Practical smooth minimum time trajectory planning for path following robotic manipulators*

Qiang Zhang[1] , Shu-Rong Li[1], Xiao-Shan Gao[2]

Abstract— **Previous computation approaches of smooth minimum time trajectory for path following robotic manipulators either cannot fully utilize the maximum ability of machines, fail to minimum machine time, or cannot be computed efficiently because of the nonlinearity of problem formulations. In this paper, an efficient computation approach is proposed that has been designed as solving a convex optimization problem to generate smooth minimum time trajectory while utilizing the maximum ability of machines.**

I. INTRODUCTION

Computation of minimum time trajectory has been wildly studied for robotic manipulators because of its effectiveness to improve the productivity, such as Pfeiffer and Johanni[1], Bobrow, et al. [2], Shiller [3], Verscheure, et al. [4] and Ardeshiri, et al. [5]. However, the torque strategy of the minimum time trajectory has been proven to be "bang-bang" or "bang-singular-bang" which is not physically realizable (Chen and Desrochers [6]). Direct implement of "bang-bang" torques can induce tool vibrations and overshoot of the nominal torque limits.

The well-known smooth minimum time trajectory planning approaches are implemented by limiting the ratio of the torques or jerks of all the manipulator joints. Gregory et al. [7] and Gasparetto and Zanotto [8] realized the smoothness of trajectory by considering modified weighted objective function, such as time-square of jerks or time-square of joint torques. Constantinescu and Croft [9] obtained the smooth minimum time trajectory by limiting the changes of joint torques. Jamhour and Andr [10], Dong et al. [11] and Mattmller and Gisler [12] introduced jerk limits into the planning problem to realize the smoothness of trajectory. Since the formulation of the jerk or the ratio of joint torque is non convex (see Verscheure et al. [4]), the ordinary smooth minimum time trajectory planning approaches are fail to execute efficiently.

In this paper, a practical computation approach is proposed to generate smooth minimum time trajectory. The proposed approach works by adding linear constraints to limit the changes of the joint torques, and we name the proposed linear constraint as pseudo torque rate constraint. Then by properly choosing state variables, the smooth minimum time trajectory planning problem can be formulated as a convex optimization problem, which can be solved efficiently by the ordinary gradient-based optimization techniques.

The rest of the paper is organized as follows. In Section II, the dynamics system description of the manipulator and the nominal minimum time trajectory planning approach are stated. In Section III, the smooth minimum time trajectory planning approach is given. The details of the numerical solution are presented in Section IV. In Section V, a numerical example is presented. Conclusions are summarized in Section VI.

II. BACKGROUND

A. Dynamics System Description

In this paper, we assume the dynamics model of an n DOF robotic manipulator satisfies the following equation (Constantinescu and Croft [9]).

$$\tau = \mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{C}(\mathbf{q})\,\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}), \tag{1}$$

where $\mathbf{q} \in \mathbf{R}^n$ is the vector of joint angular positions and is a function of parameter s, $s \in [0, 1]$, $\tau \in \mathbf{R}^n$ is the vector of actuator torques, $\mathbf{M}(\mathbf{q}) \in \mathbf{R}^{n \times n}$ is the inertia matrix of the manipulator, $\mathbf{C}(\mathbf{q}) \in \mathbf{R}^{n \times n \times n}$ is a third order tensor representing the coefficients of the centrifugal and Coriolis forces, $\mathbf{G}(\mathbf{q}) \in \mathbf{R}^n$ is the vector of gravitational torques. For simplicity, viscous and static friction terms have been neglected in the dynamics model.

Then the joint kinematics can be formulated as follows:
Joint velocity:

$$\dot{\mathbf{q}}(t) = \mathbf{q}'(s(t))\,\dot{s}(t).$$

Joint acceleration:

$$\ddot{\mathbf{q}}(t) = \mathbf{q}'(s(t))\,\ddot{s}(t) + \mathbf{q}''(s(t))\,\dot{s}^2(t).$$

The dynamics model of the robotic manipulator can be written as a function of the path parameter(Verscheure et al. [4]):

$$\mathbf{m}(s)\,\ddot{s} + \mathbf{c}(s)\,\dot{s}^2 + \mathbf{g}(s) = \tau, \tag{2}$$

where

$$\mathbf{m}(s) = \mathbf{M}(\mathbf{q}(s))\,\mathbf{q}'(s) \in \mathbf{R}^n,$$
$$\mathbf{c}(s) = \mathbf{M}(\mathbf{q}(s))\,\mathbf{q}''(s) + \mathbf{q}'(s)^{\mathrm{T}}\mathbf{C}(\mathbf{q}(s))\,\mathbf{q}'(s) \in \mathbf{R}^n,$$
$$\mathbf{g}(s) = \mathbf{G}(\mathbf{q}(s)) \in \mathbf{R}^n.$$

[1] Q. Zhang and S.L. Li are with the College of Information and Control Engineering, China University of Petroleum (East China), 266580 Qingdao, China automatic_zhangqiang@126.com; lishuron@upc.edu.cn
[2] X.S. Gao is with the KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 Beijing, China xgao@mmrc.iss.ac.cn

B. Original Minimum Time Trajectory Planning Problem Statement

The well known minimum time trajectory planning approaches solve the following optimal control problem:

Optimize the minimum time objective,

$$\min_{\tau} \ J_{obj} = \int_0^{t_{\mathrm{f}}} 1 \mathrm{d}t, \tag{3}$$

while subjects to the manipulator dynamics (1), the boundary conditions of the joint velocities,

$$\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \ \dot{\mathbf{q}}(t_{\mathrm{f}}) = \dot{\mathbf{q}}_{\mathrm{f}}, \tag{4}$$

the path constraints $\mathbf{q}(s)$ for all $s \in [0, 1]$ and the actuator torque constraints,

$$\tau_{\min} \le \tau \le \tau_{\max}. \tag{5}$$

Besides, the joint velocity and acceleration constraints can also be considered.

As mentioned in Verscheure et al. [4], the path constrained minimum time trajectory planning problem can be described as the following convex optimal control problem:

$$\begin{aligned}
\min_{b(s)} \quad & J_{obj} = \int_0^1 \frac{1}{\sqrt{a(s)}} \mathrm{d}s \\
\mathrm{s.t.} \quad & a'(s) = 2b(s), a(0) = a_0, a(1) = a_1, \\
& \mathbf{m}(s)b(s) + \mathbf{c}(s)a(s) + \mathbf{g}(s) = \tau(s), \\
& \tau_{\min} \le \tau(s) \le \tau_{\max},
\end{aligned} \tag{6}$$

where $b = \ddot{s}$ acts as the control variable and $a = \dot{s}^2$ is the corresponding state, and they satisfy $a'(s) = \frac{d\dot{s}^2}{ds} = 2b$.

C. The Properties of the Minimum Time Trajectory

Since problem (6) is an equivalent formulation of the original minimum time trajectory planning problem, the properties analysis of the minimum time trajectory is based on this formulation.

In order to realize our approach, the following properties of minimum time trajectory must be stated:

- Property 1: bang-bang control structure.

  For equivalent problem (6), let b^* be the feasible optimal control with corresponding optimal state a^*. Then, optimal b^* takes on the extreme values $b_{\min}(a^*, s)$ or $b_{\max}(a^*, s)$ at every point along the path except at singular points where optimal p-velocity $\dot{s}^*$ touches the velocity limit curve $\dot{s}_{\max}(s)$, where $b_{\min}(a^*, s)$ and $b_{\max}(a^*, s)$ are the state-based control bounds and can be evolved from torque constraints.

- Property 2: bang-bang constraints structure.

  For the robotic manipulator system, the optimal minimum time trajectory requires the structure of joint torques is "bang-bang", i.e., there exists at least one of the joint torques is saturation on each finite time interval.

  The detail proof of the property 1 and 2 can be found at Chen,and Desrochers [6].

  Based on the property 1 and 2, we can directly give the following property of minimum time trajectory without proof.

- Property 3:

  The velocity trajectory $\dot{s}(s)$ of the original minimum time motion is maximum at everywhere among all the feasible trajectories.

  The property 3 can be deduced easily from the works of Bobrow et al. [2] and Chen and Desrochers [6].

III. SMOOTH MINIMUM TIME TRAJECTORY PLANNING

The joint motion can be smoothed by limiting the torque rate, that is, the following new constraints are added too the optimal control problem (6).

$$\dot{\tau}_{\min} \le \dot{\tau}(s) \le \dot{\tau}_{\max} \tag{7}$$

where $\dot{\tau}_{\min}$ and $\dot{\tau}_{\max}$ are the lower and upper bounds for the torque rate $\dot{\tau}$.

The parameter formulation of torque rate function can be

$$\begin{aligned}
\dot{\tau} &= \tau' \dot{s} \\
&= \mathbf{m} \, \dddot{s} + (\mathbf{m}' + 2\mathbf{c}) \, \dot{s}\ddot{s} + \mathbf{c}' \dot{s}^3 + \mathbf{g}' \dot{s}.
\end{aligned} \tag{8}$$

with

$$\begin{aligned}
\mathbf{m}'(s) &= \mathbf{M}'\mathbf{q}' + \mathbf{M}\mathbf{q}'', \\
\mathbf{c}'(s) &= \mathbf{M}'\mathbf{q}'' + \mathbf{M}\mathbf{q}''' + \mathbf{q}'^{\mathrm{T}}\mathbf{C}'\mathbf{q}' + \mathbf{q}''^{\mathrm{T}}\mathbf{C}\mathbf{q}' + \mathbf{q}'^{\mathrm{T}}\mathbf{C}\mathbf{q}'', \\
\mathbf{g}'(s) &= \mathbf{G}'.
\end{aligned}$$

Define a new state as u, which satisfies $u = \frac{\dddot{s}}{\dot{s}}$. Then there exists $b'(s) = u(s)$.

Thus the torque rate can be formulated as a nonconvex functions w.r.t (a, b, u), denoted by

$$\dot{\tau}(t) = \sqrt{a}\left(\mathbf{m}u + (\mathbf{m}' + 2\mathbf{c})b + \mathbf{c}'a + \mathbf{g}'\right). \tag{9}$$

Since the torque rate function is non-convex, the efficient solution of smooth minimum time trajectory planning is always a challenging problem.

From the property 3 in Section II-C, we have known that for any feasible trajectory there always exists

$$a(s) \le a_{\mathrm{tq}}(s), s \in [0, 1], \tag{10}$$

where $a_{\mathrm{tq}}(s)$ denotes the optimal state a of problem (6).

For our problem, when we assume the torque rate bounds are set to infinite, the smooth MTTP problem can degenerate to an original MTTP problem and the corresponding state a satisfies $(a_{\mathrm{tq}}(s) - a(s)) \to 0_+$, $s \in [0, 1]$. On the contrary, when the torque rate bounds are set small enough, the optimal trajectory would be completely charged by the torque rate constraints and the optimal a would be away from $a_{\mathrm{tq}}(s)$, but the optimal a can still be large enough for keeping the minimum motion time.

So for the common specified torque rate bounds, we can give a reasonable assumption that the optimal smooth $a(s)$ only have little decrease from the original minimum time $a_{\mathrm{tq}}(s)$. And the subsequential example in the section V can also support this.

We can define a reasonably pseudo torque rate as

$$\begin{aligned}
&\dot{\tau}_{\mathrm{pseudo}}(s) \\
&= \sqrt{a_{\mathrm{tq}}(s)} \left(\begin{array}{c} \mathbf{m}(s)u(s) + (\mathbf{m}'(s) + 2\mathbf{c}(s))b(s) \\ + \mathbf{c}'(s)a(s) + \mathbf{g}'(s) \end{array} \right),
\end{aligned} \tag{11}$$

and $\dot{\tau}_{\mathrm{pseudo}}$ is linear w.r.t (a, b, u). Then we have

$$\dot{\tau}(s) = \sqrt{\frac{a(s)}{a_{\mathrm{tq}}(s)}}\dot{\tau}_{\mathrm{pseudo}}(s). \qquad (12)$$

When the optimal smooth minimum time motion is achieved, we have

$$\sqrt{\frac{a_{\mathrm{opm}}(s)}{a_{\mathrm{tq}}(s)}} \leq 1, \text{ and } \sqrt{\frac{a_{\mathrm{opm}}(s)}{a_{\mathrm{tq}}(s)}} \to 1 \text{ for s} \in [0,1]. \quad (13)$$

So the real torque rate constraints (7) can be approximated by

$$\dot{\tau}_{\min} \leq \dot{\tau}_{\mathrm{pseudo}}(s) \leq \dot{\tau}_{\max}, \qquad (14)$$

where $\dot{\tau}_{\min}, \dot{\tau}_{\max}$ denote the nominal minimum and maximum bounds of torque rate respectively.

Let $u(s)$ act as control variable, the smooth minimum time trajectory can be obtained by solving the following convex optimal control problem:

$$
\begin{aligned}
\min_{u(s)} \quad & J_{obj} = \int_0^1 \frac{1}{\sqrt{a(s)}}\mathrm{d}s \\
\text{s.t.} \quad & a'(s) = 2b(s), \ a(0) = a_0, \ a(1) = a_1, \\
& b'(s) = u(s), \ b(0) = b_0, \ b(1) = b_1, \\
& \tau(s) = \mathbf{m}(s)b(s) + \mathbf{c}(s)a(s) + \mathbf{g}(s), \\
& \dot{\tau}_{\mathrm{pseudo}}(s) \\
& = \sqrt{a_{\mathrm{tq}}(s)}\left(\begin{array}{l}\mathbf{m}(s)u(s) + (\mathbf{m}'(s) + 2\mathbf{c}(s))b(s) \\ + \mathbf{c}'(s)a(s) + \mathbf{g}'(s)\end{array}\right) \\
& \tau_{\min} \leq \tau(s) \leq \tau_{\max}, \\
& \dot{\tau}_{\min} \leq \dot{\tau}_{\mathrm{pseudo}}(s) \leq \dot{\tau}_{\max}, \\
\end{aligned}
\qquad (15)
$$

where b_0, b_1 are determined by the initial and final torque constraints, respectively.

From (11) and (12), we know the pseudo torque rate function is an overestimation of the real torque rate, meaning

$$
\begin{aligned}
\dot{\tau}(s) \leq \dot{\tau}_{\mathrm{pseudo}}(s) \leq \dot{\tau}_{\max}, \quad \text{for} \quad \dot{\tau}(s) \geq 0, \\
\dot{\tau}(s) \geq \dot{\tau}_{\mathrm{pseudo}}(s) \geq \dot{\tau}_{\min}, \quad \text{for} \quad \dot{\tau}(s) < 0.
\end{aligned}
$$

So the smooth trajectory generated by solving problem (15) also satisfies the nominal torque rate constraints.

Taken together, the smooth minimum time trajectory can be obtained by executing the following two steps:

1. First solve the original minimum time motion problem (6) to obtain the feasible maximum velocity trajectory $a_{\mathrm{tq}}(s)$ for $s \in [0,1]$. And $a_{\mathrm{tq}}(s)$ is unique.

2. Then use the $a_{\mathrm{tq}}(s)$ to construct linear pseudo torque rate constraints $\dot{\tau}_{\mathrm{pseudo}}(s)$ to limit the changes of the joint torques. And further construct convex optimization problem to obtain the smooth minimum time trajectory.

IV. NUMERICAL SOLUTION

In this section, B-spline curve is applied to parameterize the formulations (15) into a semi-infinite optimization problem. And refer to the common optimization approaches (Buskens and Maurer [13] and Chen and Vassiliadis [14]), the process constraints are considered only at certain finite points and are assumed no be violated between any two adjacent points.

A. Trajectory Parameterization by B-spline

The state $a(s)$ of problem (15) is approximated by the following B-spline curve in parameter interval $[0, 1]$,

$$a(s) \approx \tilde{a}(s) = \sum_{i=0}^{K} N_{i,p}(s)\hat{a}_i, \qquad (16)$$

where $N_{i,p}(s)$ denotes the i-th basis function which is a p order polynomial, and is defined as

$$N_{i,0}(s) = \begin{cases} 1, & \text{if } s_i \leq s \leq s_{i+1} \\ 0, & \text{otherwise} \end{cases},$$

$$N_{i,p}(s) = \frac{s - s_i}{s_{i+p} - s_i}N_{i,p-1}(s) - \frac{s_{i+p+1} - s}{s_{i+p+1} - s_{i+1}}N_{i+1,p-1}(s).$$

Since we need the torques and the torque rates are bounded, the joint velocities of robotic manipulator should be at least C^1 continuity. Here we choose $p = 2$ meaning 3-order B-spline is available for our problem.

Because the state at initial or final points is fixed, written as $a(0) = a_0, a(1) = a_1$, the spline knot sequence must to be clamped written as

$$\left[\underbrace{0, \ldots, 0}_{p+1}, s_{p+2}, \ldots, s_k, \ldots, s_{K+p-1}, \underbrace{1, \ldots, 1}_{p+1}\right]. \qquad (17)$$

According to the properties of B-spline, we can obtain the derivatives of state $a(s)$ as follows:

$$b(s) = \frac{1}{2}a'(s) = \frac{1}{2}\frac{\mathrm{d}a(s)}{\mathrm{d}s} = \frac{1}{2}\sum_{i=0}^{K}N_{i,p}'(s)\hat{a}_i, \qquad (18)$$

$$u(s) = b'(s) = \frac{\mathrm{d}b(s)}{\mathrm{d}s} = \frac{1}{2}\sum_{i=0}^{K}N_{i,p}''(s)\hat{a}_i, \qquad (19)$$

where

$$N_{i,p}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}N_{i,p}(s)$$

$$= \frac{p}{s_{i+p}-s_i}N_{i,p-1}(s) - \frac{p}{s_{i+p+1}-s_{i+1}}N_{i+1,p-1}(s).$$

For the infinite process torque constraints, we use the pointwise constraints to replace the process constraints. The constraints are evaluated in the middle between any two adjacent knot points as

$$\bar{s}_j = \begin{cases} \frac{1}{2}s_{p+2}, j = 1 \\ \frac{1}{2}(s_{j+p} + s_{j+p+1}), j = 2, \ldots, K-2 \\ \frac{1}{2}(s_{K+p-1} + 1), j = K-1 \end{cases}, \qquad (20)$$

and we name $[\bar{s}_1, \ldots, \bar{s}_k, \ldots, \bar{s}_{K-1}]$ as constraint nodes vector.

B. Convex Programming Formulation

Let CP denote the control point vector of B-spline, and

$$CP = \{\hat{a}_0, \hat{a}_1, \ldots \hat{a}_{K-1}, \hat{a}_K\}. \tag{21}$$

For the purpose of satisfying the boundary conditions of the original problem, $\hat{a}_0 = a_0$ and $\hat{a}_K = a_1$ are forced.

Then only the following sequence of control points needs to be optimized:

$$\mathbf{X} = \{\hat{a}_1, \ldots \hat{a}_{K-1}\}. \tag{22}$$

We call this as decision vector.

The minimum time objective function can be approximated as

$$
\begin{aligned}
\min \ J_{obj} &= \int_0^1 \frac{1}{\sqrt{a(s)}} \mathrm{d}s \\
&\approx \sum_{j=1}^{K-1} \left(\frac{\Delta \bar{s}_j}{\sqrt{\sum_{i=0}^{K} N_{i,p}(\bar{s}_j)\hat{a}_i}} \right).
\end{aligned} \tag{23}
$$

The joint torques and torque rates satisfy the following equations at constraint node $\bar{s}_j$,

$$
\begin{aligned}
\tau(\bar{s}_j) = &\frac{1}{2}\mathbf{m}(\bar{s}_j) \sum_{i=0}^{K} N'_{i,p}(\bar{s}_j)\hat{a}_i + \\
&\mathbf{c}(\bar{s}_j) \sum_{i=0}^{K} N_{i,p}(\bar{s}_j)\hat{a}_i + \mathbf{g}(\bar{s}_j),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&\dot{\tau}_{\text{pseudo}}(\bar{s}_j) \\
&= \sqrt{a_{\text{tq}}(\bar{s}_j)} \left(
\begin{array}{l}
\frac{1}{2}\mathbf{m}(\bar{s}_j) \sum_{i=0}^{K} N''_{i,p}(\bar{s}_j)\hat{a}_i \\
+\frac{1}{2}(\mathbf{m}'(\bar{s}_j) + 2\mathbf{c}(\bar{s}_j)) \sum_{i=0}^{K} N'_{i,p}(\bar{s}_j)\hat{a}_i \\
+\mathbf{c}'(\bar{s}_j) \sum_{i=0}^{K} N_{i,p}(\bar{s}_j)\hat{a}_i + \mathbf{g}'(\bar{s}_j)
\end{array}
\right).
\end{aligned} \tag{25}
$$

So the smooth minimum time trajectory planning problem can be reduced as the following convex programming problem:

$$
\begin{aligned}
\min_{\{\hat{a}_1, \ldots \hat{a}_{K-1}\}} \quad & \sum_{j=1}^{K-1} \left(\frac{\Delta \bar{s}_j}{\sqrt{\sum_{i=0}^{K} N_{i,p}(\bar{s}_j)\hat{a}_i}} \right) \\
\text{s.t.} \quad & \sum_{i=0}^{K} N'_{i,p}(\bar{s}_1)\hat{a}_i = 2b_0, \\
& \sum_{i=0}^{K} N'_{i,p}(\bar{s}_{K-1})\hat{a}_i = 2b_1, \\
& \tau_{\min} \leq \sum_{i=0}^{K} \mathbf{d}1(j)\hat{a}_i + \mathbf{d}2(j) \leq \tau_{\max}, \\
& \dot{\tau}_{\min} \leq \sum_{i=0}^{K} \mathbf{d}3(j)\hat{a}_i + \mathbf{d}4(j) \leq \dot{\tau}_{\max}, \\
& j = 1, 2, \cdots, K-1,
\end{aligned} \tag{26}
$$

where

$$\mathbf{d}1(j) = \frac{1}{2}\mathbf{m}(\bar{s}_j) N'_{i,p}(\bar{s}_j) + \mathbf{c}(\bar{s}_j) N_{i,p}(\bar{s}_j),$$

$$\mathbf{d}2(j) = \mathbf{g}(\bar{s}_j),$$

$$\mathbf{d}3(j) = \sqrt{a_{\text{tq}}(\bar{s}_j)} \left(
\begin{array}{l}
\frac{1}{2}\mathbf{m}(\bar{s}_j) N''_{i,p}(\bar{s}_j) \\
+\frac{1}{2}(\mathbf{m}'(\bar{s}_j) + 2\mathbf{c}(\bar{s}_j)) N'_{i,p}(\bar{s}_j) \\
+\mathbf{c}'(\bar{s}_j) N_{i,p}(\bar{s}_j)
\end{array}
\right),$$

$$\mathbf{d}4(j) = \sqrt{a_{\text{tq}}(\bar{s}_j)}\mathbf{g}'(\bar{s}_j). \tag{27}$$

Since the description (26) is convex, any local optimum is also globally optimal. Hence, any general purpose nonlinear solver (such as SQP in Buskens and Maurer [13]) can solve the problem efficiently.

V. NUMERICAL EXAMPLE

In this section, we present an example to verify the effectiveness of the proposed approach. The experimental manipulator is a three-DOF elbow manipulator as in Pfeiffer and Johanni [1]. A double parabola in the robot work space is chosen as the predefined path.

The numerical implements of convex optimization problems (6) and (15) for the test path are based on the sequential quadratic programming (SQP). All the numerical solutions are run on a laptop, Matlab environment, 32-bit system, 2.5 GHz Core i3 processor, 2GB RAM memory.

In this section, MTT, PTRT and RTRT are short for the minimum time trajectory, pseudo torque rate constrained minimum time trajectory and real torque rate constrained minimum time trajectory, respectively. The numerical singularity problem of the results reported by Verscheure et al. [4], Chen, and Desrochers [6] and Constantinescu and Croft [9] is solved by adding a small amount of regularization into the objective to penalize torque jumps (Verscheure et al. [4]). Fig. 1 shows the resulted torques of all three joints by solving the proposed pseudo torque rate constrained problem, the real torque rate constrained problem and the nominal problem, respectively. Fig. 1 indicates that the PTRT has almost the same joint torques as the RTRT and they are obviously smoother than those of the MTT. The resulted torque rates of the three joints are presented in Fig. 2. From Fig. 2, we can see that the torque rate trajectories of the PTRT and RTRT are also similar and only have little differences at the Acc/Dec switch points. So we can say the proposed PTRT can work efficiently instead of the RTRT in this example. The joint velocities of the MTT, PTRT and RTRT trajectories presented in Fig.3 can also confirm this.

The results presented in Fig. 1, 2 and 3 are under the nominal torque rate of 3500 N.m/s. Then for further testing the effectiveness of the proposed approach, we consider three levels of torque rate constraints: low torque rate limits of 5000 N.m/s, middle torque rate limits of 3500 N.m/s and high torque rate limits of 1500 N.m/s. The results listed in Table 1 indicate that the losses of the optimal objective of the proposed approach relative to the real torque rate approach are 2.5%, 3.7% and 8.5% for the nominal low, middle and high torque rate constraints respectively. But since the pseudo torque rate is an overestimation of the real torque rate,
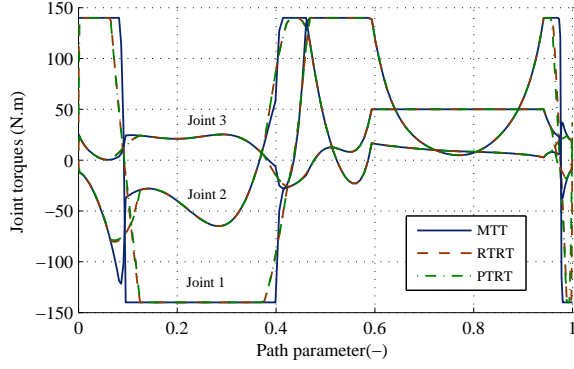
Fig. 1. Optimal joint torques for the MTT, PTRT and RTRT, respectively for the double parabola path
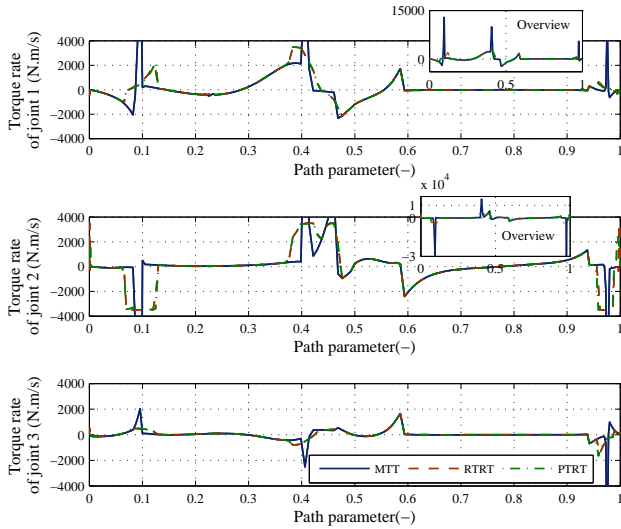

Fig. 2. Optimal joint torque rates for the MTT, PTRT and RTRT, respectively for the double parabola path
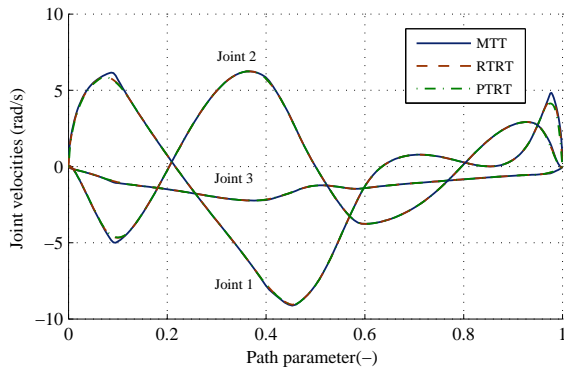

Fig. 3. Optimal joint velocities for the MTT, PTRT and RTRT, respectively for the double parabola path

the trajectory generated by the proposed approach can be smoother than that obtained by solving the real torque rate constrained problem. In the solution processes, we also find that the calculation stability of solving the real torque rate constrained problem is very bad. In contrast, the solution of the proposed pseudo torque rate constrained problem do not exist this weakness.

TABLE I

COMPUTATION PERFORMANCE COMPARISON AMONG MINIMUM TIME, PSEUDO TORQUE RATE CONSTRAINED, AND REAL TORQUE RATE CONSTRAINED TRAJECTORIES

Strategy	Trajectory	Objective	Computation time*
Non torque rate limit	MTT	1.662s	4.62s
Low torque rate limits	PTRT	2.602s	4.62s+6.72s
	RTRT	2.538s	Fault/29.4s**
Middle torque rate limits	PTRT	2.648s	4.62s+6.99s
	RTRT	2.553s	Fault/36.04s**
High torque rate limits	PTRT	2.866s	4.62s+8.51s
	RTRT	2.642s	Fault/49.95s**

In Table I, * notes the planning time is calculated while the number of the parameterized grids is 200 and the default initial value of decision vector $\mathbf{X}_0$ is 0.01. And ** notes that the optimized decision vector of the PTRT is used as the initial guess for calculating the RTRT when the RTRT can not be obtained by using the default initial $\mathbf{X}_0$.

VI. CONCLUSIONS

In this paper, a practical smooth minimum time trajectory planning approach has been presented. By approximating the joint torque rate function with a linear function, the smooth minimum time trajectory is obtained by solving a convex optimal control problem. Due to the convexity of the problem formulation, any local optimal solution is also global optimum and the solution process can be numerical efficiency. since the constructed pseudo torque rate is an overestimation of the real torque rate function, the smooth trajectory generated by the proposed approach is also a feasible trajectory for the real torque rate constrained problem. The planning results have indicated that the smooth trajectory generated by the proposed approach is similar to the real torque rate constrained trajectory and even smoother. In the planing process, we have also find that the calculation stability of solving the real torque rate constrained problem is very bad but our approach do not exist this weakness.

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