Representing Perspective Projections as Rotors in the Homogeneous Model of the Clifford Algebra for 3–Dimensional Euclidean Space

Xiaohong Jia and Ron Goldman

Abstract We show that perspective projections in 3-dimensions can be represented by rotors in the homogeneous model of the Clifford Algebra for R^3. We also show that the only rotors in this Clifford Algebra, when interpreted as transformations on R^3, are rotations, reflections, perspective projections and their composites.

1 Motivation

Perspective projections are fundamental in contemporary 3-dimensional Computer Graphics for generating photorealistic images. Therefore any algebraic representation of geometry that aspires to become a computational model for modern Computer Graphics must necessarily provide a simple technique for representing perspective projections.

Currently the standard mathematical representations for affine and projective transformations in Computer Graphics are 4×4 matrices applied to points and vectors each represented by four homogeneous coordinates [2]. These 4×4 transformation matrices include translations, rotations, uniform and non-uniform scalings, shears, and orthogonal and perspective projections. Recently several authors have advocated Geometric Algebra as an alternative to matrices as an algebraic foundation for Computer Graphics[1, 7]. But these authors have yet to incorporate perspective projections as one of the natural transformations in their computational models.

Xiaohong Jia
KLMM, AMSS & NCMIS,
Chinese Academy of Sciences, Beijing, China. e-mail: xhjia@amss.ac.cn

Ron Goldman
Department of Computer Science, Rice University,
6100 Main St., MS-132, Houston, TX 77005, USA. e-mail: rng@rice.edu

The purpose of this paper is to correct this omission by showing that perspective projections in 3-dimensions can be represented by rotors in the standard homogeneous model of the Clifford Algebra for 3-dimensional Euclidean space.

Quaternions were the first successful alternative computational model for representing perspective projections in 3-dimensions. Goldman showed that perspective projection on a point P can be computed by sandwiching the vector from the eye point E to the point P between two copies of a unit quaternion [3, 4].

Shortly thereafter, Goldman extended this quaternion approach to perspective projection to the Clifford Algebra for R^3 [4, 5]. Quaternions are a natural subalgebra of the Clifford Algebra for R^3. Therefore Goldman was again able to show how to compute perspective projections on a point P by sandwiching the vector from the eye point E to the point P between two copies of a unit quaternion. But in Clifford Algebra this particular sandwiching operation is not natural, since this transformation is not represented by a versor or a rotor – that is, perspective projection is not one of the natural transformations in this model of Clifford Algebra. Indeed in these quaternion representations for perspective projection, the composite of a pair of perspective projections is not represented by the product of the two quaternions that represent the two individual perspective projections.

Once again our goal in this paper is to correct this anomaly by showing that perspective projections in 3-dimensions are indeed represented by rotors, not in the Clifford Algebra for R^3, but rather in the standard homogeneous model of the Clifford Algebra for R^3. We will also show that the only rotors in this homogeneous model of the Clifford Algebra for R^3, when interpreted as transformations on R^3, are rotations, reflections, perspective projections and their composites.

We shall proceed in the following fashion. We begin in Section 2 by providing a brief review of the standard homogeneous model of the Clifford Algebra for R^3. In Section 3, we study the rotors in this model and we show that rotations, reflections, and perspective projections on R^3 can all be represented by rotors in this homogeneous model. In Section 4 we show that there are no other natural transformations on R^3 in this model – that is, the only rotors in this homogeneous model of the Clifford Algebra for R^3, when interpreted as transformations on R^3, are rotations, reflections, perspective projections and their composites. Finally we close in Section 4 with a short summary of our work along with a brief discussion of the deficiencies of this homogeneous model and our plans for future research.

2 The standard homogeneous model for R^3

The standard homogeneous model of Clifford Algebra associated with 3-dimensional Euclidean space is just the standard elliptical Clifford Algebra for R^4 – that is, the Clifford Algebra with signature $(+,+,+,+)$ – where the generators are interpreted as points and vectors in R^3.

Consider then an orthonormal basis e_1, e_2, e_3, e_4 for R^4 with the standard inner product and with the Clifford product defined by the following rules:

1. $e_1^2 = e_2^2 = e_3^2 = e_4^2 = +1$,
2. $e_i e_j = -e_j e_i$ for $i \neq j$,

where multiplication is associative and distributes through addition.

In this algebra, the Clifford product of two arbitrary vectors u, v is given by the expression

$$uv = u \cdot v + u \wedge v, \tag{1}$$

where $u \cdot v$ is the standard symmetric dot product and $u \wedge v$ is the classical Grassmann antisymmetric wedge product. It follows from Equation (1) that

$$u \text{ is a unit vector} \Rightarrow \quad u^2 = 1 \tag{2}$$

$$u \perp v \Rightarrow uv = -vu. \tag{3}$$

To provide a homogeneous model for R^3, we interpret e_4 to be the point at the origin in R^3 and the vectors e_1, e_2, e_3 to be unit vectors along the coordinate axes in R^3. In general, a vector $u = u_1 e_1 + u_2 e_2 + u_3 e_3 + u_4 e_4$ in R^4 has the following interpretation in 3-dimensional Euclidean space:

1. $u_4 = 0 \Rightarrow u$ is a vector in R^3 with coordinates (u_1, u_2, u_3) relative to the coordinate axes.
2. $u_4 = 1 \Rightarrow u$ is a point in R^3 located at $e_4 + u_1 e_1 + u_2 e_2 + u_3 e_3$.
3. $u_4 \neq 0, 1 \Rightarrow u$ is a point in R^3 located at $e_4 + \frac{u_1 e_1 + u_2 e_2 + u_3 e_3}{u_4}$. This point is called a weighted point or mass point with mass $= u_4$ and has coordinates $(\frac{u_1}{u_4}, \frac{u_2}{u_4}, \frac{u_3}{u_4})$.

3 Transformations

The fundamental transformations in Clifford Algebra are the sandwiching maps generated by versors and rotors. A versor is the product of any number of invertible vectors in R^4, and a rotor is the product of an even number of unit vectors.

Definition 1. Let u, v be two unit vectors in R^4, and let w also be a vector in R^4. Then

$$V_v(w) = -vwv, \qquad R_{vu}(w) = (vu)w(uv) = V_v \circ V_u(w).$$

We are now going to show how to represent rotations, reflections, and perspective projections on R^3 using rotors in the standard homogeneous Clifford Algebra for R^3.

3.1 Reflections

Lemma 1. *Let u be a unit vector in R^4, and let w also be a vector in R^4 with $w \perp u$. Then $V_u(w) = w$.*

Proof. This result follows immediately from Equations (2) and (3). □

Theorem 1. *Let u be a unit vector in R^3, and let w be a vector in R^3. Then $V_u(w)$ represents the reflection of w about the plane perpendicular to the vector u.*

Proof. Decompose w into $w = w_{\|} + w_{\perp}$, where $w_{\|} = \lambda u$ for some constant λ and $w_{\perp} \perp u$. Then by Equation (2) and Lemma 1 we have

$$V_u(w) = V_u(w_{\|}) + V_u(w_{\perp}) = -w_{\|} + w_{\perp},$$

which is the reflection of w about the plane perpendicular to the vector u. □

From now on for two vectors u, v in R^4, let $\angle(u,v)$ denote the directed angle from u to v, and let $\|v\|$ denote the length of the vector v. Then by Theorem 1 we have the following result.

Corollary 1. *Let u be a unit vector. Then*

1. $\angle(v,u) = \angle(-u, V_u(v))$.
2. $\|V_u(v)\| = \|v\|$.

3.2 Rotations

Proposition 1. *Let u, v be two unit vectors in R^4, and let w be a vector in R^4. If $w \perp u, v$, then $R_{vu}(w) = w$, i.e., $R_{vu}(\cdot)$ is the identity on all vectors perpendicular to the plane of u, v.*

Proof. Since $w \perp u, v$, by Lemma 1 we have $R_{vu}(w) = V_v(V_u(w)) = V_v(w) = w$. □

Proposition 2. *Let u, v be two unit vectors in R^4 with $\angle(u,v) = \theta$, and let w be a vector in the plane of u, v. Then $R_{vu}(w)$ rotates w in the plane of u, v by the angle 2θ.*

Proof. Again we shall use $R_{vu}(w) = V_v(V_u(w))$. By Corollary 1,

$$\angle(w, V_v(V_u(w))) = \angle(w,u) + \angle(u,v) + \angle(v, V_v(V_u(w))) \tag{4}$$

$$= \angle(-u, V_u(w)) + \angle(u,v) + \angle(V_u(w), -v) \tag{5}$$

$$= \angle(-u,-v) + \angle(u,v) = 2\theta. \tag{6}$$

Moreover, the length of w is not altered by R_{vu} since again by Corollary 1,

$$\|R_{vu}(w)\| = \|V_v(V_u(w))\| = \|V_u(w)\| = \|w\|.$$

Therefore, $R_{vu}(w)$ rotates w in the plane of u, v by the angle 2θ. □

Theorem 2. *Let u, v be two unit vectors in R^4 with $\angle(u,v) = \theta$. Then R_{vu} represents a simple rotation in the plane of u, v by the angle 2θ.*

Proof. Let w be a vector in R^4. Decompose w into $w = w_\parallel + w_\perp$, where $w_\parallel$ is a vector in the plane spanned by u, v, and $w_\perp$ is perpendicular to the plane of u, v. Then by Proposition 1 $R_{vu}(w_\perp) = w_\perp$ and by Proposition 2 $R_{vu}(w_\parallel)$ rotates $w_\parallel$ in the plane of u, v by the angle 2θ. Therefore $R_{vu}(w) = R_{vu}(w_\parallel) + R_{vu}(w_\perp)$ represents a simple rotation in the plane of u, v by the angle 2θ. □

Corollary 2. *Let u, v be unit vectors in R^3 with $\angle(u,v) = \theta$, and let w be a vector in R^3. Then $R_{vu}(w)$ rotates w in the plane of u, v by the angle 2θ.*

3.3 Perspective projections

We begin by investigating certain canonical positions for the eye point and the perspective plane.

Theorem 3. *Let v be a unit vector in R^3, and let $u = \cos(\theta)e_4 + \sin(\theta)v$, $0 < \theta \leq \frac{\pi}{2}$. Place the eye point at*

$$E = e_4 + (\cot(2\theta) - \csc(2\theta))v.$$

Then for any point P, the map $R_{e_4u}(P-E)$ represents the perspective projection of the point P from the eye point E to the plane perpendicular to v at a distance $d = \csc(2\theta)$ from E.

Proof. Since $P - E$ is a vector in R^3, we can write $P - E = \lambda v + v_\perp$, where λ is a constant and $v_\perp \perp v$ is also a vector in R^3, so $v_\perp \perp e_4$. Since $v_\perp \perp e_4, u$ and $\angle(u, e_4) = -\theta$, it follows by Propositions 2 and 1 that

$$\begin{aligned} R_{e_4u}(P-E) &= \lambda R_{e_4u}(v) + R_{e_4u}(v_\perp) && (7)\\ &= \lambda(\cos(\frac{\pi}{2} - 2\theta)e_4 + \sin(\frac{\pi}{2} - 2\theta)v) + v_\perp && (8)\\ &= \lambda\big(\sin(2\theta)e_4 + \cos(2\theta)v\big) + v_\perp && (9)\\ &\equiv e_4 + \cot(2\theta)v + \frac{\csc(2\theta)}{\lambda}v_\perp, && (10)\end{aligned}$$

which by similar triangles is the perspective projection of the point P from the eye point E into the plane perpendicular to v at a distance $d = \csc(2\theta)$ from the eye (see Figure 1). □

Given a fixed distance $d = \csc(2\theta)$ between the eye point and the perspective plane and a unit vector v normal to the perspective plane, Theorem 3 shows how to position the eye point so that we can use a rotor to represent perspective projection. We can extend this result to arbitrary positions of the eye point and the perspective

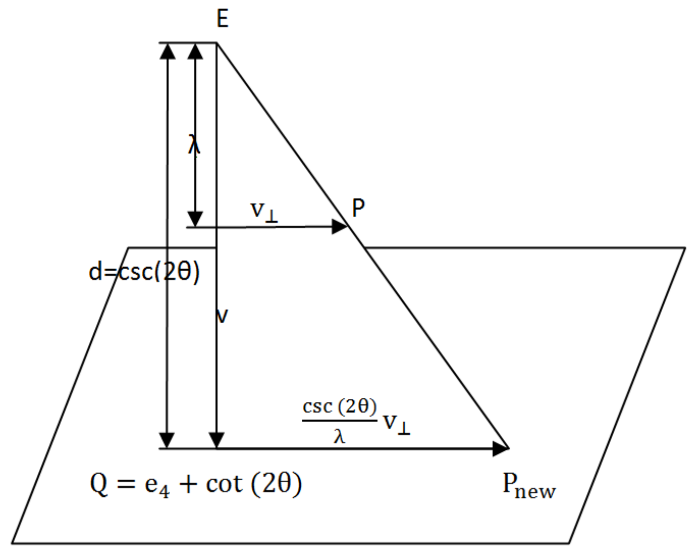

Fig. 1 Perspective projection. Here E is the eye point, Q is the orthogonal projection of the eye point E onto the perspective plane, and P_{new} is the perspective projection of the point P onto the perspective plane.

plane because translation and perspective projection commute. Indeed, let E^* be an arbitrary eye point at a distance $d = \csc(2\theta)$ from a perspective plane with unit normal v, and let $t = E - E^*$. If we project from the eye point E^* and then translate the image by t, we get the same result as when we first translate the entire scene, including the eye point and the perspective plane, by the vector t and then project from the eye point E to the plane normal to v at a distance $d = \csc(2\theta)$ from E. Therefore we have the following result.

Corollary 3. *Let v be a unit vector in R^3, and let $u = \cos(\theta)e_4 + \sin(\theta)v$, $0 < \theta \leq \frac{\pi}{2}$. Then for any point P^*, the map $R_{e_4u}(P^* - E^*)$ represents the perspective projection of the point P^* from the eye point E^* to the plane perpendicular to v at a distance $d = \csc(2\theta)$ from E^*, but the image appears in the canonical plane with unit normal v at a distance $d = \csc(2\theta)$ from the canonical eye point $E = e_4 + (\cot(2\theta) - \csc(2\theta))v$.*

Finally note that Theorem 3 and Corollary 3 are valid only when the distance d from the eye point to the perspective plane is at least one, since $\csc(2\theta) \geq 1$. Nevertheless, when this distance $d < 1$, we can still compute perspective projection using rotors because perspective projection and uniform scaling from the eye point commute. Thus when $d < 1$, we first scale the entire scene from the eye point by $\frac{1}{d}$, then use our rotors to compute perspective projection, and lastly we scale the image from the eye point by d to get the final image.

3.4 Composites

Theorem 4. *Let u, v be two unit vectors in R^4. Then the sandwiching map R_{vu} induced by the rotor vu corresponds to one of the following three transformations on R^3:*

1. *R_{vu} represents a rotation in R^3.*
2. *R_{vu} represents a reflection on vectors in R^3, and rotates all the points in R^3 by the angle π around the axis through the origin and parallel to the normal to the reflection plane.*
3. *$R_{vu} = R_1 \circ R_2$, where R_1 is a perspective projection in R^3, and R_2 is a rotation in R^3.*

Proof. Suppose $\angle(u, v) = \theta$. Let p be a unit vector in R^4. Then by Theorem 2 $p_{new} = R_{vu}(p)$ rotates p in the plane of u, v by the angle 2θ. Now let $p = e_4$.

1. If $p_{new} = e_4$, then the rotation plane spanned by u, v is perpendicular to e_4. Hence $u, v \perp e_4$ are both vectors in R^3. Therefore, by Corollary 2 R_{vu} is actually a rotation in R^3.

2. If $p_{new} = -e_4$, decompose e_4 into $e_4 = e_\parallel + e_\perp$, where $e_\parallel$ is in the plane of u, v, and $e_\perp \perp u, v$. Then by linearity,

$$R_{vu}(e_4) = R_{vu}(e_\parallel) + R_{vu}(e_\perp) = \widetilde{e_\parallel} + e_\perp = -e_4 = -e_\parallel - e_\perp,$$

where $\widetilde{e_\parallel}$ is rotation of $e_\parallel$ in the plane of u, v. Hence

$$\widetilde{e_\parallel} + e_\parallel + 2e_\perp = 0.$$

Since $e_\perp \perp e_\parallel$ and $e_\perp \perp \widetilde{e_\parallel}$, we have $e_\perp = 0$ and $\widetilde{e_\parallel} = -e_\parallel = -e_4$. Therefore R_{vu} represents rotation in the plane of u, v (in R^4) by the angle π, and e_4 is contained in the plane of u, v. Hence there is a vector $\alpha \in R^3$ in the plane of u, v and perpendicular to e_4. Since R_{vu} represents rotation by the angle π in the plane of u, v, clearly $R_{vu}(\alpha) = -\alpha$.
Now for any vector w in R^3, we can decompose w into $w = w_\parallel + w_\perp$, where $w_\parallel$ is parallel to the vector α, and $w_\perp$ is perpendicular to α. Then $w_\perp \perp u, v$, since $w_\perp \perp \alpha, e_4$. Therefore by Proposition 1 and the fact that $R_{vu}(\alpha) = -\alpha$, it follows that

$$R_{vu}(w) = R_{vu}(w_\parallel) + R_{vu}(w_\perp) = -w_\parallel + w_\perp,$$

which reflects w in the plane perpendicular to α.
On the other hand, for a point Q in R^3, we can write $Q = e_4 + (Q - e_4)$. Hence

$$R_{vu}(Q) = R_{vu}(e_4) + R_{vu}(Q - e_4) = -e_4 + R_{vu}(Q - e_4) \equiv e_4 - R_{vu}(Q - e_4).$$

Since $R_{vu}(Q - e_4)$ reflects the vector $Q - e_4$ in the plane perpendicular to α, the map $-R_{vu}(Q - e_4)$ rotates the vector $Q - e_4$ by the angle π around the axis α.

Therefore in R^3, $R_{vu}(Q)$ essentially rotates the point Q by the angle π around the axis line through the origin e_4 parallel to the vector α.

3. If $p_{new} \neq \pm e_4$, suppose that $\angle(e_4, p_{new}) = \phi$, and let w be the unit vector in the plane of p_{new} and e_4 with $\angle(w, e_4) = \frac{\phi}{2}$. By Proposition 2

$$R_{vu}(e_4) = p_{new} = R_{e_4 w}(e_4).$$

Note that $R_{e_4 w}(*)$ is a rotation in the plane spanned by e_4, w by the angle ϕ, so R_{we_4} is rotation in the plane spanned by e_4, w by the angle $-\phi$. Hence

$$R_{we_4} \circ R_{vu}(e_4) = R_{we_4}(p_{new}) = e_4.$$

Therefore, $R_{we_4} \circ R_{vu}(*)$ is a rotation by some angle 2ψ in R^3. Suppose this rotation plane is spanned by two unit vectors w_1 and w_2 with $\angle(w_1, w_2) = -\psi$. Then

$$R_{we_4} \circ R_{vu}(*) = R_{w_1 w_2}(*).$$

Hence

$$R_{vu}(*) = R_{we_4}^{-1} \circ R_{w_1 w_2}(*) = R_{e_4 w} \circ R_{w_1 w_2}(*),$$

where by Theorem 3 $R_{e_4 w}$ is a perspective projection in R^3 and by Corollary 2 $R_{w_1 w_2}$ is a rotation in R^3.

4 Conclusions and future research

We have shown that perspective projections on R^3 can indeed be represented by rotors in the standard homogeneous model of the Clifford Algebra for R^3. Nevertheless this homogeneous representation is a bit clumsy because translation and uniform scaling are not represented by rotors in this homogeneous model. Thus when the eye is not in special position, the perspective image appears in a translated plane rather than in the given perspective plane. Moreover, when the distance between the eye point and the perspective plane is less than one, we need to perform uniform scaling before and after we perform perspective projection, but uniform scaling is not presented by a rotor in this Clifford Algebra.

To overcome these deficiencies, several authors have suggested the conformal model [1, 7] as a more appropriate computational model for contemporary Computer Graphics because in the conformal model both translation and uniform scaling can be represented by rotors. In the future using some of the insights developed here we plan to show that we can also represent perspective projections by rotors in the conformal model [6].

Acknowledgements This work was partially supported by a National Key Basic Research Project of China (2011CB302404) and by grants from NSFC (60821002) and (91118001).

References

1. Dorst, L. Fontijne, D., and Mann, S. (2007), Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry, Morgan Kaufmann, Amsterdam.
2. Foley, J., van Dam, A., Feiner, S., and Hughes, J. (1990), Computer Graphic: Principles and Practice, Second Edition, Addison Wesley.
3. Goldman, R. N. (2010), Understanding Quaternions, Graphical Models, Vol. 73, pp. 21-49.
4. Goldman, R. N. (2010), Rethinking Quaternions: Theory and Computation, Synthesis Lectures on Computer Graphics and Animation, ed. Brian A. Barsky, No. 13. San Rafael: Morgan & Claypool Publishers.
5. Goldman, R.N. (2011), A Homogeneous Model for 3-Dimensional Computer Graphics Based on the Clifford Algebra for R^3 , Guide to Geometric Algebra in Practice, L. Dorst, J. Lasenby (eds.), Chapter 16, pp. 329-352, Springer-Verlag, London.
6. Goldman, R. N. (2012), Computing Perspective Projections in 3-Dimensions using Rotors in the Conformal Model of Clifford Algebra, submitted for publication.
7. Perwass, C. (2009), Geometric Algebra with Applications in Engineering, Springer-Verlag, Berlin.