Determining the Topology of Real Algebraic Surfaces

Jin-San Cheng[1], Xiao-Shan Gao[1], and Ming Li[2]

[1] Key Lab of Mathematics Mechanization,
Institute of Systems Science, AMSS, Academia Sinica, Beijing 100080, China
jcheng@amss.ac.cn, xgao@mmrc.iss.ac.cn
[2] School of Computer Science, Cardiff University, Cardiff, CF24 3AA, UK
m.li@cs.cardiff.ac.uk

Abstract. An algorithm is proposed to determine the topology of an implicit real algebraic surface in $\mathbb{R}^3$. The algorithm consists of three steps: surface projection, projection curve topology determination and surface patches composition. The algorithm provides a curvilinear wireframe of the surface and the surface patches of the surface determined by the curvilinear wireframe, which have the same topology as the surface. Most of the surface patches are curvilinear polygons. Some examples are used to show that our algorithm is effective.

1 Introduction

An implicit real algebraic surface (or curve, or hypersurface) $\mathcal{S}$ in $\mathbb{R}^u$ with degree d is defined by $f(x_1, x_2, \cdots, x_u) = 0$ where $f(x_1, x_2, \cdots, x_u) \in \mathbb{Q}[x_1, x_2, \cdots, x_u]$ is a polynomial of degree d, and $\mathbb{R}$ and $\mathbb{Q}$ are the fields of real and rational numbers, respectively. Determining the topology of an algebraic surface is not only an interesting mathematical problem, but also a key issue in computer graphics and CAGD [4, 5, 20, 22].

When $u = 1$, $\mathcal{S}$ is a set of discrete points on a line. When $u = 2$, $\mathcal{S}$ is a plane algebraic curve. Topology determination for plane algebraic curves has been studied thoroughly [1, 3, 6, 7, 9, 12, 13, 14, 16, 21]. Algorithms to determine the topology of spatial algebraic curves are also proposed in the following papers [4, 6, 9, 10]. When $u = 3$, the problem is more complex. The topology of $\mathcal{S}$ with $d = 2$ is well known. They are quadratic surfaces. But when $d \geq 3$, there are only some special surfaces whose topology can be efficiently determined [11, 12]. Fortuna et al presented an algorithm to determine the topology of non-singular, orientable real algebraic surfaces in the projective space [8]. Morse theory is used to represent an implicit algebraic surface by polyhedra in theory by Hart et al [15, 20, 22]. Theoretically, the CAD (Cylindrical Algebraic Decomposition) method proposed by Collins can be used to provide information about the topology of an algebraic surface [2, 3]. But in the general case, there exist no complete algorithms to determine the topology of an implicit algebraic surface.

In this paper, we present an algorithm to determine the topology of $\mathcal{S}$ for $u = 3, d \geq 3$. In the rest of this paper, we replace $f(x_1, x_2, x_3) = 0$ with $f(x, y, z) = 0$.

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We obtain a curvilinear wireframe of the surface. The surface patches of the surface are determined by the curvilinear wireframe. Most of the surface patches are curvilinear polygons. The wireframe and the surface patches have the same topology as the surface. If needed, we can easily modify our algorithm to ensure that all the surface patches are curvilinear triangles.

The basic idea of our algorithm is as follows. We first ensure that the surface is a normal surface by performing certain transformations. We then project $\mathcal{S}$: $f(x, y, z) = 0$ to a proper plane and obtain a plane algebraic curve $\mathcal{C}$: $g(x, y) = 0$. Thirdly, we analyze the topology of $\mathcal{C}$ in a finite box, by finding its singularities, dividing the curve into plane curve segments, and dividing the box in the plane into cells. At the fourth step, we divide the spatial curve defined by $\{f(x, y, z) = 0, g(x, y) = 0\}$ into spatial curve segments and compute the number of surface patches connected with each spatial curve segment. This is the key step of the algorithm. In order to determine the number of curve segments connected with a singular point and the number of surface patches connected with a curve segment, we introduced certain minimal circles and find these numbers from the information of the intersections of the circle with the surface. The main steps of the algorithm are similar to Collins' CAD method. But, the purpose of our algorithm is different from that of the CAD method, and many aspects of the algorithm are totally new. Main parts of the algorithm are implemented in Maple and nontrivial examples are used to show that the algorithm is effective.

This paper is divided into six sections. The aim of the second section is to obtain projection curve of the surface. The third section presents an algorithm to determine the topology of the plane projection curve. Space curve segmentation, surface patch composition and the surface topology representation are discussed in the fourth section. The fifth section presents the main algorithm to obtain the topology of a given algebraic surface. Then we draw a conclusion in the last section.

2 Projection Curve of a Surface

In the following, we always assume $\mathcal{S}$ is an algebraic surface: $f(x, y, z) = 0$, where $f(x, y, z) \in \mathbb{Q}[x, y, z]$. Suppose that

$$f(x, y, z) = f_1(x, y, z)^{m_1} \cdots f_n(x, y, z)^{m_n}, \tag{1}$$

where $f_i(x, y, z) \in \mathbb{Q}[x, y, z](i = 1, \cdots, n)$ are irreducible polynomials. If a component contains variable z only, it represents some parallel planes. We can delete this kind of components before we compute the projection curve and add these planes into the topology structure and compute the intersection curve with other components after we finish the analysis. So we suppose that there does not exist this kind of components. It is clear that $f(x, y, z) = 0$ and $f_1(x, y, z) \cdots f_n(x, y, z) = 0$ have the same topology. We still denote

$$f(x, y, z) = f_1(x, y, z) \cdots f_n(x, y, z). \tag{2}$$

Let

$$g(x,y) = Res(f(x,y,z), \frac{\partial f(x,y,z)}{\partial z}, z), \tag{3}$$

where $Res(f(x,y,z), \frac{\partial f(x,y,z)}{\partial z}, z)$ is the Sylvester resultant [26] of $f(x,y,z)$ and $\frac{\partial f(x,y,z)}{\partial z}$ with respect to z. Suppose $g(x,y) = g_1(x,y)^{n_1} \cdots g_m(x,y)^{n_m}$, where $g_i(x,y)(i = 1, \cdots, m)$ are irreducible polynomials. Still denote

$$g(x,y) = g_1(x,y) \cdots g_m(x,y). \tag{4}$$

Then the *projection curve* of the surface $\mathcal{S} : f(x,y,z) = 0$ is a plane curve defined by $g(x,y) = 0$. In this section, we will prove some properties of the projection curve of a given surface $\mathcal{S}$.

In order to determine the topology of $\mathcal{S}$ effectively and efficiently, we assume that

C1. There exist no points $P_0(x_0, y_0)$ satisfying $f(x_0, y_0, z) \equiv 0$.

C2. $\sum_{i+j+k=d} a_{i,j,k} \cdot x^i \cdot y^j \cdot z^k$ has no factors like $T(x,y)$, where d is the total degree of $f(x,y,z)$, $a_{i,j,k}$ is the coefficient of the term $x^i \cdot y^j \cdot z^k$ in $f(x,y,z)$. $T(x,y)$ is a bivariate polynomial.

A *normal surface* is an algebraic surface defined by a square-free polynomial (the multiple of the irreducible factors of the polynomial is no more than 1) satisfying conditions *C1* and *C2.*

If condition *C1* does not hold, represent $f(x,y,z)$ as follows.

$$f(x,y,z) = c_k(x,y) \cdot z^k + c_{k-1}(x,y) \cdot z^{k-1} + \cdots + c_0(x,y), \tag{5}$$

where $c_i(x,y) \in \mathbb{Q}[x,y](i = 1, \cdots, k)$ and $c_k(x,y)$ is a nonzero polynomial. Then, the variety $\{c_0(x,y) = 0, c_1(x,y) = 0, \cdots, c_k(x,y) = 0\}$ has real roots, and the line $\{x = x_0, y = y_0\}$ is on the surface $\mathcal{S}$. In this case, it is difficult to analyze the topology of the surface near this line. Here is an example.

$$f(x,y,z) = x^2 \cdot y^2 + z^2 \cdot y^2 + x^2 \cdot z^2 - 7/2 \cdot x \cdot y \cdot z. \tag{6}$$

We have $f(0,0,z) \equiv 0$ and $\{x = 0, y = 0\}$ is a line on the surface.

Represent $f(x,y,z)$ as follows.

$$f(x,y,z) = L_d(x,y,z) + L_{d-1}(x,y,z) + \cdots + L_0, \tag{7}$$

where $L_t(x,y,z) = \sum_{i+j+k=t} a_{i,j,k} \cdot x^i \cdot y^j \cdot z^k (t = 0, \cdots, d)$. It is clear that all the asymptotic surfaces are contained in the surface defined by the equation $L_d(x,y,z) = 0$. If condition *C2* does not hold, there exists an asymptotic surface of $f(x,y,z) = 0$ of the form $T(x,y) = 0$, which is vertical to XY-plane. For example, the surface $f(x,y,z) = x \cdot y \cdot z - 1 = 0$ does not satisfy condition *C2,* because $x = 0, y = 0$ are asymptotic planes of it.

Lemma 1. *If a surface is not normal, we can find a coordinate transformation like (8) such that the surface obtained with this transformation has the same topology as the original one and is normal.*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{8}$$

where (X, Y, Z) and (x, y, z) are points in the new and old coordinate systems, respectively, and a, b are rational numbers.

Proof. Taking the coordinate transformation as (8) and representing $f(x, y, z)$ as (7), we have

$$\begin{aligned} & F(X, Y, Z) \\ = & f(X + a \cdot Z, Y + b \cdot Z, Z) \\ = & L_d(X + a \cdot Z, Y + b \cdot Z, Z) + L_{d-1}(X + a \cdot Z, Y + b \cdot Z, Z) + \cdots + L_0 \\ = & L_d(a, b, 1) \cdot Z^d + C_{d-1}(X, Y) Z^{d-1} + \cdots + C_0, \end{aligned}$$

where $C_i(X, Y), i = 0, 1, \cdots, d-1$ is the coefficients of $F(X, Y, Z)$ in variable Z. We can find rational numbers a_0, b_0, such that, $L_d(a_0, b_0, 1) \neq 0$. Denote the corresponding $F(X, Y, Z)$ as $F_0(X, Y, Z)$. We will show that $F_0 = 0$ is a normal surface. Since $L(a_0, b_0, 1)$ is a nonzero constant, $F_0 = 0$ satisfies condition *C1*. It is clear that all the asymptotic surface of $F_0 = 0$ are hidden in $L_d(X + a_0 \cdot Z, Y + b_0 \cdot Z, Z) = 0$. There is a term $L_d(a, b, 1) \cdot Z^d$ in $L_d(X + a_0 \cdot Z, Y + b_0 \cdot Z, Z)$, so there are no factors like $T(X, Y)$ hidden in it. $F_0 = 0$ satisfies condition *C2*. So $F_0 = 0$ is a normal surface. □

For the non-normal surface $f(x, y, z) = x \cdot y \cdot z - 1 = 0$, we choose $a = 1, b = 1$. The new surface is $F(X, Y, Z) = Z^3 + (Y + X) \cdot Z^2 + X \cdot Y \cdot Z - 1 = 0$. It is a normal surface.

For the surface defined by (6), we can choose $(a, b) = (1, 1)$. The new surface is $F(X, Y, Z) = 3 \cdot Z^4 + 4 \cdot Y \cdot Z^3 + 4 \cdot X \cdot Z^3 - 7/2 \cdot Z^3 + 2 \cdot Y^2 \cdot Z^2 + 2 \cdot X^2 \cdot Z^2 - 7/2 \cdot Y \cdot Z^2 - 7/2 \cdot X \cdot Z^2 + 4 \cdot X \cdot Y \cdot Z^2 - 7/2 \cdot X \cdot Y \cdot Z + 2 \cdot X \cdot Y^2 \cdot Z + 2 \cdot X^2 \cdot Y \cdot Z + X^2 \cdot Y^2 = 0$. It is a normal surface.

Following the discussion above, we can derive the following algorithm to obtain a *normal projection curve* (the projection curve of a normal surface) for a given irreducible surface $f(x, y, z) = 0$.

Algorithm 1. *The input is an irreducible polynomial $f(x, y, z)$. The output is a normal projection curve $g(x, y) = 0$ of the surface $f(x, y, z) = 0$.*

1. Represent $f(x, y, z)$ as (5) and check whether the variety $\{c_k(x, y), c_{k-1}(x, y), \cdots, c_0(x, y)\}$ has a real solution. If it has, go to 3.
2. Represent $f(x, y, z)$ as (7) and check whether $L_d(x, y, z)$ has a factor which does not involve variable z. If it does not have this kind of factors, go to 4.
3. Apply the transformation (8), choose a rational number pair (a, b) such that (a, b) is not a point on curve $L_d(x, y, 1) = 0$, where $L_d(x, y, z)$ is the sum of terms whose degrees equal the total degree of $f(x, y, z)$, and compute the corresponding new surface $F(X, Y, Z) = 0$ in the new coordinate system. Still denote $F(X, Y, Z)$ as $f(x, y, z)$.
4. Compute $g(x, y) = Res(f(x, y, z), \frac{\partial f(x,y,z)}{\partial z}, z)$.
5. If $g(x, y)$ is irreducible, return $g(x, y) = 0$. Else, factor it as $g(x, y) = g_1(x, y)^{m_1} \cdot g_2(x, y)^{m_2} \cdot \cdots \cdot g_t(x, y)^{m_t}$, where $g_i(x, y)$ is irreducible. Still denote $g(x, y) = g_1(x, y) \cdot g_2(x, y) \cdot \cdots \cdot g_t(x, y)$ and return $g(x, y) = 0$.

When $f(x,y,z)$ is reducible as (2), the problem is more complex. We can use Algorithm 1 to compute its projection curve, but the computation takes much time. We can check whether each component $f_i(x,y,z)$ is a normal surface. If all components are normal surfaces, we can compute the projection curve of $f(x,y,z)=0$ as follows.

Lemma 2. *Let $\mathcal{S}: f(x,y,z)=0$, where $f(x,y,z)$ is defined by (2), $n \geq 2$. The projection curve of $\mathcal{S}$ is the curve defined by the square-free part of the following polynomial.*

$$g(x,y) = \prod_{1\le i\le j\le n} T_{i,j}(x,y), \tag{9}$$

where $T_{i,i}(x,y) = Res(f_i(x,y,z), \frac{\partial f_i(x,y,z)}{\partial z}, z)$, $T_{i,j}(x,y) = Res(f_i(x,y,z), f_j(x, y,z), z)$, $i,j = 1,\cdots,n, i \neq j$.

Proof. By (3) and the property of resultant [25], we can derive that

$$\begin{aligned}
&Res(f(x,y,z), \frac{\partial f(x,y,z)}{\partial z}, z))\\
&= Res(\prod_{1\le j\le n} f_j(x,y,z), \sum_{i=1}^{n} \frac{f(x,y,z)}{f_i(x,y,z)} \cdot \frac{\partial f_i(x,y,z)}{\partial z}, z)\\
&= \prod_{1\le j\le n} Res(f_j(x,y,z), \sum_{i=1}^{n} \frac{f(x,y,z)}{f_i(x,y,z)} \cdot \frac{\partial f_i(x,y,z)}{\partial z}, z)\\
&= c\cdot \prod_{1\le j\le n} Res(f_j(x,y,z), \frac{f(x,y,z)}{f_j(x,y,z)} \cdot \frac{\partial f_j(x,y,z)}{\partial z}, z)\\
&= c\cdot \prod_{1\le j\le n} (Res(f_j(x,y,z), \frac{\partial f_j(x,y,z)}{\partial z}, z) \cdot \prod_{1\le i\le n, i\neq j} Res(f_i(x,y,z), f_j(x,y,z), z))\\
&= c\cdot \prod_{1\le i\le n} T_{i,i}(x,y) \cdot \prod_{1\le i,j\le n, i\neq j} T_{i,j}(x,y),
\end{aligned}$$

where c is a constant. It is clear that $g(x,y)$ is a factor of $Res(f(x,y,z), \frac{\partial f(x,y,z)}{\partial z}, z)$ and any irreducible factor of $Res(f(x,y,z), \frac{\partial f(x,y,z)}{\partial z}, z)$ is contained in $g(x,y)$. So the projection curve of $\mathcal{S}$ is defined by the square-free polynomial whose components are all the irreducible components of $g(x,y)$. So the lemma holds. □

If there exists any component which is not a normal surface, take a transformation of coordinate system as (8) to insure that all components are normal surfaces in the new coordinate system. Then compute the projection curve of the new surface with the method mentioned above. For any surface $f(x,y,z)=0$, we present the following algorithm to compute its projection curve.

Algorithm 2. *The input is a polynomial $f(x,y,z)$. The output is a square-free polynomial $g(x,y)$, where $\mathcal{C}$: $g(x,y)=0$ is the normal projection curve of $f(x,y,z)$.*

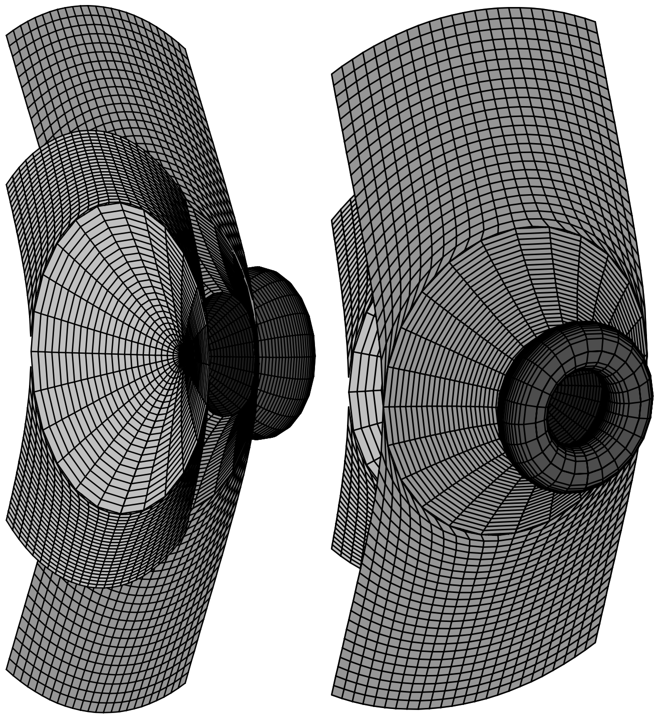

Fig. 1. An irreducible surface

1. Factor $f(x, y, z)$. Suppose $f(x, y, z)$ has a representation as (1), still denote $f(x, y, z)$ as (2).
2. If $n = 1$, compute the projection curve of $\mathcal{S}$ by Algorithm 1 and return it.
3. Else ($n > 1$), do
   (a) Check whether $f_i(x, y, z)$ is a normal surface for all i. If there exists a component which is not a normal surface, it is clear that we can find a transformation as (8), such that each component is a normal surface in the new coordinate system. Still denote the surface as $f(x, y, z) = 0$.
   (b) Compute the projection curve of $f(x, y, z)$ by Lemma 2 and return its square-free part.

Example 1. Let us consider the following surface.

$$f(x, y, z) = (y^2 + z^2 - x^2 + 1/2 \cdot x^3 - 4)^2 - 16 \cdot x^2 + 8 \cdot x^3 = 0. \qquad (10)$$

It is irreducible and normal. As is shown in Fig. 2. We can compute its projection curve by Algorithm 1.

$$Res(f(x, y, z), \frac{\partial f(x, y, z)}{\partial z}, z) = 4096 \cdot g_1(x, y)^4 \cdot g_2(x, y)^2 \cdot g_3(x, y) \cdot g_4(x, y),$$

where $g_1(x, y) = x$, $g_2(x, y) = x - 2$, $g_3(x, y) = 2 \cdot y^2 + 8 \cdot y - 2 \cdot x^2 + x^3 + 8$, $g_4(x, y) = 2 \cdot y^2 - 8 \cdot y - 2 \cdot x^2 + x^3 + 8$. So we can derive its projection curve as follows.

$$g(x, y) = g_1(x, y) \cdot g_2(x, y) \cdot g_3(x, y) \cdot g_4(x, y). \tag{11}$$

3 Projection Curve Topology Determination

In this section, we will present algorithms to determine the topology of the normal projection curve obtained in the preceding section. Such algorithms already exist([14, 16]). But their outputs do not satisfy the requirement of our algorithm for the surface topology determination. Also, our algorithm gives an intrinsic representation for the topology of the given curve.

3.1 Notations

Definition 1. *A point $P_0(x_0, y_0)$ is said to be a singularity of an implicit algebraic curve $\mathcal{C}$: $g(x, y) = 0$ if $g(x_0, y_0) = g_x(x_0, y_0) = g_y(x_0, y_0) = 0$, where $g(x, y)$ is square-free [23].*

Let $\mathcal{C} : g(x, y) = 0$ be the normal projection curve. We will consider the part of $\mathcal{C}$ inside a bounding box

$$\mathcal{B} = \{(x, y)|x_l \leq x \leq x_r, y_b \leq y \leq y_u\}$$

to be determined later. The intersection points of $\mathcal{C}$ and the boundary of $\mathcal{B}$ are called boundary points of $\mathcal{C}$. The part of $\mathcal{C}$ inside $\mathcal{B}$ (including the boundaries of $\mathcal{B}$) is denoted as $\mathcal{C}_{\mathcal{B}}$.

Definition 2. *A plane algebraic curve segment in a finite box is said to be a complete curve segment(CCS) if it is one of the following cases:*

1. *An isolated singularity P_i of $\mathcal{C} : g(x, y) = 0$, denoted as C_{P_i}.*
2. *A continuous curve segment from a singularity or a boundary point to a singularity or a boundary point, such that there is no singularities of $\mathcal{C}$ on the curve segment between the two endpoints. Denote $C^k_{i,j}$ to be the k-th curve segment from the singularity P_i to the singularity P_j; or $C_{i,j}$: the curve segment from the singularity P_i to the boundary point B_j; or $B^{-1}_{i,j}$: the curve segment from the boundary point B_i to the boundary point B_j. Note that $C^k_{i,j} = C^k_{j,i}$, $B^{-1}_{i,j} = B^{-1}_{j,i}$, $C_{i,j} \neq C_{j,i}$.*
3. *A closed continuous curve without singularities, denoted as C_Q, where Q is a point on the curve.*

Definition 3. *A cell of a plane curve in a bounding box is a closed region whose boundaries are CCSes or part of the boundaries of the box.*

Definition 4. *A curve segment sequence of a singularity of a plane curve is an ordered sequence of CCSes originating from the singularity. They are listed from left-up in the counter-clockwise order.*

Definition 5. *The topological representation of a plane algebraic curve within a bounding box consists of the following information.*

A bounding box: $\mathcal{B} = \{(x, y)|x_l \le x \le x_r, y_b \le y \le y_u\}$*;*
Boundary points: $\{B_i(x_{B_i}, y_{B_i})(b_i)[B_{i,j_1}, C_{j,i}(\text{ or } B^{-1}_{i,j}), B_{i,j_2}], i \in I_B\}$*;*
Singularities: $\{P_i(x_{P_i}, y_{P_i})(r_i)[\text{ Curve segment sequence of } P_i], i \in I_S\}$*;*
CCSes: $\{C^k_{i,j}(C_{c_1}, C_{c_2}), c_1, c_2 \in I_C\}$*;*
Cells: $\{C_k[\text{ The ordered boundaries of the cell}], k \in I_C\}$*.*

Here $(x_{B_i}, y_{B_i}), (x_{P_i}, y_{P_i})$ *are coordinates of points* B_i, P_i*, respectively;* I_S, I_B, I_C *are indexes of singularities, boundary points and cells, respectively;* $r_i(b_i)$ *is the discriminate distance (a positive number which will be defined below) of singularity* $P_i(B_i)$*;* C_{c_1}, C_{c_2} *are two cells beside the CCS* $C^k_{i,j}$*, here* $C^k_{i,j}$ *can also be CCS* $B^{-1}_{i,j}, C_{i,j}, C_{P_i}$ *or* C_Q*.*

3.2 Topology Determination

If the normal projection curve $\mathcal{C} : g(x, y) = 0$ of $\mathcal{S} : f(x, y, z) = 0$ is irreducible, then we use the following plane curve topology determination algorithm to compute the topology of $\mathcal{C}$, which is based on the algorithms in [9, 16].

Algorithm 3 (Irreducible algebraic curve topology determination). *The input is an irreducible plane algebraic curve* $\mathcal{C} : g(x, y) = 0$*. The output is the topological representation of* $\mathcal{C}_{\mathcal{B}}$*.*

1. Compute the discriminant $D(y) = \sum_{i=0}^{m} d_i y^i$ of $g(x, y)$ with respect to x and let y_u be a rational number which is larger than $\frac{\max\{|d_0|,\cdots,|d_{m-1}|\}}{|d_m|}$. Then by Cauchy's inequality, all the roots of $D(y) = 0$ are in the interval $(y_b = -y_u, y_u)$.
2. Compute the discriminant $\bar{D}(x)$ of $g(x, y)$ with respect to y and determine its real roots: $\alpha_1 < \ldots < \alpha_{s-1}$. Select two rational numbers x_l and x_r such that $x_l < \alpha_1$ and $x_r > \alpha_{s-1}$ and let $\alpha_0 = x_l$, $\alpha_s = x_r$. Now we have determined the bounding box $\mathcal{B}$. Then all the finite singularities of the curve are in the box.
3. Compute the real intersection points of $g(x, y) = 0$ and the lines $x - \alpha_0 = 0$ and $x - \alpha_s = 0$ in the interval $[y_b, y_u]$ and compute the real intersection points of $g(x, y) = 0$ and the lines $y - y_b = 0$ and $y - y_u = 0$ in the interval (x_l, x_r). The four vertexes of the box are $(x_l, y_u), (x_l, y_b), (x_r, y_u), (x_r, y_b)$. Denote these points in order as $B_i, i \in I_B$. Insure that the four endpoints are not on $\mathcal{C}$. If B_i is between its two adjacent boundary points B_{i_1}, B_{i_2}, then the discriminate distance of B_i is $b_i = \min\{\| B_i B_{i_1} \|, \| B_i B_{i_2} \|\}$. Compute the discriminate distance for each B_i (not including the vertexes).
4. For every $\alpha_i (i = 1, \cdots, s - 1)$, do
   (a) Compute within $\mathcal{B}$ the real roots of $g(\alpha_i, y)$, $\beta_{i,0} < \ldots < \beta_{i,t_i}$.
   (b) For each point $P_{i,j} = (\alpha_i, \beta_{i,j})$, do
      i. Count the numbers of branches of $\mathcal{C}$ in $\mathcal{B}$ to the left and to the right. Denote $P_{i,j}$ as $P_l (l \in I_S)$ in order if $g_x(\alpha_i, \beta_{i,j}) = g_y(\alpha_i, \beta_{i,j}) = 0$, label $r_l = \min\{\alpha_i - \alpha_{i-1}, \alpha_{i+1} - \alpha_i, \beta_{i,j} - \beta_{i,j-1}, \beta_{i,j+1} - \beta_{i,j}\}(\beta_{i,-1} =$

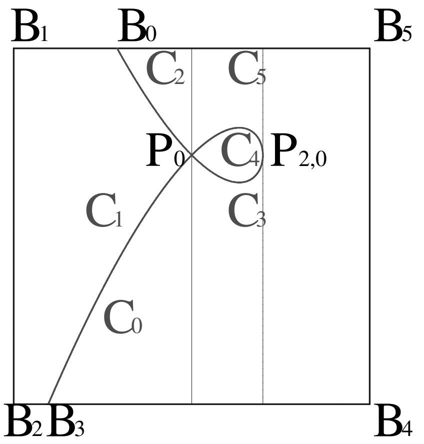

Fig. 2. Determine the topology of an irreducible curve

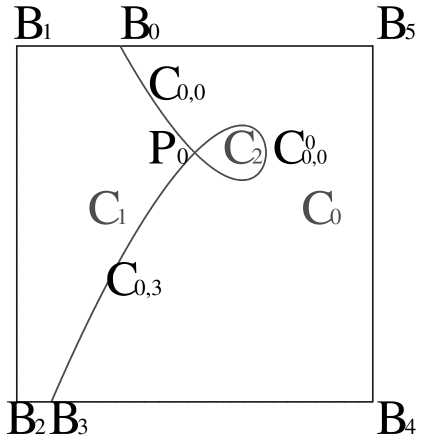

Fig. 3. The final topological figure of an irreducible plane curve

$y_b, \beta_{i,t_i+1} = y_u$) and record an ordered sequence of branches originating from the singularity from left-up to right-up in the counter-clockwise order, transform the branches to corresponding CCSes in the end.

ii. Label each cell in $\mathbf{D}_i = (\alpha_{i-1}, \alpha_i) \times (y_b, y_u)$; combine the two closed regions sharing the line segment $\overline{P_{i-1,j}P_{i-1,j+1}}$ and relabel the closed region. If any closed region is a cell, denote it as $C_k (k \in I_C)$ in order.

iii. Label each curve segment in the interval $\mathbf{D}_i$ and record the cells besides it, combine two curve segments(one in $\mathbf{D}_{i-1}$, the other in $\mathbf{D}_i$) if their unique common point $P_{i,j}$ is non-singular; relabel the new curve segment and record the cell(s) besides it. Now, we can obtain a set of CCSes and the corresponding cell(s) besides them.

5. Return corresponding information.

Example 2. Let us consider a component of the projection curve $\mathcal{C}$ defined by (11). Its equation is $g_3(x, y) = 2 \cdot y^2 - 8 \cdot y - 2 \cdot x^2 + x^3 + 8$ (Fig. 3).

Following Algorithm 3, we can obtain a finite box $\mathcal{B} = [-5, 5] \times [-5, 5]$. The boundary points are B_0, B_3 and four endpoints of the box are B_1, B_2, B_4, B_5. We can obtain $b_0 = 5 + a_1, b_3 = 5 + a_2$ (where a_1, a_2 will be defined in Example 3). And $\alpha_0 = -5, \alpha_1 = 0, \alpha_2 = 2, \alpha_3 = 5$.

Solve $g(\alpha_1, y) = 0$. We obtain one real root $y_{1,0} = 0$. The point $P_0 = (\alpha_1, y_{1,0})$ is a singularity of the curve. There are two branches originating from it on the left side and right side, respectively. The discriminate distance for P_0 is 2 ($\min\{\alpha_1 - \alpha_0, \alpha_2 - \alpha_1, y_u - y_{1,0}, y_{1,0} - y_d\}$). The closed region in $\mathbf{D}_1$ are C_0, C_1, C_2 as shown in Fig. 2. We can check that C_1 is a cell.

Solving $g(\alpha_2, y) = 0$, we obtain one point $P_{2,0}$. It is not a singularity. There are two branches originating from it on its left side and no branches originating from it on its right side. There is no boundary points in $\mathbf{D}_2$. Therefore we can connect the branches in order in $\mathbf{D}_2$. And the two curve segments from P_0 to $P_{2,0}$ compose a CCS of the given curve. Denote it as $C^0_{0,0}$. The closed region in $\mathbf{D}_2$ are C_3, C_4, C_5. C_0 and C_3, C_2 and C_5 share common line segments, and we can combine them as C_0, C_2, respectively. Since there is no boundary point on the boundary of $\mathbf{D}_3$, and there is no branches originating from $P_{2,0}$ in $\mathbf{D}_3$, the curve has no points in $\mathbf{D}_3$. C_0, C_2 share common line segments with $\mathbf{D}_3$ and we can combine them as C_0. In the end, we obtain the decomposition of the curve in Fig. 3.

The outputs about the topological representation of the curve are as follows.
The bounding box: $\mathcal{B} = \{(x,y) | -5 \le x \le 5, -5 \le y \le 5\}$.
Boundary points: $\{B_0(a_1, 5)(5+a_1)[B_{0,1}, C_{0,0}, B_{0,5}]$, $B_1(-5,5)$, $B_2(-5,-5)$, $B_3(a_2,-5)(5+a_2)[B_{3,2}, C_{0,3}, B_{3,4}]$, $B_4(5,-5)$, $B_5(5,5)\}$.
Singularities: $\{P_0(0,2)(2)[C_{0,0}, C_{0,3}, C^0_{0,0}, C^0_{0,0}]\}$.
CCSes: $\{C_{0,0}(C_0, C_1), C_{0,3}(C_0, C_1), C^0_{0,0}(C_0, C_2)\}$.
Cells: $\{C_0[B_{3,4}, B_{4,5}, B_{5,0}, C_{0,0}, C^0_{0,0}, C_{0,3}]$, $C_1[B_{0,1}, B_{1,2}, B_{2,3}, C_{0,3}, C_{0,0}]$, $C_2[C^0_{0,0}]\}$.

The following algorithm is to determine the topology of any square-free algebraic curve.

Algorithm 4 (Plane curve topology determination). *The input is $\mathcal{C}$: $g(x,y) = 0$. The output is the same as Algorithm 3.*

1. If $g(x,y)$ is irreducible, determine the topology of $\mathcal{C}$ by Algorithm 3.
2. Else ($g(x,y)$ is reducible), suppose $g(x,y)$ has a representation as (4).
   (a) Compute a bounding box for each component $g_i(x,y)(i = 1, \cdots, m)$; Compute the intersection points of any two components $g_i(x,y), g_j(x,y)$ $(i,j = 1, \cdots, m, i \ne j)$. Choose a box which contains all boxes and intersection points as the bounding box of $g(x,y) = 0$. Compute the boundary points of $g(x,y) = 0$. Compute the discriminate distance for each boundary points.
   (b) Separate the vertical lines which have a form as $A \cdot x + B = 0$ from $g(x,y)$ if they exist. Of course, we can denote them as $L_t(x,y) = x - c_t = 0$ $(t = 0, \cdots, L)$. Denote all the remainder components of $g(x,y)$ as $g_0(x,y)$. Suppose it is $g_0(x,y) = g_1(x,y) \cdots g_s(x,y)$, where $s = m - L$.
   (c) Solve $Res(g_i(x,y), \frac{\partial g_i(x,y)}{\partial y}, y) = 0$ and $Res(g_i(x,y), g_j(x,y), y) = 0$ for all $i,j = 0, \cdots, m_L (i \ne j)$. Put their roots and $c_t (t = 0, \cdots, L)$ together and rewrite them as $\alpha_k (k = 1, \cdots, l-1, \alpha_k < \alpha_{k+1})$. Let $\alpha_0 = x_l, \alpha_l = x_r$ be rational numbers such that $\alpha_0 \le \alpha_1, \alpha_l \ge \alpha_{l-1}$.
   (d) For every α_k, to $g_0(x,y)$, we can do the same work as Algorithm 3 in step 4. Note that when $\alpha_k = c_t (t = 0, \cdots, L)$, all the real intersection points of $g_0(x,y) = 0$ and the line $x - \alpha_k = 0$ in the interval (y_b, y_u), denoted as $P_{k,j} (j = 0, \cdots, t_k)$, are singularities of $g(x,y) = 0$, and line segments $\overline{P_{k,j} P_{k,j+1}} (j = 0, \cdots, t_k - 1)$, $\overline{B_{k,0} P_{k,0}}$, $\overline{P_{k,t_k} B_{k,1}}$ are CCSes of $g(x,y) = 0$, where $B_{k,0}, B_{k,1}$ are intersection points of the line $x - \alpha_k = 0$ and the boundary of the box. We obtain the topology information of $g(x,y) = 0$ in the end.
3. Return the corresponding topological information of $\mathcal{C}$.

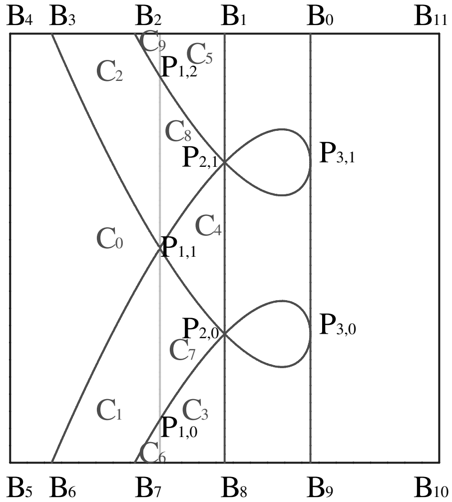

Fig. 4. Topology determination of a curve

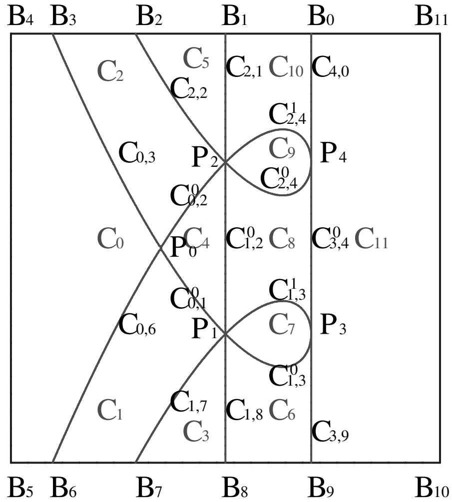

Fig. 5. The topology of a curve

Remark. If $\mathcal{C}$ has no critical points (the points which satisfy $g(x,y){=}g_y(x,y){=}0$) on $\mathcal{C}$, we can solve $g(0,y)=0$. If $g(0,y)=0$ has no real roots, solve $f(0,0,z)=0$. The surface $\mathcal{S}$ has no real parts if it has no real roots. And $\mathcal{S}$ is topologically equivalent to n parallel planes if the equation $f(0,0,z)=0$ has n real roots. If $g(0,y)=0$ has real roots $y_0,\cdots,y_m$, let the finite box be $\mathcal{B}=[-1,1]\times[y_0-1,y_m+1]$, we can obtain the boundary points $B_i(i=0,\cdots,2m+1)$ and CCSes $B^{-1}_{0,2m+1},B^{-1}_{1,2m},\cdots,B^{-1}_{m,m+1}$ of $\mathcal{C}$.

Example 3. Consider the projection curve defined by (11) as an example of a reducible curve.

Following Algorithm 4, here $g_1(x,y)$ and $g_2(x,y)$ are vertical lines, we remove them from $g(x,y)$.

$Res(g_3(x,y),\frac{\partial g_3(x,y)}{\partial y},y) = 16\cdot x^3-32\cdot x^2=0$, whose real roots are 2, 0. $Res(g_4(x,y),\frac{\partial g_4(x,y)}{\partial y},y) = 16\cdot x^3-32\cdot x^2=0$, whose real roots are also 2, 0. $Res(g_3(x,y),g_4(x,y),y) = -2\cdot x^2+x^3+8=0$, whose real root is $x_{3,4} = -\frac{1}{3}\cdot\sqrt[3]{100+12\cdot\sqrt{69}}-\frac{4}{3\cdot\sqrt[3]{100+12\cdot\sqrt{69}}}+\frac{2}{3}$. So we have $\alpha_0=-5$, $\alpha_1=x_{3,4}$, $\alpha_2=0$, $\alpha_3=2$, $\alpha_4=5$.

Then we can obtain the bounding box $\mathcal{B}=[-5,5]\times[-5,5]$ and the boundary points: $B_0(2,5)$, $B_1(0,5)$, $B_2(a_1,5)$, $B_3(a_2,5)$, $B_6(a_2,-5)$, $B_7(a_1,-5)$, $B_8(0,-5)$, $B_9(2,-5)$. Add the endpoints $B_4(-5,5)$, $B_5(-5,-5)$, $B_{10}(5,-5)$, $B_{11}(5,5)$ in the boundary point list. And $b_0=2,b_1=2,b_2=a_1-a_2,b_3=5+a_2,b_6=5+a_2,b_7=a_1-a_2,b_8=2,b_9=2$. Here $a_1=-\frac{\sqrt[3]{235+9\cdot\sqrt{681}}}{3}-\frac{4}{3\cdot\sqrt[3]{235+9\cdot\sqrt{681}}}+\frac{2}{3}$, $a_2=-\frac{\sqrt[3]{1315+21\cdot\sqrt{3921}}}{3}-\frac{4}{3\cdot\sqrt[3]{1315+21\cdot\sqrt{3921}}}+\frac{2}{3}$.

Solving $g_3(\alpha_1, y) \cdot g_4(\alpha_1, y) = 0$, we can get $y = -4, 0, 4$. They correspond to $P_{1,0}, P_{1,1}, P_{1,2}$ in Fig. 4. We can easily find that only point $P_{1,1}$ is a singularity. We rename it as P_0. Its corresponding positive number is $\min\{4-0, 0-(-4), \alpha_2 - \alpha_1, \alpha_1 - \alpha_0\} = -\alpha_1$. We can show that the curve segments in $\mathbf{D}_1 = [\alpha_0, \alpha_1] \times [y_b, y_u]$ are $\widetilde{P_{1,0}B_7}, \widetilde{P_{1,1}B_6}, \widetilde{P_{1,1}B_3}, \widetilde{P_{1,2}B_2}$. And $C_{0,6} = \widetilde{P_{1,1}B_6}, C_{0,3} = \widetilde{P_{1,1}B_3}$ are CCSes of the given curve. The regions in the interval $\mathbf{D}_1$ are C_6, C_1, C_0, C_2, C_9. Their boundaries are as shown in Fig. 4. And only C_0 is a cell. Its boundaries are $C_{0,3}, B_{3,4}, B_{4,5}, B_{5,6}, C_{0,6}$.

Solve $g_3(\alpha_2, y) \cdot g_4(\alpha_2, y) = 0$. Its real roots are $-2, 2$. We can find two singularities $P_{2,0}, P_{2,1}$, whose corresponding positive numbers are $-\alpha_1$. We rename them as P_1, P_2. In the interval $\mathbf{D}_2 = [\alpha_1, \alpha_2] \times [y_b, y_u]$, the curve segments and regions are shown in Fig. 4. We can combine curve segment $\widetilde{P_{1,0}B_7}$ in $\mathbf{D}_1$ and curve segment $\widetilde{P_{2,0}P_{1,0}}$ in $\mathbf{D}_2$ as a CCS $C_{1,7} = \widetilde{P_{2,0}B_7}$ and combine regions C_3, C_6 as C_3. Combine C_1, C_7 as C_1. Combine C_2, C_8 as C_2, and combine C_5, C_9 as C_5. Since $x - \alpha_2$ is $g_1(x, y)$, the line segments $\overline{P_{2,0}B_6}, \overline{P_{2,0}P_{2,1}}, \overline{P_{2,1}B_1}$ are CCSes $C_{1,6}, C^0_{1,2}, C_{2,1}$ of $g(x, y) = 0$. C_4 is a cell. The real roots of $g_3(\alpha_3, y) \cdot g_4(\alpha_3, y) = 0$ are -2, 2. And $x - \alpha_3$ is $g_2(x, y)$. Following Algorithm 4, we can derive the topology of $g(x, y) = 0$ as Fig. 5. The positive numbers of the five singularities are $-\alpha_1, -\alpha_1, -\alpha_1$, 2 and 2 respectively.

The output are as follows.

The bounding box: $\mathcal{B} = [-5, 5] \times [-5, 5]$.

Boundary points: $\{B_0(2, 5)(2)[B_{0,1}, C_{4,0}, B_{0,11}]$, $B_1(0, 5)(2)[B_{1,2}, C_{2,1}, B_{1,0}]$, $B_2(a_1, 5)(a_1 - a_2)[B_{2,3}, C_{2,2}, B_{2,1}]$, $B_3(a_2, 5)(5 + a_2)[B_{3,4}, C_{0,3}, B_{3,2}]$, $B_4(-5, 5)$, $B_5(-5, -5)$, $B_6(a_2, -5)(5 + a_2)[B_{6,5}, B_{6,7}, C_{0,6}]$, $B_7(a_1, -5)(a_1 - a_2)[B_{7,6}, B_{7,8}, C_{1,7}]$, $B_8(0, -5)(2)[B_{8,7}, B_{8,9}, C_{3,9}]$, $B_9(2, -5)(2)[B_{8,9}, B_{8,10}, C_{3,9}]$, $B_{10}(5, -5)$, $B_{11}(5, 5)\}$. B_4, B_5, B_{10}, B_{11} are vertexes of the box.

Singularities: $\{P_0(\alpha_1, 0)(-\alpha_1)[C_{0,3}, C_{0,6}, C^0_{0,1}, C^0_{0,2}]$, $P_1(0, -2)(-\alpha_1)[C^0_{0,1}, C_{1,7}, C_{1,8}, C^0_{1,3}, C^1_{1,3}, C^0_{1,2}]$, $P_2(0, 2)(-\alpha_1)[C_{2,2}, C^0_{0,2}, C^0_{1,2}, C^0_{2,4}, C^1_{2,4}, C_{2,1}]$, $P_3(2, -2)(2)[C^1_{1,3}, C^0_{1,3}, C_{3,9}, C^0_{3,4}]$, $P_4(2, 2)(2)[C^1_{2,4}, C^0_{2,4}, C^0_{3,4}, C_{4,0}]\}$.

CCSes: $\{C_{0,6}(C_1, C_0)$, $C_{0,3}(C_0, C_2)$, $C_{1,7}(C_1, C_3)$, $C^0_{0,2}(C_2, C_4)$, $C^0_{0,1}(C_4, C_1)$, $C_{2,2}(C_2, C_5)$, $C_{1,8}(C_3, C_6)$, $C^0_{1,2}(C_4, C_8)$, $C_{2,1}(C_5, C_{10})$, $C^0_{1,3}(C_6, C_7)$, $C^1_{1,3}(C_7, C_8)$, $C^0_{2,4}(C_8, C_9)$, $C^1_{2,4}(C_9, C_{10})$, $C_{3,9}(C_6, C_{11})$, $C^0_{3,4}(C_8, C_{11})$, $C_{4,0}(C_{10}, C_{11})\}$.

Cells: $\{C_0[B_{3,4}, B_{4,5}, B_{5,6}, C_{0,6}, C_{0,3}]$, $C_1[B_{6,7}, C_{1,7}, C^0_{0,1}, C_{0,6}]$, $C_2[C_{0,3}, C^0_{0,2}, C_{2,2}, B_{2,3}]$, $C_3[B_{7,8}, C_{1,8}, C_{1,7}]$, $C_4[C^0_{0,2}, C^0_{0,1}, C^0_{1,2}]$, $C_5[C_{2,2}, C_{2,1}, B_{1,2}]$, $C_6[B_{8,9}, C_{3,9}, C^0_{1,3}, C_{1,8}]$, $C_7[C^0_{1,3}, C^1_{1,3}]$, $C_8[C^1_{1,3}, C^0_{3,4}, C^0_{2,4}, C^0_{1,2}]$, $C_9[C^0_{2,4}, C^1_{2,4}]$, $C_{10}[C^1_{2,4}, C_{4,0}, B_{0,1}, C_{2,1}]$, $C_{11}[B_{9,10}, B_{10,11}, B_{11,0}, C_{4,0}, C^0_{3,4}, C_{3,9}]\}$.

4 Space Curve Segmentation and Surface Patch Composition

In this section, we will determine the position of each space curve segment and each surface patch of $\mathcal{S}$. The algorithm works as follows:

First, we need to determine the points of $\mathcal{S}$ on each line lifted from a boundary point $B_i(\bar{x}_i, \bar{y}_i)(i \in I_B)$ or a singularity $P_i(x_i, y_i)(i \in I_S)$ of $\mathcal{C}$: $g(x, y) = 0$.

These points are the endpoints of the space curve segments. Second, we need to determine how many space curve segments originating from each endpoint. Then we can determine all space curve segments of $\mathcal{S}$. Third, we need to compute the number of surface patches originating from each space curve segment. Finally, we can determine the surface patches in each region from bottom to top by pointing out their boundaries.

4.1 Notations

In order to describe our algorithm clearly, we present the following definitions. Let us assume that we have already obtained a topological representation for the projection curve of $\mathcal{S}$.

Definition 6. *A complete cylindrical patch (CCP) $SC_{i,j}^k$ is a cylindrical patch lifted from a CCS $C_{i,j}^k$ obtained in section 3. Then $SC_{i,j}^k = C_{i,j}^k \times [-N, N]$, where N is a positive number that will be defined later. $SC_{i,j}$, $SB_{i,j}$, $SB_{i,j}^{-1}$, SC_{P_i}, SC_{Q_i} can be defined similarly.*

Definition 7. *A cell body is a body lifted from a cell obtained in section 3. We can denote it as CC_i, where C_i is a cell of the projection curve. Two cell bodies share a CCP as a boundary. When a CCS is an isolated singularity, there is only one cell body beside the CCP corresponding to the CCS.*

Definition 8. *A complete space curve segment (CSCS) of $\mathcal{S}$: $f(x, y, z) = 0$ is a space curve segment which is an intersection of a CCP and $\mathcal{S}$. We denote it as $C_{i,j}^{k,l}(C_{i,j}^{-1,l}, B_{i,j}^l, B_{i,j}^{-1,l}, V_{i,l}, C_{Q_i}^l)$ if its corresponding CCS in the plane is $C_{i,j}^k(C_{i,j}, B_{i,j}, B_{i,j}^{-1}, C_{P_i}, C_{Q_i})$, where l is an index starting from bottom to up.*

Definition 9. *A complete surface patch (CSP) of $\mathcal{S}$: $f(x, y, z) = 0$ is a surface patch which is part of $\mathcal{S}$, and its boundaries are several CSCSes. We can denote it as S_i^l if it is the l-th surface patch in cell body CC_i from bottom to up.*

Definition 10. *A critical curve of a surface is a space curve satisfying $f(x, y, z) = f_z(x, y, z) = 0$, where (x, y, z) is any point on the curve.*

Definition 11. *A singular curve of a surface is a space curve, which satisfies $f(x, y, z) = f_x(x, y, z) = f_y(x, y, z) = f_z(x, y, z) = 0$ for any point (x, y, z) on the curve.*

Let $\mathcal{S}$ be the surface, $\mathcal{C}$ the projection curve of $\mathcal{S}$, $\mathcal{B}$ the bounding box of $\mathcal{C}$, $\mathcal{C}_\mathcal{B}$ part of the projection curve within $\mathcal{B}$, $V_{i,j}$(or $V_{i,j}^0$)$(j = 0, \cdots, t_i)$ the points of $\mathcal{S}$ on the line lifted from a singularity P_i (or a boundary point B_i)

Definition 12. *The topological information of a surface include the following information:*

The point lists: $\{V_{i,j}$ (or $V_{i,j}^0)(j = 0, \cdots, t_i, i \in I_S$ (or $i \in I_B))\}$, which are corresponding to certain singularities (or boundary points) of $\mathcal{C}_\mathcal{B}$. For example,

$B_i[V_{i,0}^0, \cdots, V_{i,t_i}^0], P_j[V_{j,0}, \cdots, V_{j,t_j}]$ *are point lists corresponding to a boundary point and a singularity.*

The CSCS lists: {*The CSCS list corresponding to each line segment* $B_{i,j}$ *and each CCS of* $\mathcal{C}_{\mathcal{B}}$}. *For instance,* $B_{i,j}[B_{i,j}^0(V_{i,0}^0, V_{j,0}^0), \cdots, B_{i,j}^p(V_{i,i_1}^0, V_{j,j_1}^0)]$, $C_{i,j}^k[C_{i,j}^{k,0}(V_{i,0}, V_{j,0}), \cdots, C_{i,j}^{k,l}(V_{i,i_2}, V_{j,j_2})]$, *where the two points for each CSCS are their endpoints.*

The CSP lists: {*The CSPs (including their boundary CSCSes) corresponding to each cell*}. *For instance,* $C_i(n)\{S_i^0$[*The ordered boundary CSCSes of the surface patch*]$, \cdots, S_i^{n-1}$[*The ordered boundary CSCSes of the surface patch*]$\}$.

The CSCSes lists form a curvilinear wireframe of the surface. The boundaries of the CSPs in the CSP lists are the CSCSes in the CSCS lists. So they are determined by the curvilinear wireframe.

4.2 Basic Theorems and Algorithms

Theorem 1. *Let* $\mathcal{S} : f(x, y, z) = 0$ *be a normal surface,* $\mathcal{C} : g(x, y) = 0$ *the projection curve computed by Algorithm 2,* $\mathcal{B}$ *the finite box obtained in Algorithm 4 for* $g(x, y)$. *For any point* (x_0, y_0) *inside* $\mathcal{B}$, *the real roots of* $f(x_0, y_0, z) = 0$ *are finite, that is, there exists a positive number* N *such that for any real root* z_0 *of* $f(x_0, y_0, z) = 0$, $-N < z_0 < N$.

Proof. It is clear that there is no surface patch of $\mathcal{S}$ which is approaching to infinity inside $\mathbf{B}$. This is guaranteed by conditions *C1* and *C2*. So the theorem holds. □

In fact, we can compute the number N. We can compute the maximum and minimum of the z-coordinate inside $\mathcal{B}$ (including its boundary). We use equation (10) as an example. As we know, $\mathcal{B} = [x_l, x_r] \times [y_b, y_f] = [-5, 5] \times [-5, 5]$. Compute the maximum and minimum in z-direction of $f(x, y, z) = 0$ for $(x, y) \in [-5, 5] \times [-5, 5]$. We can use Wu's finite kernel method([24]). The number with the largest absolute value is $2 + 5/2 \cdot \sqrt{14}$. Choose N to be a rational number which is larger than the absolute value of the computed number. Here we can choose $N = 12$.

Theorem 2. *All the notations are with the same meaning as Theorem 1.* $\mathcal{S}$ *and the part of* $\mathcal{S}$ *in the cube* $\mathbf{B} = \mathcal{B} \times [-N, N]$ *have the same topology.*

Proof. Denote the part of $\mathcal{S}$ inside the cube $\mathbf{B}$ as $\mathcal{S}_{\mathbf{B}}$. Let $\mathbf{B}_1$ be a cube containing $\mathbf{B}$ strictly. We will show that the part of $\mathcal{S}$ inside $\mathbf{B}$ and the part of $\mathcal{S}$ inside $\mathbf{B}_1$ have the same topology. This is because, the part of $\mathcal{S}$ between $\mathbf{B}$ and $\mathbf{B}_1$ can be seen as surfaces (or lines) lifted from the intersection of $\mathcal{S}$ and $\mathbf{B}$ without adding new intersections. Topologically, they are the same with cylindrical surfaces. Hence, adding these surfaces does not change the topology of $\mathcal{S}_{\mathbf{B}}$. So the theorem holds. □

We further assume that there is no singularities on the CCP lifted from any CCS of $\mathcal{C}$. In fact, we can find a new coordinate system such that the isolated singularities and the intersection points of the critical curves of the surface are projected onto the singularities of the projection curve.

Definition 13. *Given a univariate function $P(x)$, let $P_0(x) = P(x), P_1(x) = P'(x)$ and define the Sturm functions by*

$$P_i(x) = -(P_{i-2}(x) - P_{i-1}(x)[\frac{P_{i-2}(x)}{P_{i-1}(x)}]),$$

where $[\frac{P_{i-2}(x)}{P_{i-1}(x)}]$ is a polynomial quotient. The chain is terminated when $P_n(x)$ is a constant. Then $P_0(x)$, $P_1(x)$, $\cdots$, $P_n(x)$ is the Sturm functions (more details can be found in [17, 26]) of $P(x)$.

Definition 14. *Sign-changing number of Sturm functions of $P(x)$ at point $x = a$ is the number of sign changes on the Sturm functions of $P(x)$ evaluated at point $x = a$. That is, the number of sign changes of $P_0(a), P_1(a), \cdots, P_n(a)$.*

The following algorithm is to isolate the real roots of a polynomial $T(x) \in \mathbb{R}[x]$. The difference between the algorithm and general algorithm is that the isolated points of our algorithm is not a root of $T(x)$. For more detail, one can see [17, 18, 26].

Algorithm 5. *(Real Root-Isolating) The input are Sturm functions of a polynomial $T(x)$ and an interval (a, b) (where a, b are rational numbers, $T(a) \neq 0, T(b) \neq 0$, $T(x) \in R[x]$). The output is a series of ordered rational numbers in (a, b), such that there is a real root of $T(x) = 0$ between each pair of adjacent numbers.*

1. Compute the sign-changing numbers $V(a), V(b)$ of the Sturm functions of $T(x)$ at $x = a, x = b$, respectively. $V(a) - V(b)$ is the number of real roots between (a, b) by Sturm theorem. Let the rational number set be $N_s := \{a, b\}$. If $V(a) - V(b) = 0$, return $\emptyset$. If $V(a) - V(b) = 1$, return N_s.
2. When $V(a) - V(b) > 1$, if $T(\frac{a+b}{2}) \neq 0$, let $c = \frac{a+b}{2}$, else choose another rational number c near $\frac{a+b}{2}$ in (a, b) insuring that $T(c) \neq 0$.
   (a) If $V(a)-V(c) > 1$ and $V(c)-V(b) > 1$, $N_s := N_s \bigcup \{c\}$; let i. $a = a, b = c$, respectively, ii. $a = c, b = b$, respectively, go to 2.
   (b) Else if $V(a) - V(c) = 1$ and $V(c) - V(b) > 1$, $N_s := N_s \bigcup \{c\}$; let $a = c, b = b$, respectively, go to 2.
   (c) Else if $V(a) - V(c) > 1$ and $V(c) - V(b) = 1$, $N_s := N_s \bigcup \{c\}$; let $a = a, b = c$, respectively, go to 2.
   (d) Else if $V(a) - V(c) = 0$ and $V(c) - V(b) > 1$, let $a = c, b = b$, go to 2.
   (e) Else if $V(a) - V(c) > 1$ and $V(c) - V(b) = 0$, let $a = a, b = c$, go to 2.
3. Return the ordered rational numbers N_s.

Example 4. Continuing from Example 3, we want to isolate the points on the line lifted from $P_3(2, -2)$ in Fig. 5. Here the input are $f(2, -2, z) = z^4$ and $(a, b) = (-12, 12)$. The equation has only one real root $z = 0$. We can obtain its isolated points $W_{3,0}(2, -2, -12), W_{3,1}(2, -2, 12)$. There is a point $V_{3,0}$ of the surface on the line segment $\widetilde{W_{3,0}W_{3,1}}$ between $W_{3,0}, W_{3,1}$.

Given a point, a positive number and a plane curve (the point can be on the curve or not on the curve), the following algorithm is to find the circle whose center is the point, which is the minimal circle among the circles tangent to the curve.

Algorithm 6. *The inputs are a plane algebraic curve* $T(x, y) = 0$*, a positive number* r *and a point* $P_0(x_0, y_0)$*. The output is a positive number which is equal to half of the minimal of the extremum distance* r_{min} *from* P_0 *to the curve* $T(x, y) = 0$ *and* r*.*

1. Let $L(x, y, \lambda) = (x - x_0)^2 + (y - y_0)^2 + \lambda T(x, y)$.
2. Eliminating x and λ from $\{2(x-x_0)+\lambda T_x(x, y), 2(y-y_0)+\lambda T_y(x, y), T(x, y)\}$ in the order $\{\lambda \succ x \succ y\}$, we can obtained a univariate polynomial $P(y)$.
3. Solve $P(y)$ in the interval (y_0-r, y_0+r). If there is no real root in the interval, return $r/2$; Else, get corresponding $x_{i,j}$ for each real root y_i in the interval $(x_0 - r, x_0 + r)$. If there is no real roots in the interval, return $r/2$; else, let $R = min_{i,j} \sqrt{(x - x_{i,j})^2 + (y_0 - y_i)}$, if $R \le r$, return $R/2$, else, return $r/2$.

Remark. The step 2 of this algorithm is based on a method of Wu to find extremal values. One can find more details in [24].

Example 5. Continuing from Example 4, let $g(x, y)$ be the curve defined by (11), $P_3 = (2, -2)$. The input is $g(x, y)$ and the positive number 2 corresponding to P_3. With this algorithm, we can find that the minimal positive extremum from P_3 to $g(x, y)$ is 2. So the output is 1.

4.3 Compute the Space Curve Segments

To each singularity $P_i(x_i, y_i)(i \in I_S)$ (or boundary point $B_i(x_i, y_i)(i \in I_B)$)of $g(x, y) = 0$, there is a sequence of CCSes $C^{k_1}_{i,j_1}, \cdots, C^{k_t}_{i,j_t}$ originating from it. Here the CCSes in the sequence can also be $C_{i,j}$ or boundary line segments $B_{i,j}$ (for B_i only). Lifting them up, we can obtain a sequence of CCPs $SC^{k_1}_{i,j_1}, \cdots, SC^{k_t}_{i,j_t}$. The point $P_i(x_i, y_i)$ corresponds to a vertical line $\{x = x_i, y = y_i\}$. There are some points $V_{i,j}(j = 0, \cdots, s_i)$ of $\mathcal{S}$ on the line. There are some CSCSes $C^{k_l,m}_{i,j_l}(m = 0, \cdots, t_{i,j,k})$ on each CCP $SC^{k_l}_{i,j_l}(l = 1, 2, \cdots, t)$ originating from $V_{i,j}$. We need to determine the CSCSes originating from each $V_{i,j}$ on each CCP. The following algorithm is to do this.

Algorithm 7. *The inputs are a real algebraic surface* $\mathcal{S} : f(x, y, z) = 0$*, its projection curve* $\mathcal{C} : g(x, y) = 0$*, a point* $P_i(x_i, y_i)$ *on* $\mathcal{C}$*, the discriminate distance* r_i *of* P_i *and a sequence of CCSes* $\{C^{k_1}_{i,j_1}, \cdots, C^{k_t}_{i,j_t}\}$ *originating from* P_i*. The outputs are a sequence of points* $V_{i,j}(j = 0, \cdots, s_i)$ *of* $\mathcal{S}$ *on the line lifted from* P_i*, a set of sequences of CSCSes* $\{C^{k_l,m}_{i,j_l}, m = 0, \cdots, t_{i,j,k}\}$ *for each* $C^{k_l}_{i,j_l}$*. Note that we only know one endpoint of the CSCSes. But we can compute the corresponding information for the other endpoint by this algorithm, then the CSCS is determined.*

1. Isolate the real roots of $f(x_i, y_i, z) = 0$ by Algorithm 5 and obtain the isolating values $z_{i,0}, z_{i,1}, \cdots, z_{i,s_i}$. Denote $(x_i, y_i, z_{i,j})$ as $W_{i,j}$. There exists a point of $\mathcal{S}$, $V_{i,j}$, which is on the line $\{x = x_i, y = y_i\}$ and between points $W_{i,j}$ and $W_{i,j+1}$. For an instance, please see Fig. 6.
2. From $r_i, P_i, g(x, y) = 0$, we can obtain a positive number R_i by Algorithm 6. It is clear that the number of intersection points of the circle $(x - x_i)^2 + (y - y_i)^2 = r^2 (0 < r \leq R_i)$ and $\mathcal{C}$ is equal to the number of the CCSes in the input sequence.
3. In plane $z = z_{i,j} (j = 0, 1, \cdots, s_i)$, from $r_i, P_i, f(x, y, z_{i,j}) = 0$, we can obtain a positive number $r_{i,j}$ by Algorithm 6. Still denote the minimal among $\{R_i, r_{i,0}, \cdots, r_{i,s_i}\}$ as $r_i (r_i \leq R_i)$.
4. Compute the real intersection points of the equations $\{(x-x_i)^2+(y-y_i)^2 = r_i^2, g(x, y) = 0\}$. We can determine a point $P_{i,j_l}^{k_l}$ on $C_{i,j_l}^{k_l}, l = 1, \cdots, t$. Denote them as $\{P_{i,j_1}^{k_1}, P_{i,j_2}^{k_2}, \cdots, P_{i,j_t}^{k_t}\}$.
5. For each $P_{i,j_l}^{k_l}(x_{i,j_l,k_l}, y_{i,j_l,k_l})(l = 1, \cdots, t)$, compute the number of real roots of $f(x_{i,j_l,k_l}, y_{i,j_l,k_l}, z) = 0$ in the interval $(z_{i,j}, z_{i,j+1})(j = 0, 1, \cdots, s_i - 1)$. It is the number of CSCSes originating from $V_{i,j}$ on the CCP $SC_{i,j_l}^{k_l}$. So we can determine the CSCSes on each CCP: one of their two endpoints is on the line lifted from P_i. Their order on the CCPs is from bottom to top. Denote them as $C_{i,j_l}^{k_l,m}$. If there does not exist a real root in the interval $(z_{i,0}, z_{i,s_i})$, delete the CCS from the topology information and combine the cells divided by it.
6. Return the corresponding information.

Remark. In Algorithm 7, if the singularity is an isolated point of $\mathcal{C}$, we need not to compute it by this algorithm. If the input sequence of CCSes may include CCS like $C_{i,j}$ (the endpoints are a singularity and a boundary point), the algorithm is also valid. The CSCSes on the CCP $SC_{i,j}$ are determined by computing P_i with this algorithm. For a boundary point, the algorithm is also valid. Since the numbers of CSCSes originating from the points of $\mathcal{S}$ on the line lifted from P_i, P_j are the same, we can determine all the CSCSes on $SC_{i,j}^k$ after we compute P_i, P_j for the surface with this algorithm.

Theorem 3. *Algorithm 7 provides the correct output.*

Proof. We will prove that we can obtain what we want from Algorithm 7. From step 2 and step 3, it is clear that there is no other critical curves of $\mathcal{S}$ in the cylindrical body $\mathbf{D} = \{(x, y, z) | (x - x_i)^2 + (y - y_i)^2 \leq r_i^2, -N < z < N\}$, which can be projected onto the XY-plane except $\{C_{i,j_1}^{k_1}, \cdots, C_{i,j_t}^{k_t}\}$. And the discs $\{(x, y, z_{i,j}) | (x - x_i)^2 + (y - y_i)^2 \leq r_i^2\} (j = 0, \cdots, s_i)$ isolate the CSCSes originating from each $Vi, j (j = 0, \cdots, s_i - 1)$ on each CCP $SC_{i,j_l}^{k_l}(l = 1, \cdots, t)$ in $\mathbf{D}$. Then we will prove that the number of CSCSes originating from $V_{i,j}(j = 0, \cdots, s_i - 1)$ on the CCP $SC_{i,j_l}^{k_l}(l = 1, \cdots, t)$ is equal to the number of real roots of equation $f(x_{i,j_l,k_l}, y_{i,j_l,k_l}, z) = 0$ in the interval $(z_{i,j}, z_{i,j+1})(j = 0, 1, \cdots, s_i - 1)$. Since the total number of CSCSes originating from $V_{i,j}$ for each j is equal to the number of CSCSes originating from the points of $\mathcal{S}$ on the line lifted

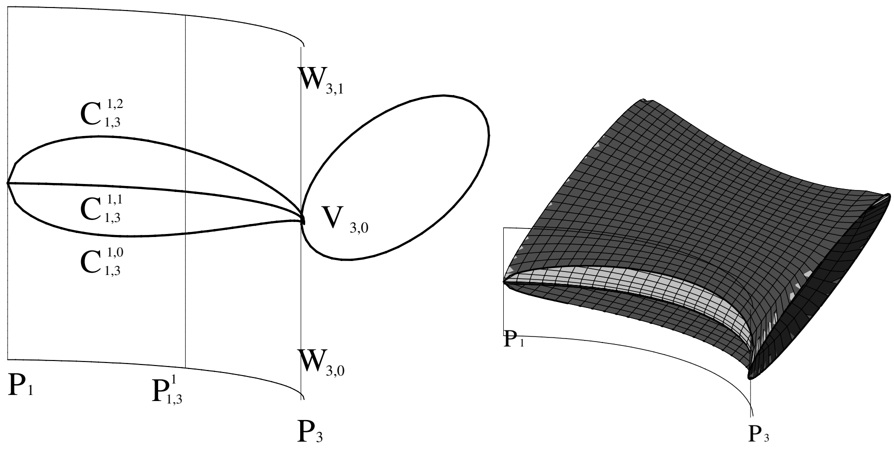

Fig. 6. Compute P_3 with Algorithm 7

form P_{j_l}, each CSCS originating from $V_{i,j}$ should connect one point on the line lifted from P_{j_l}. So if the conclusion is not right, there must exist a point on one CSCS originating from $V_{i,j}$ in $\mathbf{D}$, which is also a point on a critical curve of $\mathcal{S}$. Projecting the critical curve onto the XY-plane, it must share a singular point with CCS $C_{i,j_l}^{k_l}$. This is in contradiction with the given condition. So the algorithm is valid. $\square$

Example 6. Continuing from Example 5, let us consider P_3 with this algorithm. The inputs are $f(x,y,z)=0, g(x,y)=0, P_3(2,-2)(2)[C_{1,3}^1, C_{1,3}^0, C_{3,9}, C_{3,4}^0]$. We have known that there is only one real point $V_{3,0}$ of the surface on the line lifted from P_3 and R_3 equals 1. Its isolated points are $W_{3,0}(2,-2,-12), W_{3,1}(2,-2,12)$ (Fig. 6). In step 3, we can obtain 1 by Algorithm 6 if the input is 2 and $f(x,y,-12)$(or $f(x,y,12)$). So $r_3=1$. In order to illustrate our method simply, we choose the discriminate distance as a number less than 1: $\sqrt{13}/4$. Solving the equations $\{(x-2)^2+(y+2)^2-13/16=0, g(x,y)=0\}$, we can obtain the following points: $(\frac{3}{2},-\frac{5}{4})$, $(\frac{3}{2},-\frac{11}{4})$, $(2,-2-\frac{\sqrt{13}}{4})$ and $(2,-2+\frac{\sqrt{13}}{4})$. Comparing their coordinates and the curve segment sequence of P_3, we can find that they correspond to the CCSes $C_{1,3}^1, C_{1,3}^0, C_{3,9}, C_{3,4}^0$, respectively. Denote them as $P_{1,3}^1, P_{1,3}^0, P_{3,9}^{-1}, P_{3,4}^0$. Then compute the number of real roots of $f(\frac{3}{2},-\frac{5}{4},z)=0$, $f(\frac{3}{2},-\frac{11}{4},z)=0$, $f(2,-2-\frac{\sqrt{13}}{4},z)=0$ and $f(2,-2+\frac{\sqrt{13}}{4},z)=0$ in the interval $(-12,12)$. They are 3, 1, 0, 2, respectively. This is shown in the left part of Fig. 6. That means the numbers of the CSCSes originating from $V_{3,0}$ on the CCPs $SC_{1,3}^1, SC_{1,3}^0, SC_{3,9}, SC_{3,4}^0$ are 3, 1, 0, 2, respectively. There is no real points of the surface on the line lifted from the point $P_{3,9}^{-1}$ which is on the CCS $C_{3,9}$. So we need to delete the boundary point B_9, CCS $C_{3,9}$ from the topology information of $\mathcal{C}: g(x,y)=0$ and combine the cells C_6 and C_{11} as C_6. $V_{3,0}$ is one endpoint of the CSCSes $C_{1,3}^{1,0}, C_{1,3}^{1,1}, C_{1,3}^{1,2}, C_{1,3}^{0,0}, C_{3,4}^{0,0}, C_{3,4}^{0,1}$. As is shown in the right part of Fig. 6.

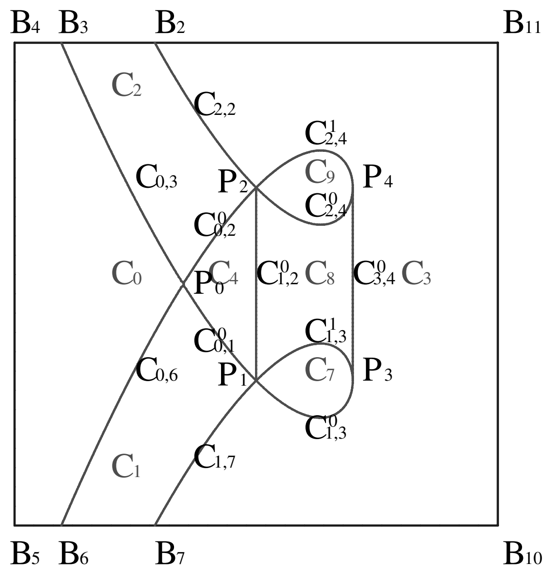

Fig. 7. Topology determination of the projection curve of a surface

The output is $\{V_{3,0}\{C^1_{1,3}(P^1_{1,3}(\frac{3}{2},-\frac{5}{4}))[C^{1,0}_{1,3},C^{1,1}_{1,3},C^{1,2}_{1,3}]$, $C^0_{1,3}(P^0_{1,3}(\frac{3}{2},-\frac{11}{4}))[C^{0,0}_{1,3}]$, $C^0_{3,4}(P^0_{3,4}(2,-2-\frac{\sqrt{13}}{4}))[C^{0,0}_{3,4},C^{0,1}_{3,4}]\}\}$.

After computing all boundary points and singularities of $\mathcal{C}$ by Algorithm 7, we can determine the position of all CSCSes of $\mathcal{S}$. And the projection curve of the surface is simplified as Fig. 7.

4.4 Compute the Surface Patches

Now, we need to compute the numbers of CSPs originating from each CSCS in the two cell bodies connected with the CCP which the CSCS lies on respectively. The following algorithm is for the purpose.

Algorithm 8. *The inputs are a real algebraic surface $\mathcal{S} : f(x,y,z) = 0$, the projection curve $\mathcal{C} : g(x,y) = 0$ of $\mathcal{S}$, a CCS $C^k_{i,j}$ (or $C_{i,j}$) on $\mathcal{C}$ and two cells beside it: C_{k_1}, C_{k_2}, a non-singular point on the CCS: $P^k_{i,j}$ (or $P^{-1}_{i,j}$)(x_0,y_0) and a sequence of CSCSes $\{C^{k,m}_{i,j}, m = 0,\cdots,l_{i,j,k}-1\}$ on the CCP lifted from the CCS. The output are two ordered number lists of CSPs originating from each CSCS of the sequence in the two cell bodies from bottom to top respectively.*

1. Compute the tangent line of $\mathcal{C}$ at point $P^k_{i,j}$; compute the vertical line of the tangent line at $P^k_{i,j}$ and parameterize it as $(ta+x_0, tb+y_0)$.
2. Compute the real roots of the equation $g(ta+x_0, tb+y_0) = 0$. Record the root whose absolute value is the minimal among the nonzero real root(s). If the root does not exist, denote r as a constant, such as 1, else denote r as the absolute value of the root with minimal absolute value.

3. Isolate the real roots of $f(x_0, y_0, z) = 0$ by Algorithm 5, to obtain a sequence of rational number $\{z_0, z_1, \cdots, z_{l_{i,j,k}}\}$.
4. Compute the real roots of the equation $f(ta + x_0, tb + y_0, z_i) = 0$ for each $i = 0, 1, \cdots, l_{i,j,k}$. Record the root whose absolute value is the minimal among the real root(s). Denote the absolute value of the root as r_i. Let $R = \min\{r, r_0, r_1, \cdots, r_{l_{i,j,k}}\}/2$.
5. Compute the number of real roots of $f(Ra+x_0, Rb+y_0, z) = 0$ and $f(-Ra+x_0, -Rb + y_0, z) = 0$ in the interval $(z_m, z_{m+1})(m = 0, \cdots, l_{i,j,k} - 1)$ respectively. They are the numbers of CSPs originating from the CSCS $C_{i,j}^{k,m}$ in the cell bodies CC_{k_1}, CC_{k_2}.
6. Return the corresponding information.

Remark. If the CCS is an isolated singularity of $\mathcal{C}$, we need only to lift the point up, isolate the real roots of S on the line obtained in Algorithm 5, and find a line segment (its direction is parallel to XY-plane) which passes through the point as Algorithm 8. Then we can easily determine the number of CSPs originating from the points of S on the lifted line. If the CCS is a closed curve, Q is a point on the CCS, we can also easily compute the number of CSPs originating from the CSCSes on the CCP lifted from the CCS like Algorithm 8.

Theorem 4. *Algorithm 8 provides the correct output.*

Proof. The proof for this algorithm is same to the one for Algorithm 7 and is much easier. In this algorithm, we just replace the discs in Algorithm 7 with line segments. □

Example 7. Continuing from Example 6, we will compute the number of CSPs originating from the CSCSes on the CCP $SC_{0,3}$ as an example for this algorithm. The inputs are $g(x, y) = 0$, $f(x, y, z) = 0$, $C_{0,3}(C_0, C_2)[C_{0,3}^{-1,0}(V_{0,0}, V_{3,0}^0)$, $C_{0,3}^{-1,1}(V_{0,1}, V_{3,1}^0)$, $C_{0,3}^{-1,2}(V_{0,2}, V_{3,2}^0)]$, $P_{0,3}^{-1}(-2, -2 + 2 \cdot \sqrt{2})$. In step 1, we can obtain the line $(5 \cdot t - 2, 2 \cdot \sqrt{2} \cdot t - 2 + 2 \cdot \sqrt{2})$. Isolating $f(-2, -2 + 2 \cdot \sqrt{2}, z) = 0$, we can obtain $-12, -2, 5, 12$. We can find that R is a positive number more than $\frac{1}{20}$. In order to simplify our illustration, here we choose $\frac{1}{20}$ as R. Computing the number of real roots of $f(\frac{1}{20}a + x_0, \frac{1}{20}b + y_0, z) = 0$ and $f(-\frac{1}{20}a + x_0, -\frac{1}{20}b + y_0, z) = 0$ in the interval $(-12, -2), (-2, 5), (5, 12)$, respectively, we can obtain $\{1, 0, 1\}, \{1, 2, 1\}$. It means that there are 1, 0, 1(1,2,1) CSP(s) originating from the CSCSes $C_{0,3}^{-1,0}, C_{0,3}^{-1,1}, C_{0,3}^{-1,2}$ in the cell body lifted from $C_2(C_0)$, respectively. As is shown in Fig. 8. The output is $C_{0,3}(C_{0,3}^{-1,0}, C_{0,3}^{-1,1}, C_{0,3}^{-1,2})\{C_0[1, 2, 1], C_2[1, 0, 1]\}$.

Computing all the CSCSes of $\mathcal{S}$ with Algorithm 8, we can determine all the CSCSes and the number of CSPs originating from each CSCS in the two cell bodies beside it. Then we can form the CSPs of $\mathcal{S}$.

For each cell body lifted from a cell of $\mathcal{C}$, because the number of CSPs originating from all the CSCSes on each CCPs of the cell body is the same, we can determine each CSP in the cell body by pointing out its boundaries: CSCSes.

The following algorithm is to determine the CSPs of $\mathcal{S}$ by the topology information of $\mathcal{C}$ obtained by Algorithm 4.

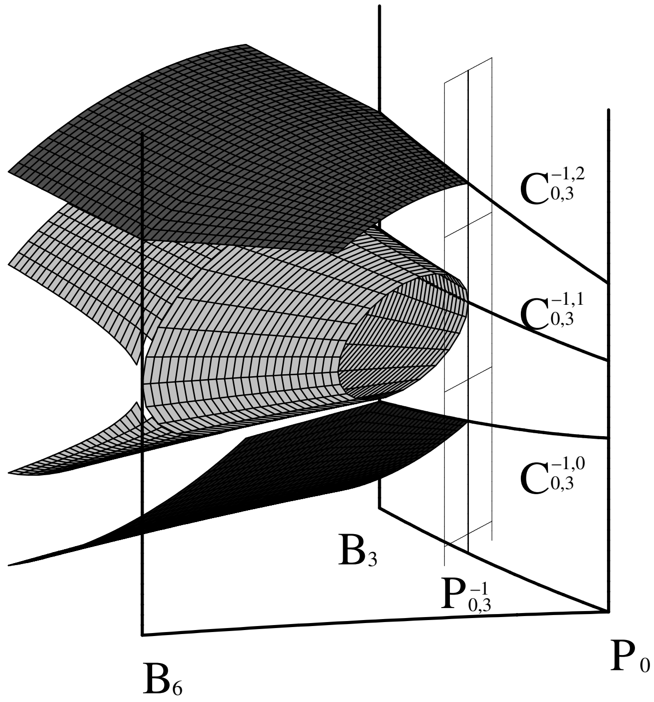

Fig. 8. Compute the CCS $C_{0,3}$ with Algorithm 8

Algorithm 9. *The inputs are* $\mathcal{S}$: $f(x, y, z) = 0$ *and the output of Algorithm 4. The output is the topological information of* $\mathcal{S}$.

1. Compute all the singularities and boundary points of $\mathcal{C}$ by Algorithm 7; determine all the CSCSes on each CCP lifted from the CCS of $\mathcal{C}$.
2. Compute the number of CSPs originating from each CSCS in two cell bodies beside it by Algorithm 8.
3. For each cell body lifted from a cell of $\mathcal{C}$, since the number of CSPs originating from the CSCSes on each CCP of the cell body is the same, we can determine each CSP by point out its boundaries–CSCSes.
4. Return the corresponding information of $\mathcal{S}$.

Example 8. Continuing from Example 7, we have determined all the CSPs of $\mathcal{S}$. The set of CSPs of $\mathcal{S}$ obtained by Algorithm 9 has the same topology as $\mathcal{S} : f(x, y, z) = 0$.

For the same example above, the outputs of the surface with Algorithm 9 are as follows. The figure of this surface is in Fig. 2. The figure of its real projection curve is as Fig. 7.

Points:
$\{\{B_2[V_{2,0}^0],\ B_3[V_{3,0}^0, V_{3,1}^0, V_{3,2}^0],\ B_4[V_{4,0}^0, V_{4,1}^0, V_{4,2}^0, V_{4,3}^0],\ B_5[V_{5,0}^0, V_{5,1}^0, V_{5,2}^0, V_{5,3}^0],$

$B_6[V^0_{6,0}, V^0_{6,1}, V^0_{6,2}]$, $B_7[V^0_{7,0}]$, $B_{10}[\,]$, $B_{11}[\,]\}$, $\{P_0[V_{0,0}, V_{0,1}, V_{0,2}]$, $P_1[V_{1,0}]$, $P_2[V_{2,0}]$, $P_3[V_{3,0}]$, $P_4[V_{4,0}]\}\}$.

For example, $B_3[V^0_{3,0}, V^0_{3,1}, V^0_{3,2}]$ means that there are three points of $\mathcal{S}$ on the line lifted from B_3. They are $V^0_{3,0}, V^0_{3,1}, V^0_{3,2}$, from bottom to top, respectively.

CSCSes:

$\{\{B_{2,3}[B^0_{2,3}(V^0_{2,0}, V^0_{3,0}), B^1_{2,3}(V^0_{2,0}, V^0_{3,2})]$,
$B_{3,4}[B^0_{3,4}(V^0_{3,0}, V^0_{4,0}), B^1_{3,4}(V^0_{3,1}, V^0_{4,1}), B^2_{3,4}(V^1_{3,1}, V^2_{4,2}), B^3_{3,4}(V^0_{3,2}, V^3_{4,3})]$,
$B_{4,5}[B^0_{4,5}(V^0_{4,0}, V^0_{5,0}), B^1_{4,5}(V^0_{4,1}, V^0_{5,1}), B^2_{4,5}(V^0_{4,2}, V^0_{5,2}), B^3_{4,5}(V^0_{4,3}, V^3_{5,3})]$,
$B_{5,6}[B^0_{5,6}(V^0_{5,0}, V^0_{6,0}), B^1_{5,6}(V^0_{5,1}, V^0_{6,1}), B^2_{5,6}(V^0_{5,2}, V^0_{6,1}), B^3_{5,6}(V^0_{5,3}, V^2_{6,2})]$,
$B_{6,7}[B^0_{6,7}(V^0_{6,0}, V^0_{7,0}), B^1_{6,7}(V^0_{6,2}, V^0_{7,0})]$,
$B_{7,10}[\,]$,
$B_{10,11}[\,]$,
$B_{11,2}[\,]\}$,
$\{C_{0,3}[C^{-1,0}_{0,3}(V_{0,0}, V^0_{3,0}), C^{-1,1}_{0,3}(V_{0,1}, V^0_{3,1}), C^{-1,2}_{0,3}(V_{0,2}, V^0_{3,2})]$,
$C^0_{0,1}[C^{0,0}_{0,1}(V_{0,0}, V_{1,0}), C^{0,1}_{0,1}(V_{0,1}, V_{1,0}), C^{0,2}_{0,1}(V_{0,2}, V_{1,0})]$,
$C_{0,6}[C^{-1,0}_{0,6}(V_{0,0}, V^0_{6,0}), C^{-1,1}_{0,6}(V_{0,1}, V^0_{6,1}), C^{-1,2}_{0,6}(V_{0,2}, V^0_{6,2})]$,
$C^0_{0,2}[C^{0,0}_{0,2}(V_{0,0}, V_{2,0}), C^{0,1}_{0,2}(V_{0,1}, V_{2,0}), C^{0,2}_{0,2}(V_{0,2}, V_{2,0})]$,
$C_{1,7}[C^{-1,0}_{1,7}(V_{1,0}, V^0_{7,0})]$,
$C_{2,2}[C^{-1,0}_{2,2}(V_{2,0}, V^0_{2,0})]$,
$C^0_{1,2}[C^{0,0}_{1,2}(V_{1,0}, V_{2,0}), C^{0,1}_{1,2}(V_{1,0}, V_{2,0})]$,
$C^0_{3,4}[C^{0,0}_{3,4}(V_{3,0}, V_{4,0}), C^{0,1}_{3,4}(V_{3,0}, V_{4,0})]$,
$C^0_{1,3}[C^{0,0}_{1,3}(V_{1,0}, V_{3,0})]$,
$C^1_{1,3}[C^{1,0}_{1,3}(V_{1,0}, V_{3,0}), C^{1,1}_{1,3}(V_{1,0}, V_{3,0}), C^{1,2}_{1,3}(V_{1,0}, V_{3,0})]$,
$C^0_{2,4}[C^{0,0}_{2,4}(V_{2,0}, V_{4,0})]$,
$C^1_{2,4}[C^{1,0}_{2,4}(V_{2,0}, V_{4,0}), C^{1,1}_{2,4}(V_{2,0}, V_{4,0}), C^{1,2}_{2,4}(V_{2,0}, V_{4,0})]\}\}$.

For example, $B_{2,3}[B^0_{2,3}(V^0_{2,0}, V^0_{3,0}), B^1_{2,3}(V^0_{2,0}, V^0_{3,2})]$ means that there are two CSCSes on the CCP $SB_{2,3}$: $B^1_{2,3}$, whose endpoints are $V^0_{2,0}, V^0_{3,0}$ and $B^1_{2,3}$, whose endpoints are $V^0_{2,0}, V^0_{3,2}$.

CSPs:

$\{C_0(4)\{S^0_0[C^{-1,0}_{0,3}, B^0_{3,4}, B^0_{4,5}, B^0_{5,6}, C^{-1,0}_{0,6}]$, $S^1_0[C^{-1,1}_{0,3}, B^1_{3,4}, B^1_{4,5}, B^1_{5,6}, C^{-1,1}_{0,6}]$,
$S^2_0[C^{-1,1}_{0,3}, B^2_{3,4}, B^2_{4,5}, B^2_{5,6}, C^{-1,1}_{0,6}]$, $S^3_0[C^{-1,2}_{0,3}, B^3_{3,4}, B^3_{4,5}, B^3_{5,6}, C^{-1,2}_{0,6}]\}$,
$C_1(2)\{S^0_1[C^{-1,0}_{0,6}, B^0_{6,7}, C^{-1,0}_{1,7}, C^{0,0}_{0,1}]$, $S^1_1[C^{-1,2}_{0,6}, B^1_{6,7}, C^{-1,0}_{1,7}, C^{0,2}_{0,1}]\}$,
$C_2(2)\{S^0_2[C^{-1,0}_{0,3}, C^{0,0}_{0,2}, C^{-1,0}_{2,2}, B^0_{2,3}]$, $S^1_2[C^{-1,2}_{0,3}, C^{0,2}_{0,2}, C^{-1,0}_{2,2}, B^1_{2,3}]\}$,
$C_3(0)\{\,\}$,
$C_4(4)\{S^0_4[C^{0,0}_{0,2}, C^{0,0}_{0,1}, C^{0,0}_{1,2}]$, $S^1_4[C^{0,1}_{0,2}, C^{0,1}_{0,1}, C^{0,0}_{1,2}]$,
$S^2_4[C^{0,1}_{0,2}, C^{0,1}_{0,1}, C^{0,1}_{1,2}]$, $S^3_4[C^{0,2}_{0,2}, C^{0,2}_{0,1}, C^{0,1}_{1,2}]\}$,
$C_7(2)\{S^0_7[C^{0,0}_{1,3}, C^{1,0}_{1,3}]$, $S^1_7[C^{0,0}_{1,3}, C^{1,2}_{1,3}]\}$,
$C_8(4)\{S^0_8[C^{0,0}_{2,4}, C^{0,0}_{1,2}, C^{1,0}_{1,3}, C^{0,0}_{3,4}]$, $S^1_8[C^{0,1}_{2,4}, C^{0,0}_{1,2}, C^{1,1}_{1,3}, C^{0,0}_{3,4}]$,
$S^2_8[C^{0,1}_{2,4}, C^{0,1}_{1,2}, C^{1,1}_{1,3}, C^{0,1}_{3,4}]$, $S^3_8[C^{0,2}_{2,4}, C^{0,1}_{1,2}, C^{1,2}_{1,3}, C^{0,1}_{3,4}]\}$,
$C_9(2)\{S^0_9[C^{0,0}_{2,4}, C^{1,0}_{2,4}]$, $S^1_9[C^{0,2}_{2,4}, C^{1,0}_{2,4}]\}\}$.

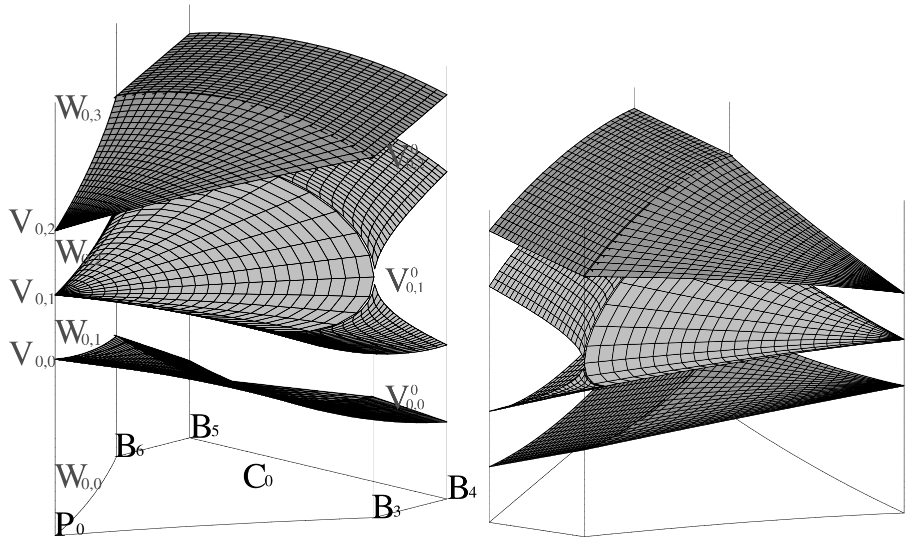

Fig. 9. The CSPs in the cell body lifted from C_0

For example, $C_0(4)\{S_0^0[C_{0,3}^{-1,0}, B_{3,4}^0, B_{4,5}^0, B_{5,6}^0, C_{0,6}^{-1,0}]$, $S_0^1[C_{0,3}^{-1,1}, B_{3,4}^1, B_{4,5}^1, B_{5,6}^1, C_{0,6}^{-1,1}]$, $S_0^2[C_{0,3}^{-1,1}, B_{3,4}^2, B_{4,5}^2, B_{5,6}^2, C_{0,6}^{-1,1}]$, $S_0^3[C_{0,3}^{-1,2}, B_{3,4}^3, B_{4,5}^3, B_{5,6}^3, C_{0,6}^{-1,2}]\}$ means that there are four CSPs, S_0^0, S_0^1, S_0^2, S_0^3 in the cell body CC_0 from bottom to up. $[C_{0,3}^{-1,0}, B_{3,4}^0, B_{4,5}^0, B_{5,6}^0, C_{0,6}^{-1,0}]$, $[C_{0,3}^{-1,1}, B_{3,4}^1, B_{4,5}^1, B_{5,6}^1, C_{0,6}^{-1,1}]$, $[C_{0,3}^{-1,1}, B_{3,4}^2, B_{4,5}^2, B_{5,6}^2, C_{0,6}^{-1,1}]$, $[C_{0,3}^{-1,2}, B_{3,4}^3, B_{4,5}^3, B_{5,6}^3, C_{0,6}^{-1,2}]$ are the boundaries of S_0^0, S_0^1, S_0^2, S_0^3, respectively. The CSPs in the cell body CC_0 are shown in Fig. 9.

5 Main Algorithm

By the discussion in the previous sections, we can present the main algorithm to determine the topology of an implicit algebraic surface.

Algorithm 10. *The input is an implicit algebraic surface* $\mathcal{S} : f(x, y, z) = 0$. *The output is a set of surface patches which have the same topology as the original surface* $\mathcal{S}$.

1. Compute the projection curve $\mathcal{C}$: $g(x, y) = 0$ of $\mathcal{S}$ by Algorithm 2.
2. Determine the topology of $\mathcal{C}$ by Algorithm 4.
3. Space curve and surface patch segmentation of $\mathcal{S}$ by Algorithm 9.
4. Return the corresponding topology information of $\mathcal{S}$.

Example 9. We will illustrate the algorithm with another example defined by

$$f(x, y, z) = f_1(x, y, z) \cdot f_2(x, y, z),$$

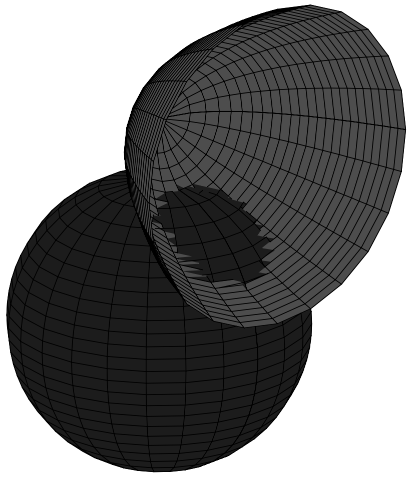

Fig. 10. A reducible surface

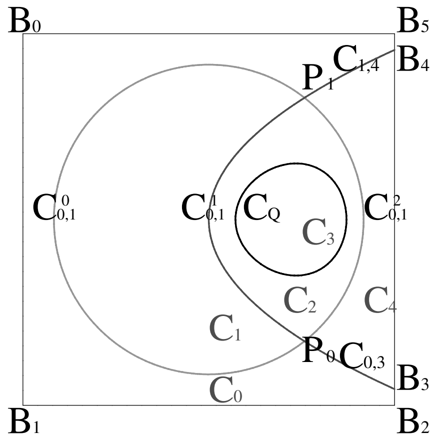

Fig. 11. Topology determination of the projection curve of a reducible surface

where $f_1(x,y,z) = y^2 + (z-11)^2 - 5\cdot x$, $f_2(x,y,z) = x^2 + y^2 + (z-4)^2 - 25$. Its figure is in Fig. 10.

Since $f_1(x,y,z), f_2(x,y,z)$ are normal surfaces, we can derive the projection curve of $\mathcal{S}$ by Algorithm 2.

$$g(x,y) = \prod_{1\le i\le j\le 2} T_{i,j}(x,y)$$

where $T_{1,1}(x,y) = 4\cdot(y^2 - 5\cdot x)$, $T_{1,2}(x,y) = 73\cdot x^2 + 196\cdot y^2 + 576 - 740\cdot x + 10\cdot x^3 + x^4$, $T_{2,2}(x,y) = 4\cdot(x^2+y^2-25)$.

With Algorithm 4, we can get the topological information of the projection curve as Fig. 11. We can derive the following information by Algorithm 9.

Points:

$\{\{B_0[\,], B_1[\,], B_2[\,], B_3[V^0_{3,0}], B_4[V^0_{4,0}], B_5[\,]\}, \{P_0[V_{0,0}, V_{0,1}], P_1[V_{1,0}, V_{1,1}]\}\}$.

CSCSes:

$\{\{B_{0,1}[\,], B_{1,2}[\,], B_{2,3}[\,], B_{3,4}[B^0_{3,4}(V^0_{3,0}, V^0_{4,0}), B^1_{3,4}(V^0_{3,0}, V^0_{4,0})], B_{4,5}[\,], B_{5,0}[\,]\}$,
$\{C^0_{0,1}[C^{0,0}_{0,1}(V_{0,0}, V_{1,0})]$, $C^1_{0,1}[C^{1,0}_{0,1}(V_{0,0}, V_{1,0}), C^{0,1}_{0,1}(V_{0,0}, V_{1,0}), C^{0,2}_{0,1}(V_{0,1}, V_{1,1})]$,
$C^2_{0,1}[C^{2,0}_{0,1}(V_{0,0}, V_{1,0}), C^{2,1}_{0,1}(V_{0,1}, V_{1,1}), C^{2,2}_{0,1}(V_{0,1}, V_{1,1})]$, $C_{1,4}[C^{-1,0}_{1,4}(V_{1,1}, V^0_{4,0})]$,
$C_{0,3}[C^{-1,0}_{0,3}(V_{0,1}, V^0_{3,0})], C_Q[C^0_Q, C^1_Q, C^2_Q]\}\}$.

CSPs:

$\{C_0(0)\{\,\}$,
$C_1(2)\{S^0_1[C^{0,0}_{0,1}, C^{1,0}_{0,1}], S^0_1[C^{0,0}_{0,1}, C^{1,1}_{0,1}]\}$,
$C_2(4)\{S^0_2[[C^{1,0}_{0,1}, C^{2,0}_{0,1}], [C^0_Q]], S^1_2[[C^{1,1}_{0,1}, C^{2,0}_{0,1}], [C^1_Q]]$,
$S^2_2[[C^{1,2}_{0,1}, C^{2,1}_{0,1}], [C^1_Q]]$, $S^3_2[[C^{1,2}_{0,1}, C^{2,2}_{0,1}], [C^2_Q]]\}$,

$C_3(4)\{S_3^0[C_Q^0], S_3^1[C_Q^1], S_3^2[C_Q^1], S_3^3[C_Q^2]\}$,
$C_4(2)\{S_4^0[C_{0,1}^{2,0}, C_{0,3}^{-1,0}, B_{3,4}^0, C_{1,4}^{-1,0}],\ S_4^1[C_{0,1}^{2,1}, C_{0,3}^{-1,0}, B_{3,4}^1, C_{1,4}^{-1,0}]\}\}$.

According to our experiments, the topology determination of the projection curve is the most time-consuming phase of the algorithm.

6 Conclusion

In this paper, we present an algorithm, which can be used to give a representation for the topology of an implicit algebraic surface $f(x, y, z) = 0$. We give a curvilinear wireframe of the surface and the surface patches of the surface determined by the curvilinear wireframe, which present the same topology as the surface. Most of the surface patches are curvilinear polygons. If needed, we can easily modify our algorithm to give a polyhedron which has the same topology as the surface.

The algorithm mainly involves computation of resultants, determination of the topology of plane curves, computation of singularities of surfaces and curves, isolating real roots of univariate equations. Many aspect of the algorithm could be further improved. This will be done in our later work.

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