_i$ are interminates. The output is $\bar{a}_i$ for $i = n, \cdots, 0$ in Step 2 of Algorithm 6.1.

1. Let $\bar{a}_n = -\frac{b}{n^a}$, $\bar{a}_{n-1} = 0$.

2. Let $i = n - 2$
   while $i \geq 0$ do
      (a) $\bar{y} := \bar{a}_n x^n + \cdots + \bar{a}_{i+1} x^{i+1}$.
      (b) $C_i$ := the coefficient of $x^{(n-1)^2 + i - 1}$ in $F(\bar{y})$ by Algorithm 6.4.
      (c) $\bar{a}_i := -\frac{C_i}{(-b/n^a)^{n-2(n-1-i)b}}$.
It is easy to know that the complexity of Algorithm 6.5 is \(O(n^5)\). In Step 3 of Algorithm 6.1, we verify whether \(\bar{y}\) is a polynomial solution of \(F(y) = 0\). If \(\bar{y}\) is not a polynomial solution of \(F(y) = 0\), that is said \(F(y) \neq 0\), then \(F(y)\) will be a polynomial in \(x\) with degree not greater than \(n(n-1)\). Hence, if \(F(\bar{y}) \neq 0\), then \(F(\bar{y}) = 0\) as an equation in \(x\) has \(n(n-1)\) roots at most. So we can verify it by numerical computation. If \(F(\bar{y})(k) = 0\) for several values of \(k\), then \(F(y) = 0\). Otherwise, \(F(\bar{y}) \neq 0\).

Algorithm 6.6 The inputs are \(F(y)\) as the form (5) and \(\bar{y} = \sum_{i=0}^{n} a_i x^i\) as in Algorithm 6.1. The output is “Yes” or “No” where “Yes” means \(\bar{y}\) is a solution of \(F(y) = 0\) and “No” means \(\bar{y}\) is not a solution of \(F(y) = 0\).

1. Write \(F(y)\) as the form: \(F(y) = \sum_{j=0}^{n} (d_{0,j} + d_{1,j} y + \cdots + d_{n-j,j} y^{n-j}) y^j\) where \(d_{i,j} \in C\).
2. For \(k\) from \(-n(n-1)/2\) to \(n(n-1)/2\)
   (a) Compute \(k^i\) for \(i = 1 \cdots n\) by the same way as Step 1 of Algorithm 6.4.
   (b) Substitute \(x^i\) by \(k^i\) in \(\bar{y}\) and \(\bar{y}_1\). Then we get the values of \(\bar{y}\) and \(\bar{y}_1\) at \(k\) denoted by \(\bar{y}(k)\) and \(\bar{y}_1(k)\). Compute \(\bar{y}_1(k)^i\) and \(\bar{y}(k)^i\) for \(i = 1 \cdots n\).
   (c) result:=0
   For \(j\) from 0 to \(n\), compute \(p_j(k) = d_{0,j} + d_{1,j} \bar{y}(k) + \cdots + d_{n-j,j} \bar{y}(k)^j\)
   result:=result+\(p_j(k)\bar{y}_1(k)^j\)
   (d) If result\(\neq 0\) then return(No)
3. return(Yes)

It is easy to see that the complexity of Algorithm 6.6 is \(O(n^3)\). So we have the following theorem.

**Theorem 6.7** We can decide whether \(F(y) = 0\) has a polynomial general solution and compute one if it exists with \(O(n^5)\) multiplications.

Table 1 shows the statistic results of running our algorithm for ten differential equations. Differential polynomials in our experiments are given in the Appendix of this paper. We only give the total degrees and terms of these differential polynomials here. In the table, \(F_1\) and \(G_i\) denote differential polynomials. Here, the coefficients of \(F_1\) and \(G_j\) are integers. The coefficients of \(F_i\) are less than \(10^6\) but that of \(G_1\) may be very large. The unit of running time is seconds. The column of “solution” means whether they have a polynomial general solution or not. The program is written in Maple. The running time is collected on a computer with Pentium 4, 2.66GHzCPU and and 256M memory.

## 7 Conclusion

We give a polynomial-time algorithm to compute a polynomial general solution of first order ODEs with constant coefficients. Our experiments show that the algorithm can be used to solve very large ODEs.

It is interesting to see whether the result can be extend to the case when the coefficients of first order ODEs are not constant or the case of high order ODEs with constant coefficients.
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Table 1: Statistics on Solving Differential Equations

References


\[ F_0 = 33491^y - 279967y^5 + 194478 - 198389^y y_1 + 113804y^4 y_1^4 - 381141y^2 y_1^2 - 219431y + 209023y + 167749y^3 y_1^3 - 382588y^2 y_1^2 - 72424^y y_1^4 + 220632y^3 y_1 + 213577^y y_1^2 + 279767y^2 y_1^3 - 267548y - 148190y^2 + 90647y^3 + 37783^y + 46515y - 24863y^4 - 38516y^5 \]

\[ F_1 = 34579 + 25778y^2 y_1 + 80625y y_1^4 + 78679y^2 y_1^2 + 43313^y y_1 + 46139y - 73145y - 50398y + 218y - 65 \]

\[ F_2 = 65821 - 25460y^2 y_1 + 61809y y_1^4 - 74708y^2 y_1^2 + 117y + 301473y - 219739y^2 y_1^4 + 96281y^6 + 6542^y y_1^3 - 119686y y_1^4 - 98034^y y_1^5 + 69685y^3 y_1^3 + 93551y^2 y_1^2 + 74603y^2 y_1^3 + 45035^y y_1^4 + 32219y^5 y_1^5 + 21648y^4 y_1^4 + 111359^y y_1^6 + 36109y y_1^7 + 51768y^3 + 4297y - 147755y^3 y_1^3 - 33025y^3 y_1^4 + 111279y^3 - 89315y^4 \]

\[ F_3 = 29746 \quad 89y - 171479y^2 y_1 + 66023y y_1^4 - 169859y^2 y_1^2 - 15692y - 212470y^3 y_1 + 18496y^3 y_1^2 - 905339y y_1^5 - 458442y^2 y_1^2 + 3009y^3 y_1^4 \]

\[ F_4 = 165y^10^y + 113925y^9 y_1 + (65389 + 46498)^y y_1^7 + 123249y^9 y_1^2 + 914y - 176941y y_1^9 + (136118 - 7376y + 74818y^9 + 130026y y_1^9) y_1^6 + (6437 + 117228y - 155272y^2 + 340028y^3 y_1^3 + (1 - 216102y + 29929y^4) y_1^5 \]

\[ F_{10} = 181y^11^y + 3849y^9 y_1^2 + 94780 + 12878y y_1^3 + (22493y + 11288)^y y_1^4 + (35340y - 72422y^3 + 4368 + 9846y^9 y_1^7 + (1 - 9922y^2 + 71119 + 100156)^y y_1^3 + (1 - 93819y + 14667y^2 + 22174y^3 - 72274y^4 - 138109y^5 + 77309y y_1^3 + (107129y^2 + 2483y - 28027y + 13839^y y_1 + 68997y y_1^2 + 55960y y_1^3 + (75561 + 149929y^2 - 8166y^2 + 57498y + 24843y^2 + 112586y y_1^2 + 83916^y y_1^3 + 71778y y_1^4 + (36767y - 83589y y_1 + 30821 + 58285y^2 - 50548y^2 - 196780y y_1^4 + 128189y - 16486 + 1893y^9 y_1^2 + 125790y^2 y_1 + 45710y y_1^3 + 235415y + 40485y^3 + 112299y^3 + 88202y^3 - 6533^y y_1^4 + 9988 + 90092y^2 y_1 + 71113y^2 y_1^2 + 77724y^3 + 111626y^4 + 35568y^5 + 30872y + 12149y y_1^5 + 23335y^4 y_1^2 + 28742y^5 \]

8 Appendix