

# Interpolation of Parametric CNC Machining Path Under Confined Jounce\*

Wei Fan, Xiao-Shan Gao<sup>†</sup>, Wei Yan, Chun-Ming Yuan  
KLMM, Academy of Mathematics and Systems Science  
Chinese Academy of Sciences, Beijing 100190, China

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## Abstract

Aiming at reducing the CNC machining vibration and increasing machining quality, an interpolation method for parametric tool paths with confined jounce, jerk, acceleration, and speed is proposed. An acceleration/deceleration profile with confined jounce, jerk, acceleration, and speed is proposed and it is shown that this profile is time-optimal to change the speed from one value to another under the given constraints. For a given parametric tool path, the velocity function is obtained by first computing the critical points of the tool path where the radius of curvature reaches extremal values, then determining the feasible maximal speeds at the critical points, and finally using the jounce confined acceleration/deceleration profile to connect the speeds at the adjacent critical points. A vibration experiment is conducted, which shows that vibration of the CNC machine decreases significantly for motions under confined jounce than that under confined acceleration and jerk. Simulation for two real CNC models are given to show the feasibility of the method.

**Keywords.** Interpolation, velocity planning, confined jounce, time-optimal, parametric curve, vibration, high-precision CNC machining.

## 1 Introduction

Interpolation algorithms, which control how the CNC machine tool moves along the machining path, is one of the important factors in high speed and high precision CNC machining. In particular, proper kinematic controls and interpolation algorithms are essential to reduce vibration and achieve high-precision CNC machining [13]. An interpolation algorithm in the CNC controller usually consists of two phases: velocity planning and parameter computation. Let  $C(u)$   $u \in [0, 1]$  be the machining path. The phase to determine the velocity function  $v(u)$  along  $C(u)$  is called velocity planning. When the velocity function  $v(u)$  is known, the phase of sampling or computing the next interpolation point at parametric value  $u_{i+1} = u_i + \Delta u$  during one sampling period is called parameter computation.

A key factor to compute the velocity function during velocity planning is to choose an acceleration/deceleration (abbr. AD) profile. An AD profile is the procedure to use different acceleration and deceleration modes to change the speed from one value to another under certain constraints.

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<sup>†</sup>To whom correspondence should be addressed. Email: [xgao@mmrc.iss.ac.cn](mailto:xgao@mmrc.iss.ac.cn), <http://www.mmrc.iss.ac.cn/~xgao/>.

The simplest AD profile is the linear AD profile, where the acceleration and speed are bounded or confined. Using a linear AD profile for each axis, Borow [2], Shiller et al [14], and Farouki and Timar [15] presented a time-optimal velocity planning method for a parametric path. Zhang et al simplified the method in [15] for quadratic B-splines and realized real-time machining on an industrial CNC machine [22]. Yuan et al presented a time-optimal velocity planning method with confined acceleration and chord error [19].

In a linear AD profile, the acceleration can change instantaneously from its maximal value  $A$  to  $-A$ , resulting in large vibration of the CNC machine and decreasing the machining quality. A better AD profile is the *S-shape* profile, where the jerk is confined. The S-shape AD profile is widely used in velocity planning [3, 6, 8, 10, 11, 12, 16, 20] to generate motion with confined jerk. Another method to obtain velocity functions with confined jerk is to use trigonometric function profiles [7]. Comparing to the S-shape profile, the acceleration function obtained with trigonometric function profiles is not differentiable. It also does not have the time-optimal property of the S-shape profile similar to Theorem 2.5 in this paper.

The S-shape AD profile increases the smoothness of the velocity function. However, the jerk can jump from its maximal value to its minimal value. Since change of the jerk reflects the non-smooth change of acceleration, the profile could still result in vibration and decrease machining quality. Gai et al [5] used an average filter to improve the S-shape AD profile to obtain a continuous jerk in some extent. But this method fails to guarantee the continuity of acceleration and is not time-optimal.

A natural way to obtain smoother velocity functions is to use confined jounce. Let  $F$  be the driving force of the CNC servo and  $F_c$  the combined cutting force and friction force. Denote the mass of the axes to be  $m$  and let  $j$  be jerk. Differentiate  $j$  to obtain

$$\frac{d^2(F - F_c)}{d^2t} = m \frac{dj}{dt} = ms, \quad (1)$$

where  $s$  is called the *jounce* or *snap*, which reflects the instantaneous change of jerk. If the jounce is confined, a motion with continuous jerk, differentiable acceleration, and  $C^2$  velocity will be obtained.

The main contribution of this paper is to design and investigate an AD profile with confined jounce, and to present an interpolation algorithm for parametric tool paths based on the jounce confined AD profile.

The key component of the AD profile is a 7-period acceleration profile to increase the speed from one value to another. The whole AD profile first uses a 7-period acceleration profile to increase the speed from the initial value to its maximal value, then stays at the maximal value for a while, and finally uses a 7-period deceleration profile to decrease the speed from its maximal value to the end value. This paper analyzes the properties of the AD profile in detail and prove that this profile is time-optimal to change the speed from one value to another under the given constraints.

For a curved tool path given by a set of parametric equations, the sensitive corner approach similar to that used in [7, 18] is used to compute its velocity function. Firstly, the critical points of the tool path, where the radius of curvature reaches extremal values, are computed. From these extremal values and a given sampling period, the maximal speeds that can be reached in these critical points are determined. Secondly, a backtracking procedure is used to check the reachability of the maximal speeds at each pair of neighboring critical points, that is, whether it is possible to change the maximal speed at one critical point to the maximal speed at the next critical point within the given tool path length. In the unreachable case, the speeds at the critical points are adjusted to make them reachable. Finally, between each pair of critical points, use the jounce confined AD profile to bridge their speeds, and the final velocity function is the combination of these AD profiles.

A vibration test experiment is carried out to compare AD profiles with confined acceleration, jerk, and jounce, which shows that the AD profile with confined jounce indeed can be used to reduce

the vibration of the CNC machine tools and hence is useful to improve CNC machining quality. Furthermore, simulation results are given for two CNC models to show the feasibility of the proposed method.

The rest of this paper is organized as follows. In Section 2, the 7-period AD profile with confined jounce is designed and analyzed. In Section 3, the interpolation algorithm for parametric tool paths is proposed. In Section 4, experimental and simulation results are given. In Section 5, the paper is concluded.

## 2 An AD profile with confined jounce, jerk, acceleration, and speed

In this section, an AD profile for the speed to change from a start value  $v_s$  to an end value  $v_e$  within a distance  $d_m$  and under confined jounce will be presented. The profile is time optimal under the given conditions. Since the obtained velocity function has confined jounce, it is  $C^2$  continuous.

### 2.1 A 7-period acceleration profile

In this section, a 7-period acceleration profile to increase the speed from zero to a given value  $v_m > 0$  is introduced. The profile will serve as a basic step of our AD profile to be presented in Section 2.4.

The 7-period acceleration profile is given in Figure 1, where the horizontal axis is time  $t$ . The profile is used to increase the speed from zero to  $v_m > 0$  in seven periods as marked in the figure. If the speed is decreased from  $v_m$  to zero, then the reverse procedure can be used.

Make the basic assumption that the jounce is bounded by a given value  $S_m$ :

$$|s(t)| \leq S_m. \quad (2)$$

At the start and end points, the acceleration and jerk are made to be zero, in order to form  $C^2$  continuous velocity function with similar velocity profiles. That is

$$a_s = a_e = 0; j_s = j_e = 0.$$

As shown in Figure 1, the first period with duration  $t_1$  is used to increase the jerk from zero to its maximal value  $j_m$  with the maximal jounce  $S_m$ . In the second period with duration  $t_2$ , the jerk has constant value  $j_m$  and the acceleration is further increased. In the third period with duration  $t_1$ , the minimal jounce is used to decrease the jerk from  $j_m$  to zero and at the same time to increase the acceleration to its maximal value  $a_m$ . The fourth period with duration  $t_3$  and with a constant acceleration  $a_m$  is used to further increase the speed. The last three periods are anti-symmetric with respect to the first three periods, which are used to increase the speed to its maximal value  $v_m$  and at the same time to decrease the acceleration from  $a_m$  to zero. The whole procedure lasts  $t_m = 4t_1 + 2t_2 + t_3$ . It is clear that the 7-period acceleration profile is “bang-bang” in the sense that at any time, at least one of the quantities  $v(t), a(t), j(t), s(t)$  reaches its boundary value.

The 7-period profile is uniquely determined by the three parameters  $(t_1, t_2, t_3)$  and the maximal jounce  $S_m$ . Therefore, the triplet  $(t_1, t_2, t_3)$  is used to represent such a profile. After a triplet  $(t_1, t_2, t_3)$  is given, it is easy to obtain the expressions for  $v_0(t), a(t), j(t), s(t), (t \in [0, t_m])$ , which can be found in (28) to (31) in the appendix of this paper. It is clear that these expressions have the following form:

$$\begin{aligned} s(t) &= 6m_0 \\ j(t) &= 6m_0t + 2m_1 \\ a(t) &= 3m_0t^2 + 2m_1t + m_2 \\ v_0(t) &= m_0t^3 + m_1t^2 + m_2t + m_3 \end{aligned}$$

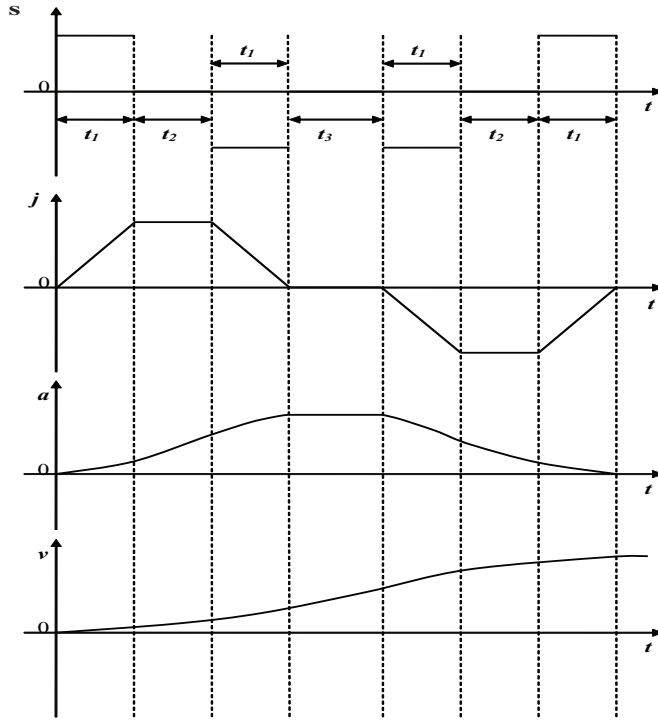


Figure 1: jounce( $s$ ), jerk( $j$ ), acceleration( $a$ ), and speed( $v$ )

where  $m_i$  are constants depending on  $S_m$ ,  $s(t)$  is piecewise constant with three values  $0, S_m, -S_m$ ,  $j(t)$  is piecewise linear,  $a(t)$  is the integration of  $j(t)$  about  $t$  and is  $C^1$  piecewise linear or quadratic polynomials in  $t$ ,  $v(t)$  is the integration of  $a(t)$  about  $t$  and is  $C^2$  continuous piecewise linear, quadratic, or cubic polynomials in  $t$ .

From Figure 1 and the formulas (28) to (31), the maximal values  $j_m, a_m, v_m$  for  $j(t), a(t), v_0(t)$  can be respectively obtained:

$$j_m = S_m t_1, \quad (3)$$

$$a_m = S_m(t_1 t_2 + t_1^2), \quad (4)$$

$$v_m = 2S_m t_1^3 + S_m t_1 t_2^2 + 3S_m t_1^2 t_2 + S_m t_1 t_2 t_3 + S_m t_1^2 t_3 \quad (5)$$

where  $S_m$  is from (2). As mentioned above,  $j_m$  is reached at  $t = t_1$ ,  $a_m$  is reached at  $t = 2t_1 + t_2$ , and  $v_m$  is reached at  $t = t_m = 4t_1 + 2t_2 + t_3$ . The distance needed for the procedure can be computed as follows

$$\begin{aligned} d_m &= 4S_m t_1^4 + 8S_m t_1^3 t_2 + S_m t_1 t_2^3 + 5S_m t_1^2 t_2^2 + \frac{9}{2}S_m t_1^2 t_2 t_3 + \frac{3}{2}S_m t_1 t_2^2 t_3 \\ &+ \frac{1}{2}S_m t_1 t_2 t_3^2 + 3S_m t_1^3 t_3 + \frac{1}{2}S_m t_1^2 t_3^2 \\ &= \frac{v_m}{2}(4t_1 + 2t_2 + t_3). \end{aligned} \quad (6)$$

Now, suppose that the starting speed is  $v_s$  and the end speed is  $v_e$ . From (5),

$$v_e = v_s + 2S_m t_1^3 + S_m t_1 t_2 t_3 + S_m t_1 t_2^2 + 3S_m t_1^2 t_2 + S_m t_1^2 t_3. \quad (7)$$

Furthermore, it is easy to show that the distance of finishing the 7-period process is

$$d_{min} = 4S_m t_1^4 + 8S_m t_1^3 t_2 + S_m t_1 t_2^3 + 5S_m t_1^2 t_2^2 + \frac{9}{2}S_m t_1^2 t_2 t_3 + \frac{3}{2}S_m t_1 t_2^2 t_3$$

$$+ \frac{1}{2}S_m t_1 t_2 t_3^2 + 3S_m t_1^3 t_3 + \frac{1}{2}S_m t_1^2 t_3^2 + v_s(4t_1 + 2t_2 + t_3) \quad (8)$$

$$= \frac{v_s + v_e}{2}(4t_1 + 2t_2 + t_3) = \frac{v_s + v_e}{2}t_m. \quad (9)$$

## 2.2 The acceleration profile for the speed to increase from $v_s$ to $v_e$

This section will determine how to use the 7-period acceleration profile to increase the speed from  $v_s$  to  $v_e$  when the basic parameters of the CNC machine are given. It will be shown that the acceleration procedure is time-optimal.

Three *basic parameters* of the CNC machine will be considered: the maximal jounce  $S_m$ , the maximal jerk  $J_m$ , and the maximal acceleration  $A_m$ . So, the constraints are

$$|a(t)| \leq A_m, |j(t)| \leq J_m, |s(t)| \leq S_m. \quad (10)$$

Since the 7-period acceleration profile is used, the initial conditions are

$$a_s = a(0) = a_e = a(t_m) = 0; \quad j_s = j(0) = j_e = j(t_m) = 0. \quad (11)$$

Without loss of generality, assume that  $v_s \leq v_e$ .

The purpose of this section is to determine a 7-period acceleration profile to increase the speed from  $v_s$  to  $v_e$  under the constraints (10) and (11). The distance needed for the acceleration procedure is also given. But the constraint on the distance is not considered in this section. As mentioned before, to determine the 7-period acceleration profile, one just need to determine  $(t_1, t_2, t_3)$ .

Firstly, check whether  $t_2 = 0$  based on the values of  $S_m, J_m, A_m$ . From (3) and (4),

$$S_m a_m = S_m j_m t_2 + j_m^2. \quad (12)$$

Three cases are considered. If  $J_m^2 > S_m A_m$ , then the jerk cannot reach  $J_m$  and  $t_2$  must be zero. Assume the contrary, from (12),  $J_m^2 + S_m t_2 J_m = S_m a_m$ . Since  $0 \leq a_m \leq A_m$ ,  $t_2 = (S_m a_m - J_m^2)/(S_m J_m)$  must be negative, which is impossible. As a consequence,  $t_2 = 0$ .

If  $J_m^2 = S_m A_m$ ,  $j$  reaches  $J_m$  at  $t_1 = J_m/S_m$  and  $t_2 = 0$ . When  $t = 2t_1$ , the acceleration reaches  $A_m$ .

If  $J_m^2 < S_m A_m$  and under the assumption that there exist no constraints on the final speed and the distance, then from (3) and (12), the jerk reaches  $J_m$  at  $t_1 = J_m/S_m$ , the acceleration reaches  $A_m$  at  $t = 2t_1 + t_2$ , and  $t_2 = (S_m A_m - J_m^2)/(S_m J_m) > 0$ . Of course, if there exist constraints on the final speed  $v_m$ , then  $t_2$  might be zero, which will be discussed later.

Summarize the above analysis as the following proposition.

**Proposition 2.1** 1. If  $J_m^2 \geq S_m A_m$ , then  $t_2 = 0$ . In this case, the jerk  $j$  cannot reach the maximal value  $J_m$  unless the equal sign holds.

2. If  $J_m^2 < S_m A_m$  and there exist no other constraints, then the jerk reaches  $J_m$  at  $t_1 = J_m/S_m$ , the acceleration reaches  $A_m$  at  $t = 2t_1 + t_2$ , and  $t_2 = (S_m A_m - J_m^2)/(S_m J_m) > 0$ .

In the above analysis, the constraint raised from the final speed  $v_m$  is not considered. Now consider how to increase the speed from zero to  $v_m$  with the 7-period acceleration profile by taking into account of  $v_m$ . Two cases are considered due to Proposition 2.1.

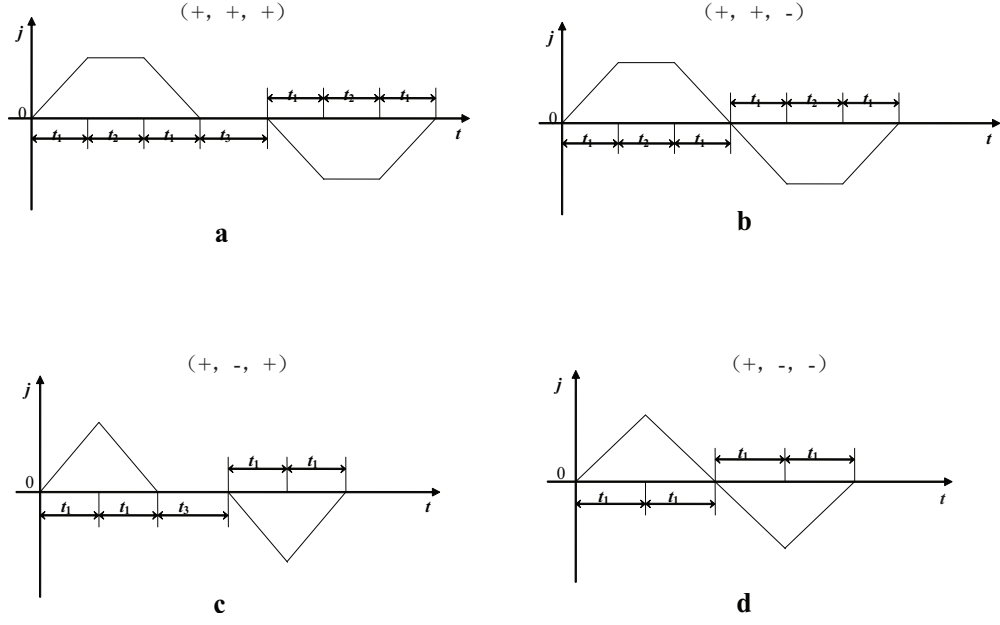


Figure 2: Jerk functions for all possible acceleration profiles

**Case 1.**  $J_m^2 \geq S_m A_m$ . By Proposition 2.1,  $t_2 = 0$ . If the acceleration reaches  $A_m$ , then from (4)  $t_1 = \sqrt{A_m/S_m}$ . Let

$$v_0 = 2S_m t_1^3 = 2S_m \left(\frac{A_m}{S_m}\right)^{\frac{3}{2}}.$$

Then  $(t_1, t_2, t_3)$  can be determined as follows

**Case 1.1** If  $v_m > v_0$ , then  $a(t)$  reaches  $A_m$  at  $t = 2t_1 = 2\sqrt{\frac{A_m}{S_m}}$ . In order to reach  $v_m$ , there exists a constant acceleration phase  $t_3$  which can be computed from (5) as follows

$$t_3 = \frac{v_m - 2S_m t_1^3}{S_m t_1^2} = \frac{v_m - 2S_m t_1^3}{A_m} = \frac{v_m - v_0}{A_m} > 0, \quad (13)$$

where  $t_1 = \sqrt{\frac{A_m}{S_m}}$ .

For  $i = 1, 2, 3$ , introduce the symbol  $\hat{t}_i = +$  if  $t_i \neq 0$  and  $\hat{t}_i = -$  if  $t_i = 0$ . Then use  $(\hat{t}_1, \hat{t}_2, \hat{t}_3)$  to denote the shape of the 7-period acceleration profile. In case 1.1, the acceleration profile is  $(+, -, +)$  which is shown in Figure 2(c).

**Case 1.2** If  $v_m \leq 2S_m \left(\frac{A_m}{S_m}\right)^{\frac{3}{2}}$ , then the acceleration will not reach  $A_m$  unless the equality holds. Assume the contrary, from (13),  $t_3$  will be negative, which is impossible. Therefore  $t_3 = 0$  and  $t_1$  will be determined by  $v_m$  with formula (5):  $t_1 = \sqrt[3]{\frac{v_m}{2S_m}}$ . Then, the acceleration profile is  $(+, -, -)$  which is shown in Figure 2(d).

**Case 2.**  $J_m^2 < S_m A_m$ . By Proposition 2.1, if the jerk reaches  $J_m$  then  $t_1 = J_m/S_m$  and if the acceleration reaches  $A_m$  then  $t_2 = (S_m A_m - J_m^2)/(S_m J_m)$ . Under the above mentioned conditions,

set

$$\begin{aligned} v_1 &= 2S_m t_1^3 = \frac{2J_m^3}{S_m^2}. \\ v_2 &= 2S_m t_1^3 + S_m t_1 t_2^2 + 3S_m t_1^2 t_2 = \frac{A_m(S_m A_m + J_m^2)}{S_m J_m}. \end{aligned}$$

Then  $(t_1, t_2, t_3)$  can be determined as follows

**Case 2.1** If  $v_m > v_2$ , then both the jerk and the acceleration reach their maximal values and  $t_3 \neq 0$  can be computed from (5) as follows

$$t_3 = \frac{v_m - v_2}{A_m} = \frac{v_m}{A_m} - \left( \frac{J_m}{S_m} + \frac{A_m}{J_m} \right).$$

The acceleration profile is  $(+, +, +)$  which is shown in Figure 2(a).

**Case 2.2** If  $v_1 < v_m \leq v_2$ , then  $t_3 = 0$ , the jerk can reach  $J_m$ , and  $a(t)$  cannot reach  $A_m$  unless  $v_m = v_2$ . Furthermore,  $t_2$  is determined by  $v_m$  from (5) with the following formula

$$t_2 = \frac{\sqrt{t_1^2 + \frac{4v_m}{J_m}} - 3t_1}{2} = \frac{\sqrt{J_m^3 + 4v_m S_m^2} - 3J_m^{\frac{3}{2}}}{2S_m \sqrt{J_m}}$$

The acceleration profile is  $(+, +, -)$  which is shown in Figure 2(b).

**Case 2.3** If  $v_m \leq v_1$ , then neither the jerk (unless  $v_m = v_1$ ) nor the acceleration reaches their maximal values and  $t_2 = t_3 = 0$ . Furthermore,  $t_1$  is determined by  $v_m$  from (5) as follows:

$$t_1 = \sqrt[3]{\frac{v_m}{2S_m}}.$$

The acceleration profile is  $(+, -, -)$  which is shown in Figure 2(d).

Based on the above analysis, the algorithm to increase the speed from  $v_s$  to  $v_e$  with the 7-period acceleration profile can be given. If  $v_s > v_e$ , a decrease profile can be designed similarly. Also assume that the distance is long enough for the procedure to carry out. The constraints on the distance will be considered in Section 2.4.

### Algorithm 2.2 PRO\_VV( $v_s, v_e$ )

**Input:** Parameters  $S_m, J_m, A_m$  in (10), the start speed  $v_s$  and end speed  $v_e$ .

**Output:** The parameters  $(t_1, t_2, t_3)$  of the 7-period acceleration profile.

1. Let  $v^* = |v_e - v_s|$ . If  $J_m^2 \geq S_m A_m$ , then go to Step 2, else go to Step 3.
2. Compute  $t_1, t_2, t_3$  by the following formulas and output  $(t_1, t_2, t_3)$ .

$$\begin{cases} t_1 = \sqrt{\frac{A_m}{S_m}}, t_2 = 0, t_3 = \frac{v^* - 2S_m t_1^3}{A_m}, & \text{if } v^* > 2S_m \left(\frac{A_m}{S_m}\right)^{\frac{3}{2}}, \\ t_1 = \sqrt[3]{\frac{v^*}{2S_m}}, t_2 = 0, t_3 = 0, & \text{if } v^* \leq 2S_m \left(\frac{A_m}{S_m}\right)^{\frac{3}{2}} \end{cases}$$

3. Compute  $t_1, t_2, t_3$  by the following formulas and output  $(t_1, t_2, t_3)$ .

$$\begin{cases} t_1 = \frac{J_m}{S_m}, t_2 = \frac{A_m - \frac{J_m^2}{S_m}}{J_m}, t_3 = \frac{v^* - (2S_m t_1^3 + 3S_m t_1^2 t_2 + S_m t_1 t_2^2)}{A_m}, & \text{if } v^* > \frac{A_m^2}{J_m} + \frac{A_m J_m}{S_m}, \\ t_1 = \frac{J_m}{S_m}, t_2 = \frac{\sqrt{t_1^2 + \frac{4v^*}{J_m}} - 3t_1}{2}, t_3 = 0, & \text{if } \frac{2J_m^3}{S_m^2} < v^* \leq \frac{A_m^2}{J_m} + \frac{A_m J_m}{S_m}, \\ t_1 = \sqrt[3]{\frac{v^*}{2S_m}}, t_2 = 0, t_3 = 0, & \text{if } v^* \leq \frac{2J_m^3}{S_m^2} \end{cases}$$

Use the notation  $(\hat{t}_1, \hat{t}_2, \hat{t}_3)$ , Algorithm **PRO\_VV** can be summarized as the flowchart in Figure 3.

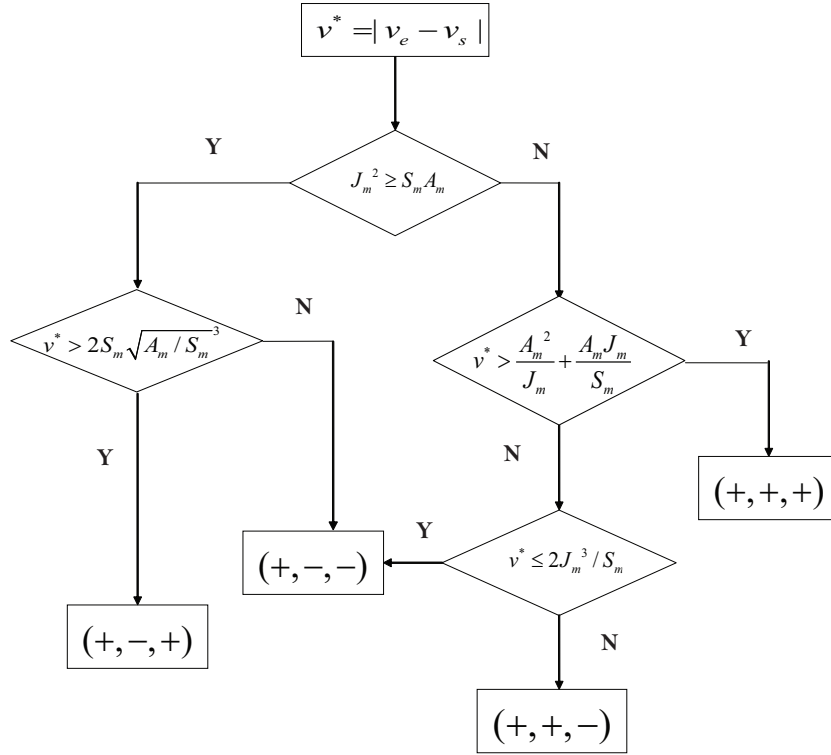


Figure 3: Flowchart for Algorithm **PRO\_VV**

It is clear that the selections of the parameters  $t_1, t_2, t_3$  are not unique. For instance, in Case 1, we can increase the jerk to a value  $j_m < J_m$  and then include a constant jerk part  $t_2 > 0$ . Also, in Case 2, if mode  $(+, +, +)$  fails, it seems that both  $(+, +, -)$  and  $(+, -, +)$  can be chosen. The following theorem will show that our choosing method for the parameters leads to a time-optimal solution to the problem. The proof of the theorem can be found in Appendix A.2 of this paper.

**Theorem 2.3** *Algorithm **PRO\_VV** is a time-optimal procedure to increase the speed from  $v_s$  to  $v_e$  under the constraints (10) and (11).*

### 2.3 The acceleration profile with initial speed $v_s$ and distance $d_m$

In this section, an acceleration profile will be given to compute the maximal speed that can be reached with a start speed  $v_s$  and a given distance  $d_m$ .

Similar to Section 2.2, we need to compute the parameters  $(t_1, t_2, t_3)$  for the 7-period acceleration profile by considering two cases.

**Case 1.**  $J_m^2 \geq S_m A_m$ . By Proposition 2.1,  $t_2 = 0$ . If the acceleration reaches  $A_m$ , then from (4)  $t_1 = \sqrt{A_m / S_m}$ . Let

$$d_0 = 4S_m t_1^4 + 4v_s t_1 = 4 \frac{A_m^2}{S_m} + 4v_s \sqrt{\frac{A_m}{S_m}}. \quad (14)$$

Then  $(t_1, t_2, t_3)$  can be determined as follows

**Case 1.1** If  $d_m > d_0$ , then  $A_m$  is reachable and  $t_1 = \sqrt{A_m / S_m}$ . Furthermore, there exists a constant acceleration period  $t_3 \neq 0$ . From (9),  $t_3$  can be computed by solving the following



quadratic equation in  $t_3$ :

$$d_m = 4S_m t_1^4 + 3S_m t_1^3 t_3 + \frac{1}{2} S_m t_1^2 t_3^2 + v_s(4t_1 + t_3). \quad (15)$$

It is easy to show that the above equation has a unique positive solution due to the fact  $d_m > d_0$ . The acceleration profile is  $(+, -, +)$  which is shown in Figure 2(c).

**Case 1.2** If  $d_m \leq d_0$ , then  $A_m$  is not reachable (unless  $d_m = d_0$ ) and  $t_3 = 0$ . Furthermore,  $t_1$  is determined by  $d_m$  based on (9) by solving the following equation in  $t_1$ :

$$d_m = 4S_m t_1^4 + 4v_s t_1. \quad (16)$$

The acceleration profile is  $(+, -, -)$  which is shown in Figure 2(d).

**Case 2.**  $J_m^2 < S_m A_m$ . By Proposition 2.1, if the jerk reaches  $J_m$  then  $t_1 = J_m/S_m$  and if the acceleration reaches  $A_m$  then  $t_2 = (S_m A_m - J_m^2)/(S_m J_m)$ . Under the above mentioned conditions, set

$$\begin{aligned} d_1 &= 4S_m t_1^4 + 4v_s t_1 = 4\frac{J_m^4}{S_m^3} + 4v_s \frac{J_m}{S_m} \\ d_2 &= 4S_m t_1^4 + 8S_m t_1^3 t_2 + S_m t_1 t_2^3 + 5S_m t_1^2 t_2^2 + v_s(4t_1 + 2t_2) \\ &= A_m \left( \frac{A_m}{J_m} + \frac{J_m}{S_m} \right)^2 + 2v_s \left( \frac{A_m}{J_m} + \frac{J_m}{S_m} \right). \end{aligned} \quad (17)$$

Then  $(t_1, t_2, t_3)$  can be determined as follows

**Case 2.1** If  $d_m > d_2$ , then both the jerk and acceleration can reach their maximal values and the constant acceleration phase  $t_3 \neq 0$  which can be computed by solving the quadratic equation (9) in  $t_3$ . The acceleration profile is  $(+, +, +)$  which is shown in Figure 2(a).

**Case 2.2** If  $d_1 < d_m \leq d_2$ , then the jerk can reach  $J_m$  and the acceleration cannot reach  $A_m$  unless  $d_m = d_2$ . Therefore, the stage of constant acceleration  $t_3 = 0$  and  $t_2$  is determined from (9) with  $d_m$  by solving the quadratic equation in  $t_2$

$$d_m = 4S_m t_1^4 + 8S_m t_1^3 t_2 + S_m t_1 t_2^3 + 5S_m t_1^2 t_2^2 + v_s(4t_1 + 2t_2).$$

The acceleration profile is  $(+, +, -)$  which is shown in Figure 2(b).

**Case 2.3** If  $d_m \leq d_1$ , then neither the jerk (unless  $d_m = d_1$ ) nor the acceleration reaches their maximal value and  $t_2 = t_3 = 0$ . Further,  $t_1$  is determined from (9) with  $d_m$  by solving the equation:  $d_m = 4S_m t_1^4 + 4v_s t_1$ . The acceleration profile is  $(+, -, -)$  which is shown in Figure 2(d).

Based on the above analysis, we give the algorithm to determine the end speed with a start speed  $v_s$  and a distance  $d_m$ . Again, the output of the algorithm is the parameters  $(t_1, t_2, t_3)$ , with which the end speed  $v_e$  can be computed with (7) and the speed function  $v(t)$  can be computed as  $v(t) = v_s + v_0(t)$  where  $v_0(t)$  is defined in (31).

**Algorithm 2.4 PRO\_VD**( $v_s, d_m$ )

**Input:** Parameters  $S_m, J_m, A_m$  in (10), the start speed  $v_s$ , and a distance  $d_m$ .

**Output:** The parameters  $(t_1, t_2, t_3)$  of the 7-period acceleration profile.

1. If  $J_m^2 \geq S_m A_m$ , then go to Step 2, else go to Step 3.

2. Let  $t_2 = 0$  and  $d_0$  be given by (14). If  $d_m > d_0$ , set  $t_1 = \sqrt{A_m/S_m}$  and compute  $t_3$  with equation (15); otherwise set  $t_3 = 0$  and compute  $t_1$  with equation (16). Output  $(t_1, t_2, t_3)$ .
3. Let  $d_1$  and  $d_2$  be given in (17).

$$\begin{cases} t_1 = \frac{J_m}{S_m}, t_2 = \frac{A_m - \frac{J_m^2}{S_m}}{J_m}, \text{ solve equation (9) to find } t_3, & \text{if } d_m > d_2 \\ t_1 = \frac{J_m}{S_m}, t_3 = 0, \text{ solve equation (9) to find } t_2, & \text{if } d_1 < d_m \leq d_2 \\ t_2 = t_3 = 0, \text{ solve equation (9) to find } t_1, & \text{if } d_m \leq d_1 \end{cases}$$

Output  $(t_1, t_2, t_3)$ .

Similar to Theorem 2.3, it can be shown that the algorithm is optimal in that it can give the maximal end speed when using the 7-period acceleration profile.

**Theorem 2.5** *Algorithm PRO\_VD outputs the maximal speed that can be reached with the start speed  $v_s$  and the distance  $d_m$  with the 7-period acceleration profile.*

Algorithm **PRO\_VD**( $v_s, d_m$ ) can be modified easily to compute the minimal speed that can be reached from  $v_s$  within distance  $d_m$ .

## 2.4 The AD profile with confined jounce, jerk, acceleration, and speed

With the preparations given in Sections 2.2 and 2.3, now a complete AD profile can be given to change the speed from  $v_s$  to  $v_e$  with distance  $d_m$  and under the given jounce, jerk, acceleration, and speed bounds.

As shown in Figure 4, the AD profile consists of three phases composed by 15 periods. Firstly, the speed is increased from the start speed  $v_s$  to the maximal speed  $V_m$  with a 7-period acceleration profile; secondly, the speed keeps the constant value  $V_m$ ; and thirdly, the speed decreases from  $V_m$  to the end speed  $v_e$  with a 7-period deceleration profile. Let  $(t_1, t_2, t_3)$  be the parameters for the 7-period acceleration profile in the first phase,  $t_4$  the time for the constant speed phase, and  $(t_5, t_6, t_7)$  the parameters for the 7-period deceleration profile in the third phase. Then the AD profile can be uniquely determined by the seven parameters  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$  and its velocity function can be computed with formula (18). Due to the initial conditions (11) of the 7-period profile, the velocity function of the AD profile is  $C^2$  continuous.

$$\bar{v}(t) = \begin{cases} v_s + v_0(t), v_0(t) \text{ is from (31) with } (t_1, t_2, t_3), & t \in [0, t_{m_1}] \\ V_m, & t \in [t_{m_1}, t_{m_1} + t_4] \\ V_m - v_0(t - t_{m_1} - t_4), v_0(t) \text{ is from (31) with } (t_5, t_6, t_7), & t \in [t_{m_1} + t_4, t_m] \end{cases} \quad (18)$$

where  $t_{m_1} = 4t_1 + 2t_2 + t_3$ ,  $t_{m_2} = 4t_5 + 2t_6 + t_7$ ,  $t_m = t_{m_1} + t_4 + t_{m_2}$ . From (9), the distance for the AD profile is

$$d = \frac{(v_s + V_m)t_{m_1}}{2} + t_4 V_m + \frac{(v_e + V_m)t_{m_2}}{2}. \quad (19)$$

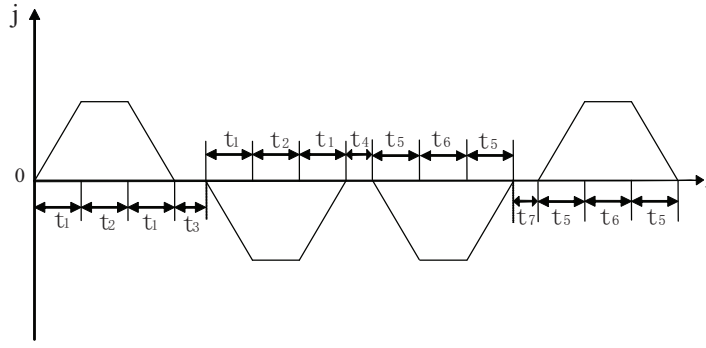


Figure 4: Jerk function for 15-period AD profile

First, a brief introduction to the algorithm is given. Without loss of generality, assume  $v_s \leq v_e$ . Firstly, check the reachability of  $v_e$  from  $v_s$  within distance  $d_m$ . The minimal distance  $d_{min}$  for accelerating from  $v_s$  to  $v_e$  can be determined using Algorithm **PRO\_VV**( $v_s, v_e$ ). If  $d_m \geq d_{min}$ , then  $v_e$  is reachable; otherwise the algorithm terminates.

Now suppose  $v_e$  is reachable. Algorithms **PRO\_VV**( $v_s, V_m$ ) and **PRO\_VV**( $v_e, V_m$ ) can be used to compute the minimal distances  $d_i$  and  $d_d$  for the speed to increase from  $v_s$  to  $V_m$  and to decrease from  $V_m$  to  $v_e$ . If  $d_i + d_d \leq d_m$ , then  $V_m$  is reachable and there is a phase of constant speed  $V_m$ .

If  $d_i + d_d > d_m$ , then  $V_m$  is not reachable and one need to compute the practical reachable maximal speed  $v_m$ . The speed  $v_m$  can be computed by solving a set of algebraic equations. But due to the complicated selection procedure for the parameters  $t_i, i = 1, 2, 3, 5, 6, 7$ , this approach is not practical. Instead, an iteration procedure will be used to compute  $v_m$ . Set  $v_0 = V_m, v_1 = v_e$ , and  $v_2 = (v_0 + v_1)/2$ . Then compute the distances  $d_i$  and  $d_d$  for the speed to increase from  $v_s$  to  $v_2$  and to decrease from  $v_2$  to  $v_e$ . If  $d_i + d_d \leq d_m$ , then  $v_2$  is reachable and set  $v_0 = \max\{v_0, v_1\}, v_1 = v_2$ . Otherwise,  $v_2$  is unreachable and set  $v_1 = \min\{v_0, v_1\}, v_0 = v_2$ . Repeat the above bisection procedure until  $|d_{i0} + d_{d0} - d_m|$  is smaller than a given precision, and use the last computed value of  $v_2$  as  $v_m$ .

With the above analysis, the algorithm to find the AD profile is given below.

**Algorithm 2.6** **PRO\_AD**( $v_s, v_e, d_m$ )

**Input:** Parameters  $S_m, J_m, A_m, V_m$ , the start speed  $v_s$ , the end speed  $v_e$ , and the distance  $d_m$ .

**Output:** The speed  $v_e$  is unreachable, or the parameters  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$  for the AD profile.

1. Compute  $(t_1, t_2, t_3)$  with Algorithm **PRO\_VV**( $v_s, v_e$ ). Set  $d_{min}$  to be the distance computed with formula (9) and  $(t_1, t_2, t_3)$ . If  $d_{min} > d_m$ , then  $v_e$  is unreachable and the algorithm terminates.
2. Let  $(t_1, t_2, t_3)$  and  $(t_5, t_6, t_7)$  be the outputs of Algorithm **PRO\_VV**( $v_s, V_m$ ) and Algorithm **PRO\_VV**( $v_e, V_m$ ) respectively, and  $d_i$  and  $d_d$  the distances for the speed to increase from  $v_s$  to  $V_m$  and to decrease from  $V_m$  to  $v_e$  computed with formula (9).
3. If  $d_i + d_d \leq d_m$ , then  $V_m$  is reachable. Compute  $t_4 = (d_m - d_i - d_d)/V_m$ . Outputs  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ .
4. Now  $d_i + d_d > d_m$  and  $V_m$  is unreachable. Set  $t_4 = 0, v_0 = V_m, v_1 = v_e, \epsilon = 10^{-4}$ .
  - (a) Set  $v_2 = (v_0 + v_1)/2$ .

- (b) Let  $(t_1, t_2, t_3)$  and  $(t_5, t_6, t_7)$  be the outputs of Algorithms **PRO\_VV** $(v_s, v_2)$  and **PRO\_VV** $(v_e, v_2)$  respectively, and  $d_i$  and  $d_d$  the distances for the speed to increase from  $v_s$  to  $v_2$  and to decrease from  $v_2$  to  $v_e$  computed with formula (9).
- (c) If  $|d_i + d_d - d_m| < \epsilon$ , outputs  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ .
- (d) If  $d_i + d_d \leq d_m$ , then  $v_2$  is reachable. Let  $v_0 = \max\{v_0, v_1\}$ ,  $v_1 = v_2$ . Goto Step (a).
- (e) Now  $d_i + d_d > d_m$  and  $v_2$  is unreachable. Let  $v_1 = \min\{v_0, v_1\}$ ,  $v_0 = v_2$ . Goto Step (a).

Based on Theorems 2.3 and 2.5, it can be shown that the above algorithm is time-optimal.

**Theorem 2.7** *Algorithm PRO\_AD outputs a time-optimal procedure for the speed to change from  $v_s$  to  $v_e$  with distance  $d_m$  under the constraints (10) and (11).*

### 3 Interpolation of parametric tool path with confined jounce

In this section, an interpolation method along a parametric tool path with confined jounce, jerk, acceleration, and speed will be proposed.

The algorithm works as follows. Firstly, by a transversal of the tool path, the critical points of the tool path, where the radius of curvature reaches local extremal values are obtained. Due to the relation between the chord error and speed, the maximal machining speed also reaches local extremal values at these points, which are called limit speeds. Secondly, by a backtracking algorithm, the limit speeds of the critical points are adjusted to make them reachable. Finally, Algorithm **PRO\_VV** or Algorithm **PRO\_AD** is used to interpolate a segment of the tool path between two adjacent critical points.

#### 3.1 Critical points on the tool path

This section will show how to compute the points on the tool path where the limit machining speed reaches local extremal values.

Let the tool path be a  $k$ th-order B-spline curve with expression

$$C(u) = \{x(u), y(u), z(u)\} = \sum_{i=0}^n Q_i N_{i,k}(u), \quad 0 \leq u \leq 1 \quad (20)$$

where  $Q_i$  ( $i = 0, 1, \dots, n$ ) are the  $n + 1$  control points,  $N_{i,k}(u)$  are the basis functions of  $k$ th-order B-splines based on the knot vector  $\mathbf{T} = \{t_0, t_1, \dots, t_{n+k+1}\}$ ,  $u$  is the parameter. Note that other parametric curves such as NURBS can be treated similarly.

Denote the parametric speed of  $C(u)$  to be

$$\sigma(u) = \frac{ds}{du} = |C'(u)|,$$

where  $'$  is the derivative w.r.t.  $u$ . The curvature and radius of curvature are defined to be

$$\kappa(u) = \frac{|C'(u) \times C''(u)|}{\sigma(u)^3}, \quad \rho(u) = \frac{1}{\kappa(u)}. \quad (21)$$

It is well known that at a point  $u = u_0$  of the tool path  $C(u)$ , the interpolation speed is limited by the given chord error  $\delta$ . Expanding  $C(u)$  in the neighborhood of  $C(u_0)$  with the following Taylor formula:

$$C(u_0 + \Delta u) = C(u_0) + \mathbf{a}_0 \Delta u + \frac{1}{2} \kappa_0 \mathbf{b}_0 (\Delta u)^2 + \frac{1}{6} (-\kappa_0^2 \mathbf{a}_0 + \dot{\kappa}_0 \mathbf{b}_0 + \kappa_0 \tau_0 \mathbf{c}_0 + \epsilon) (\Delta u)^3, \quad (22)$$

where  $\kappa_0, \tau_0$  are the curvature and torsion respectively,  $\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0$  are the values of tangential vector  $\mathbf{a}$ , the principal normal vector  $\mathbf{b}$ , the binormal vector  $\mathbf{c}$  at  $u = u_0$ , and  $\epsilon$  is a tiny quantity.

According to (22), when using line segments to approximate the spline curve, the second-order residual term is only related with the curvature and the third-order term is related with the torsion. Therefore, a circle whose radius is  $\rho(u_0)$  can approximate the curve locally. The contour error of the interpolation is limited by the chord error  $\delta$  which is caused by approximating the real curve using the short line segment within one sampling period.

Let  $\delta$  be the chord error bound, and  $\Delta u$  the maximal possible increase of the parameter  $u$  without violating the chord error bound. Thus the following well-known relationship is obtained [19]:

$$\Delta L(u) = |C(u)C(u + \Delta u)| \approx \sqrt{8\rho\delta}.$$

Let  $T$  be the sampling period. So the maximal possible speed at point  $C(u)$  constrained by the chord error bound is

$$v_{lim}(u) = \Delta L(u)/T = \sqrt{8\rho(u)\delta}/T. \quad (23)$$

The speed  $v_{lim}(u)$  is called the *limit speed* at point  $C(u)$ .

According to (23),  $v_{lim}(u)$  has the same extremal points with the radius of curvature function  $\rho(u)$ , which has the same extremal points with the curvature function  $\kappa(u)$  by (21). These extremal points are called *critical points* of the tool path. The extremal points of  $v_{lim}(u)$  can be computed by solving the following algebraic equations in  $u$  for each piece of the curve  $C(u)$ .

$$\kappa(u)' = 0. \quad (24)$$

Furthermore, if the continuity of the spline at the connection point is less than  $C^2$ , then these points need to be treated as critical points and the minimal value of the limit speeds of the two curve segments at the connect point is taken as the limit speed.

The extremal points can also be computed in a discrete way as follows. First make a traversal of the curve  $C(u), u \in [0, 1]$  with the speed  $v_{lim}(u)$  and a given sampling period  $T$  to find interpolation points  $u_0 = 0, u_1, \dots, u_n = 1$ . Then we may find the extremal points from the values  $v_{lim}(u_i)$ . Based on the above analysis, the following algorithm is given to compute the extremal points.

**Algorithm 3.1** CR\_PTS( $C(u), u \in [0, 1]$ )

**Input:**  $B$ -spline curve  $C(u), u \in [0, 1]$ , the sampling period  $T$ , the maximal speed  $V_m$  of the CNC machining.

**Output:** Parametric values of the critical points  $u_i, i = 0, \dots, N$ .

1. Use Algorithm 3.5 with inputs  $C(u), u \in [0, 1]$  and  $v_{lim}(u), u \in [0, 1]$  (step size at  $u = u_0$  can be found as  $\Delta L = \min(v_{lim}(u_0), V_m)T$ ) to find the interpolation points  $\bar{u}_0 = 0, \bar{u}_1, \dots, \bar{u}_m = 1$ .
2. Select the parameters  $\bar{u}_k$  correspond to the extremal points. That is, the  $\bar{u}_k$  satisfying

$$(v_{lim}(\bar{u}_k) - v_{lim}(\bar{u}_{k-1}))(v_{lim}(\bar{u}_k) - v_{lim}(\bar{u}_{k+1})) \geq 0.$$

Denote these parameters as  $u_1, \dots, u_{N-1}$ . Output  $u_0 = 0, u_1, \dots, u_{N-1}, u_N = 1$ .

**Remark 3.2** When the parametric curve is simple, the extremal points can be found by directly solving equation (24). For parametric curves with high degrees, use Algorithm 3.1 to compute the extremal points.

Let  $C(u_i), i = 0, \dots, N$  be the points found with the above algorithm. If the limit speed  $v_{lim}(u)$  achieves a local minimal (maximal) value at  $u_i$ , then  $u_i$  is called a *minimal (maximal) critical point*. If the curve  $C(u)$  is at least second order differentiable, minimal and maximal critical points occur alternatively.

### 3.2 Reachability test

The limit speeds at the critical points might not be reachable due to the constraints of distance between two adjacent critical points. It is necessary to check reachability of the speeds at two adjacent critical points and adjust the unreachable speeds to make them reachable.

Consider the reachability from point  $C(u_k)$  with limit speed  $v_k$  to  $C(u_{k+1})$  with limit speed  $\hat{v}_{k+1}$ . Let  $L_k$  be the length of the curve segment  $C(u), u \in [u_k, u_{k+1}]$ . With Algorithm **PRO\_VD**( $v_k, L_k$ ), compute the maximal speed  $v^+$  that can be reached from  $v_k$  within distance  $L_k$  and the minimal speed  $v^-$  that can be reached from  $v_k$  within distance  $L_k$ . If  $v^- \leq \hat{v}_{k+1} \leq v^+$ , then  $v_{k+1} = \hat{v}_{k+1}$  is reachable. If  $\hat{v}_{k+1} > v^+$ , then  $v_k$  cannot increase to  $\hat{v}_{k+1}$  and set  $v_{k+1} = v^+$ . If  $v^- > \hat{v}_{k+1}$ , then  $v_k$  cannot decrease to  $v_{k+1} = \hat{v}_{k+1}$  and it is necessary to do a backtracking by checking the reachability starting from point  $u_{k+1}$  in the backward direction.

Based on the above analysis, the reachability test and adjustment algorithm is given below.

**Algorithm 3.3 REACH**( $u_i, i = 0, \dots, N$ )

**Input:** Critical points  $u_i, i = 0, \dots, N$  of the tool path  $C(u)$  assuming  $u_0 = 0$  and  $u_N = 1$ . The sampling period  $T$  and the maximal speed  $V_m$  of the CNC machining.

**Output:** Reachable speeds  $v_k, i = 0, \dots, N$  at the critical points.

1. Set  $i = 0, v_0 = 0, v_N = 0$ .
2. If  $i = N$ , then terminate the algorithm; otherwise let  $L_i$  be the length of the curve segment  $C(u), u \in [u_i, u_{i+1}]$ .
3. Using Algorithm **PRO\_VD**( $v_i, L_i$ ) to compute the maximal speed  $v^+$  and minimal speed  $v^-$  that can be reached from  $v_i$  with distance  $L_i$ .
4. If  $i + 1 = N$  then set  $\hat{v}_{i+1} = 0$ , else set  $\hat{v}_{i+1} = \min\{v_{lim}(u_{i+1}), V_m\}$ , where  $v_{lim}$  is defined in (23).
5. If  $v^- \leq \hat{v}_{i+1} \leq v^+$ , then  $\hat{v}_{i+1}$  is reachable, set  $v_{i+1} = \hat{v}_{i+1}$ ,  $i = i + 1$ , goto Step 2.
6. If  $\hat{v}_{i+1} > v^+$ , then set  $v_{i+1} = v^+$ ,  $i = i + 1$ , goto Step 2.
7. If  $\hat{v}_{i+1} < v^-$ , then set  $v_{i+1} = \hat{v}_{i+1}$ ,  $i = i + 1$ ,  $j = i$ , and execute the following backtracking steps.
  - (a) Using Algorithm **PRO\_VD**( $v_j, L_{j-1}$ ) to compute the maximal speed  $v^*$  that can be reached from  $v_j$  with distance  $L_{j-1}$ .
  - (b) If  $v_{j-1} \leq v^*$ , then from  $v_{j-1}$  to  $v_j$  is reachable. Goto Step 2.
  - (c) If  $v_{j-1} > v^*$ , then from  $v_{j-1}$  to  $v_j$  is not reachable. Set  $v_{j-1} = v^*$ ,  $j = j - 1$ , and if  $j \geq 2$ , then goto step (a); else goto Step 2.

Let  $C(u_i), i = 0, \dots, N$  be the *critical points*. After the adjustments, the speeds at certain critical points are decreased and the property of alternating minimal and maximal speeds is not true anymore. So delete some of the critical points to make the speed alternatively minimal and maximal again. That is, a point  $u_i$  is removed if  $(v_i - v_{i-1})(v_i - v_{i+1}) < 0$ . Note that this strategy enhances the interpolation speed.

### 3.3 Interpolation with confined jounce

In this section, an algorithm will be given to achieve interpolation with confined jounce, jerk, acceleration, and speed. And the chord error is kept as small as possible. As usual, the interpolation procedure is divided into two phases: the velocity planning phase to find the velocity function and the parameter computing phase to find the interpolation points.

First, consider the velocity planning phase. Let  $C(u_i), i = 0, \dots, N$  be the critical points of the tool path  $C(u)$ . Then, from point  $C(u_i)$  to  $C(u_{i+1})$ , the limit speed is either monotonously increasing or monotonously decreasing. As a consequence, Algorithm **PRO\_VV** can be used to interpolate the tool path segments between two adjacent critical points.

Let  $v(t), t \in [0, t_m]$  be the speed function obtained with Algorithm **PRO\_VV**( $v_i, v_{i+1}$ ),  $d_m$  computed with (9), and  $L_i$  the length of the curve segment  $C(u), u \in [u_i, u_{i+1}]$ . Then  $\bar{v}(t) = v(\frac{td_m}{L_i}), t \in [0, \frac{t_m L_i}{d_m}]$  is a speed function which increases the speed from  $v_i$  to  $v_{i+1}$ . Since the distance for this procedure is  $L_i$ ,  $\bar{v}(t)$  interpolates the curve segment  $C(u), u \in [u_i, u_{i+1}]$ . Note that between two adjacent critical points,  $\bar{v}(t)$  and the velocity limit curve  $v_{lim}(u)$  are both monotone and are equal at the end points. Then, the chord error is approximately kept within the give bound. Due to the initial conditions (11) of the 7-period profile, the speed function thus obtained is  $C^2$  continuous.

If the maximal speed  $V_m$  is reached at certain critical point, the above algorithm does not use the full power of the machine. Let us explain this in detail. Suppose the  $v_{lim}(u_i) < V_m$  and  $v_{lim}(u_{i+1}) > V_m$  and there is a  $u^* \in (u_i, u_{i+1})$  such that the limit speed at each point in  $[u^*, u_{i+1}]$  is larger than  $V_m$ . Then the above algorithm cannot reach  $V_m$  at any point in  $[u^*, u_{i+1}]$ . In general, to find  $u^*$  is computationally costs. A simple and improved idea is to use Algorithm **PRO\_AD** instead of Algorithm **PRO\_VV**. If  $v_i < V_m$  and  $v_j = V_m, j = i + 1, \dots, i + s$ , then Algorithm **PRO\_AD** is used to interpolate  $C(u), u \in [u_i, u_{i+s+1}]$ . Notice that Algorithm **PRO\_AD** will find an approximate value  $u^*$  after which the maximal speed  $V_m$  will be used.

Based on the above analysis, the following velocity planning algorithm with confined jounce is given.

**Algorithm 3.4 VP\_CJ**( $C(u), u \in [0, 1]$ )

**Input:** *B-spline curve  $C(u), u \in [0, 1]$  as the tool path, the parameters  $S_m, J_m, A_m, V_m, T$ , and the chord error bound  $\delta$ .*

**Output:** *A velocity function for the tool path, which satisfies the constraints.*

1.  $s = 1, u_1 = 0, k = 1$ .
2. Use Algorithm **CR\_PTS**( $C(u), u \in [0, 1]$ ) to compute the critical points  $\hat{u}_i, i = 1, \dots, p$ .
3. Use Algorithm **REACH**( $\hat{u}_i, i = 1, \dots, p$ ) to compute the reachable speeds  $v_i, i = 1 \dots, p$ , at these points.
4. If  $v_{k+1} < V_m$ , goto step 5, else goto step 6.
5. Interpolation with Algorithm **PRO\_VV**.
  - (a) Use Algorithm **PRO\_VV**( $v_k, v_{k+1}$ ) to find  $(t_1, t_2, t_3)$ .
  - (b) Let  $v_0(t)$  be the speed function in (31) determined with parameters  $(t_1, t_2, t_3)$ ,  $t_m = 4t_1 + 2t_2 + t_3$ , and  $d_m$  computed with (9). If  $v_k \leq v_{k+1}$ , the velocity function is  $v(t) = v_k + v_0(t)$ ; else the velocity function is  $v(t) = v_k - v_0(t)$ .

- (c) Let  $L$  be the length of the curve segment  $C(u), u \in [u_k, u_{k+1}]$ .
- (d) Let  $\bar{t}_m = \frac{t_m L}{d_m}$  and set  $\bar{v}(t) = v(\frac{t d_m}{L}), t \in [0, \bar{t}_m]$  to be the velocity function for the curve segment  $C(u), u \in [u_k, u_{k+1}]$ .
- (e) Set  $k = k + 1$  and goto step 4.

6. *Interpolation with Algorithm PRO\_AD.*

- (a) Let  $w$  be the largest number satisfying  $v_{k+j} = V_m, j = 1, \dots, w$ .
- (b) Let  $L$  be the length of the curve segment  $C(u), u \in [u_k, u_{k+w+1}]$ .
- (c) Use Algorithm **PRO\_AD**( $v_k, v_{k+w+1}, L$ ) to find  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ .
- (d) Let  $d_c = (4t_1 + 2t_2 + t_3)(v_k + V_m)/2 + V_m t_4 + (4t_5 + 2t_6 + t_7)(v_{k+w+1} + V_m)/2$  be the distance of the AD profile from (19).
- (e) Let  $v(t)$  be the speed function computed with (18) with parameters  $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ .
- (f) Let  $t_m = 4t_1 + 2t_2 + t_3 + t_4 + 4t_5 + 2t_6 + t_7$  and  $\bar{t}_m = \frac{t_m L}{d_c}$ . Set  $\bar{v}(t) = v(\frac{d_c t}{L}), t \in [0, \bar{t}_m]$  to be the velocity function for the curve segment  $C(u), u \in [u_k, u_{k+w+1}]$ .
- (g) Set  $k = k + w + 1$  and goto step 4.

After the velocity function is obtained, the parameter values of the interpolation points need to be computed. Suppose the curve segment to be interpolated is  $C(u), u \in [u_1, u_2]$  and a velocity function  $\bar{v}(t), t \in [0, t_m]$  is obtained with Algorithm **VP\_CJ**. Let  $T$  be the sampling period. Then the speed at the interpolation points can be obtained as  $v_j = \max(\bar{v}(T \cdot j), 0.1)$  for  $j = 0, 1, \dots$ . Note that we need to take 0.1 as the minimal interpolation velocity to ensure that the cutter can start to move at the beginning of a spline curve.

Suppose that the current parameter value is  $u_0$  which corresponds to the interpolation speed  $v_j$ . To find the next parametric value  $u_0 + \Delta u$ , first compute the step size

$$\Delta L = v_j T \approx |C(u_0)C(u_0 + \Delta u)|,$$

where  $T$  is the sampling period. Because the interpolation distance  $\Delta L$  in one period can be approximately treated as the arc length, the approximate relationship between two adjacent parameters about arc distance is [17]

$$\Delta u = \frac{\Delta L}{\sigma(u_0)} - \frac{C'(u_0) \cdot C''(u_0) \Delta L^2}{2\sigma(u_0)^4} + o(\Delta L^2). \quad (25)$$

Since  $\Delta L$  is a small quantity, from (25) we can compute  $\Delta u$  with the first order Taylor approximation

$$\Delta u = \frac{\Delta L}{\sigma(u_0)} \quad (26)$$

or with the second order Taylor approximation

$$\Delta u = \frac{\Delta L}{\sigma(u_0)} - \frac{D \Delta L^2}{2\sigma(u_0)^4} \quad (27)$$

where  $D = (\frac{dx}{du} \frac{d^2x}{du^2} + \frac{dy}{du} \frac{d^2y}{du^2} + \frac{dz}{du} \frac{d^2z}{du^2})_{u=u_0}$ . If the approximation fails, then use Newton's iterative method or the dichotomy to find  $u_0 + \Delta u$ .

Based on the above analysis, the following algorithm to compute the next interpolation point is given.



**Algorithm 3.5** NEXT\_U( $C(u), u_0, \Delta L$ )**Input:**  $B$ -spline curve  $C(u)$   $u \in [0, 1]$ , the current parameter  $u_0$ , and the next step size  $\Delta L$ .**Output:** The next interpolation parameter  $u_n$ .

1. Compute the second order approximate  $\Delta u$  with (27).
2. If  $\Delta u > 0$ , set  $u_n = u_0 + \Delta u$  and return  $u_n$ .
3. If  $\Delta u < 0$ , then use Newton iterative method to find the next parametric value.

(a) Construct the following objective function in  $u$ 

$$f(u) = (x(u) - x(u_0))^2 + (y(u) - y(u_0))^2 + (z(u) - z(u_0))^2 - \Delta L^2.$$

(b) Use Newton's iterative method to find the next interpolation parametric value with initial value  $u = u_0 + \frac{\Delta L}{\sigma(u_0)}$  (first order approximation)

$$u^* = u - \frac{f(u)}{f'(u)}.$$

(c) If  $f(u^*) > \varepsilon$ ,  $u^* < u_0$ , or  $u^* \gg u_0$ , then the method fails, where  $\varepsilon$  is a given tiny positive number; otherwise set  $u_n = u^*$  and return  $u_n$ .

4. If the Newton method fails, then use the dichotomy to compute a solution  $u_n$  of  $f(u) = 0$  in  $(u_0, u_r)$ , where  $u_r = u_0 + k \frac{\Delta L}{\sigma(u_0)}$  and  $k$  is the smallest natural number such that  $f(u_r) > 0$ . Return  $u_n$ .

In most cases, formula (27) gives a nice approximation and the iterative methods are not needed.

**Remark 3.6** The proposed method can be implemented in CNC controllers in the following way. Firstly, the velocity functions are computed off-line with Algorithm **VP\_CJ**. Then the velocity functions are used as parts of the input to the CNC controllers to achieve real-time interpolation. Of course, the CNC controller must be modified to accept the new G codes. This strategy is adopted by many existing work such as [6, 19, 20, 22]. In particular, industrial CNC machining is realized in [19, 22] using this approach. The idea of the above approach is to use a pretreatment to add the velocity function to the G codes and then use the new G codes for CNC machining.

### 3.4 An illustrative example

An example is used to illustrate the algorithm. The tool path in Figure 5(a) is a planar quadratic B-spline  $(x(u), y(u))$ ,  $u \in [0, 1]$  with  $C^1$  continuity and consisting of six pieces of quadratic curve segments. Use the following parameters:  $V_m = 50mm/s$ ,  $A_m = 1500mm/s^2$ ,  $J_m = 2 \times 10^5 mm/s^3$ ,  $S_m = 2 \times 10^8 mm/s^4$ ,  $T = 1ms$ ,  $\delta = 0.2\mu m$ .

Firstly, compute the critical points of the B-spline curve with Algorithm **CR\_PTS**, and the parameter values  $u_i$  of the critical points are given in the first column of Table 1. The  $y$  and  $z$  coordinates of critical points are listed respectively in the second and third columns.  $v_i = v_{lim}(u_i)$  is listed in the fourth column. At  $u = 0$ , the velocity is set to be  $0.1mm/s$  to ensure that cutter can move at the beginning. In Step 3, check the reachability of the limit speeds  $v_{lim}(u_i)$  at the critical points using Algorithm **REACH**, and the adjusted velocity is shown in the fifth column of Table 1. Note that the speeds at  $u = 0.7208, 0.8354, 0.9069, 0.9438$  are obtained by the backtracking step of Algorithm **REACH** due to the final speed at  $u = 1.0$ .

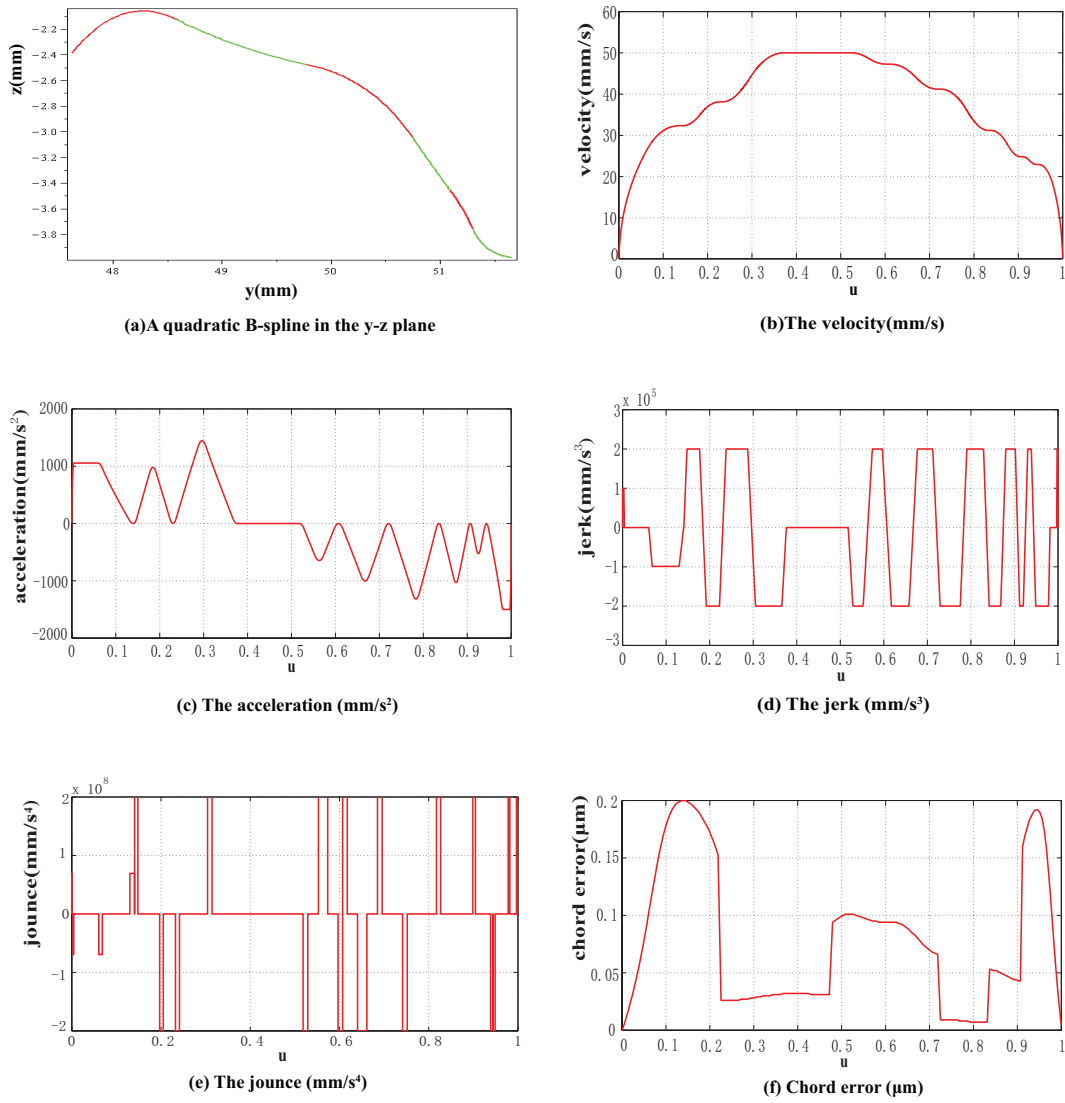


Figure 5: Velocity, acceleration, jerk, jounce and chord error functions of a quadratic spline. The horizontal axis is the parameter  $u$  for the tool path in (b), (c), (d), (e), and (f).

The speed function obtained is shown in Figure 5(b). Its acceleration, jerk, and jounce are shown in Figures 5(c), 5(d), 5(e) respectively. From Figures 5(b-e), it is not difficult to see that either the jounce, jerk, acceleration, or the velocity reaches their maximal values except in  $I_1 = [0, 0.1407]$ . By the results proved in Section 3, this means that the velocity is time-optimal except in  $I_1$ . In the parameter interval  $I_1$ , the tool-path length  $L$  is larger than the length  $d_m$  of the 7-period profile to increase the speed from zero to the limit speed at  $u = 0.1407$  in Step 6(d) of Algorithm **VP\_CJ**. As a consequence, the velocity function over  $I_1$  is not time optimal. Figure 5(f) gives the theoretic chord error under the planned velocity and the sampling period  $T$ , which is less than the given bound  $\delta$  for all parameter values.

Table 1: Information of critical points (unit for speed:  $mm/s$ )

$u_i$	$y(u_i)$ (mm)	$z(u_i)$ (mm)	$v_i$	adjusted $v_i$
0	47.62	-2.39	49.654	0.1
0.1407	48.2112	-2.06188	32.3387	32.3387
0.2303	48.6154	-2.13848	93.2816	38.124
0.4017	49.4231	-2.40066	91.7538	50
0.4724	49.7656	-2.47271	92.0354	50
0.6071	50.3573	-2.69218	47.2607	47.2607
0.7208	50.7513	-3.04704	140.729	41.2157
0.8354	51.1049	-3.46825	48.4228	31.1962
0.9069	51.3000	-3.75227	51.5066	24.8385
0.9438	51.4077	-3.88634	23.9501	22.9357
1.0000	51.66	-3.98	31.4245	0.1

The velocity of the key point given by Algorithm **REACH** can be improved. Before doing the reachability test, delete some useless key points according to whether its velocity is constrained by the others. Precisely, if the velocity of parameter  $u_i$  is  $v(u_i)$ , then from this point, under a given length  $l$ , the velocity is limited. So use an acceleration profile from the start point  $(u_i, v(u_i))$ , with  $j(u_i) = a(u_i) = 0$ , form a forward velocity function  $v^+(t)$  or  $v^+(u)$ . Check whether  $v^+(u_{i+1}) < v(u_{i+1})$ . If  $v^+(u_{i+1}) < v(u_{i+1})$  and  $v^+(u_{i+1}) < V_m$ , then the key point  $(u_{i+1}, v(u_{i+1}))$  can be deleted since the final velocity at  $u_{i+1}$  will never exceed  $v(u_{i+1})$ . Finally, use the renewed key points to do the velocity planning.

By this improved method, the selected critical points and new velocities are given in Table 2. The key points  $u = 0.2303$ ,  $u = 0.8354$  and  $u = 0.9069$  are deleted according to the above rule. The velocity functions of the improved method and Algorithm 3.4 are given in Figure 6, which shows that the velocity function obtained with the improved method is better than the previous one.

Table 2: Information of critical points with the improved method (unit for speed:  $mm/s$ )

$u_i$	$y(u_i)$ (mm)	$z(u_i)$ (mm)	$v_i$	selected or deleted	adjusted $v_i$
0	47.62	-2.39	49.654	selected	0.1
0.1407	48.2112	-2.06188	32.3387	selected	32.3387
0.2303	48.6154	-2.13848	93.2816	deleted	-
0.4017	49.4231	-2.40066	91.7538	selected	50
0.4724	49.7656	-2.47271	92.0354	selected	50
0.6071	50.3573	-2.69218	47.2607	selected	47.2607
0.7208	50.7513	-3.04704	140.729	selected	50
0.8354	51.1049	-3.46825	48.4228	deleted	-
0.9069	51.3000	-3.75227	51.5066	deleted	-
0.9438	51.4077	-3.88634	23.9501	selected	22.9357
1.0000	51.66	-3.98	31.4245	selected	0.1

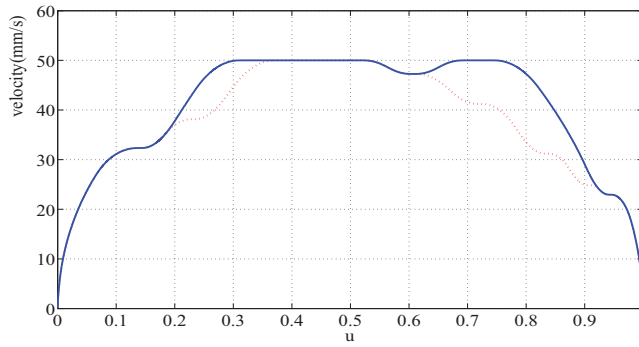


Figure 6: Red one: velocity function of Algorithm 3.4. Blue one: velocity function of the improved method

## 4 Experimental results

In this section, experimental results will be presented. The first experiment shows that confined jounce indeed can be used to reduce vibration of the CNC machine tools. The second experiment shows the feasibility of the interpolation method by simulating two real CNC machining models.

### 4.1 Vibration test

In this section, we will compare three AD profiles with confined acceleration, confined jerk, and confined jounce in a CNC machine to show that confined jounce indeed can be used to reduce vibration comparing to confined jerk and acceleration.

The equipment used to test vibration is LMS SCADAS3 which is a fast signal acquisition and analysis system, including hardware and software components. The hardware component consists of sensors, excitors, data acquisition tools, and a laptop PC. The software component can be used for real-time signal acquisition, calculation, analysis, display, record, and output. Figure 7 shows the LMS SCADAS3 sensor attached to a CNC machine.

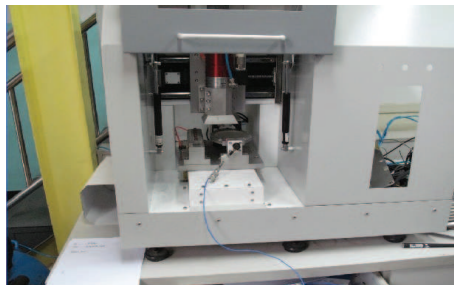


Figure 7: The CNC machine and the sensor attached on the machine

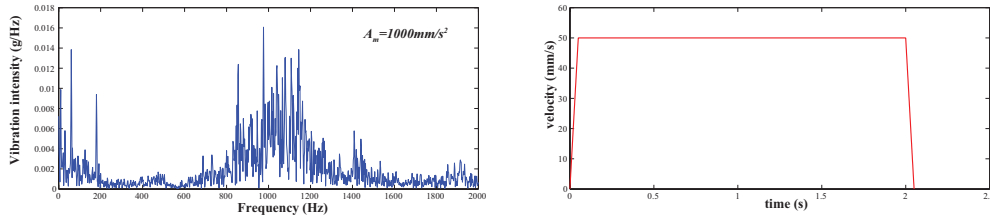
In the vibration test, the tool path is a line segment of  $100\text{mm}$ , and the CNC machine tool will idly move along the line segment. Three AD profiles are used respectively to generate the velocity function: the linear AD profile with confined acceleration, the S-splate AD profile with confined jerk, and the AD profile with confined jounce introduced in this paper. The maximal speed  $V_m$ , acceleration  $A_m$ , jerk  $J_m$ , jounce  $S_m$ , and the machining time  $t$  are given in Table 3. The three velocity functions

are shown in the right hand side of Figure 8.

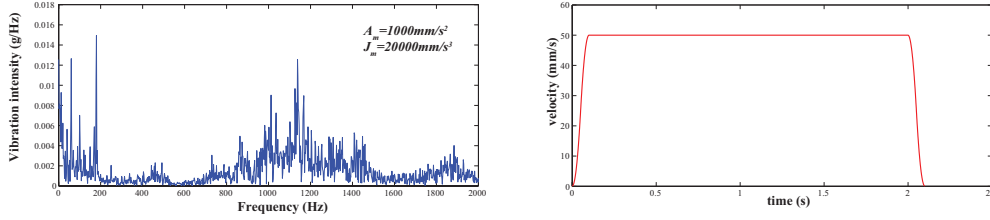
Table 3: Parameters of tests (a), (b), (c) and the interpolation time

Test	$S_m(mm/s^4)$	$J_m(mm/s^3)$	$A_m(mm/s^2)$	$V_m(mm/s)$	$d(mm)$	$t(s)$
(a)	—	—	1000	50	100	2.051
(b)	—	20000	1000	50	100	2.101
(c)	200000	20000	1000	50	100	2.201

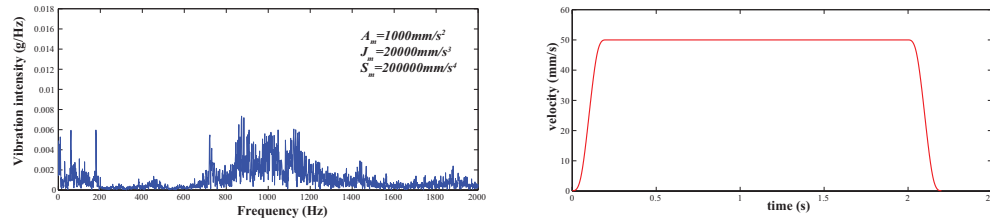
The sampling time of LMS SCADAS3 is one second and the sampling frequency is 4096Hz. Since the machining time is about two seconds, we record the signals from machining the first half of the line segment only. The vibration frequency spectrum diagrams of the three tests are given in the left hand side of Figure 8, where the vertical axis is the vibration intensity whose unit is g/Hz and g is the gravitational acceleration, and the unit of horizontal axis is Hz. The spectrum diagram gives the distribution of the intensity of the vibration at difference frequencies. For instance, from Figure 8(a), the strongest vibration with intensity 0.016g/Hz occurs when the machine tool vibrates at frequency 980Hz.



(a) Vibration frequency spectrum and velocity of motion profile under confined acceleration



(b) Vibration frequency spectrum and velocity of motion profile under confined jerk



(c) Vibration frequency spectrum and velocity of motion profile under confined jounce

Figure 8: The vibration frequency spectrum diagrams and the corresponding velocity functions

From Figure 8, one can see that for confined acceleration, the maximal vibration is 0.016g/Hz, and

the machine vibrates relatively strong ( $> 0.004g/Hz$ ) in the frequency intervals  $[0, 200]$  and  $[820, 1450]$ . For confined jerk, the maximal vibration is  $0.0145g/Hz$ , and the machine vibrates relatively strong ( $> 0.004g/Hz$ ) in the frequency intervals  $[0, 180]$  and  $[850, 1430]$ . For confined jounce, the maximal vibration is  $0.008g/Hz$ , and the machine vibrates relatively strong ( $> 0.004g/Hz$ ) in the frequency intervals  $[0, 180]$  and  $[870, 1220]$ . It is very clear that in the case of confined jerk, the machine vibrates significantly less than that of confined acceleration, and in the case of confined jounce, the machine vibrates significantly less than that of confined jerk.

From Table 3, the machining times for confined acceleration, confined jerk, and confined jounce are 2.051s, 2.101s, 2.201s respectively. Comparing to the confined acceleration and confined jerk, the machining time with confined jounce increases about 7.3% and 4.8% respectively.

In summary, using confined jounce, with the loss of not much machining time, we can reduce the vibration significantly comparing with confined acceleration and confined jerk.

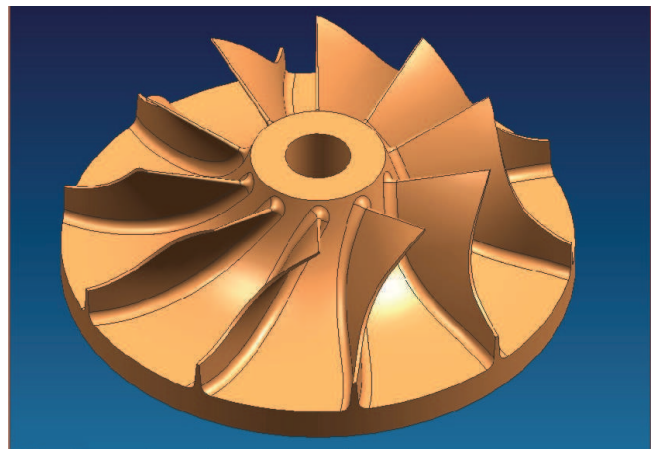
It is known that the machining quality is affected by many factors, and vibration of the machining tool is certainly one of the important factors. We choose to test vibration instead of machining quality directly for the following reason. By running the CNC machine tool idle, the affect of the adopted AD profile on the vibration becomes more significant, and from which we can see more clearly the difference of the three AD profiles.

## 4.2 Simulation results on two CNC machining models

In this section, simulation results on two CNC machining models will be presented to show the feasibility of the interpolation algorithm. All the computations are carried out using C language on a PC with Windows XP, a 2.13GHz CPU, and 1.98GB memory.



(a) The vase model



(b) The impeller model

Figure 9: Two CNC machining models

The first model is the “vase” shown in Figure 9(a). The vase is a CNC model consisting of more than 116000 G01 codes with total tool path length of 46.67m. Its B-spline representation consists of more than 42000 quadratic curve segments [22]. The G01 codes and the spline representation of the model can be found in <http://www.mmrc.iss.ac.cn/~xgao/cnc/vase.html>. The tool path of the vase

Table 4: The machining times of “vase” model under different constraints. The unit for  $S_m$  is  $mm/s^4$ .

$A_m$ ( $mm/s^2$ )	$J_m = 5 \times 10^5 mm/s^3$		$J_m = 2 \times 10^5 mm/s^3$	
	$S_m = 2 \times 10^8$	$S_m = 1 \times 10^8$	$S_m = 2 \times 10^8$	$S_m = 1 \times 10^8$
1000	25.1167min	26.3298min	26.4793min	27.2322min
2000	22.3648min	23.5293min	24.0420min	24.8345min
3000	21.4223min	23.1568min	23.4963min	24.2998min
6000	20.7693min	22.5608min	23.2368min	24.0440min

Table 5: The machining time of “impeller” model under different constraints. The unit for  $S_m$  is  $mm/s^4$ .

$A_m$ ( $mm/s^2$ )	$J_m = 5 \times 10^5 mm/s^3$		$J_m = 2 \times 10^5 mm/s^3$	
	$S_m = 2 \times 10^8$	$S_m = 1 \times 10^8$	$S_m = 2 \times 10^8$	$S_m = 1 \times 10^8$
1000	0.5757min	0.5816min	0.5830min	0.5856min
2000	0.4583min	0.4647min	0.4704min	0.4759min
3000	0.3982min	0.4153min	0.4259min	0.4297min
6000	0.3549min	0.3650min	0.3997min	0.4037min

fluctuates violently making optimal speed velocity planning difficult. The spline curve in Figure 5 is from this model.

The second model is one piece of blade in the “impeller” shown in Figure 9(b), which is a cubic B-spline with  $C^2$  continuity and consisting of 394 cubic curve segments [22]. The total tool path length of the blade is 3.167m. The G01 codes and the spline representation of the model can be found in <http://www.mmrc.iss.ac.cn/~xgao/cnc/blade.html>. Note that this is a 5-axis model:  $(x(u), y(u), z(u), A(u), C(u)) u \in [0, 1]$ . For simplicity, the velocity for the  $(x, y, z)$  tool path is planned with the method introduced in this paper and the parameter values  $u_i$  thus obtained are used to compute the angular interpolation positions  $(A(u_i), C(u_i))$ .

In both experiments, the following parameters are used:  $V_m = 200mm/s$ ,  $T = 1ms$ ,  $\delta = 0.2\mu m$  and  $S_m$ ,  $J_m$  and  $A_m$  are taken different values for comparison, which are shown in Table 4.

Note that in our method, the speed decreases to zero at the connection points of two splines. This could be further improved using the method given in [7].

Table 4 and Table 5 give the machining timings for the “vase” model and the “impeller” model under different bounds of acceleration, jerk, and jounce.

Table 6 and Table 7 give the times needed to compute the velocity function and the interpolation points for the “vase” and “impeller” models respectively. In Table 6 and Table 7, “ $t_V$ ” and “ $t_P$ ” denote the times to compute the velocity function and the interpolation points respectively.

From Table 6 and Table 7, we can see that the computation time of the algorithm is much less than the machining time. As mentioned in Remark 3.6, the velocity function is computed off-line and the interpolation points are computed in real time. For the “vase” model and the “impeller” model, the times to compute the interpolation point are 13%-18% and 5%-8% of the machining times, indicating that the algorithm is feasible for real time implementation.

Table 6: Time to compute velocity functions ( $t_V$ ) and interpolation points ( $t_P$ ) for “vase” model. The unit for  $S_m$  is  $mm/s^4$ .

$A_m$ ( $mm/s^2$ )	$J_m = 5 \times 10^5 mm/s^3$				$J_m = 2 \times 10^5 mm/s^3$			
	$S_m = 2 \times 10^8$		$S_m = 1 \times 10^8$		$S_m = 2 \times 10^8$		$S_m = 1 \times 10^8$	
	$t_V$	$t_P$	$t_V$	$t_P$	$t_V$	$t_P$	$t_V$	$t_P$
1000	117.06s	205.42s	119.03s	207.40s	120.59s	210.21s	120.37s	211.36s
2000	131.98s	207.93s	135.73s	208.08s	136.08s	209.04s	141.16s	217.21s
3000	139.59s	204.47s	143.06s	209.28s	142.20s	210.98s	143.45s	210.13s
6000	143.34s	204.05s	147.91s	208.89s	142.57s	206.69s	147.32s	211.80s

Table 7: Time to compute velocity functions ( $t_V$ ) and interpolation points ( $t_P$ ) for “impeller” model. The unit for  $S_m$  is  $mm/s^4$ .

$A_m$ ( $mm/s^2$ )	$J_m = 5 \times 10^5 mm/s^3$				$J_m = 2 \times 10^5 mm/s^3$			
	$S_m = 2 \times 10^8$		$S_m = 1 \times 10^8$		$S_m = 2 \times 10^8$		$S_m = 1 \times 10^8$	
	$t_V$	$t_P$	$t_V$	$t_P$	$t_V$	$t_P$	$t_V$	$t_P$
1000	0.594s	1.844s	0.609s	1.813s	0.690s	1.716s	0.638s	1.800s
2000	0.702s	1.735s	0.595s	1.827s	0.598s	1.855s	0.530s	1.907s
3000	0.658s	1.779s	0.671s	1.767s	0.629s	1.855s	0.592s	1.876s
6000	0.577s	1.798s	0.595s	1.795s	0.549s	1.888s	0.591s	1.877s

## 5 Conclusion

In this paper, an acceleration/deceleration profile with confined jounce, jerk, acceleration, and speed is designed and its properties are studied. In particular, it is shown that the profile is time-optimal to change the speed from one value to another under the given constraints. Based on the jounce confined acceleration/deceleration profile, an interpolation method for parametric tool paths is given. The idea is to compute the maximal speeds at the critical points of the tool path where the radius of curvature reaches extremal values and to use the jounce confined acceleration/deceleration profile to connect the speeds of two adjacent critical points. The final velocity function has  $C^2$  continuity. A vibration test is given to show that comparing with confined acceleration and jerk, the motion profile under confined jounce can reduce vibration more effectively.

In most existing work on interpolation, the friction force and the cutting force in (1) are assumed to be independent of the velocity. An interesting problem is to plan the velocity by assuming the existence of certain relation between the cutting force and the velocity. Another problem is to plan the velocity under confined chord error and confined jounce in each axis as done in [15, 19, 20] in simpler cases such as confined acceleration or confined jerk.

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## A Appendix.

In this appendix, the 7-periods acceleration profile and the proof of its time-optimal property are given.

### A.1 Formulas for the 7-periods acceleration profile

Assume that initial velocity  $v_s = 0$ . The following formulas are the expressions of the jounce  $s(t)$ , jerk  $j(t)$ , acceleration  $a(t)$ , and speed  $v(t)$ ,  $t \in [0, t_m]$  for the 7-period acceleration profile in Figure 1.

$$s(t) = \begin{cases} S_m, & 0 \leq t < t_1, \\ 0, & t_1 \leq t < t_1 + t_2, \\ -S_m, & t_1 + t_2 \leq t < 2t_1 + t_2, \\ 0, & 2t_1 + t_2 \leq t < 2t_1 + t_2 + t_3, \\ -S_m, & 2t_1 + t_2 + t_3 \leq t < 3t_1 + t_2 + t_3, \\ 0, & 3t_1 + t_2 + t_3 \leq t < 3t_1 + 2t_2 + t_3, \\ S_m, & 3t_1 + 2t_2 + t_3 \leq t < 4t_1 + 2t_2 + t_3 \end{cases} \quad (28)$$

$$j(t) = \begin{cases} S_m t, & 0 \leq t < t_1, \\ S_m t_1, & t_1 \leq t < t_1 + t_2, \\ S_m(2t_1 + t_2 - t), & t_1 + t_2 \leq t < 2t_1 + t_2, \\ 0, & 2t_1 + t_2 \leq t < 2t_1 + t_2 + t_3, \\ S_m(2t_1 + t_2 + t_3 - t), & 2t_1 + t_2 + t_3 \leq t < 3t_1 + t_2 + t_3, \\ -S_m t_1, & 3t_1 + t_2 + t_3 \leq t < 3t_1 + 2t_2 + t_3, \\ S_m(t - 4t_1 - 2t_2 - t_3), & 3t_1 + 2t_2 + t_3 \leq t < 4t_1 + 2t_2 + t_3 \end{cases} \quad (29)$$

$$a(t) = \begin{cases} \frac{1}{2}S_m t^2, & 0 \leq t < t_1, \\ S_m t_1 t - \frac{1}{2}S_m t_1^2, & t_1 \leq t < t_1 + t_2, \\ -\frac{1}{2}S_m(2t_1 + t_2 - t)^2 + S_m(t_1 t_2 + t_1^2), & t_1 + t_2 \leq t < 2t_1 + t_2, \\ S_m(t_1 t_2 + t_1^2), & 2t_1 + t_2 \leq t < 2t_1 + t_2 + t_3, \\ S_m(t_1 t_2 + t_1^2) - \frac{1}{2}S_m(2t_1 + t_2 + t_3 - t)^2, & 2t_1 + t_2 + t_3 \leq t < 3t_1 + t_2 + t_3, \\ 2S_m t_1 t_2 + \frac{7}{2}S_m t_1^2 + S_m t_1 t_3 - S_m t_1 t, & 3t_1 + t_2 + t_3 \leq t < 3t_1 + 2t_2 + t_3, \\ \frac{1}{2}S_m(t - 4t_1 - 2t_2 - t_3)^2, & 3t_1 + 2t_2 + t_3 \leq t < 4t_1 + 2t_2 + t_3 \end{cases} \quad (30)$$

$$v_0(t) = \begin{cases} \frac{1}{6}S_m t^3, & 0 \leq t < t_1, \\ \frac{1}{2}(t - t_1)^2 S_m t_1 + \frac{1}{2}(t - t_1)S_m t_1^2 + \frac{1}{6}S_m t_1^3, & t_1 \leq t < t_1 + t_2, \\ \frac{1}{3}S_m t_1^3 + \frac{1}{2}S_m t_1^2 t_2 - S_m t_1^2 t + \frac{1}{2}S_m t_1 t_2^2 - S_m t_1 t_2 t \\ + S_m t_1 t^2 + \frac{1}{6}S_m t_2^3 - \frac{1}{2}S_m t_2^2 t + \frac{1}{2}S_m t_2 t^2 - \frac{1}{6}S_m t^3, & t_1 + t_2 \leq t < 2t_1 + t_2, \\ -S_m t_1^3 - \frac{3}{2}S_m t_1^2 t_2 - \frac{1}{2}S_m t_1 t_2^2 + S_m t_1 t_2 t + S_m t_1^2 t, & 2t_1 + t_2 \leq t < 2t_1 + t_2 + t_3, \\ S_m(t_1 t_2 + t_1^2)(t - 2t_1 - t_2 - t_3) + \frac{1}{6}S_m(2t_1 + t_2 + t_3 - t)^3 \\ + S_m t_1^3 + \frac{3}{2}S_m t_1^2 t_2 + \frac{1}{2}S_m t_1 t_2^2 + S_m t_1 t_2 t_3 + S_m t_1^2 t_3, & 2t_1 + t_2 + t_3 \leq t < 3t_1 + t_2 + t_3, \\ 2S_m t_1 t_2 t + \frac{7}{2}S_m t_1^2 t + S_m t_1 t_3 t - \frac{1}{2}S_m t_1 t^2 - 4S_m t_1^2 t_2 \\ - S_m t_1 t_2^3 - S_m t_1 t_2 t_3 - \frac{25}{6}S_m t_1^3 - \frac{5}{2}S_m t_1^2 t_3 - \frac{1}{2}S_m t_1 t_3^2, & 3t_1 + t_2 + t_3 \leq t < 3t_1 + 2t_2 + t_3, \\ -\frac{26}{3}S_m t_1^3 + 4S_m t_1 t_3 t - 7S_m t_1 t_2 t_3 + 8S_m t_1 t_2 t + 2S_m t_2 t_3 t \\ - 7S_m t_1 t_2^2 + 8S_m t_1^2 t - 2S_m t_1 t^2 - 13S_m t_1^2 t_2 - 7S_m t_1^2 t_3 \\ - 2S_m t_1 t_3^2 - S_m t_2 t^2 - \frac{1}{2}S_m t_3 t^2 + 2S_m t_2^2 t + \frac{1}{2}S_m t_3^2 t \\ - 2S_m t_2^2 t_3 - S_m t_2 t_3^2 + \frac{1}{6}S_m t^3 - \frac{4}{3}S_m t_2^3 - \frac{1}{6}S_m t_3^3, & 3t_1 + 2t_2 + t_3 \leq t < 4t_1 + 2t_2 + t_3 \\ 2S_m t_1^3 + S_m t_1 t_2 t_3 + S_m t_1 t_2^2 + 3S_m t_1^2 t_2 + S_m t_1^2 t_3, & 4t_1 + 2t_2 + t_3 \leq t. \end{cases} \quad (31)$$

## A.2 Proof for Theorem 2.3

In order to prove Theorem 2.3, the following result is needed.

**Theorem A.1** [1][p.24-26] *Let  $y, z$  be solutions of the following differential equations*

$$y' = F(x, y), z' = G(x, z),$$

*respectively, where  $F(x, y) \leq G(x, y), a \leq x \leq b$ , and  $F$  or  $G$  satisfies Lipschitz's condition. If  $y(a) = z(a)$ , then  $y(x) \leq z(x)$  for any  $x \in [a, b]$ .*

*Proof. of Theorem 2.3.*

First of all, prove the 7-period jerk-profile is time-optimal. Denote by  $v(t), t \in [0, t_m]$  the velocity curve of the 7-period jerk-profile, where  $t_m = 4*t_1 + 2*t_2 + t_3$ , and denote by  $j(t), a(t)$  its corresponding jerk and acceleration. Claim that for any other velocity curve  $v^*(t), t \in [0, t^*]$  which satisfies the constraints (10) and (11),  $t^* \geq t_m$ . Also, denote by  $j^*(t), a^*(t), s^*(t)$  the jerk and acceleration and

jounce of  $v^*(t)$  respectively. Assume on the contrary, there exists a  $t_4 \in [0, t^*]$ , such that  $v(t_4) < v^*(t_4)$  (especially take  $t_4 = t^*$ ). Show that this  $t_4$  does not exist for  $t_4 \in [0, t_m]$ .

Case 1,  $t_4 \notin [0, t_1 + t_2]$ . Since for  $t \in [0, t_1 + t_2]$ ,  $j(t) \geq j^*(t)$ , then  $a(t) \geq a^*(t)$  and  $v(t) \geq v^*(t)$ .

Case 2,  $t_4 \notin (t_1 + t_2, 2 * t_1 + t_2]$ . If this is not the case, then there exists a  $t_5 \in (t_1 + t_2, t_4]$  such that  $a(t_5) < a^*(t_5)$ . Claim that  $j(t_5) \leq j^*(t_5)$ . If this is not the case, that is  $j(t_5) > j^*(t_5)$ . Then, by Theorem A.1,  $j(t) = j(t_5) + \int_t^{t_5} S_m dt_s > j^*(t) + \int_t^{t_5} s^*(t_s) dt_s = j^*(t)$  for  $t \in [t_1 + t_2, t_5]$ . Then  $a(t_5) < a^*(t_5)$  is impossible by case 1. Let  $t_6 = 2 * t_1 + t_2 - t_5$ , then by Theorem A.1,  $j^*(t) = j^*(t_5) - \int_{t_5}^t s^*(t_s) dt_s \geq j(t)$ , for  $t \in [t_5, t_5 + t_6]$ . Then, by Theorem A.1,

$$\begin{aligned} a^*(t_5 + t_6) &= a^*(t_5) + \int_{t_5}^{t_5+t_6} j^*(t) dt \\ &\geq a^*(t_5) + \int_{t_5}^{t_5+t_6} j(t) dt \\ &> a(t_5) + \int_{t_5}^{t_5+t_6} j(t) dt \\ &> A_m \end{aligned}$$

This contradicts to the constraint (10).

Case 3,  $t_4 \notin (2 * t_1 + t_2, 2 * t_1 + t_2 + t_3]$ . If this is not the case, there exists a  $t_5 \in [0, 2 * t_1 + t_2)$  such that  $a(t_5) < a^*(t_5)$ . This is impossible by the discussion in Cases 1 and 2.

Case 4,  $t_4 \notin (2 * t_1 + t_2 + t_3, t_m)$ . Denote  $\bar{v}^*(t)$  by

$$\bar{v}^*(t) = \begin{cases} 0, & t \in [0, t_m - t^*] \\ v^*(t + t^* - t_m), & t \in [t_m - t^*, t_m] \end{cases}$$

Similar to the Cases 1, 2, 3, for  $t \in [2 * t_1 + t_2, t_m]$ , from the  $-t$  direction,  $v(t)$  uses the extremal deceleration capability to decrease its velocity. So,  $v(t) \leq \bar{v}^*(t)$ ,  $t \in [2 * t_1 + t_2, t_m]$ . Since  $v(t)$  is monotone, let  $t_5 = 2 * t_1 + t_2 + t^* - t_m \in [0, 2 * t_1 + t_2]$ , then  $v(t_5) < v(t_5 + t_m - t^*) \leq \bar{v}^*(t_5 + t_m - t^*) = v^*(t_5)$ . This contradicts to Cases 1 and 2.

The time-optimal properties of other type acceleration profile can be proved in a similar way by adding the different constraints.

So far, it has been proved that Algorithm **PRO\_VV** is a time-optimal procedure to increase the speed from  $v_s$  to  $v_e$  under the constraints (10) and (11).

It needs to be proved that the selections of concrete jerk-profiles given in Section 2 are reasonable. For simplicity, consider only the transitions from  $(+, +, +)$  to  $(+, +, -)$  or  $(+, -, +)$  and from  $(+, +, -)$  to  $(+, -, -)$  or  $(+, -, +)$ . Without loss of generality, let  $v_s = 0$ .

First it is obvious to find that  $(+, +, +)$  is the most optimal profile if feasible. But on condition of  $J_m^2 < S_m A_m$ , if  $v_e < \frac{A_m^2}{J_m} + \frac{A_m J_m}{S_m}$ , then  $v_e$  is not large enough for profile  $(+, +, +)$  and the profile transition should be considered. If profile  $(+, -, +)$  is adopted, then  $t_2 = 0$  and  $t_3 > 0$ ,  $j_m \leq J_m < \sqrt{S_m A_m}$ .

Obviously  $t_1 = \frac{j_m}{S_m}$ . By (8),  $v_e$  is of the following form

$$v_e = 2S_m t_1^3 + S_m t_1 t_2 t_3 + S_m t_1 t_2^2 + 3S_m t_1^2 t_2 + S_m t_1^2 t_3 = 2S_m t_1^3 + S_m t_1^2 t_3,$$

so  $t_3 = \frac{v_e S_m}{j_m^2} - \frac{2j_m}{S_m}$  and the time of profile  $(+, -, +)$  is  $t_{(+,-,+)} = t(j_m) = \frac{v_e S_m}{j_m^2} + \frac{2j_m}{S_m}$ . So according to the value of  $v_e$ , when  $v_e < \frac{A_m^2}{J_m} + \frac{A_m J_m}{S_m}$ , there are two cases to be discussed.

Case a:  $v_e > \frac{2J_m^3}{S_m^2}$ . Since  $t(j_m)$  is monotonically decreasing on interval  $j_m \in (0, (v_e S_m^2)^{\frac{1}{3}}]$  and  $j_m \leq J_m < (\frac{v_e S_m^2}{2})^{\frac{1}{3}}$ ,  $t(j_m) \geq \frac{v_e S_m}{J_m^2} + \frac{2J_m}{S_m}$ . It is easy to find that  $\frac{v_e S_m}{J_m^2} + \frac{2J_m}{S_m} > \frac{J_m}{S_m} + \sqrt{\frac{J_m^2}{S_m^2} + \frac{4v_e}{J_m}} = t_{(+,+, -)}$ . So  $t_{(+,-,+)} > t_{(+,+, -)}$ . The time of profile  $(+, +, -)$  is less than that of profile  $(+, -, +)$ .

Case b:  $v_e \leq \frac{2j_m^3}{S_m^2}$ . In this case, profile  $(+, +, -)$  is invalid otherwise  $j_m$  exceeds  $J_m$ . Now show that the time of profile  $(+, -, -)$  is less than that of profile  $(+, -, +)$ . If profile  $(+, -, +)$  is adopted, then  $t_1 = \frac{j_m}{S_m}$  and  $t_2 = 0$  and  $t_3 = \frac{v_e S_m}{j_m} - \frac{2j_m}{S_m} > 0$ . So  $j_m < (\frac{v_e S_m^2}{2})^{\frac{1}{3}}$ . Denote by  $t(j_m) = 4 * t_1 + t_3$  the total acceleration time with maximal jerk  $j_m$ . Since  $t(j_m)$  is monotonically decreasing on interval  $j_m \in (0, (v_e S_m^2)^{\frac{1}{3}}]$ ,  $t_{(+, -, +)} = t(j_m) > t((\frac{v_e S_m^2}{2})^{\frac{1}{3}}) = 4(\frac{v_e}{2S_m})^{\frac{1}{3}} = t_{(+, -, -)}$ . So the time of profile  $(+, -, -)$  is less than that of profile  $(+, -, +)$ .

Thus the optimality of profile transition in Algorithm **PRO\_VV** have been proved. Other cases can be proved similarly. Now Theorem 2.3 is proved. ■

If the acceleration profile is symmetric, then time-optimal means the distance to accelerate  $v_s$  to  $v_e$  is the shortest one since the distance is  $d = \frac{v_s + v_e}{2} t^*$ , where  $t^*$  is the total acceleration time.