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Topological classification of non-degenerate intersections of two ring tori

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Highlights
▶ We give the topological classification of all non-degenerate intersection curves of two arbitrary ring tori;
▶ An algebraic method is presented for determining the topological type of the non-degenerate intersection curve of two given ring tori;
▶ The algebraic method for computation needs only to analyze the real roots of two univariate quartic polynomials constructed from the two tori.
Topological Classification of Non-degenerate Intersections of Two Ring Tori

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Abstract

We study the topological classification of all non-degenerate intersection curves of two arbitrary ring tori. Firstly, all topologically different types of such curves are enumerated. Secondly, an algebraic method is presented for determining the topological type of the non-degenerate intersection curve of two given ring tori. The method works by analyzing the real roots of two univariate quartic polynomials constructed from the two tori.

Keywords: torus; intersection curve; inverse geometry.

1. Introduction

Computing surface-surface intersection is an important problem in geometric modeling and CAD/CAM. Existing studies on this problem either present numerical algorithms for tracing the intersection curve of two surfaces or consider the topological classification of the intersection curve. For example, [BFJR87] develops an adaptive algorithm for computing the intersection curve of a pair of rectangular parametric surface patches; [BHLH88] provides a numerical procedure for tracing the intersection curves of two surfaces; and [GK97] computes the intersection loci of surfaces by solving a differential algebraic equation. Detailed information about other numerical algorithms of computing the intersection curve of two surfaces can be found in [PG86] and [Pat93].

Tori, as well as cyclides, which form a superset of tori, are widely used in geometric modeling, especially as blending surfaces. In the present paper we shall study the topological classification of non-degenerate intersection curves of two ring tori. Topological classification is important because it facilitates the computation of a topologically correct representation of an intersection curve in the presence of numerical errors. So far, topological classification is possible only for the intersection curves of some low degree surfaces, such as quadric surfaces (see [DLLP08] and [TWMW09]). More study on computing the intersection of two quadrics can be found in [Mil87], [MG95],[WWK01], [WGT03], [WJG03].

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There have been previous works on torus-related intersection problems. For example, the intersection curve of a torus and a sphere is studied in [KKO98], and a symbolic method based on circle decomposition to compute the intersection curves of a cyclide with a surface swept out by a circle is presented in [Joh93]. There is also extensive literature on geometric modeling with cyclides, e.g. [Pra90], [Pra95] and [FG04].

We make two contributions in this paper: (1) all topologically different types of non-degenerate intersection curves of two ring tori are enumerated; (2) an algebraic method is presented for determining the topological type of the intersection curve of any two given ring tori, assuming that the intersection curve is non-degenerate. We need to stress two key simplifying assumptions made in this study. Firstly, only ring tori are considered. Secondly, it is assumed that the input tori are in generic relative positions such that their intersection curve is non-degenerate, i.e., free of any singular point. Hence, more research is needed in the future in an extension to solve the more general problem of classifying the intersection curves (especially degenerate ones) of two arbitrary tori or even cyclides. In this regard, the results of the present paper provide useful ideas and insights for such an extension.

The remainder of the paper is organized as follows. In Section 2 we present some preliminaries on tori and the key idea behind our approach. Section 3 provides an enumeration of the planar toric section which is the intersection curve of a torus with a plane, which paves the way for the enumeration in Section 4 of all combinations of planar toric sections of two nested tori. In Section 5 we go through all the intersection curves of two tori based on the enumeration of the two nested planar toric sections given in Section 4 and conclude that there are only seven topologically different types of non-degenerate intersection curves of two ring tori. In Section 6 we present an algorithm for determining the topological type of the intersection curve of any two ring tori, assuming that the curve is non-degenerate. In Section 7 we conclude the paper with discussions on possible future work.

2. Outline of the main idea

2.1. Preliminaries

A torus is a surface of revolution generated by revolving a circle in three dimensions about an axis coplanar with the circle. We call the axis the symmetric axis of the torus and the circle the revolving circle of the torus. The center of the revolving circle moves along a circle, called the main circle of the torus. See Figure 1. The radius \( r \) of the revolving circle is called the minor radius of the torus and the radius \( R \) of the main circle is called the major radius of the torus. There are three basic types of tori, depending on the relative sizes of the minor radius and the major radius. The torus is called a ring torus if \( r < R \), a horn torus if \( r = R \), and a spindle torus if \( r > R \) (see Figure 2.) As mentioned before, since we shall consider only non-degenerate intersection curves of two ring tori, and the two tori are assumed throughout to be in generic positions and orientations such that their intersection is free of singularity or self-intersection.
A torus $T$ is said to be in a canonical position if it has the $z$-axis as its symmetric axis and its main circle lying in the $x$-$y$ plane. The implicit equation of a canonical torus $T$ with minor radius $r$ and major radius $R$ is given by the following quartic equation

$$T(x, y, z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0. \tag{1}$$

Note that any torus can be transformed into such a canonical position via a Euclidean transformation, that is, rotation and translation.

2.2. Relative positions of two tori

The topological type of an intersection curve of two tori is determined by the relative positions of the two tori. Now we shall first use a simple example of two circles in the plane to illustrate the basic idea behind our approach to classifying the relative positions of two tori.

Consider the problem of classifying the relative positions of two circles in the plane. Let $C = \{(x, y) \in \mathbb{R}^2 | F(x, y) = (x - x_0)^2 + (y - y_0)^2 - r^2 = 0\}$ be a circle with center $(x_0, y_0)$ and radius $r$. The interior of
Figure 3: Relative positions of two circles. The three cases of separation, intersection and containment are respectively shown in the three columns from left to right.

\(\mathcal{C}\) is denoted by \(\text{Int}(\mathcal{C}) = \{(x, y) \in \mathbb{R}^2 | F(x, y) < 0\}\), and the disk bounded by \(\mathcal{C}\) is denoted by \(\text{Disc}(\mathcal{C}) = \{(x, y) \in \mathbb{R}^2 | F(x, y) \leq 0\}\). Let \(\mathcal{C}_1\) and \(\mathcal{C}_2\) be two circles in the same plane. There are the following four generic relative positions for \(\mathcal{C}_1\) and \(\mathcal{C}_2\): (1) the circles \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are separate if \(\text{Disc}(\mathcal{C}_1) \cap \text{Disc}(\mathcal{C}_2) = \emptyset\); (2) \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are said to intersect if \(\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset\); (3) \(\mathcal{C}_1\) is contained in \(\mathcal{C}_2\) if \(\text{Disc}(\mathcal{C}_1) \subset \text{Int}(\mathcal{C}_2)\); and (4) the similar case of \(\mathcal{C}_1\) containing \(\mathcal{C}_2\). Note that the cases of external and internal touching are regarded as the intersecting case.

Referring to Figure 3, consider two circles \(\mathcal{C}_1\) and \(\mathcal{C}_2\) with radii \(r_1\) and \(r_2\), respectively. Without loss of generality, assume that \(r_1 \leq r_2\). From the circle \(\mathcal{C}_2\), we obtain two of its concentric circles, \(\mathcal{C}_2^O\) of radius \(r_2 + r_1\) and \(\mathcal{C}_2^I\) of radius \(r_2 - r_1\). In other words, \(\mathcal{C}_2^O\) and \(\mathcal{C}_2^I\) are obtained by respectively dilating and shrinking \(\mathcal{C}_2\) by the same amount \(r_1\), the radius of the first circle \(\mathcal{C}_1\).

Then it is easy to see that the relative position of the circles \(\mathcal{C}_1\) and \(\mathcal{C}_2\) is indicated by the position of the center of the circle \(\mathcal{C}_1\), denoted by \(O\), with respect to the two circles \(\mathcal{C}_2^O\) and \(\mathcal{C}_2^I\), as summarized in the following theorem.

**Theorem 2.1.**

1. \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are separate if and only if the point \(O\) is outside \(\mathcal{C}_2^O\);
2. \(\mathcal{C}_1\) and \(\mathcal{C}_2\) intersect if and only if the point \(O\) is inside \(\mathcal{C}_2^O\) and outside \(\mathcal{C}_2^I\) or on \(\mathcal{C}_2^O\) or on \(\mathcal{C}_2^I\);
3. \(\mathcal{C}_1\) is contained in \(\mathcal{C}_2\) if and only if the point \(O\) is inside \(\mathcal{C}_2^I\).

Next we consider the extension of the above approach to determining the relative positions of two ring tori. Note that a similar method for determining the intersection curve of a torus and a sphere is presented.
Figure 4: Relative positions of two ring tori. The three cases of separation, intersection and containment are respectively shown in the three columns from left to right.

in [KKO98]. Let \( T \) denote a torus represented by an implicit equation \( T(x, y, z) = 0 \). We may assume that \( T(x, y, z) \) is appropriately chosen so that the interior of the torus, \( \text{Int}(T) \), is defined by \( \{(x, y, z) \in \mathbb{R}^3 | T(x, y, z) < 0 \} \) and the toroid bounded by \( T \) is defined by \( \text{Vol}(T) = \{(x, y, z) \in \mathbb{R}^3 | T(x, y, z) \leq 0 \} \).

Let \( T_1 \) and \( T_2 \) be two ring tori. Similar to the above analysis on two circles, we have the four cases of the relative positions of the two tori, that is, separation, intersection and the two cases of containment. Specifically, \( T_1 \) and \( T_2 \) are separate if \( \text{Vol}(T_1) \cap \text{Vol}(T_2) = \emptyset \); \( T_1 \) and \( T_2 \) intersect if \( T_1 \cap T_2 \neq \emptyset \); and \( T_1 \) is contained in \( T_2 \) if \( \text{Vol}(T_1) \subset \text{Int}(T_2) \), or vice versa. Again the external or internal touching case of the two ring tori belongs to the intersection category.

Refer Figure 4 showing two ring tori \( T_1 \) and \( T_2 \) with minor radii \( r_i \) and major radii \( R_i \), \( i = 1, 2 \), respectively. Suppose that \( r_1 \leq r_2 \). Now we reduce \( T_1 \) to its main circle \( S \). Meanwhile, from \( T_2 \) we derive two tori having the same main circle as that of \( T_2 \) – the torus \( T_2^O \) with minor radius \( r_2 + r_1 \) and the torus \( T_2^I \) with minor radius \( r_2 - r_1 \). In other words, \( T_2^O \) and \( T_2^I \) are obtained by expanding and shrinking respectively the minor radius of torus \( T_2 \) by \( r_1 \). We will show that the relative position of the two original tori \( T_1 \) and \( T_2 \) can be told from the relative position of the main circle \( S \) of \( T_1 \) with respect to the two nested tori \( T_2^O \) and \( T_2^I \). Being regarded as a torus with zero minor radius, the relative position of a circle with respect to a torus is defined in the same away as for two ring tori.

**Theorem 2.2.** The relative position of the ring tori \( T_1 \) and \( T_2 \) can be deduced from the relative position of the circle \( S \) and the tori \( T_2^O \) and \( T_2^I \) as follows:
1. \( T_1 \) and \( T_2 \) are separate if and only if the circle \( S \) and the torus \( T_2^O \) are separate;

2. \( T_1 \) and \( T_2 \) intersect if and only if there exists a point \( P \) on the circle \( S \) that is inside the torus \( T_2^O \) and outside the torus \( T_2^I \) or on \( T_2^O \) or on \( T_2^I \);

3. \( T_1 \) is contained in \( T_2 \) if and only if the circle \( S \) is contained in the torus \( T_2^I \).

Proof. We take the tori \( T_1 \) as the envelopes of the moving sphere \( SP_1(t) \), whose center is a moving point \( P_1(t) \).

1. \( T_1 \) and \( T_2 \) are separate \( \iff \) for arbitrary \( t \) the moving sphere \( SP_1(t) \) is always separate from the torus \( T_2 \) \( \iff \) the distance between the sphere center \( P_1(t) \) and the torus \( T_2 \) is always bigger than \( r_1 + r_2 \) \( \iff \) the circle \( S \) and the torus \( T_2^O \) are separate.

2. \( T_1 \) and \( T_2 \) intersect \( \iff \) there exists a parameter \( t \) such that the sphere \( SP_1(t) \) intersects with the torus \( T_2 \) \( \iff \) the distance from the point \( P_1(t) \) to the torus \( T_2 \) is smaller than or equal to \( r_1 + r_2 \), but bigger than or equal to \( r_2 - r_1 \) \( \iff \) the point \( P_1(t) \) is inside the torus \( T_2^O \) and outside the torus \( T_2^I \) or on \( T_2^O \) or on \( T_2^I \).

3. \( T_1 \) is contained in \( T_2 \) \( \iff \) each moving sphere \( SP_1(t) \) is contained in \( T_2 \) \( \iff \) each center point \( P = P_1(t) \) is contained in the torus \( T_2^I \).

Remark 2.3. Note that \( T_2^O \) is not necessarily a ring torus, since its minor radius \( r_1 + r_2 \) can be equal to or greater than its major radius \( R_2 \); \( T_2^I \) is either a ring torus or degenerates to its main circle, since \( 0 \leq r_2 - r_1 < r_2 < R_2 \).

2.3. Outline of enumerating the intersection curve of two ring tori

Case 1 and case 3 in Theorem 2.2 are trivial, since in these cases the intersection curve \( C \) of \( T_1 \) and \( T_2 \) contains no real point. In case 2, \( T_1 \) and \( T_2 \) intersect along a real curve, which needs more elaborate analysis. By Theorem 2.2, the intersection problem of \( T_1 \) and \( T_2 \) has now been reduced to analyzing the relative position of the circle \( S \) with respect to the new tori \( T_2^O \) and \( T_2^I \).

Since the circle \( S \) is a planar curve, its intersection with tori \( T_2^O \) and \( T_2^I \) lies in the plane \( \pi \) containing \( S \). We call the intersection curve of a torus with a plane a toric section (TS). The second row of Figure 5 shows the circle \( S \) together with the two nested toric sections (NTS) of \( T_2^O \) and \( T_2^I \) cut by the plane \( \pi \), which is called the intersection discriminant chart (IDC) of \( T_1 \) and \( T_2 \).

Since an IDC is derived from two ring tori, we ask whether a classification of IDCs can lead to an classification of the intersection curve of the two ring tori. We shall take the following course of study. To enumerate all different IDCs, we shall first enumerate all the possible configurations of the two nested toric sections (NTS) of \( T_2^O \) and \( T_2^I \). Then all IDCs can be enumerated by considering the relative position of the circle \( S \) with respect to each NTS.
The above procedure is illustrated below:

\[
\text{TS (a Toric Section)} \quad \downarrow \\
\text{NTS (two Nested Toric Sections)} \quad \downarrow \\
\text{IDC (Intersection Discriminant Chart)}
\]

3. Enumeration of TS

In this section we shall enumerate different types of the intersection curve of a torus with an arbitrary plane. Let \( \mathcal{T} \) be a torus in the canonical position and orientation with minor radius \( r \) and major radius \( R \). Consider the toric section of \( \mathcal{T} \) cut by the translating plane \( \mathcal{P}(t) : P(t) = ax + by + cz + t = 0 \) with an arbitrary but fixed normal vector \( \mathbf{0} \neq (a, b, c) \in \mathbb{R}^3 \). See Figure 6. Since \( \mathcal{T} \) is symmetric about \( z \)-axis, a rotation of the \( xy \) plane about \( z \)-axis does not change the relative configuration of the plane and the torus. Hence, without loss of generality, we assume that \( (a, b, c) = (a, 0, c) \). Now the moving plane is \( \mathcal{P}(t) : P(t) = ax + cz + t = 0 \).

As \( t \) varies from \( -\infty \) to \( +\infty \), \( \mathcal{P}(t) \) first comes close to, then intersects with, and finally separates from the torus \( \mathcal{T} \). If \( a = 0 \), the cases that can arise are easy to enumerate – the translating plane is perpendicular to the symmetric axis of the torus, and the TS will change from null to two nested loops via a critical topology, i.e., a double loop that is perpendicular to the symmetric axis. In the following we assume that \( a \neq 0 \).
3.1. TS of a ring torus

We first suppose $\mathcal{T}$ to be a ring torus. As the parameter $t$ goes from $-\infty$ to $+\infty$, the shape of the toric section of $\mathcal{T}$ with the plane $\mathcal{P}(t) = 0$ varies continuously, while the topology of the toric section changes at some critical instants. The topology change of the toric section has two possibilities, depending on the angle $\theta$ between the normal vector $(a, 0, c)$ and the $z$-axis.

To be more precise, as $t$ goes from $-\infty$ to $+\infty$, the plane $\mathcal{P}(t) = 0$ sequentially touches the torus $\mathcal{T}$ at four points $Q_i = (x_i, 0, z_i)$ at four critical instants $t_i, i = 1, 2, 3, 4$, with $t_1 < t_2 < t_3 < t_4$. There are only two cases, depending on the value of $\theta$. In the first case (see Figure 6), the $x$-coordinates of the four sequential touching points, denoted $x_i, i = 1, 2, 3, 4$, form an increasing sequence, i.e., $x_1 < x_2 < x_3 < x_4$. In this case, the TS starts from a loop, then is gradually pinched at the middle, and finally splits into two disconnected loops; the above process then repeats in a symmetric reverse order. In the second case (see Figure 7), the order of the $x$-coordinates of the two touching points in the middle will swap to give $x_1 < x_3 < x_2 < x_4$. In this case, the TS passes from a loop, then through a U-shape, and finally into two nested loops; again the process then repeats in a symmetric reverse order.

**Remark 3.1.** Since the shape of the TS varies continuously with $t$, the topology of the degenerate TS at critical points can be derived from the TS in its neighboring time intervals. That is to say, the degenerate configurations of the TS can be deduced from the sequence shown in Figures 6 and 7.

Note that when $a = 0$, i.e., the translating plane is perpendicular to the symmetric axes of the ring torus, the TS will change from null to two nested loops via a critical topology, hence this situation provides

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$^1$The plane touches the torus at the point $Q$ if the normals of the plane and the torus at $Q$ are parallel.
Figure 7: $x_1 < x_2 < x_3 < x_4$. $R = 5, r = 2, \theta = \frac{\pi}{5}$. The toric section passes from a loop, and then through a U-shape, and into two nested loops; the process then repeats in a symmetric reverse order. The inset figure for $t = t_3$ in the first row shows the same TS when looked from downside to highlight the singular point on the TS.

Figure 8: Three possible topologies of a non-degenerate ring TS.

only the two nested loops as a non-degenerate ring TS. We now summarize with an enumeration of the non-degenerate ring toric sections by the two processes in Figures 6 and 7:

**Theorem 3.2.** A non-degenerate ring TS can be null or have three possible topologies: one loop, two disconnected loops, or two nested loops, as listed in Figure 8.

3.2. TS of spindle torus

When studying the TS of the torus $T_2^O$, we must consider the case where $T_2^O$ is a spindle torus. Now suppose that $T$ is a spindle torus. We still examine its TS cut by the moving plane $P(t) : P(t) = ax + cz + t = 0$. The TS changes in a more complicated process than that for a ring torus, since a spindle torus has two self-intersection points $O_1$ and $O_2$, which play an important role when the spindle TS switches to another topology. Similar to the ring torus, when $t$ varies continuously from $-\infty$ to $+\infty$, the moving plane $P(t)$ also sequentially touches the torus $T$ at four points $Q_i$, $i = 1, 2, 3, 4$. Considering the self-intersection points $O_i$, $i = 1, 2$, and the touching point $Q_j$, $j = 1, 2, 3, 4$ together, when $t$ goes from $-\infty$ to $+\infty$ the moving plane $P(t)$ sequentially touches the torus $T$ at $Q_1, Q_2, O_1, O_2, Q_3, Q_4$ as shown in Figure 9 and Figure 10, with the
Figure 9: $R = 3$, $r = 6$, $\theta = \frac{\pi}{6}$. The critical points are $Q_1, Q_2, O_1, O_2, Q_3, Q_4$, with $x_1 < x_2 < x_3 < x_4$ for $Q_i, i = 1, 2, 3, 4$.

Figure 10: $R = 3$, $r = 3.5$, $\theta = \frac{\pi}{3}$. The critical points are $Q_1, Q_2, O_1, O_2, Q_3, Q_4$, with $x_1 < x_3 < x_2 < x_4$ for $Q_i, i = 1, 2, 3, 4$.

difference that the $x$-coordinates of $Q_i$ is arranged by $x_1 < x_2 < x_3 < x_4$ in Figure 9 and $x_1 < x_3 < x_2 < x_4$ in Figure 10. The corresponding topologies of the intersection curves are also shown in Figure 10.

Note that when $a = 0$, i.e., the translating plane is perpendicular to the symmetric axes of the ring torus, the TS again provides only the two nested loops as a non-degenerate ring toric section. Now we can extract all the non-degenerate topological possibilities of the TS of a spindle torus from Figure 9 and Figure 10.

**Theorem 3.3.** A non-degenerate TS of a spindle torus can be null or the two possible topologies: one loop or two nested loops shown as the first and third case in Figure 8.
4. Enumeration of NTS

Considering the two tori $T_O^2$ and $T_I^2$ that have the same major circle but different minor radii $r^O$ and $r^I$ with $r^O > r^I$. By a coordinate transformation these two tori are in the standard position and orientation. We now use an arbitrary plane $P$ to cut the two tori.

Definition 4.1. Let $P$ be a plane and $T$ be a torus with the implicit equation $T(x,y,z) = 0$ in Eqn. (1). The interior of the TS cut by the plane $P$ is defined as $\text{Int}_P(T) := \{(x,y,z) \in P | T(x,y,z) < 0\}$, and the exterior of the TS cut by the plane $P$ is defined as $\text{Ext}_P(T) := \{(x,y,z) \in P | T(x,y,z) > 0\}$.

We first assume that the outer torus $T_O^2$ is a ring torus. The case when the outer torus $T_O^2$ is spindle will be discussed later. By Theorem 3.2, both TS of $T_O^2$ and TS of $T_I^2$ can be in one of the four possible cases: null, one loop, two disconnected loops or two nested loops. By taking combinations of these four possibilities, we obtain the 12 candidates of possible NTS as shown in Table 1, excluding the four trivial cases where the TS of $T_O^2$ is null (which means the NTS is null). Note that these 12 candidates are enumerated according to the following criterion that an NTS naturally satisfies.

Lemma 4.1. Containment criterion: If $T_O^2$ is a ring torus, $\text{Int}_P(T_I^2) \subset \text{Int}_P(T_O^2)$.

In fact, not all these 12 candidate cases can be realized. We have the following further necessary condition that an NTS needs to satisfy. Recall that the cutting plane for generating an NTS has the equation $ax + cz + t = 0$ (see the beginning of Section 3). Then we have the following lemma.

Lemma 4.2. Symmetry criterion: When viewed along the $x$-axis, the NTS of $T_I^2$ is symmetric in the vertical line which is the projection of $z$-axis.

Some examples of symmetric TS and non-symmetric ones are shown in Figure 11. The proofs of the conditions in Lemma 4.1 and Lemma 4.2 follow from straightforward geometric observations, so are skipped.

![Figure 11](image.png)

(a) non-symmetric       (b) non-symmetric       (c) symmetric

Figure 11: The TS of $T_I^2$ (solid line) should be topologically symmetric with respect to the TS of $T_O^2$ (dashed line).

We list the candidates of NTS in Table 1, where the dashed lines are the TS of $T_O^2$ and the solid lines are the TS of $T_I^2$. The validity of the NTS candidate is denoted by “Y” for “yes” or “N” for “no”, checked in accordance with Lemma 4.2 and other consideration.
Table 1: Enumeration of NTS candidates.

\begin{tabular}{|c|c|c|c|c|}
\hline
(1) & Y & (2) & Y & (3) & Y \\
\hline
(5) & N & (6) & N & (7) & Y \\
\hline
(9) & Y & (10) & Y & (11) & Y \\
\hline
(12) & Y & & & & \\
\hline
\end{tabular}

**Theorem 4.3.** The NTS can be null or is in one of the eight cases (1), (2), (3), (7), (9), (10), (11), and (12) in Table 1.

Proof. We now explain the validity of all the cases in Table 1. We will provide an example for each of those valid NTS.

1. \( R = 5, r^O = 2.5, r^I = 1; \) \( \mathcal{P} : x - 7 = 0; \)
2. \( R = 5, r^O = 2.5, r^I = 1; \) \( \mathcal{P} : x - 5 = 0; \)
3. \( R = 5, r^O = 2.5, r^I = 1; \) \( \mathcal{P} : x - 3 = 0; \)
4. The inner TS is a pair of two nested loops, which can only happen for the case between \( t = t_2 \) and \( t = t_3 \) in Figure 7. Since the torus \( T^O_2 \) contains \( T^I_2 \), the outer TS can only be two nested loops. This is a contradiction.
5. The outer TS consists of two separate loops only when the plane cut the torus as in the case between time instants \( t = t_2 \) and \( t = t_3 \) in Figure 6. In this case the inner TS cannot be null. This is a contradiction.
6. This configuration is not symmetric, contradicting Lemma 4.2;
7. \( R = 5, r^O = 2.5, r^I = 1; \) \( \mathcal{P} : x = 0; \)
8. This configuration is not symmetric, contradicting Lemma 4.2;
9. \( R = 5, r^O = 4, r^I = 0.5; \) \( \mathcal{P} : -\cos(\frac{\pi}{20})z + \sin(\frac{\pi}{20})x + 2/\cos(\frac{\pi}{20}) = 0; \)
10. \( R = 5, r^O = 4, r^I = 2; \) \( \mathcal{P} : -\cos(\frac{\pi}{20})z + \sin(\frac{\pi}{20})x + 2/\cos(\frac{\pi}{20}) = 0; \)
11. \( R = 5, r^O = 3, r^I = 1.5; \) \( \mathcal{P} : -\cos(\frac{\pi}{20})z + \sin(\frac{\pi}{20})x = 0; \)
12. \( R = 5, r^O = 4, r^I = 2; \) \( \mathcal{P} : -\cos(\frac{\pi}{20})z + \sin(\frac{\pi}{20})x = 0. \)
Now we consider the case when $T^O_2$ is a spindle torus. By Theorem 3.3, its TS can only be null, one loop or two nested loops. Since no new TS occurs, we need only to check whether Lemma 4.1 and Lemma 4.2 still hold. Clearly, Lemma 4.2 is still true when the outer torus is spindle; however, the analysis is more complicated for Lemma 4.1. For example, referring to Figure 12, consider two standard tori $T^O_2$ and $T^I_2$ with the same major radius $R = 3$, but different minor radii $r^O = 6$ and $r^I = 2$. We use the plane $\mathcal{P} : x = 0$ to simultaneously cut these two tori. From both of the 3D and the 2D views, one can see that $\text{Int}_\mathcal{P}(T^I_2) \not\subset \text{Int}_\mathcal{P}(T^O_2)$ (the shadow area). We shall show that that our $T^O_2$ and $T^I_2$ cannot be in this case.

Note that $T^O_2$ and $T^I_2$ have the same major radius $R$, but different minor radii $r^O > r^I$. However, since $T^O_2$ and $T^I_2$ are constructed from the original ring tori $T_1$ and $T_2$, we have $r^O = r_1 + r_2$ and $r^I = r_2 - r_1$, where $r_1$ and $r_2$ are the minor radii of $T_1$ and $T_2$. Hence $r^O + r^I = 2r_2 < 2R$. But in the NTS shown in Figure 12, since $|O_1O_2| = 2R$, we have $O_1A = 2R - r^O < r^I$, which yields $r^O + r^I > 2R$. This leads to a contradiction. Hence such complicated cases will not occur with our construction of $T^O_2$ and $T^I_2$. Hence, when $T^O_2$ is spindle, Lemma 4.1 still holds. Therefore, when $T^O_2$ is spindle, it does not create any NTS that is essentially different from those in Table 1 and so we do not need to treat the spindle cases separately.

5. Enumeration of IDC and Intersection Curve

Based on the preceding enumeration of NTS, we next study the position of the circle $S$ with respect to an NTS to enumerate all possible topologies of IDC. We adopt the following convention. Let $\mathcal{P}$ be the plane containing the circle $S$. We encode the regions induced by the NTS on $\mathcal{P}$ as follows (see Figure 13): the region $\text{Ext}_\mathcal{P}(T^O_2) \cap \text{Ext}_\mathcal{P}(T^I_2)$ is referred to regions 1; the region $\text{Int}_\mathcal{P}(T^O_2) \cap \text{Ext}_\mathcal{P}(T^I_2)$ is referred to as region 2; the region $\text{Int}_\mathcal{P}(T^O_2) \cap \text{Int}_\mathcal{P}(T^I_2)$ is referred to as region 3. We define the region sequence (RS) of the circle $S$ with respect to the NTS to be the sequence of the region numbers that the circle $S$ traverses through counterclockwise, and assume it always starts from the smallest region number it contains. For example, the RS of the circle in thick solid line in Figure 13 is $[1232321]$.
Since $S$ is a closed curve, once it goes into a region it also has to go out, i.e., the number of the region changes is even, hence the length of RS must be an odd number. On the other hand, a circle can penetrate (outside $\rightarrow$ inside $\rightarrow$ outside) a torus at most twice. By an inverse map ([DABG99]) we can open a circle to a line while the torus is mapped to a cyclide which is also of algebraic degree four. By Bézout theorem the line intersects the cyclide in at most four real space points, which correspond to the four endpoints of the two segments of the circle inside the torus. Hence, the most complex RS occurs when the circle $S$ penetrates both $T_2^O$ and $T_2^I$ twice, i.e., RS is $[123212321]$ of length 9. Other cases when the length is 1,3,5,7 can be derived from the above four rules. Specifically, we have the following theorem.

![Figure 13: Plane partition by NTS; the RS of the black circle is [1232321].](image)

**Theorem 5.1.** There are in total the eleven RS’s, as follows.

$$[1], [2], [3], [121], [232], [12321], [23232], [12121], [1232121], [1232321], [123212321].$$

Proof. The enumeration is derived by applying the following observations: (1) the length of RS is odd, and is no greater than 9; (2) the RS starts from the smallest region number it contains; (3) every two consecutive region numbers in the RS differ by 1; and (4) the first region number is the same as the last one, since the circle is a closed curve.

Now we study the topology of the intersection curve corresponding to each of the above eleven RSs.

**Theorem 5.2.** If the RS of the circle $S$ is $[1]$ or $[3]$, then the tori $T_1$ and $T_2$ have no real intersection.

Proof. By Theorem 2.2, if the RS is $[1]$, the tori $T_1$ and $T_2$ are separate. Hence $T_1$ and $T_2$ have no real intersection. If the RS is $[3]$, the torus $T_1$ is contained in the torus $T_2$. Hence, again $T_1$ and $T_2$ have no real intersection.
Definition 5.1. Let $\mathcal{O}$ be a connected component of the intersection curve $C$ of the tori $T_1$ and $T_2$. We call $\mathcal{O}$ an $\alpha$-loop if $\mathcal{O}$ can continuously contract on both tori $T_1$ and $T_2$ to a point; we call $\mathcal{O}$ a $\beta$-loop if $\mathcal{O}$ can continuously contract on only one of the tori $T_1$ and $T_2$ but not on the other to a point; we call $\mathcal{O}$ a $\gamma$-loop if $\mathcal{O}$ cannot continuously contract on either $T_1$ or $T_2$ to a point (See Figure 14 for illustration).

The way in which the circle $S$ penetrates the NTS determines the topology of the intersection curve $C$. We next divide the circle $S$ into curve segments whose RSs are triplets (i.e. subsequence of 3 elements) such that the last number of one triplet overlap with the first number of the next triplet. For example, the circle $S$ in Figure 13 divides into triplets $[123], [323], [321]$. For the RS given in Theorem 5.1 with length longer than 3, we divide them into the following triplets:

$[123] : [123], [321]$  
$[232] : [232], [232]$  
$[121] : [121], [121]$  
$[123121] : [123], [321], [121]$  
$[123231] : [123], [323], [321]$  
$[123212321] : [123], [321], [123], [321]$.  

The following theorems tell which type of loops these triplets correspond to.

Theorem 5.3. 1. The triplet $[123]$ (or $[321]$) corresponds to a $\beta$-loop in the intersection curve $C$;  
2. The triplet $[121]$ corresponds to an $\alpha$-loop in $C$;  
3. The triplet $[232]$ corresponds to an $\alpha$-loop in $C$;  
4. The triplet $[323]$ corresponds to an $\alpha$-loop in $C$;

Proof. By Theorem 2.2, if the region sequence is $[123]$ then the corresponding segment of torus $T_1$ is at first separated from, and then intersects with, and is finally contained in the torus $T_2$. Hence, it produces a $\beta$-loop which can continuously contract on $T_2$ but not on $T_1$ to a point.
Similarly, if the region sequence is [121], then the corresponding segment of torus $T_1$ is at first separate from, and then intersects with, and is again separate from the torus $T_2$. Hence it produces an $\alpha$-loop.

The other cases can be verified similarly.

We now examine the RS of the circle $S$ for each valid NTS listed in Table 1 and list all the corresponding IDCs in Table 2 through Table 9. For each RS, we provide an example of the two tori $T_1$ and $T_2$, which realizes the circle $S$ and the two nested tori $T_1^O$ and $T_2^I$. The topologies of the intersection curve of these examples can be analyzed via their RS by Theorem 5.2 and 5.3. Note that in the column “IDC” of in Table 2 through Table 9, the thickened solid circle is the circle $S$, the dashed line is the TS of $T_1^O$ and the solid line is the TS of $T_2^I$.

<table>
<thead>
<tr>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>$C$</th>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2]</td>
<td>[2]</td>
<td>β</td>
<td>2</td>
<td>[121]</td>
<td>[121]</td>
<td>[121]</td>
<td>1</td>
</tr>
<tr>
<td>[2121]</td>
<td></td>
<td></td>
<td></td>
<td>[232]</td>
<td>[232]</td>
<td>[232]</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Enumeration for IDCs associated with NTS (1).
Table 3: Enumeration for IDCs associated with NTS (2).

<table>
<thead>
<tr>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>[12121]</td>
<td></td>
<td></td>
<td>2 α</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>[1232321]</td>
<td></td>
<td></td>
<td>1 α</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2 β</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Enumeration for IDCs associated with NTS (3).

<table>
<thead>
<tr>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[2]</td>
<td></td>
<td></td>
<td>2 β</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>[121]</td>
<td></td>
<td></td>
<td>1 α</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>[12121]</td>
<td></td>
<td></td>
<td>2 α</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>[12321]</td>
<td></td>
<td></td>
<td>2 β</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>[1232321]</td>
<td></td>
<td></td>
<td>1 α</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2 β</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4 β</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5: Enumeration for IDCs associated with NTS (7).

<table>
<thead>
<tr>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[2]</td>
<td></td>
<td>2β</td>
</tr>
<tr>
<td>3</td>
<td>[121]</td>
<td></td>
<td>1α</td>
</tr>
<tr>
<td>4</td>
<td>[232]</td>
<td></td>
<td>2α</td>
</tr>
<tr>
<td>5</td>
<td>[12121]</td>
<td></td>
<td>1α</td>
</tr>
<tr>
<td>6</td>
<td>[1321]</td>
<td></td>
<td>2β</td>
</tr>
<tr>
<td>7</td>
<td>[2332]</td>
<td></td>
<td>1α</td>
</tr>
<tr>
<td>8</td>
<td>[12321]</td>
<td></td>
<td>2β</td>
</tr>
<tr>
<td>9</td>
<td>[1232321]</td>
<td></td>
<td>1α</td>
</tr>
<tr>
<td>10</td>
<td>[123212321]</td>
<td></td>
<td>2β</td>
</tr>
</tbody>
</table>

Table 6: Enumeration for IDCs associated with NTS (9).

<table>
<thead>
<tr>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[2]</td>
<td></td>
<td>2β</td>
</tr>
<tr>
<td>3</td>
<td>[121]</td>
<td></td>
<td>1α</td>
</tr>
<tr>
<td>4</td>
<td>[12121]</td>
<td></td>
<td>2α</td>
</tr>
</tbody>
</table>

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Table 7: Enumeration for IDCs associated with NTS (10).
<table>
<thead>
<tr>
<th></th>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>$C$</th>
<th></th>
<th>IDC</th>
<th>RS</th>
<th>Example</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="image1" alt="Diagram" /></td>
<td>[2]</td>
<td><img src="image2" alt="Diagram" /></td>
<td>2 $\beta$</td>
<td>2.</td>
<td><img src="image3" alt="Diagram" /></td>
<td>[2]</td>
<td><img src="image4" alt="Diagram" /></td>
<td>2 $\gamma$</td>
</tr>
<tr>
<td>3.</td>
<td><img src="image5" alt="Diagram" /></td>
<td>[121]</td>
<td><img src="image6" alt="Diagram" /></td>
<td>1 $\alpha$</td>
<td>4.</td>
<td><img src="image7" alt="Diagram" /></td>
<td>[232]</td>
<td><img src="image8" alt="Diagram" /></td>
<td>1 $\alpha$</td>
</tr>
<tr>
<td>5.</td>
<td><img src="image9" alt="Diagram" /></td>
<td>[12321]</td>
<td><img src="image10" alt="Diagram" /></td>
<td>2 $\beta$</td>
<td>6.</td>
<td><img src="image11" alt="Diagram" /></td>
<td>[23232]</td>
<td><img src="image12" alt="Diagram" /></td>
<td>2 $\alpha$</td>
</tr>
<tr>
<td>7.</td>
<td><img src="image13" alt="Diagram" /></td>
<td>[1232321]</td>
<td><img src="image14" alt="Diagram" /></td>
<td>1 $\alpha$ 2 $\beta$</td>
<td>8.</td>
<td><img src="image15" alt="Diagram" /></td>
<td>[123212321]</td>
<td><img src="image16" alt="Diagram" /></td>
<td>4 $\beta$</td>
</tr>
</tbody>
</table>

Table 8: Enumeration for IDCs associated with NTS (11).
Table 9: Enumeration for IDCs associated with NTS (12).
intersection curve of two given ring tori.

Let $Q$ be an arbitrary point on the main circle $\mathcal{S}$ of the torus $T_1$ (refer to Fig. 15). Let $\mathcal{S}$ be the sphere centered at $Q$ with radius $R_1$, the major radius of $T_1$. Then the inverse map with respect to the sphere $\mathcal{S}$ ([DABG99]) is defined by:

$$
\eta : X' = Q + \frac{R_1^2(X - Q)}{\|X - Q\|^2}.
$$

This inverse map transforms the circle $\mathcal{S}$ to a line $l$ not passing through $Q$, and simultaneously maps the two tori $T_2^O$ and $T_2^I$ to two cyclides $\mathcal{C}_a$ and $\mathcal{C}_b$. Denote the linear parametric form of the line $l$ by $\ell(t)$. Let the implicit equations of $\mathcal{C}_a$ and $\mathcal{C}_b$ be $f(x, y, z) = 0$ and $g(x, y, z) = 0$ respectively, where $f$ and $g$ are both degree four polynomials.

The following theorem is obvious.

**Theorem 6.1.** $f(\ell(t_0)) = 0$ if and only if $\eta^{-1}(\ell(t_0))$ is an intersection point of the circle $\mathcal{S}$ and the torus $T_2^O$. Similarly, $g(\ell(t_0)) = 0$ if and only if $\eta^{-1}(\ell(t_0))$ is an intersection point of the circle $\mathcal{S}$ and the torus $T_2^I$.

Since Theorem 6.1 reduces the intersection of the circle $\mathcal{S}$ with the NTS to the roots of $f(\ell(t)) = 0$ and $g(\ell(t)) = 0$, the correspondence between the IDC and the topology of the intersection curve $\mathcal{C}$ listed in Table 10 can be represented in terms of the numbers of real roots of $f(\ell(t)) = 0$ and $g(\ell(t)) = 0$, as shown in Table 11.

We next use a working example to illustrate our classification algorithm.

**Example 6.1.** Consider two tori $T_1$ and $T_2$ with implicit equations (see Fig. 6.1),

$$
T_1(x, y, z) = (x^2 + (y + 6)^2 + (z + \frac{3}{2})^2 + R_1^2 - r_1^2)^2 - 4R_1^2x^2 - 4R_1^2(y + 6)^2 = 0
$$

$$
T_2(x, y, z) = (x^2 + y^2 + z^2 + R_2^2 - r_2^2)^3 - 4R_2^2x^2 - 4R_2^2y^2 = 0,
$$
Table 11: Correspondence between the numbers of real root numbers of $f(l(t)) = 0$, $g(l(t)) = 0$ (denoted $\sharp f$ and $\sharp g$, respectively) and the topology of the intersection curve $C$.

<table>
<thead>
<tr>
<th>$\sharp f$</th>
<th>$\sharp g$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>n/a or $2\gamma$ or $2\beta$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$1\alpha$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$2\alpha$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$1\alpha+2\beta$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$2\beta$</td>
</tr>
</tbody>
</table>

where

$$R_1 = 4, r_1 = 1, R_2 = 5, r_2 = 3.$$ 

The main circle $S$ of $T_1$ lies in the plane $z = -\frac{3}{2}$, which has the implicit equation:

$$\begin{align*}
    x^2 + (y + 6)^2 - 16 &= 0 \\
    z &= -\frac{3}{2}.
\end{align*}$$

Let $r_2 := r_1 + r_2 = 4$ and $r_2 := r_2 - r_1 = 2$ we get tori $T^O_2$ and $T^I_2$ with implicit equations:

$$\begin{align*}
    F(x, y, z) &= (x^2 + y^2 + z^2 + 9)^2 - 100x^2 - 100y^2 = 0 \\
    G(x, y, z) &= (x^2 + y^2 + z^2 + 21)^2 - 100x^2 - 100y^2 = 0.
\end{align*}$$

Choose point $Q = (4, -6, -\frac{3}{2})$ on the circle $S$. Define a reference sphere $S$ centered at $Q$ with radius $R_1 = 4$.

Then the inverse map defined with the sphere $S$ is

$$\eta(X) = Q + \frac{16(X - Q)}{\|X - Q\|^2},$$

which inverts the circle $S$ to a line $l$ not passing through $Q$. Pick two other points $P_1 = (\sqrt{7}, -3, -\frac{3}{2})$ and $P_2 = (1, \sqrt{15} - 6, -\frac{3}{2})$ on the circle $S$. Then the line $l$ goes through the point $\eta(P_1) = (2, \frac{2}{3}(\sqrt{7} - 5), -\frac{3}{2})$ and $\eta(P_2) = (2, -6 + 2\sqrt{15}, -\frac{3}{2})$. Hence the parametric form of the line $l$ is

$$l(t) = (2, t, -\frac{3}{2}).$$
The two tori $T_2^O$ and $T_2^I$ are inverted to two cyclides with implicit equations

\[
 f(x, y, z) = -972877959 - 467561376y + 311707584x - 163891944z - 614112y^2z^2 \\
 - 67603192x^2 - 95608312y^2 - 45482472z^2 - 307056x^4 - 307056z^4 \\
 - 307056y^4 - 614112x^2y^2 - 614112x^2z^2 - 8671872y^3 + 5781248x^3 \\
 - 3396768z^3 - 8671872x^2y - 3396768x^2z + 5781248xy^2 + 67212288xy \\
 + 5781248xz^2 + 2663472xz - 3396768y^2z - 8671872yz^2 - 39950208yz = 0, \\
 g(x, y, z) = -213989031 - 86710176y + 57806784x - 68679144z + 236832y^2z^2 - 9593848x^2 \\
 - 11425528y^2 - 5467368z^2 + 118416x^4 + 118416z^4 + 236832x^2y^2 \\
 + 236832x^2z^2 + 359808y^3 - 239872x^3 - 1138848z^3 + 359808z^3y - 1138848x^2z \\
 - 239872xy^2 + 4396032xy - 239872xz^2 + 10929408xz - 1138848y^2z + 359808y^2z^2 \\
 - 16394112yz = 0.
\]

Intersecting the line $l(t)$ with $f(x, y, z) = 0$ and $g(x, y, z) = 0$ yields

\[
 f(l(t)) = 307056t^4 + 8671872t^3 + 8278880t^2 + 327410688t + 464157696, \\
 g(l(t)) = 118416t^4 + 359808t^3 - 8716800t^2 - 51078144t - 66518016.
\]

The equation $f(l(t)) = 0$ has two real roots

\[
 t_1 = -13.52518714, \ t_2 = -6.531548243,
\]

and the equation $g(l(t)) = 0$ has four real roots

\[
 t_3 = -5.745502191, t_4 = -4.996573139, t_5 = -2.013660632, t_6 = 9.717227652.
\]

Hence, according to Table 11, the intersection curve $C$ of tori $T_1$ and $T_2$ consists of one $\alpha$-loop and two $\beta$-loops, as shown in Figure 16.

![Figure 16: The intersection curve of two tori.](image)

When both of $f(l(t)) = 0$ and $g(l(t)) = 0$ have no real roots, by Table 11 we conclude that the intersection
curve $C$ is either empty or consists of two $\beta$-loops or two $\gamma$-loops. To further distinguish between these cases, we first need the following theorem.

**Theorem 6.2.** Let $P$ be an arbitrary point on the circle $S$. Suppose $\eta(P) = l(t_0)$. Then $P$ is inside the torus $T_2^O$ if and only if $f(l(t_0)) < 0$. Similarly, $P$ is inside the torus $T_2^I$ if and only if $g(l(t_0)) < 0$.

Proof. Suppose the implicit equation of the torus $T_2^O$ to be $F(x,y,z) = 0$. By the definition of inverse map we have $f(l(t_0)) = f(\eta(P)) = F(P)$. The fact that the point $P$ is inside $T_2^O$ if and only if $F(P) < 0$ then gives the conclusion.

**Theorem 6.3.** If $f(l(t)) > 0$ for all $t \in \mathbb{R}$ or $g(l(t)) < 0$ for all $t \in \mathbb{R}$, then the intersection curve $C$ has no real component.

Proof. By Theorem 6.2, for an arbitrary $t \in \mathbb{R}$, $f(l(t)) > 0$ means that the point $P = \eta^{-1}(l(t_0))$ lies outside the torus $T_2^O$, hence the circle $S$ lies outside $T_2^O$. Therefore, by Theorem 2.2, the ring tori $T_1$ and $T_2$ are separate. Hence, the intersection curve $C$ has no real component. Similarly, for an arbitrary $t \in \mathbb{R}$, $g(l(t)) < 0$ suggests that the point $P = \eta^{-1}(l(t_0))$ lies inside the torus $T_2^I$, hence the circle $S$ lies inside $T_2^I$. Again, by Theorem 2.2, the ring tori $T_1$ and $T_2$ are separate. Hence, the intersection curve $C$ has no real component.

The next theorem summarizes all the cases that may arise.

**Theorem 6.4.** The intersection curve $C$ consists of two $\beta$-loops or two $\gamma$-loops if and only if $f(l(t)) < 0$ for all $t \in \mathbb{R}$ and $g(l(t)) > 0$ for all $t \in \mathbb{R}$.

Proof. By Theorem 6.2, for an arbitrary $t \in \mathbb{R}$, $f(l(t)) < 0$ and $g(l(t)) > 0$ means that the point $P = \eta^{-1}(l(t_0))$ on the circle $S$ is inside the outer torus $T_2^O$ and outside the inner torus $T_2^I$. Hence the region sequence of the circle $S$ is [2], which by Theorem 5.5 means that the intersection curve $C$ consists of two $\beta$-loops or two $\gamma$-loops.

The only remaining task now is to distinguish the two cases that the intersection curve $C$ consists of two $\beta$-loops or two $\gamma$-loops when $f(l(t)) < 0$ and $g(l(t)) > 0$ for all $t \in \mathbb{R}$. By Theorem 5.5, this can easily be done by checking whether there is any interior point $Q$ inside the disk bounded by the circle $S$ that is on the main circle or the symmetric axis of the torus $T_2$.

**7. Conclusion**

We have enumerated all the topologies of the non-degenerate intersection curves of two ring tori in generic positions and orientations by going through all the IDCs of the two ring tori. We have also presented a simple algebraic approach to determining the topological type of the non-degenerate intersection curve of
two ring tori by reducing the intersection problem of two ring tori to the problem of determining the root patterns of two univariate quartic polynomials.

A future problem is to extend our approach to determining the topology of the intersection curve (degenerate as well as non-degenerate) of two general cyclides.

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