Building Dynamic Mathematical Models with Geometry Expert*
II. Linkages

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Abstract. In this series of papers, we will show how to build dynamic models for learning and teaching geometry related mathematical concepts with Geometry Expert, which is a software for dynamic geometric drawing and automated geometry theorem proving and discovering. This paper focuses on constructing linkages that can be used to draw lines, conics, and in particular any plane algebraic curve according to a well-known result of Kempe. Besides education, constructing dynamic models for linkages has applications in mechanical designs.

1 Introduction

With computers become more and more popular, using computers to assist teaching and learning is changing from theoretical study to practice. There are lots of educational softwares in the market. But most of them are more like "electronic books," at most with sound and animation. They do not accept input from the user, let along solve problems proposed by the user. In our viewpoint, this kind of softwares does not use the full power of modern computers. They actually make little advance from a well-written paper book. Educators are setting higher standards for education softwares [15]. They want a software that can be used by teachers and students to operate and to generate something by themselves.

Geometry Expert (GEX) is built to answer the educators' call for more flexible and powerful educational softwares. GEX is a software for *dynamic geometric drawing* and *automated geometry theorem proving and discovering* [7]. With GEX, we can draw geometric diagrams dynamically, that is, we can

* This work is partially supported by a grant from Chinese NSF and a grant CCR-9420857 from US NSF.

change these diagrams in a continuous way and at the same time keeping the geometric relations in the figure intact. This feature of GEX provides a powerful and convenient platform for teaching mathematical concepts in an intuitive way.

GEX's theorem prover is based on recent achievements in the field of automated geometry reasoning [17, 3, 4]. With GEX, we have proved hundreds of difficult geometry theorems and generated elegant proofs for them. Furthermore, we can use GEX to build a database which contains all the properties of a given diagram that can be deduced using a set of geometric axioms. With this feature, teachers can easily make exercises and test problems; students can enhance their ability of solving problems by fully explore the properties of a given diagram. With GEX, students can learn how to write proofs in a few lines for moderately difficult theorems. The advanced part of the system can be used by geometry experts to solve challenging problems or conjectures.

In this series of papers, we are going to explore how to use GEX to build dynamic models for teaching and learning mathematical concepts related to geometry. In this paper, we will show how to construct linkages as dynamic mathematical models. Especially, we will show how to construct a linkage in GEX, which can be used to generate a given algebraic plane curve.

Related Work. Symbolic computation softwares like Mathematica [16] and Maple [2] provide powerful tools of solving mathematical problems. They also provide good graphical interfaces. But, solving geometry problems is a weak point of these softwares: they cannot prove geometry theorem efficiently let along generate readable and elegant proofs as GEX does. Also, their graphical editors lack the dynamical ability of GEX.

Dynamic geometry softwares Geometer's sketchpad [9] and Gabri are quite similar to the graphical part of GEX. But they cannot prove or discover geometry theorems.

MathLab [19] is a series of intelligent softwares quite similar to GEX which can be used to teach solid geometry, trigonometric functions, analytical geometry, and algebra.

The rest of this paper is organized as follows. In Section 2, we will describe what kinds of dynamic models can be generated with GEX. In Section 3, we will show how to generate several special linkages with GEX. In Section, 4, we will show how to build a linkage with GEX which is capable of generating any given plane algebraic curve.

2 Dynamic Models

By *dynamic models*, we mean models built by computer softwares that can be changed dynamically. In this series of papers, we shall show how to build *dynamic models* for various mathematical concepts using GEX. We will consider the following four classes of models.

1. Geometric transformations of diagrams.

2. Loci generated by diagrams constructed using ruler and compass with multiple driving points moving on a line or a circle. This class includes conics, functions $y = \frac{\sqrt[n]{f(x)}}{\sqrt[m]{g(x)}}$ where $f(x)$ and $g(x)$ are polynomials and n and m are positive integers, trigonometric functions, various curves defined in polar coordinate systems, etc.

3. Diagrams of functions using the numerical computation facility provided by the C language. This class includes functions of the form: $y = f(x)$ where $f(x)$ could be any "elementary functions"– a^x, x^a, $log(x)$, trigonometric functions and their inverse functions – and their arithmetic expressions and compositions. This part is quite similar to most "Graphic Calculator," but is more flexible and powerful.

4. Loci generated by linkages with rotating joints alone. As proved by Kempe [13], this class includes any algebraic curve $f(x, y) = 0$ where $f(x, y)$ is a polynomial of x and y.

   The use of linkages as tools can be traced back to the ancient age of human history. But, linkages became a hot research topic for mathematicians only in the late nineteen century mainly due to Watt's invention of the steam engine, which needs a mechanism that can transform a cyclic movement into a movement along a straight line. The study of linkages attracts many great mathematicians of that time such as Cayley, Kempe, Hart, Darboux, Clifford, Darwin, etc. In particular, Kempe proved the famous result that any algebraic curve can be drawn with a linkage. By given a procedure of constructing a linkage to generate a given algebraic plane curve, we actually provide a constructive proof for Kempe's result.

   Linkages are often used as drawing tools in the class room [18]. We will show that linkage models built with computer softwares are more powerful, flexible, convenient, and intuitive than models built of real materials like plastics and wood.

Dynamic models built with computer softwares have the following advantages.

Dynamic Transformation We can change the diagrams not only by standard geometric transformations like translation, rotation, dilation, and reflection, but also by dragging certain points in the diagram freely and at the same time keeping all the geometric relations in the diagram intact. Using this tool, users may easily get a variety of forms for a diagrams and may check conjectures numerically. For instance, when we want to prove that three altitudes of a triangle are concurrent, we may drag one of the vertices of the triangle to all possible positions and observe that the three altitudes always intersect in a point. This implies strongly that this statement does true. Grünbaum and Shepherd discovered many interesting geometry properties by numerical checking [8]. Dynamic softwares like GEX, certainly provide convenient tools for this kind of work.

Animation Through animation, the user may observe the generation process of curves or the figures of functions. For example, by continuously changing the three parameters a, b, c in function $y = a\sin(bx + c)$, we may get an intuitive view of the affect of these parameters on the shape of the sine function. The case for c is particularly interesting: when changing c continuously, we get a wave-like movement of the diagram for the sine function.

Generality With GEX, we can construct diagrams for almost all elementary functions and most of the commonly encountered plane curves. This seems beyond the scope of the usual graphical calculators.

More Participation Using GEX to draw diagram of functions, the users may construct the models by themselves. For one diagram, the user may give different methods to generate it. This flexibility gives students a chance to gain inside properties of a diagram and to enhance their ability of solving problems.

Openness GEX provides an open platform for the user to explore their own problems. Even for the generation of diagrams with ruler, compass, and linkages, there are still many open problems:

Problem 1. Design a linkage that can be used to draw a conic passing through five given points.

Problem 2. In a diagram drawn with ruler and compass, what kind of curves can be generated if only one driving point is allowed to move on a line or a circle.

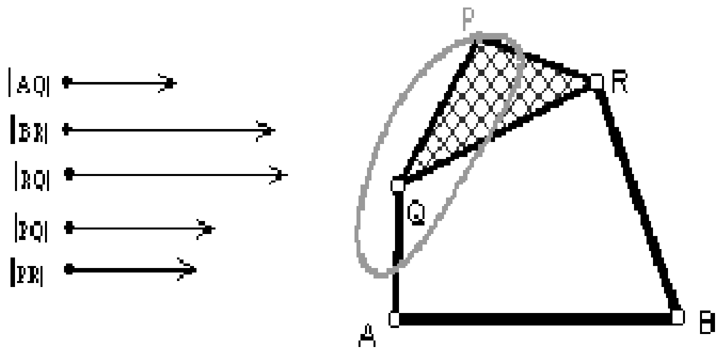
Problem 3. In a diagram drawn with ruler and compass, what kind of curves can be generated if multiple driving points are allowed to move on lines or circles. We can generate diagrams for trigonometric functions. But, can we generate the diagram for the exponential function?

3 Linkages Generating Special Plane Curves

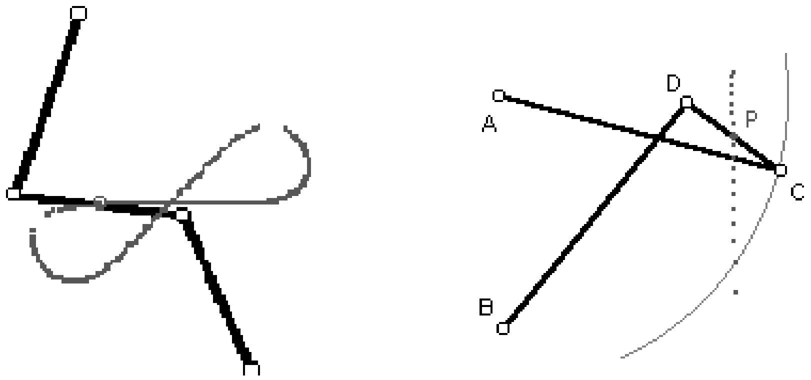
Using linkages to generate curves is a special kind of locus generation. The operation process is quite simple: first we need to draw a linkage with line (ruler) and circle (compass); then we need select a point as the driving point and assign to it a circle; finally we select a point as the locus point. When the driving point moves on the circle assigned to them, the locus point will generate a locus. More on locus generation with GEX can be found in [6].

3.1 The Four-Bar Linkage

The four-bar linkage is the simplest and one of the most important linkage [10]. In order to be more convenient for designers, there are books that give the traces generated by various kinds of four-bar linkages [12]. Using GEX, we can change the four links in a four-bar linkage continuously and show its traces at the same time. In other words, we provide an electronic book that have the traces of all possible four-bar linkages. This is obviously more convenient than fixed traces in books.



The above figure show two four-bar linkages and their traces. Triangle PQR has a fixed shape. When point Q rotates around point A, point P will generate the trace shown in the figure. On the left side of the figure, the five arrows represent the lengths of AQ, BR, RQ, PQ, and PR respectively. The user may drag these arrows to change the length of the corresponding links.



The linkage on the left side above was devised by Watt to mimic line motion. The linkage on the right side was devised by Chebyshev for line approximation. All these linkages and their traces can be obtained by properly changing the lengths of the five links.

3.2 Linkages Drawing A Straight Line

A basic question in mechanism is to design a linkage converting a cyclic motion into a line motion. The mathematical principle for such linkages is the following result.

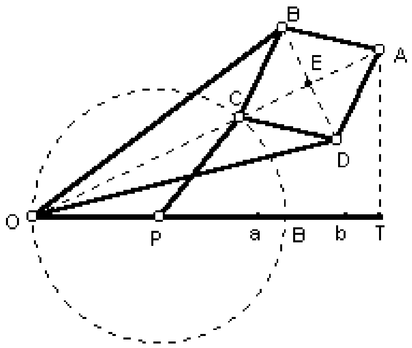
Inversion Theorem. The inversion image of a circle passing through the inversion center is a straight line. The inversion Q of a point P with a circle (O) is defined as follows: Q is a point on line OP (in the same side of O with P) such that $OP \times OQ$ equals to the square of the radius of circle O.

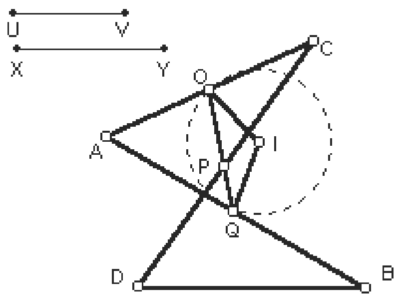
GEX provides a command to draw the inversion point directly, so it is easy to construct diagrams that can convert circles to lines with GEX. But to draw a linkage to do that is not easy, because of the restrictions on linkages. The following are two classical examples.

Example 3.1. (Peucellier linkage) In the diagram on the right side, $b = \mid OD \mid = \mid OB \mid$, $a = \mid AB \mid = \mid BC \mid = \mid CD \mid = \mid DA \mid$, and $\mid OP \mid = \mid PC \mid = c$. Let AC and BD meet in point E. Thus

$$OC \times OA = (OE - CE) \times (OE + CE) = OE^2 - CE^2 = b^2 - DE^2 - CE^2 = b^2 - a^2$$

Therefore, point A is the inversion point of point C with respect to a circle whose center is point O. Following the inversion theorem, when point C moves on a circle passing through point O, point A's trace is a line. Condition $\mid OP \mid = \mid PC \mid = c$ ensures that point C will move on such a circle.





Example 3.2. (Hart linkage) In the diagram, $ADBC$ is an isosceles trapezoid. Its two waists AC, BD and two diagonals are links. Point O is fixed and is on line AC. $Q = AB \cap$ Pline(O,AD) (i.e, the line passing trough O and parallel to AD),$P = CD \cap OQ$.I is a point on the perpendicular-bisector of OQ. Fix two points O and I. When point Q rotates around Pcircle(I,O), point P's trace is a line. In the diagram, UV and XY represent the lengths of OA and QA.

In order to prove that the Hart linkage can really draw a line, we first prove the following result.

Lemma. In an isosceles trapezoid, the product of the two parallel edges equals to the difference of the square of the diagonal and the square of the waist. (see [6] for a proof).

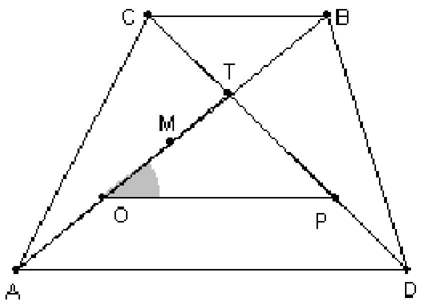
In the Hart linkage, $OQ/BC = OA/CA$; $OP/AD = BC/CA$. Following the above lemma ,

$$OP \times OQ = (BC \times AD \times OA \times OC)/(CA^2) = ((AB^2 - AC^2)OA \times OC)/(CA^2)$$

Then point P is the inverse point of point Q to a circle whose center is point O. Following the inverse theorem, the trace of point P is a line.

3.3 Linkages Drawing Conics

In the lemma in the preceding section, we have $AD \times BC = AB^2 - AC^2$. Let $AC = BD = 2b$, $AB = CD = 2a$ be four links. Fixing AB, we get a linkage $ABCD$. Select a point P on CD and a point O on AB such that OP is parallel to AD. This parallel relation will remain true as the linkage moves. Let AB and CD meet in point T, M the midpoint of AB . Let $OP = r$, $\angle POB = \angle TPO = \angle BCD = \angle CBA = \alpha$, $OM = c$, $MT = z$. Then



$r = 2(c + z)\cos\alpha$ (1)

$BC = 2BT\cos\alpha = 2(a - z)\cos\alpha$ (2)

$AD = 2AT\cos\alpha = 2(a + z)\cos\alpha$ (3)

From the last two equalities, we have $BC \times AD = 5(a^2 - z^2)cos^2\alpha$

Eliminate z from the above equations, we have

$$a^2\cos^2\alpha - (r/2 - c \times \cos\alpha)^2 = a^2 - b^2$$

Thus, point Q is the inversion point of point P with respect to a circle with center O and radius $\sqrt{2}k$. The inversion equation is $r \times s = 2k^2$ where $(s = OQ)$. The equation of the inversion point Q is :

$$a^2s^2\cos^2\alpha - (k^2 - c \times s \times \cos\alpha)^2 = s^2(a^2 - b^2)$$

Use a coordinate system whose origin is point O and whose x-axis is AB. Suppose that the coordinate of point Q in this system is (x,y). Then $x = s \times cos\alpha$, $y = s \times sin\alpha$, $s^2 = x^2 + y^2$.

Changing the equations of point Q to a equation about x,y, we have

$$(c^2 - b^2)x^2 + (a^2 - b^2)y^2 - 2ck^2x + k^4 = 0$$

Because $a \succ b$, the coefficient of y^2 is positive. The character of the conic is determined entirely by the coefficient of x^2. Thus the curve is:

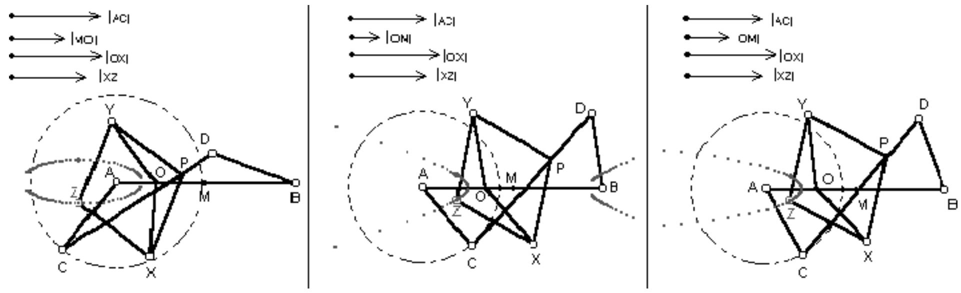
a Parabola, if $c = b$

an Ellipse, if $c \succ b$

a Hyperbola, if $c \prec b$

Following the above analysis, we can design a linkage that can be used to draw all three kinds of conics.

Example 3.3. In the isosceles trapezoid mentioned above, fix points A and B. Point P is inverted to Z by means of a Peucellier cell. In the diagram, the lengths of $AC = 2b$ and $MO = c$ can be change by dragging the arrows near them. Therefore, we can use this linkage to draw the three kinds of conics by just changing the lengths of AC and MO properly. The three linkages below are obtained in this way.

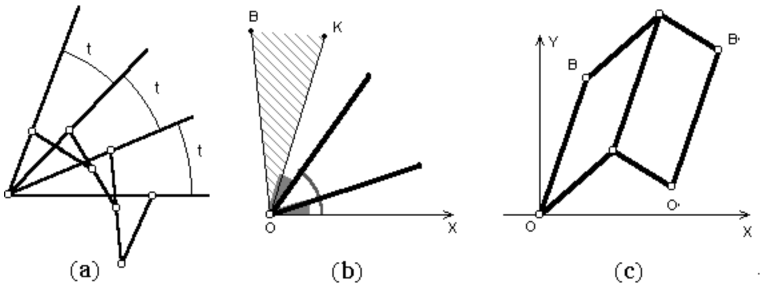


4 Linkages Generating General Algebraic Plane Curves

Kempe proved that any algebraic plane curve can be generated with a linkage [13]. In this section, we will show that for a given algebraic plane curve defined by the following equation

$$f(x,y) = 0 \tag{1}$$

how to construct a linkage with GEX that will generate the diagram of the above curve. First we need three kinds of tools.

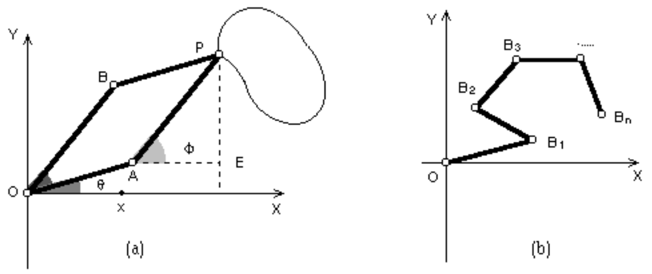


(a) (b) (c)

1. The Multiplicator shown in Figure 1 consists of similar crossed parallelograms. Using a Multiplicator, we may obtain integral multiples of any angle, e.g. $a\varphi$ or $b\theta$.

2. The Additor. Joining one Multiplicator to another will produce the combination $a\varphi \pm b\theta$. This is the mechanism shown in Figure 2 where the plate BOK with angle B is connected rigidly to the bar. Thus we build up a linkage to produce $\angle BOK = a\varphi \pm b\theta \pm \beta$. If OB is taken equal to A, then the x-coordinate of the point B is $A \times cos(a\varphi \pm b\theta \pm \beta)$.

3. The Translator shown in Figure 3 consists of parallelograms with OB pivoted at O. In this linkage, the bar $O'B'$ is always parallel to OB and can be moved freely within its limits.



Now suppose that a curve $f(x,y)=0$ is given. We construct a parallelogram (Figure 4) such that OA and OB have lengths m and n and form angles $\theta$and φ with the X-axis respectively. P is a point on the curve. Its coordinates are then given below.

$$x = m \times \cos\theta + n \times \cos\varphi; \tag{2}$$
$$y = m \times \sin\theta + n \times \sin\varphi = m \times \cos(\pi/2-\theta) + n \times \cos(\pi/2-\varphi) \tag{3}$$

Note that the products and powers of cosines can be expressed as the sum of cosines. If substituting equations (2) and (3) into (1), we shall have a sum of terms of the following form

$$f(x,y) = \sum_{i=1}^{k} A_i \times cos(a_i\varphi \pm b_i\theta \pm \beta_i) + C = 0$$

where A_i and C are constants; a_i and b_i are positive integers; and β_i equals 90, or 0. Note that if a_i and b_i are rational numbers, we can find a common denominator d for them and treat φ/d and θ/d as angle variables. We thus can always assume that a_i and b_i are integers.

For each term $A_i \times cos(a_i\varphi \pm b_i\theta \pm \beta_i)$, we can use the Multiplicator and Additor to construct a link OC_i such that OC_i is of length A_i and form an angle $a_i\varphi \pm b_i\theta \pm \beta_i$ with the X-axis. Using the Translators, we can construct a chain of links $OB_1, B_1B_2, B_2B_3, \ldots$, as shown in Figure 5, such that $B_1 = C_1$ and $OC_iB_iB_{i-1}$ is a parallelogram. Therefore, point B_n has x-coordinate:

$X = \sum_{i=1}^{n} A_i \times cos(a_i\varphi \pm b_i\theta \pm \beta_i) = f(x,y) - C$ (by equation (4))

If P is moved along the given curve, its coordinates x, y satisfy: $f(x,y)=0$.

Accordingly, the locus of the end point B_n of the chain is

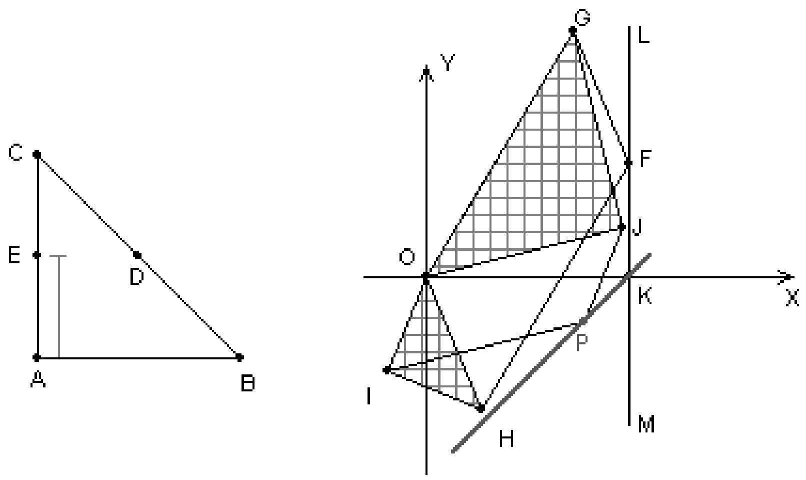
$$X + C = 0$$

a straight line parallel to the Y-axis. Conversely, if B_n is moved along this line (with the help of a Peaucellier cell, for instance) point P will generate the curve $f(x, y) = 0$.Thus we finish the construction.

Example 4.1. We will construct a linkage generating the line defined by $f(x, y) = x - y - 2 = 0$.

Let $x = 2 \times \cos\theta + cos\varphi$, $y = 2 \times \sin\theta + \sin\varphi$. Substituting them into $f(x, y) = 0$, we get a new equation

$$f(x, y) = 2\sqrt{2}cos(\theta + \pi/4) + \sqrt{2}cos(\varphi + \pi/4) - 2 = 0.$$



Following the above construction procedure, we can construct a linkage. ABC is a right isosceles triangle whose right angle edge AB is of length 2. LM is a line parallel to the y-axis and has a distance of 2 with the y -axis. F is a free point on line LM. $OGFH$ is a parallelogram such that $OG = BC = 2\sqrt{2}$, $FG = \sqrt{2}$. OGJ and OHI are right isosceles triangles. $PJOH$ is a parallelogram. When point F moves on line LM, point P draws the line $f(x, y) = 0$. Thus we have constructed a linkage drawing the line $f(x, y) = 0$.

Of course, the line LM still need to be generated by a Peaucellier linkage. The device designed here is actually to transform one line to another line. It is not difficult to see that the construction procedure in the above example can be used to generate any straight line.

For non-linear curves, the construction is much more complicated. After the construction of the chain of points $B_1, \ldots, B_n$, we need to let point B_n move on line $X + C = 0$. The problem is that point B_n is a fixed point in GEX and cannot be moved freely. To solve this problem, we need another approach: for a given position of B_n, find the positions of other points using a numerical iteration method like the Newton method. Thus the curve can be generated by combining GEX's construction procedure and a numerical solver for polynomial equations. It is readily seen that the construction procedure is too complicated to be practical. For a given curve, how to design a minimal generating linkage is still an open problem.

References

[1] C. A. Bauer, T182 Graphical Calculator, Matrices, and Tables *Proc. of ATCM'95*, pp. 89-91, 1995, Singapore.

[2] B. Char, K. Geddes, *Maple V*, Springer-Verlag, 1992.

[3] S.C. Chou, *Mechanical Geometry Theorem Proving*, D.Reidel Publishing Company, Dordrecht, Netherlands, 1988.

[4] S.C. Chou, X.S. Gao, & J.Z. Zhang, *Machine Proofs in Geometry*, World Scientific, Singapore, 1994.

[5] Jen-chung Chuan, Geometric Constructions with the Computer, *Proc. of ATCM95*, pp.329-338.(1995)

[6] X. S. Gao, C. C. Zhu, and Y. Huang, Building Dynamic Mathematical Models with Geometry Expert, preprint, March, 1998 (in Chinese).

[7] X. S. Gao, J. Z. Zhang, and S. C. Chou, *Geometry Expert*, Nine Chapter Pub., 1998 (in Chinese).

[8] B. Grünbaum & G.C. Shephard, From Menelaus to Computer Assisted Proofs in Geometry, preprint, 1993.

[9] N. Jakiw, *Geometer's Sketchpad*, User guide and Reference Manual, Key Curriculum Press, 1994.

[10] A. S. Hall, Kinematics and Linkage Design, Prentice-Hall, INC, 1962, Englewood, Cliffs.

[11] R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages McGraw-Hill Book, 1964.

[12] J. A. Hrones and G. L. Nelson, Analysis of the Four-bar Linkage, MIT Press and Wiley, New York, 1951.

[13] A. B. Kempe, On a general method of describing plane curves of the nth degree by linkwork, Proc. of L.M.S., p.213–216, 1876; voir aussi Messenger of Math., T. VI., p.143-144.

[14] F. V. Morley, American Mathematical Monthly 31 pp.71-77 (1924)

[15] R. F. Tang, The tentative Ideas and Practice of Multimedia Instruction Software, *Proc. of ATCM'95*, pp.244-251, 1995, Singapore.

[16] S. Wolfram, *Mathematica*, Addison-Wesley Pub. Comp., Inc, 1985.

[17] Wu Wen-tsün, *Basic Principles of Mechanical Theorem Proving in Geometries*, Springer-Verlag, 1993.

[18] R. C. Yates, *Geometrical Tools, A Mathematical Sketch and Model Book*, Educational Pub. Inc., Saint Louis, 1949.

[19] J.Z.Zhang: Mathematical laboratory, preprint 1998