Using Dynamic Visual and Logic Models in Education –An Introduction to Geometry Expert*

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Abstract

In this paper, we introduce a software, Geometry Expert (GEX), developed by us for automated geometry reasoning and dynamic geometry drawing and show how to use GEX to geometry, algebra, physics and mechanism (linkage) education. As a dynamic geometry software, GEX can be used to build *dynamic visual models* to assist teaching and learning of various mathematical concepts and engineering mechanisms in an intuitive way. As an automated reasoning software, we can build *dynamic logic models* which can do reasoning themselves. Logic models can be used for more intelligent educational tasks, such as automated solution of students' questions, automated generation of test problems, automated evaluation of students' answers, intelligent tutoring, etc.

1 Introduction

With the invention of dynamic geometry softwares, noticeably, Gabri [1] and Geometer's Sketchpad [2], successful experiments have been done on using dynamic geometry softwares to education [3, 4, 5, 6, 7, 8]. As compared with models built with real materials, visual models built with dynamic geometry softwares are more flexible, powerful, and more open for manipulation. These softwares represent a major step forward in modernizing the teaching of geometry in middle and high schools.

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Geometry Expert (GEX) [9, 10] is a software for dynamic diagram drawing and automated geometry theorem proving and discovering. As a dynamic geometry software, GEX can be used to build *dynamic visual models* to assist teaching and learning of various mathematical concepts. As an automated reasoning software, we can build *dynamic logic models* which can do reasoning themselves. Logic models can be used for more intelligent educational tasks, such as automated generation of test problems, automated evaluation of students' answers, intelligent tutoring, etc.

GEX has been used in middle schools to teach geometry, algebra and physics in China and Taiwan. In this paper, Examples came from classrooms are used to show how to teach some abstract mathematical concepts intuitively by using GEX. GEX can also be used to simulate physical phenomena and mechanism movements. In particular, we will show how to generate linkages with GEX automatically. Linkages were used to be the main tools to make teaching and studying mathematics intuitive [11].

Compared with visual models [7], the application of automated geometry reasoning to education is much less influential. This might be due to the inadequacy of the automated reasoning methods developed in the past. Recently, many new powerful methods of automated reasoning have been discovered [12, 13, 10, 14, 15, 16]. With these new methods, we can not only prove hundreds of geometry theorems efficiently, but also discover unknown properties of geometric configurations [17]; we can not only generate elegant and readable proofs [10] but also find multiple and shortest proofs [18]. GEX is the only software that implements all these methods. We believe that the time is mature for applying these research results to the improvement, and perhaps reform, of the teaching of geometry related topics [19].

2 Dynamic Visual Models

By *dynamic visual models*, we mean models built by computer softwares that can be changed dynamically. Their basic properties are: dynamic transformation, free dragging, animation, and generality. By doing dynamic transformation and free dragging, we can obtain various forms of diagrams easily and see the changing process vividly. Through animation, the user may observe the generation process of curves or the figures of functions.

With GEX, we can build four classes of dynamic visual models: geometric transformations, loci generation, diagrams of functions, and loci generation.

2.1 Geometric Transformations

By transformations, we mean not only standard geometric transformations like translation, rotation and dilation, but also a powerful operation: dragging points in the diagram freely and at the same time keeping all the geometric relations in the diagram intact. Transformations have several important educational implications. The two examples given below were used by Ms. Lu in her geometry classes, which demonstrates how GEX may help students understand geometry concepts more easily.

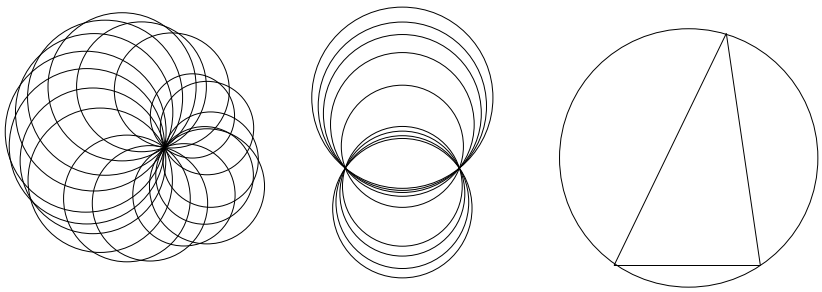

Figure 1. Circles passing through one, two, three points

We may give the concept "number of solutions" an intuitive explanation. For instance, we know that "there are infinite circles passing through one or two given points." By dragging the points themselves, students may get a deeper understanding of the concept of infinity (Figure 1).

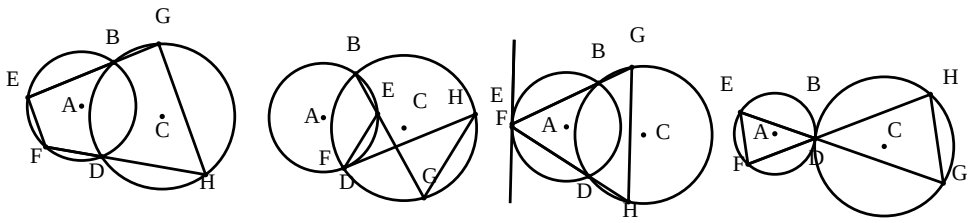

Figure 2. Multiple shapes of one theorem

By dragging points in a diagram we can demonstrate how to get multiple solutions of one problem. An example is as follows: *Two circles A and C intersect at points B, D. Passing through B, D, two lines are drawn meeting circle A and circle C at E, F and G, H. Show that EF is parallel to GH* (Figure 2). This problem has more than ten different forms when dragging the points in the diagram. For instance, F may become coincident with point D, and the secant FH becomes a tangent line of circle A; points E and F may become coincident and secant EF becomes a tangent line of circle A; etc. For each diagram, we get a "new theorem" which is closely connected with the original one and has similar proofs.

2.2 Loci Generation by Ruler and Compass Construction

Locus generation is a powerful tool of generating diagrams for functions and plane curves. We can further combine locus generation with free dragging to demonstrate the continuous change for shapes of functions.

The Geometric Approach. A typical example is to generate conics. An ellipse is the locus of points the sum of distances from which to two fixed points is a constant. In Figure 3, A and D are fixed points; C is a point on a fixed circle; F is the intersection of line AC and the perpendicular-bisector of CD. When C moves on the circle, F will generate an ellipse. Figure 3 also shows the following property of ellipse: "lights coming out from point D and reflected by the ellipse will pass through point A."

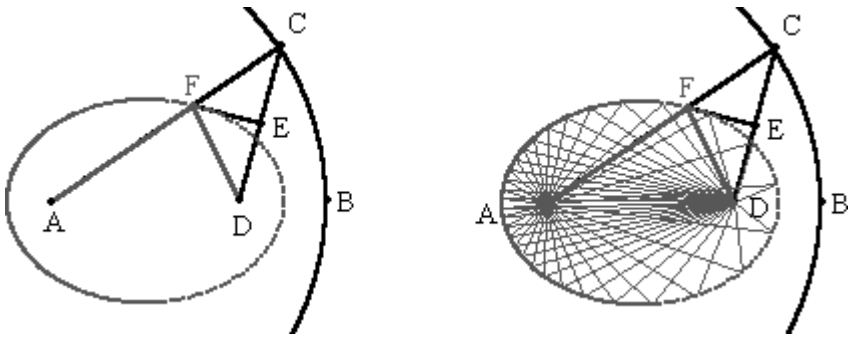

Figure 3. Ellipse and its properties

To show how to combine free dragging and locus generation, let H be the orthocenter of triangle ABC. We fix points A and B and let point C move on a circle c. We want to know the shape of the locus of point H. Let (x_0, y_0) and r be the center and radius of circle c, $A = (0, 0)$, and $B = (d, 0)$. With the method of mechanical formula derivation [12], we obtain the equation of this locus

$$((x - x_0)^2 + y_0^2 - r^2)y^2 + 2y_0(x - d)xy + (x - d)^2x^2 = 0.$$

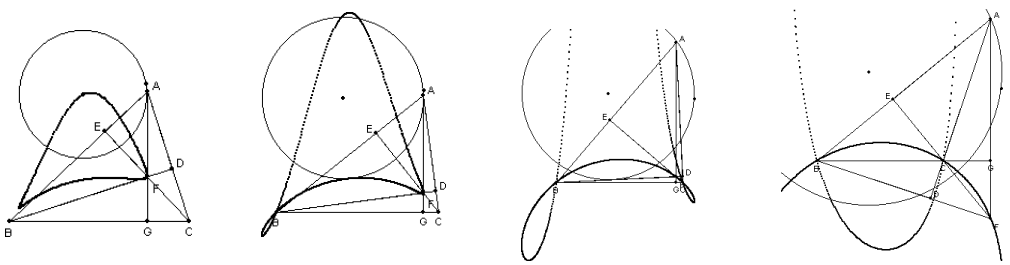

Figure 4. Shapes of the locus of the orthocenter

But from this equation, we still do not know the shape of the curve. With *GEX*, we can continuously change the radius of circle c by free dragging and observe the shape

changes of the curve (Figure 4).

The above examples use a single driving point. By using multiple driving points, we can use GEX to draw many curves, like trigonometric functions. The following diagram is to generate the *locus of the moon* when the moon rotates around the earth on a circle and the earth rotates around the sun on an ellipse. This example uses two driving points.

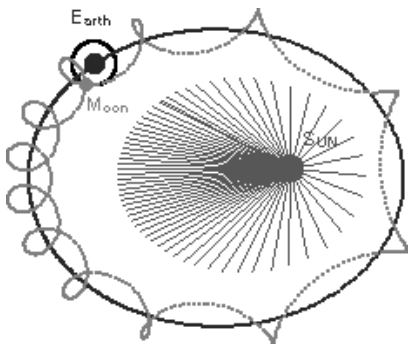

Figure 5. The locus of the moon

Algebraic Approach: Horner's Method. With Horner's method, we can draw the diagram of

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0.$$

The idea is writing the above polynomial as $y = (\ldots((a_n x + a_{n-1})x + a_{n-2})\ldots + a_0)$. We need only two operations, multiplication and addition, to construct the diagram of this curve. By extending this method, we can draw many kinds of functions and curves:

1. Functions of the form $y = \frac{\sqrt[n]{f(x)}}{\sqrt[m]{g(x)}}$.
2. Diagrams for any *rational plane curves* defined by their parametric equations:$x = \frac{P(t)}{R(t)}; y = \frac{Q(t)}{R(t)}$ where P, Q, R are polynomials in t.
3. **Interpolation polynomials.** For n distinct points $(x_i, y_i), i = 1, ..., n$, draw the diagram of a polynomial $y = f(x)$ such that $y_i = f(x_i)$. The construction is based on the *Aitken Lemma*: *if* $f(x) = g(x)$ *for all* x *in a finite set* T *and* $a \neq b$, *then* $h(x) = \frac{(b-x)f(x)-(a-x)g(x)}{a-b}$ *satisfies:* $h(x) = f(x) = g(x)$ *for* $x \in T$ *and* $h(a) = f(a), h(b) = g(b)$.

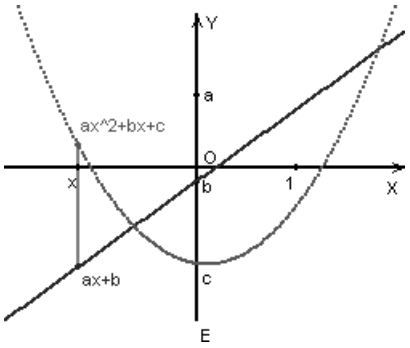

Figure 6. Linear and quadratic functions

In Figure 6, we generate the figures of $y = ax + b$ and $y = ax^2 + bx + c$. The values for a, b, c are represented by segments Oa, Ob, and Oc and hence can be changed continuously.

2.3 Figures of Elementary Functions

Using the numerical computation, we may draw diagrams for functions of the form: $y = f(x)$ where $f(x)$ could be any "elementary functions" defined below.

- Any real number or variable x is an elementary function.
- If f and g are elementary functions, $f + g$, $f - g$, $f * g$ f/g, f^g, $log(f)$, $sin(f)$, $cos(f)$, $tan(f)$, $asin(f)$, $acos(f)$, and $atan(f)$ are elementary functions.
- If f and g are elementary functions, $f(g)$ is also an elementary function.

This part is similar to a Graphic Calculator, but is more flexible and powerful.

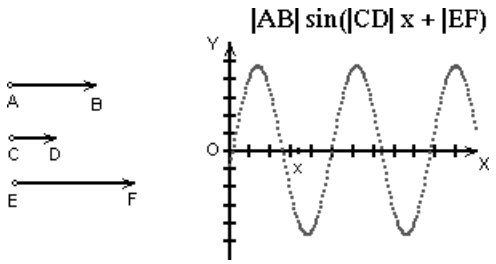

Figure 7. Diagrams of functions

Figure 7 is the diagram of $y = a\sin(bx + c)$, where $a = |AB|, b = |CD|, c = |EF$. By continuously changing the three parameters a, b, c in function $y = a\sin(bx + c)$, we may get an intuitive view of the affect of these parameters on the shape of the sine function.

2.4 Loci Generated by Linkages

Generating linkages allows GEX to be used in education and real linkage design.

The Four-Bar Linkage. The four-bar linkage is the simplest and most important linkage. In order to be convenient for designers, there are books that give the traces generated by various kinds of four-bar linkages. Using GEX, we can change the four links in a four-bar linkage continuously and show its traces at the same time. In other words, we provide an electronic book that has the traces of all possible four-bar linkages.

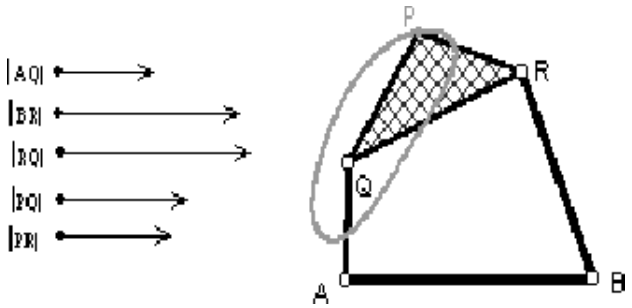

Figure 8. A dynamic four-bar linkage

The above figure shows a four-bar linkage and its trace. When point Q rotates around point A, point P will generate the trace shown in the figure. On the left side of the figure, the five arrows represent the lengths of AQ, BR, RQ, PQ, and PR respectively. The user may drag these arrows to change the lengths of the corresponding links.

Linkages Drawing a Straight Line. A basic question in the theory of mechanism is to design a linkage converting a cyclic movement into a line movement. The following are two classic examples realized in GEX.

The Peaucellier linkage. On the left side of Figure 9, $|OD| = |OB|$, $|AB| = |BC| = |CD| = |DA|$ and $|OP| = |PC|$. When point C moves on a circle passing through point O, point A's trace is a line.

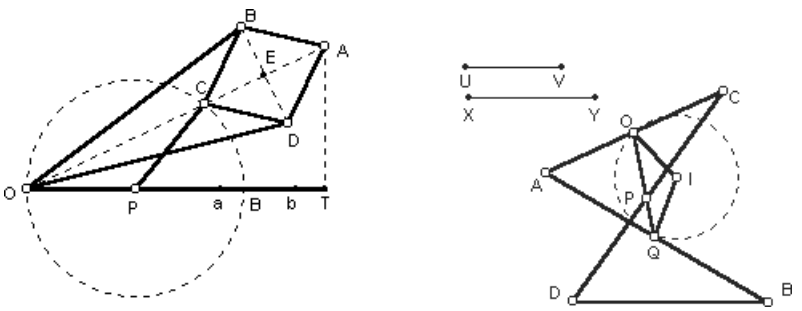

Figure 9. Linkages generating lines

The Hart linkage. On the right side of Figure 9, $ADBC$ is an isosceles trapezoid. Its two waists AC, BD and two diagonals are links. Point O is fixed. $QO \parallel AD$. I is a point on the perpendicular-bisector of OQ. Fix two points O and I. When point Q rotates around a circle with center I, point P's trace is a line. In the diagram, UV and XY represent the lengths of OA and QA.

Linkages Drawing Conics. In an isosceles trapezoid $ABCD$ (Figure 10), let $AC = BD = 2b, AB = CD = 2a$ be four links. Fixing A and B, we get a linkage. Select a point P on CD and a point O on AB such that OP is parallel to AD. Let M be the midpoint of AB, $OM = c$. Point P is inverted to Z by means of a Peaucellier cell $O - PXYZ$. We assume that the radius of this circle is $\sqrt{2}k$. Use a coordinate system whose origin is O and whose x-axis is AB. Suppose $Z = (x, y)$. Then the equations of point Z is [20]

$$(c^2 - b^2)x^2 + (a^2 - b^2)y^2 - 2ck^2x + k^4 = 0.$$

Because $a > b$, the coefficient of y^2 is positive. The character of the conic is determined entirely by the coefficient of x^2. Thus the curve is a Parabola, if $c = b$; an Ellipse, if $c \succ b$; and a Hyperbola, if $c \prec b$.

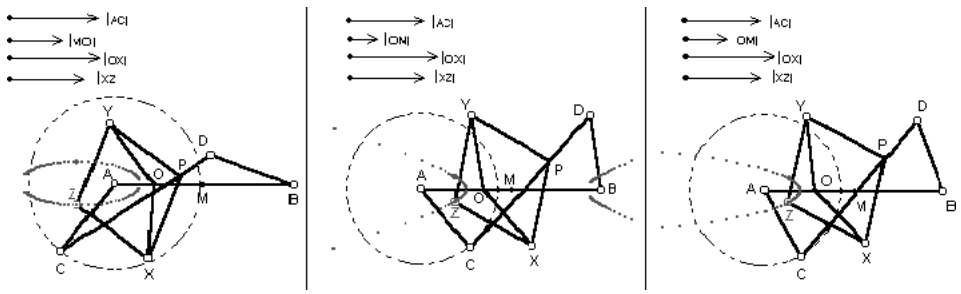

Figure 10. A linkage generates three kinds of conics

Kempe' Linkages. Kempe proved that any algebraic plane curve can be generated with a linkage [20]. We designed an algorithm in GEX to generate Kempe's linkage for any plane algebraic curve $f(x, y) = 0$. Using this method, we have constructed curves of degree up to three. For a given curve, how to design a minimal generating linkage is still an open problem.

3 Dynamic Logic Models

In this section, we will show how to use GEX to build *dynamic logic models*. Precisely speaking, for a given geometric diagram, GEX can generate a database which contains all the properties of this diagram that can be deduced from a fixed set of geometric rules or

axioms, and for each geometric property in the database, GEX can generate an elegant proof for it [17]. Our experiments show that GEX can discover most of the well-known results and often some unexpected ones for hundreds of geometry diagrams.

With this feature, GEX can be used as a Dynamic Geometry Dictionary. With it, teachers can easily make exercises and test problems; students can enhance their ability of solving problems by fully exploring the properties of a given diagram.

3.1 Fixpoint and Deductive Database

Let D_0 be the given geometric properties in a geometry configuration and R the set of geometric rules. We may use the *breadth-first forward chaining method* to find new properties of this diagram. Basically speaking, the breadth-first forward chaining method works as follows

$$\boxed{D_0} \overset{R}{\subset} \boxed{D_1} \overset{R}{\subset} \ldots \overset{R}{\subset} \boxed{D_k} \quad \text{(Fixpoint)}$$

where D_{i+1} is the union of D_i and the set of new properties obtained by applying rules in R to properties in D_i. If at certain step $D_k = D_{k+1}$, i.e.,

$$R(D_k) = D_k,$$

then we say that a *fixpoint* (of reasoning) for D_0 and R is reached.

The naive form of breadth-first forward chaining is notorious for its inefficiency. But, in the case of geometry reasoning, by introducing new search techniques, we manage to build a very effective prover based on this idea [17].

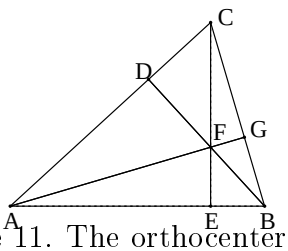

Figure 11. The orthocenter theorem

Example 3.1 (Orthocenter Theorem) *Show that the three altitudes of a triangle are concurrent.*

Let D and E be the feet of the perpendicular lines drawn from points B and C to AC and AB respectively, $F = BD \cap CE$, $G = BC \cap AF$. Then the hypotheses are:

$$D_0 = \left\{ \begin{array}{l} collinear(D,A,C), BD \perp AC, collinear(E,A,B), CE \perp AB, \\ collinear(F,B,D), collinear(F,C,E), collinear(G,B,C), collinear(G,A,F). \end{array} \right\}$$

Reaching the fixpoint for D_0 with GEX costs 0.75 second on a Spark-20 workstation. The fixpoint contains 151 geometry properties:

6 collinear point sets,
3 perpendicular pairs,
6 co-cyclic points sets,
24 equal angle pairs,
7 similar triangles sets,
105 and equal ratio pairs.

We will explain some of the 151 properties in the next section.

3.2 Automated Theorem Proving and Discovering

Forward chaining is a natural way of discovering properties for a given geometric configuration. Any thing obtained in the forward chaining may be looked as a "new" result. Our experiments show that GEX can discover most of the well-known results and often some unexpected ones.

Example 3.2 *Take the simple configuration (Figure 13) related to the orthocenter theorem as an example. Our program has discovered the most often mentioned properties about this configuration:* $AG \perp BC$ *(the orthocenter theorem) and* $\angle EGA = \angle AGD$. *The fixpoint also contains seven sets of similar triangles*

$\triangle DBA \sim \triangle DCF \sim \triangle EBF \sim \triangle ECA;$
$\triangle DCB \sim \triangle DFA \sim \triangle GFB \sim \triangle GCA;$
$\triangle EFA \sim \triangle EBC \sim \triangle GBA \sim \triangle GFC;$
$\triangle FBC \sim \triangle FED \sim \triangle GBD \sim \triangle GEC;$
$\triangle ACB \sim \triangle AED \sim \triangle GCD \sim \triangle GEB;$
$\triangle CED \sim \triangle CAF \sim \triangle GAD \sim \triangle GEF;$
$\triangle FBA \sim \triangle EBD \sim \triangle FGD \sim \triangle EGA.$

Another amazing fact is that this simple configuration contains 105 nontrivial ratios!

For each geometric property in the database, GEX can produce a proof of traditional style. The following is the proof for the Orthocenter Theorem ($AG \perp BC$) by GEX.

Notice that the proof is in an "analysis style", i.e., it starts from the conclusion and goes all the way to the hypotheses of the statement.

1. $AG \perp BC$,
   because $AC \perp BD$(hypothesis), (2)$\angle[AC, BD] = \angle[BC, AF]$.
2. $\angle[AC, BD] = \angle[BC, AF]$,
   because (3)$\angle[AC, BC] = \angle[BD, AF]$. (This is rule in GEX).
3. $\angle[CA, CB] = \angle[BD, AF]$,
   because (4)$\angle[CA, CB] = \angle[DE, AB]$, (5)$\angle[BD, AF] = \angle[DE, AB]$.
4. $\angle[CA, CB] = \angle[DE, AB]$,
   because (6)co-cyclic$[B, D, C, E]$.
5. $\angle[BD, AF] = \angle[DE, AB]$,
   because (7)co-cyclic$[A, D, E, F]$.
6. co-cyclic[B,D,C,E],
   because $DC \perp DB$(hypothesis), $EC \perp EB$(hypothesis).
7. co-cyclic$[A, D, E, F]$,
   because $DF \perp DA$(hypothesis), $EF \perp EA$(hypothesis).

The first step of the proof can be understood as follows. $AG \perp BC$ is true because $AC \perp BD$ which is a hypothesis and $\angle[AC, BD] = \angle[BC, AF]$ which will be proved in the second step. The other steps can be understood similarly.

Example 3.3 (International Mathematics Olymiad, 1985) *A, C, K, and N are four points on a circle. $B = AK \cap CN$. M is the intersection of the circumcircle of triangles BKN and BAC. Show that $BM \perp MO$ (Figure 14).*

GEX generates a database contains 480 properties including the conclusion and the following interesting results.

OO_1BO_2 is a parallelogram.

$A, N, O, M; C, O, K, M; O, O_2, O_1, M$ are co-cyclic point sets.

$\triangle MCN \sim \triangle MAK \sim \triangle OO_2O_1 \sim \triangle MO_1O_2 \sim \triangle BO_1O_2 \sim \triangle CKB$
$\sim \triangle ANB \sim \triangle ONO_2 \sim \triangle OKO_2 \sim \triangle AOO_1 \sim \triangle COO_1$.

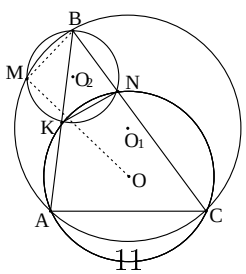

Figure 12. A problem from International Mathematics Olymiad

3.3 Constructing Auxiliary Points

We know that constructing new points or lines is one of the most basic methods of solving geometry problems. One advantage of GEX is that it can automatically add auxiliary points to prove a geometry statement if needed.

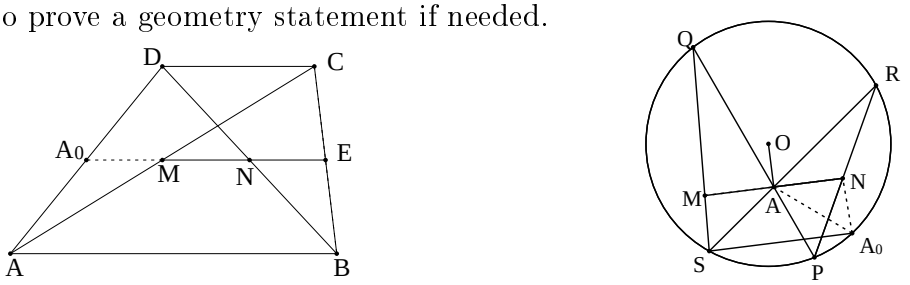

Figure 13. Adding Auxiliary points

Example 3.4 *$ABCD$ is a trapezoid such that $AB \parallel CD$. M and N are the midpoints of AC and BD. Let $E = MN \cap BC$. Show that E is the midpoint of BC. (Figure 15)*

GEX solves this problem by adding an auxiliary point A_0 which is the midpoint of AD. With this auxiliary point, GEX gives the following proof.

The Machine Proof

1. midp$[E, BC]$, because (2)para$[CD, EN]$, (hyp)midp$[N, BD]$.
2. para$[CD, EN]$, because (3)coll$[ENMA_0]$, (4)para$[CD, MA_0]$.
3. coll$[ENMA_0]$, because (hyp)line$[MNE]$, (5)line$[MNA_0]$.
4. para$[CD, MA_0]$, because (hyp)midp$[M, AC]$, (hyp)midp$[A_0, DA]$.
5. line$[MNA_0]$, because (6)para$[A_0M, A_0N]$.
6. para$[A_0M, A_0N]$, because (7)para$[A_0M, AB]$, (8)para$[A_0N, AB]$.
7. para$[A_0M, AB]$, because (4)para$[CD, MA_0]$, (hyp)para$[AB, CD]$.
8. para$[A_0N, AB]$, because (hyp)midp$[N, BD]$, (hyp)midp$[A_0, DA]$.

Example 3.5 (The Butterfly Theorem) *P, Q, R, and S are on the same circle with center O. $A = PQ \cap SR$. The line passing through A and perpendicular to OA meets PR and QS at N and M respectively. Show that A is the midpoint of NM. (Figure 15)*

The conclusion is not in the first fixpoint. The program then automatically adds an auxiliary point A_0 which is the intersection of the line passing through S and parallel to AN and the circle O. With the point A_0, it takes the program 0.4 second to reach the fixpoint which contains the conclusion.

3.4 Other Automated Reasoning Methods in GEX

Geometry Expert (GEX) is a powerful computer program for geometric reasoning. Within its domain, it invites comparison with the best of human geometry provers. It implements most of the effective methods for geometric reasoning introduced in the past twenty years. Besides the deductive database method, GEX also implements the the following methods.

Wu's method is the most powerful method in terms of proving difficult geometry theorems [12, 13]. Wu's method is a coordinate-based method. It first transfers geometry conditions into polynomial equations in the coordinates of the points involved, then deals with the polynomial equations with the characteristic set method.

The area method uses high-level geometric lemmas about geometry invariants such as the area and the Pythagorean difference as the basic tool of proving geometry theorems [10]. The method has been used to produce short, elegant, and human-readable proofs for more than 500 geometry theorems.

The Groebner basis method is also a coordinate-based method [13, 14, 15]. Instead of using the characteristic set method, it uses the Gröbner basis method to deal with the polynomial equations.

Vector method This method is a variant of the area method and is based on the calculation of vectors and complex numbers [10].

The full-angle method is based on the calculation of full-angles. Full-angle method is not a decision procedure [10]. But this method also has its advantages: all the proofs produced by the method are very short, and it has been used to prove several theorems that all the other methods fail to prove because of very large polynomials occurring in the proving process.

To use more than one method in the prover, we can produce a variety of proofs with different styles. This might be important in using GEX to geometry education, since different methods allow students to explore different and better proofs.

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