Axial moving planes and singularities of rational space curves

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\textbf{Article history:}
Received 17 April 2008
Received in revised form 24 July 2008
Accepted 2 September 2008
Available online 6 September 2008

\textbf{Keywords:}
Axial moving planes
μ-basis
Singularity
Rational space curve

\textbf{ABSTRACT}

Relationships between the singularities of rational space curves and the moving planes that follow these curves are investigated. Given a space curve $C$ with a generic 1–1 rational parametrization $F(s, t)$ of homogeneous degree $d$, we show that if $P$ and $Q$ are two singular points of orders $k$ and $k'$ on the space curve $C$, then there is a moving plane of degree $d - k - k'$ with axis $\overrightarrow{PQ}$ that follows the curve. We also show that a point $P$ is a singular point of order $k$ on the space curve $C$ if and only if there are two axial moving planes $L_1$ and $L_2$ of degree $d - k$ such that: (1) the axes of $L_1$, $L_2$ are orthogonal and intersect at $P$, and (2) the intersection of the moving planes $L_1$ and $L_2$ is the cone through the curve $C$ with vertex $P$ together with $d - k$ copies of the plane containing the axes of $L_1$ and $L_2$. In addition, we study relationships between the singularities of rational space curves and generic moving planes that follow these curves. In particular, we show that if $p(s, t), q(s, t), r(s, t)$ are a μ-basis for the moving planes that follow a rational space curve $F(s, t)$, then $P$ is a singular point of $F(s, t)$ of order $k$ if and only if $\text{deg}(\gcd(p(s, t) \cdot P, q(s, t) \cdot P, r(s, t) \cdot P)) = k$. Moreover, the roots of this gcd are the parameters, counted with proper multiplicity, that correspond to the singularity $P$. Using these results, we provide straightforward algorithms for finding all the singularities of low degree rational space curves. Our algorithms are easy to implement, requiring only standard techniques from linear algebra. Examples are provided to illustrate these algorithms.

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1. Introduction

Singularities are the most interesting and important points on algebraic curves. The number and nature of the singularities tell us a good deal about the geometry and topology of the curve. Detecting and analyzing singularity is very useful in a large variety of applications, ranging from engineering design and manufacturing to computer graphics. For example, robust rendering algorithms for algebraic curves require the location of all the singularities (Alberti and Mourrain, 2007). For rational planar curves, there is a long history of the study of singularities (see, for example, Chen and Sederberg (2002), Coolidge (1931), Fulton (1989), Hilton (1920), Pérez-Díaz (2007), Walker (1950)). The purpose of this paper is to investigate the orders of the singularities of rational space curves, i.e. rational non-planar curves in three dimensions.

Most of the literature related to singularities of space curves is concentrated on the classification of space curves by their degree and their geometric genus (see, for example, Abhyankar and Sathaye (1974), Harris (1980), Harris and Eisenbud (1982)). In this body of work, an upper bound on the number of singular points of a space curve is given by the geometric genus, a non-negative integer which is a basic birational invariant of algebraic varieties. Recently Park (2002) applied the
the parameters, counted with proper multiplicity, that correspond to the singularity follows the curve. In Section 4, we show that a point provides straightforward algorithms based only on elementary linear algebra for finding all the singularities of low degree curves. Using these results, in Section 7 we specialize these results to rational space curves using only elementary techniques from linear algebra. Similar results for rational planar curves are discussed by Chen et al. (2008), but there the emphasis is on computing singularities using implicitization matrices.

This paper is a sequel to the paper of Song et al. (2007). Our goal is to extend their main results from rational planar curves to rational space curves. As in Song et al. (2007), we will concentrate on the orders of singular points, and will not study the resolution of singularities.

We begin in Section 2 with a brief review of moving lines and moving planes. In Sections 3 and 4 we provide two extensions of the main theorem of Song et al. (2007). In Section 3 we prove that if and only if there are two axial moving planes and of degree that follow the curve such that: (1) the axes of and, then or (2) the intersection of the moving planes and is the cone (moving line) through the curve with vertex together with copies of the plane containing the axes of and.

In Section 5, we derive relationships between singularities of rational space curves and generic moving planes that follow these curves, and in Section 6 we specialize these results to -bases for rational space curves. In particular, we show that if and and are a -basis for the moving planes that follow a rational space curve , then is a singular point of of order if and only if .

Moreover, the roots of this gcd are the parameters, counted with proper multiplicity, that correspond to the singularity . Using these results, in Section 7 we provide straightforward algorithms based only on elementary linear algebra for finding all the singularities of low degree rational space curves as well as tight bounds on the number of such singularities. We also flesh out these algorithms with many examples. We close in Section 8 with a brief summary of our main results.

2. Moving lines and moving planes

Throughout this paper, we shall consider rational space curves in real or complex projective three-space given as the image of generic one-to-one rational parametrizations:

\[
F(s, t) = (f_0(s, t), f_1(s, t), f_2(s, t), f_3(s, t)), \quad (s, t) \neq (0, 0),
\]

where , , , are linearly independent homogeneous polynomials of the same degree , and . When a curve is given by a parameterization as in (1), we shall say that is a curve of degree . Though some of our results require complex parameters, our main results—Theorem 3.3, Theorem 4.3, and Corollary 6.7—are valid over the reals, so all of our examples will consist of real curves with real coefficients. To investigate the singularities of these space curves, we are going to invoke moving lines and moving planes.

A moving line is a collection of lines parametrized by . A moving line follows the curve and the line at parameter intersects the curve at the point . An axial moving line is a moving line where all the lines of the family pass through a common point . The point is called the axis point of the moving line. Thus, an axial moving line with axis that follows the curve is the family of lines that pass through the axis point and a point on the curve . Hence, an axial moving line that follows the curve is just the cone through the curve with the vertex at the axis point .

A moving plane is a family of planes with each pair of parameters corresponding to a plane:

\[
L(x_0, x_1, x_2, x_3; s, t) = \sum_{i=0}^{3} A_i(s, t)x_i = A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3,
\]

where , , , are homogeneous polynomials in of the same degree. For simplicity, sometimes we write a moving plane as . The moving plane (2) follows the parametrization (1) if
that is, if the point on the curve at the parameter \((s, t)\) lies on the plane at the parameter \((s, t)\).

A moving plane \(L\) is of degree \(k\) if \(\deg(A_i) = k\), \(0 \leq i \leq 3\); \(L\) is of minimal degree \(k\) if \(\deg(A_i) = k\), \(0 \leq i \leq 3\) and \(\gcd(A_0, A_1, A_2, A_3) = 1\). Notice that if \(L\) is a moving plane that follows a space curve, then the minimal degree of \(L\) must be at least one.

An axial moving plane is an axial moving plane where all the planes of the family pass through either a common point \(A\) or a common line \(AB\). The point \(A\) is called an axis point, and the line \(AB\) is called the axis line or axis of the moving plane. Notice that \(A\) is an axis point of a moving plane \(L\) if and only if \(A \equiv 0\) if \(L \cdot A = 0\) if there are two points \(A\) and \(B\) such that \(L \cdot B = 0\), then by linearity \(L \cdot (aA + bB) = 0\), so \(AB\) is an axis line of \(L\).

Two moving planes typically intersect in a moving line, but there may be some exceptional parameter values where the two moving planes actually coincide. If two moving planes follow a curve \(C\), then their intersection will also follow the curve \(C\). Two axial moving planes need not intersect in an axial moving line, but if the axes of two axial moving planes intersect in a point \(A\), then the moving planes intersect in an axial moving line with axis point \(A\). Moreover, if the axes of two axial moving planes intersect at a single point, then the only plane that can lie in the intersection of the two moving planes is their axial plane, that is, the plane containing their two axis lines. If two axial moving planes whose axes intersect at a point follow a rational space curve \(C\) given by a parametrization \(F(s, t)\), then their axial plane can correspond to only a finite number of parameter values, the parameter values \((s^*, t^*)\) where \(F(s^*, t^*)\) lies on their axial plane.

Example 2.1. Let \(F(s, t) = (s^3, s^3 t^2, s^3 t^3, t^5)\).

\[L_1 := (t^3, 0, -s^3, 0)\] is an axial moving plane of minimal degree 3 with axis \(\overrightarrow{AB}\) where \(A = (0, 0, 0, 1)\) and \(B = (0, 1, 0, 1)\) that follows the curve.

\[L_2 := (t^2, -s^2, 0, 0)\] is an axial moving plane of minimal degree 2 with axis \(\overrightarrow{AC}\) where \(A = (0, 0, 0, 1)\) and \(C = (0, 0, 1, 1)\) that follows the curve.

Since \(L_1\) and \(L_2\) have a common axis point \(A = (0, 0, 0, 1)\), the intersection of \(L_1\) and \(L_2\) contains an axial moving line with axis point \(A\) along with the axial plane \(x_0 = 0\) at \(s = 0\).

For a moving plane \(L\) of degree \(\mu\), we can write

\[L = (L_0(s, t), L_1(s, t), L_2(s, t), L_3(s, t)) = \sum_{i=0}^{\mu}(L_{0,i}, L_{1,i}, L_{2,i}, L_{3,i})s^{i-1}t^i = [s^\mu \quad s^{\mu-1}t \quad \ldots \quad t^\mu] \cdot M_L,\]

where \(M_L = 
\begin{bmatrix}
L_{0,0} & L_{1,0} & L_{2,0} & L_{3,0} \\
L_{0,1} & L_{1,1} & L_{2,1} & L_{3,1} \\
\vdots & \vdots & \vdots & \vdots \\
L_{0,\mu} & L_{1,\mu} & L_{2,\mu} & L_{3,\mu}
\end{bmatrix}.\]

\(M_L\) is the \((\mu + 1) \times 4\) coefficient matrix of \(L\). For example, if \(L = (-5st^2, -3st, 0, t^2)\), then the coefficient matrix

\[M_L = 
\begin{bmatrix}
-5 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.\]

Lemma 2.2. Let \(L\) be a moving plane that follows a rational space curve \(C\) given by a rational parametrization \(F(s, t)\) as in Eq. (1). Then

1. \(\text{rank}(M_L) \neq 0, 1\).
2. \(L\) has an axis line if and only if \(\text{rank}(M_L) = 2\).
3. \(L\) has an axis point, but no axis line if and only if \(\text{rank}(M_L) = 3\).
4. \(L\) is not axial if and only if \(\text{rank}(M_L) = 4\).

Proof. 1. If \(\text{rank}(M_L) = 0\), then \(M_L = 0\), so \(L\) is not a moving plane. If \(\text{rank}(M_L) = 1\), then there is a homogeneous polynomial \(f(s, t)\) and constants \(a_0, a_1, a_2, a_3\) such that \(L = f(s, t)(a_0, a_1, a_2, a_3)\), so \(L \cdot F = f(s, t)(\sum_{i=0}^{3}a_i f_i) = 0\). Hence \(\sum_{i=0}^{3}a_i f_i = 0\), contradicting the assumption that the curve \(C\) is a space curve. Thus, \(\text{rank}(M_L) \neq 0, 1\).

2. \(L\) has an axis line if and only if there exist two distinct points \(A, B\) such that \(L \cdot A = L \cdot B = 0\), that is, if and only if \(M_L \cdot A = M_L \cdot B = 0\), i.e. if and only if \(\text{rank}(M_L) = 2\).

3. \(L\) has an axis point if and only if there exists a unique point \(A\) such that \(L \cdot A = 0\), that is, if and only if \(M_L \cdot A = 0\), i.e. if and only if \(\text{rank}(M_L) = 3\).

4. \(L\) has no axis if and only if there does not exist a point \(A\) such that \(L \cdot A = 0\), that is, if and only if \(M_L \cdot A = 0\) has no non-trivial solution, i.e. if and only if \(\text{rank}(M_L) = 4\). \(\square\)
Corollary 2.3. Let \( L \) be a moving plane.

1. If \( \deg(L) = 1 \), then \( L \) has an axis line.
2. If \( \deg(L) = 2 \), then \( L \) has an axis point. Moreover, \( L \) has an axis line if and only if \( \text{rank}(M_L) = 2 \).
3. If \( \deg(L) \geq 3 \), then
   (a) \( L \) has no axis if and only if \( \text{rank}(M_L) = 4 \).
   (b) \( L \) has an axis point if and only if \( \text{rank}(M_L) = 3 \).
   (c) \( L \) has an axis line if and only if \( \text{rank}(M_L) = 2 \).

3. Singularities and axial moving planes

We are now going to present our first extension to rational space curves of the main theorem of Song et al. (2007) for rational planar curves. We begin with some simple observations about singularities.

Definition 3.1. Let \( P \) be a point on a space curve \( C \), and let \( \Pi \) be a plane containing \( P \). Then the point \( P \) is called a singular point of order \( k \geq 2 \), if the intersection multiplicity of \( C \) with \( \Pi \) at \( P \) is \( k \geq 2 \) for every generic choice of \( \Pi \). A point on the curve is non-singular if and only if \( k = 1 \). A point is not on the curve if and only if \( k = 0 \).

Remark 3.2. Let \( P \) be a singular point of order \( k \geq 2 \) on a space curve \( C \) given by a rational parametrization \( F(s, t) \) as in Eq. (1). The parameters of the intersections of \( C \) and a generic plane \( \Pi = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \), which we shall typically write as \( \Pi = (a_0, a_1, a_2, a_3) \), are given by the roots of the polynomial \( \Phi(s, t) := \Pi \cdot F(s, t) = a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 \). In particular, \( \Phi(s, t) \) has a factor \( \Phi_1(s, t) \) of degree \( k \) such that

\[
\Phi_1(s, t) = \prod_{\ell=0}^{\alpha} (t - s(t))^{k_\ell}, \quad \sum_{\ell=0}^{\alpha} k_\ell = k,
\]

where the point \( P = F(s_0, t_0) = \cdots = F(s_a, t_a) \). Thus, \( P \) is singular point of order \( k \geq 2 \) if and only if \( F \) maps to \( P \) \( k \) times counting multiplicity.

Theorem 3.3. Let \( C \) be a rational space curve of degree \( d \) given by \( F(s, t) \) as in (1). If \( P, Q \) are two points of orders \( k \geq 0 \) and \( k' \geq 0 \), then there is a moving plane of degree \( d - k - k' \) with axis \( \overline{PQ} \) that follows the curve \( C \).

Proof. By a projective transformation, we can assume without loss of generality that \( P = (1, 0, 0, 0) \) and \( Q = (0, 1, 0, 0) \) are the two points of orders \( k \) and \( k' \). Then there exist polynomials \( \Phi_1(s, t) \) and \( \Phi_2(s, t) \) such that

\[
\gcd(f_1, f_2, f_3) = \Phi_1(s, t), \quad \gcd(f_0, f_2, f_3) = \Phi_2(s, t),
\]

\[
\deg(\Phi_1) = k, \quad \deg(\Phi_2) = k', \quad \gcd(\Phi_1, \Phi_2) = 1.
\]

Therefore there are polynomials \( g_0, g_1, g_2, g_3 \) such that

\[
f_0 = \Phi_2 g_0, \quad f_1 = \Phi_1 g_1, \quad f_2 = \Phi_1 \Phi_2 g_2, \quad f_3 = \Phi_1 \Phi_2 g_3.
\]

\[
\deg(g_0) = d - k', \quad \deg(g_1) = d - k, \quad \deg(g_2) = \deg(g_3) = d - k - k'.
\]

Let \( L = (0, 0, -g_3, g_2) \). Then \( L \) is an axial moving plane of degree \( d - k - k' \) with axis \( \overline{PQ} \) that follows the curve \( C \). \( \square \)

Example 3.4. Let \( F(s, t) = (s^5, s^5 t^4, s^2 t^7, t^9) \).

\( P = (1, 0, 0, 0) \) is a singular point of order 4 on the curve, and the point \( A = (1, 1, 0, 0) \) is not on the curve. \( L_1 := (0, 0, t^5, -s^2 t^3) \) is an axial moving plane of degree 5 = 9 - 4 - 0 and minimal degree 2 with axis \( \overline{PA} \) that follows the parametrization \( F \).

\( Q = (0, 0, 1) \) is a singular point of order 2 on the curve. \( L_2 := (0, t^3, -s^3, 0) \) is an axial moving plane of minimal degree 3 = 9 - 4 - 2 with axis \( \overline{PQ} \) that follows the parametrization \( F \).

Corollary 3.5. The sum of the orders of any two singularities of a rational space curve must be strictly less than the degree of the curve. Moreover, the order of any singularity of a rational space curve can be no greater than the degree of the curve minus 2.

Proof. This result follows directly from Theorem 3.3. \( \square \)

Corollary 3.6. A rational cubic space curve has no singular points.
4. Singularities and axial moving lines

Here we are going to present our second extension of the main theorem of Song et al. (2007) to rational space curves.

**Proposition 4.1.** Let $P$ be a singular point of order $k$ on a rational space curve $C$ of degree $d$. Then the cone (axial moving line) formed by the lines joining the point $P$ to the curve $C$ has implicit degree $d - k$.

**Proof.** Let $\lambda$ be a generic line not containing the singular point $P$ such that the plane $\Pi$ determined by the singular point $P$ and the generic line $\lambda$ is transversal to the rational space curve $C$ at the singular point $P$. We will show that the generic line $\lambda$ intersects the cone in exactly $d - k$ points counting multiplicity.

Since the plane $\Pi$ is generic, it follows by Bezout’s Theorem that the plane $\Pi$ intersects the rational space curve $C$ at $d - k$ points counting multiplicity, other than the singular point $P$. Therefore, there are $d - k$ lines, counting multiplicity, through the point $P$ containing these $d - k$ points. Since these $d - k$ lines and the generic line $\lambda$ all lie on the same plane $\Pi$, these $d - k$ lines intersect the generic line $\lambda$ at $d - k$ points counting multiplicity. Therefore, the cone (axial moving line) formed by the line joining the singular point $P$ to the curve $C$ has implicit degree $d - k$. □

We can now provide an alternative proof to the result proved in Corollary 3.5.

**Corollary 4.2.** A rational space curve of degree $d$ can have no singular point of order $d - 1$.

**Proof.** Suppose there is a singularity $P$ of order $d - 1$ on a rational space curve $C$ of degree $d$. By Proposition 4.1, the cone formed by the lines joining the point $P$ to the curve $C$ would have implicit degree $d - (d - 1) = 1$. Thus the curve $C$ would be contained in a plane, which contradicts our assumption that $C$ is a space curve. Therefore, $C$ has no singular point of order $d - 1$. □

We are now ready to state our main extension of the principal theorem of Song et al. (2007).

**Theorem 4.3.** Let $C$ be a rational space curve of degree $d$ given by $F(s, t)$ as in (1). Then $P \in C$ is a singularity of order $k \geq 2$ if and only if there exist two axial moving planes $L_1$ and $L_2$ of degree $d - k$ that follow the curve with the following properties:

1. the axes of $L_1$ and $L_2$ are orthogonal and intersect at $P$;
2. the intersection of $L_1$ and $L_2$ consists of an axial moving line (cone) that follows the curve $C$ together with $d - k$ copies of the plane containing the axes of $L_1$ and $L_2$. In particular

$$\text{Res}_{s, t}(L_1, L_2) = C(x_0, x_1, x_2, x_3)R(x_0, x_1, x_2, x_3)^{d-k},$$

where $C(x_0, x_1, x_2, x_3) = 0$ is the implicit equation of the cone of degree $d - k$ formed from the vertex $P$ and the curve $C$, and

$$R(x_0, x_1, x_2, x_3) = 0$$

is the implicit equation of the plane containing the axes of $L_1$ and $L_2$.

**Proof.** ($\Rightarrow$) Any point $(x, y, z, w)$ can be transformed to the point $(1, 0, 0, 0)$ by a Householder transformation which is both orthogonal and symmetric. (See Strang (2006), page 361 for details of the construction of the Householder matrix.) Therefore without loss of generality, we can let $P = (1, 0, 0, 0)$ be the singular point on $C$ of order $k \geq 2$. Then there is a polynomial $\Phi(s, t)$ such that $\text{gcd}(f_1, f_2, f_3) = \Phi(s, t)$, $\text{gcd}(f_0, \Phi) = 1$ and $\deg(\Phi) = k$. Thus there are polynomials $h_1, h_2, h_3$ such that

$$f_1 = \Phi h_1, \quad f_2 = \Phi h_2, \quad f_3 = \Phi h_3,$$

where

$$\text{gcd}(h_1, h_2, h_3) = 1, \quad \deg(h_1) = \deg(h_2) = \deg(h_3) = d - k.$$

Define

$$L_1 = h_2x_1 - h_1x_2, \quad L_2 = h_3x_1 - h_1x_3.$$

Clearly $L_1$ and $L_2$ are axial moving planes of degree $d - k$ that follow the curve $C$ with orthogonal axes intersecting at $P$. Thus, $L_1$ and $L_2$ satisfy property 1.

If $(s_0, t_0)$ is not a root of $h_1(s, t) = 0$, then $L_1(s_0, t_0)$ and $L_2(s_0, t_0)$ are distinct planes that intersect in a line. This line passes through the point $P$ and the point $F(s_0, t_0)$, so this line lies on an axial moving line (cone) that follows the curve and the vertex of this cone is at $P$. Moreover, by Proposition 4.1, the degree of the implicit equation of this cone is $d - k$.

By construction, $\deg(h_1) = d - k$; therefore, $h_1$ has $d - k$ roots counting multiplicity. If $h_1(s_0, t_0) = 0$, then at $(s_0, t_0)$ either $L_1$ and $L_2$ are both equal to a constant multiple of $x_1$, or one of $L_1$, $L_2$ is identically zero and the other is a constant times $x_1$. In either case the intersection of $L_1$ and $L_2$ is the plane $x_1 = 0$. Therefore, the intersection of $L_1$ and $L_2$ at the roots of $h_1$ yields $d - k$ copies of the plane $(x_1 = 0)$ containing the axes of $L_1$ and $L_2$. 
In particular, \( \text{Res}_{s,t}(L_1(s, t), L_2(s, t)) \) is of degree \( 2(d - k) \) and
\[
\text{Res}_{s,t}(L_1(s, t), L_2(s, t)) = C(x_0, x_1, x_2, x_3)R(x_0, x_1, x_2, x_3)^{d-k},
\]
where \( \text{C}(x_0, x_1, x_2, x_3) = 0 \) is the implicit equation of the cone of degree \( d - k \) formed from the vertex \( P \) and the curve \( C \), and \( \text{R}(x_0, x_1, x_2, x_3) = 0 \) is the implicit equation of the plane containing the axes of \( L_1 \) and \( L_2 \). Thus, \( L_1 \) and \( L_2 \) satisfy property 2.

\((\Leftarrow)\) Let \( L_1 \) and \( L_2 \) be two moving planes of degree \( d - k \) that follow the curve \( C \) and satisfy properties 1 and 2. Again by a Householder transformation, we can, without loss of generality, choose the axes of \( L_1 \) and \( L_2 \) to be the \( x_3 \) and \( x_2 \) axes. Then \( L_1 = ax_1 - bx_2 \) and \( L_2 = ax_1 - \beta x_3 \) for some polynomials \( a, b, \alpha, \beta \) in \( s, t \) of degree \( d - k \). We need to show that the point \( P = (1, 0, 0, 0) \) where the axes intersect is a singular point of order \( k \) on the curve \( C \). First, we observe that \( \text{gcd}(a, b, \alpha, \beta) = 1 \). Otherwise, if \( g = \text{gcd}(a, b, \alpha, \beta) \) and \( \text{deg}(g) \geq 1 \), then \( g \) is a common factor of \( L_1 \) and \( L_2 \), so \( L_1 \) and \( L_2 \) both vanish on the points where \( g(s, t) \) vanishes. But this is impossible since \( L_1 \) and \( L_2 \) satisfy property 2, so their intersection cannot contain all of \( 3 \)-space. Therefore, \( \text{gcd}(a, b, \alpha, \beta) = 1 \). In addition, property 2 says that the intersection of \( L_1 \) and \( L_2 \) contains \( d - k \) copies of the plane of the \( x_2, x_3 \) axes (i.e. \( x_1 = 0 \)), which implies that \( b \) and \( \beta \) have the same roots. Therefore, we may assume that \( L_1 = ax_1 - bx_2 \) and \( L_2 = \alpha x_1 - bx_3 \) where \( \text{deg}(a) = \text{deg}(b) = \text{deg}(\alpha) = d - k \) and \( \text{gcd}(a, b, \alpha, \beta) = 1 \).

Since \( L_1 \) and \( L_2 \) follow the curve \( C \), we have
\[
a_f - b_f = 0, \quad \alpha a_f - b_f = 0.
\]
By Eq. (4), every root of \( f_1 \) that is not a root of \( b \) must be a root of both \( f_2 \) and \( f_3 \). Since \( \text{deg}(b) = d - k \), it follows that \( f_1, f_2, f_3 \) have at least \( k \) common roots counting multiplicity. Moreover, \( f_1, f_2, f_3 \) have at most \( k \) common roots counting multiplicity. Indeed if \( f_1, f_2, f_3 \) have \( k + 1 \) common roots, then \( f_1 \) and \( b \) would have \( d - (k + 1) = d - k - 1 \) roots in common. Since \( \text{deg}(b) = d - k \), at least one of the roots of \( b \) cannot be a root of \( f_1 \). Therefore, by Eq. (4), this root of \( b \) must be a root of both \( a \) and \( \alpha \), contradicting \( \text{gcd}(a, b, \alpha) = 1 \). Thus \( f_1, f_2, f_3 \) cannot have \( k + 1 \) common roots. Therefore \( f_1, f_2, f_3 \) have exactly \( k \) common roots counting multiplicity, so \( P = (1, 0, 0, 0) \) is a singular point of order \( k \) on the curve \( C \).

Example 4.4. Let \( F(s, t) = (s^2 - t^2)s^2, (s^2 - 2t^2)s^2t^2, (s^2 - 3t^2)s^2t^2, (s^2 - 4t^2)2^5 \).
\( P = (0, 0, 0, 1) \) is a singular point of order 2. There are two axial moving planes \( L_1, L_2 \) of degree 5 whose orthogonal axes intersect at \( (0, 0, 0, 1) \),
\[
L_1 = (3t^5 - s^5)x_1 + (s^2t^2 - 2st^4)x_2, \quad L_2 = (-3s^5 + s^2t^3)x_0 + (-s^5 + s^2t^3)x_2.
\]
Moreover, \( \text{Res}_{s,t}(L_1, L_2) \) is a polynomial in \( x_0, x_1, x_2, x_3 \), and \( \text{Res}_{s,t}(L_1, L_2) = 0 \) is of the form
\[
x_2^3(x_0 x_3^2 - 12x_0 x_1^4 + 36x_1^5 - x_0^2 x_2^2 + 11x_0^2 x_1 x_2^2 - 29x_0 x_3^2 x_2^2 - 9x_1^3 x_2^2 + 8x_2^3) = 0.
\]
Notice that the short factor \( x_2^3 \) corresponds to five copies of the plane containing the axes of \( L_1 \) and \( L_2 \), and the long factor corresponds to the degree 5 equation of the cone containing the curve \( F(s, t) \) with vertex at the singular point \((0, 0, 0, 1)\).

5. Singularities and generic moving planes

We are now going to investigate relationships between singularities on a rational space curve and generic moving planes that follow the curve. We begin with a simple lemma.

Lemma 5.1. Let \( C \) be a space curve with a parametrization \( F(s, t) \) as in Eq. (1), and let \( M \) be a plane through the point \( F(s_0, t_0) \). Then there is a moving plane \( L(s, t) \) that follows the parametrization \( F(s, t) \) such that \( L(s, t) \cdot F(s_0, t_0) \neq 0 \) and \( L(s_0, t_0) = M \).

Proof. Let \( M = (a_0, a_1, a_2, a_3) \) be a plane that passes through the point \( F(s_0, t_0) \). Without loss of generality, we can assume that \( f_0(s_0, t_0) \neq 0 \). Since \( M \) passes through the point \( F(s_0, t_0) \),
\[
a_0 = -\frac{-f_1(s_0, t_0)a_1 + f_2(s_0, t_0)a_2 + f_3(s_0, t_0)a_3}{f_0(s_0, t_0)}.
\]
Let
\[
L(s, t) = (-f_1(s, t)a_1 - f_2(s, t)a_2 - f_3(s, t)a_3, f_0(s, t)a_1, f_0(s, t)a_2, f_0(s, t)a_3)
\frac{f_0(s_0, t_0)}{f_0(s_0, t_0)}.
\]
It is easy to check that \( L(s, t) \cdot F(s, t) = 0 \), so \( L(s, t) \) is a moving plane that follows the curve \( F(s, t) \). Moreover, since \( F(s, t) \) is a space curve, \( L(s, t) \cdot F(s_0, t_0) = -\sum a_i f_i(s, t) = -M \cdot F(s, t) \neq 0 \). Finally,
\[
L(s_0, t_0) = \frac{-f_1(s_0, t_0)a_1 - f_2(s_0, t_0)a_2 - f_3(s_0, t_0)a_3, f_0(s_0, t_0)a_1, f_0(s_0, t_0)a_2, f_0(s_0, t_0)a_3}{f_0(s_0, t_0)}
\]
\[
= (a_0, a_1, a_2, a_3) = M.
\]
Definition 5.2. We say that a parameter pair \((s_0, t_0)\) has multiplicity \(k\) on the rational space curve \(C\) defined by a parametrization \(F(s, t)\) as in Eq. (1), if and only if for every generic plane \(P\) through the point \(F(s_0, t_0)\), the polynomial \(P \cdot F(s, t)\) has a root at \((s_0, t_0)\) of multiplicity \(k\). In this case, we write \(\text{mult}_P(s_0, t_0) = k\).

Theorem 5.3. Let \(C\) be a space curve with a rational parametrization \(F(s, t)\) as in Eq. (1). Then \(\text{mult}_P(s_0, t_0) = k\) if and only if for every moving plane \(L(s, t)\) that follows the space curve \(F(s, t)\) such that \(L(s_0, t_0)\) is a root of the polynomial \(L(s, t) \cdot F(s_0, t_0)\) of multiplicity \(k\).

Proof. Without loss of generality, we can assume that \(F(s_0, t_0) = (1, 0, 0, 0)\). Then
\[
\begin{align*}
f_0(s, t) &= a(s, t), \\
f_1(s, t) &= b(s, t)h(s, t), \\
f_2(s, t) &= c(s, t)h(s, t), \\
f_3(s, t) &= d(s, t)h(s, t),
\end{align*}
\]
where \((s_0, t_0)\) is a root of \(h(s, t)\), and \(\gcd(b, c, d) = \gcd(a, h) = 1\). Now \(\text{mult}_P(s_0, t_0) = k\) if and only if \((s_0, t_0)\) is a root of \(h(s, t)\) of multiplicity \(k\).

Let \(L(s, t) = (L_0, L_1, L_2, L_3)\) be a moving plane that follows the space curve \(F(s, t)\). Then \(L(s, t) \cdot F(s, t) = L_0a + (L_1b + L_2c + L_3d)h = 0\), so \(L_0a = -(L_1b + L_2c + L_3d)h\).

Since \((s_0, t_0)\) is a generic plane, \((L_1b + L_2c + L_3d)(s_0, t_0) \neq 0\). If \((L_1b + L_2c + L_3d)(s_0, t_0) = 0\), then the plane \(L(s_0, t_0)\) must pass through the point \((a(s_0, t_0), b(s_0, t_0), c(s_0, t_0), d(s_0, t_0))\) and hence the plane \(L(s_0, t_0)\) is not generic. Also \(\gcd(a, h) = 1\) implies that \(a(s_0, t_0) \neq 0\). Therefore, it follows from Eq. (6) that the multiplicity of \((s_0, t_0)\) as a root of \(L_0(s, t)\) is the same as the multiplicity of \((s_0, t_0)\) as a root of \(h(s, t)\). But \(L_0(s, t) = L(s, t) \cdot F(s_0, t_0)\). Thus \((s_0, t_0)\) is a root of \(h\) of multiplicity \(k\) if and only if \((s_0, t_0)\) is a root of \(L(s, t) \cdot F(s_0, t_0)\) of multiplicity \(k\). Therefore, \(\text{mult}_P(s_0, t_0) = k\) if and only if \((s_0, t_0)\) is a root of \(L(s, t) \cdot F(s_0, t_0)\) of multiplicity \(k\).

Remark 5.4. Suppose that \(A\) is a point on a rational space curve \(F(s, t)\), and let \(\{(s_i, t_i)\}, i = 0, 1, \ldots, l - 1\) be all the different parameters corresponding to the point \(A\), i.e. \(F(s_i, t_i) = A\). Then it follows by Definition 3.1 and Remark 3.2 that \(A\) is a singular point of order \(k\) on the curve \(F(s, t)\) if and only if \(\sum_{i=0}^{l-1} \text{mult}_F(s_i, t_i) = k\). In this case we shall write \(\text{order}_F(A) = k\).

Corollary 5.5. Suppose \(A\) is a singular point on the rational space curve \(F(s, t)\), and \(\{(s_0, t_0), (s_1, t_1), \ldots, (s_{l-1}, t_{l-1})\}\) are all the different parameters that correspond to the point \(A\). Then \(\text{order}_F(A) = k\) if and only if \((s_i, t_i)\) is a root of the polynomial \(L(s, t) \cdot A\) of multiplicity \(k_i\) where \(\sum_{i=0}^{l-1} k_i = k\) for every moving plane \(L(s, t)\) that follows the curve \(F(s, t)\) such that \(L(s_0, t_0)\) is a generic plane through the point \(F(s_0, t_0)\).

Proof. This result follows directly from Remark 5.4 and Theorem 5.3.

Corollary 5.6. Let \(A\) be a singular point on the rational space curve \(F(s, t)\). Then \(\deg(\gcd(L(s, t) \cdot A)) = \text{order}_F(A)\), where the gcd is taken over all moving planes \(L(s, t)\) that follow the curve \(F(s, t)\). Moreover, the roots of this gcd are the parameters with the proper multiplicity corresponding to the point \(A\) on the curve \(F(s, t)\).

Proof. This result follows immediately from Corollary 5.5.

Corollary 5.7. Let \(L(s, t)\) be a moving plane that follows the rational space curve \(F(s, t)\), and let \(A\) be a singular point of order \(k\) on the curve \(F(s, t)\). If \(\deg(L(s, t)) < k\), then \(A\) must be an axial point of the moving plane \(L(s, t)\), i.e. \(L(s, t) \cdot A = 0\).

Proof. Suppose that \(\text{order}_F(A) = k\). Then by Corollary 5.6, the polynomial \(L(s, t) \cdot A\) has at least \(k\) roots, contradicting the assumption that \(\deg(L(s, t)) < k\). Therefore \(L(s, t) \cdot A = 0\).

6. Singularities and \(\mu\)-bases

To find the singularities on a rational space curve, we cannot compute with all the moving planes that follow the curve, as required by Theorem 5.3 and Corollary 5.6. Instead we need a finite set of generators for the moving plane module. Therefore next we recall the concept of a \(\mu\)-basis for rational space curves. For the convenience of the reader, we restate the following definition given in Song and Goldman (2009).
Definition 6.1. Three moving planes \( p(s, t), q(s, t) \) and \( r(s, t) \) are called a \( \mu \)-basis of the rational space curve \( F(s, t) \) if \( p, q \) and \( r \) are moving planes that follow \( F(s, t) \) and satisfy the following two conditions:

1. \( [p, q, r] = \kappa F(s, t) \),
2. \( \deg(p) + \deg(q) + \deg(r) = \deg(F) \),

where \( \kappa \) is some non-zero constant and \( [p, q, r] \) is the outer product of \( p, q \) and \( r \).

The notion of a \( \mu \)-basis for rational space curves can be generalized in an obvious way to rational curves of arbitrary dimension. The existence of a \( \mu \)-basis for a rational curve in any dimension follows directly from the Hilbert–Burch Theorem (Eisenbud, 1994, Theorem 20.15). In particular, the proof of the existence of a \( \mu \)-basis for a rational curve in an affine \( n \)-space is given in Cox et al. (1998), Exercise 17, page 286. An alternative existence proof as well as a simple algorithm to compute a \( \mu \)-basis based solely on Gaussian elimination is presented in Song and Goldman (2009).

The \( \mu \)-basis elements \( p, q, r \) for a rational space curve are not unique. But the degrees of the \( \mu \)-basis elements \( \mu_1 = \deg(p), \mu_2 = \deg(q), \mu_3 = \deg(r) \) for a rational space curve are unique. This result and the following proposition are proved in Song and Goldman (2009).

Proposition 6.2. Suppose \( p(s, t), q(s, t), r(s, t) \) are \( \mu \)-bases of degrees \( \mu_1, \mu_2, \mu_3 \) for the rational space curve \( F(s, t) \) and let \( L(s, t) \) be a moving plane of degree \( m \) that follows the curve. Then there exist polynomials \( \alpha(s, t), \beta(s, t), \gamma(s, t) \) such that \( L = \alpha p + \beta q + \gamma r \), where \( \deg(\alpha) = m - \mu_1, \deg(\beta) = m - \mu_2, \deg(\gamma) = m - \mu_3 \).

Lemma 6.3. Let \( p, q, r \) be a \( \mu \)-basis for a space curve \( C \) given by a rational parametrization \( F(s, t) \) as in (1). Then there does not exist a point \( A \) on the curve \( C \) such that

\[
p \cdot A = q \cdot A = r \cdot A = 0.
\]

Proof. If \( p \cdot A = q \cdot A = r \cdot A = 0 \), then since \( p, q \) and \( r \) form a \( \mu \)-basis, it follows from Proposition 6.2 that \( L \cdot A = 0 \) for every moving plane \( L(s, t) \) that follows the curve \( F(s, t) \). But this is impossible by Lemma 5.1.

We are now ready to present our main theorems, which will allow us to compute the singularities of a rational space curve from a \( \mu \)-basis for the curve.

Theorem 6.4. Suppose that \( p, q \) and \( r \) are \( \mu \)-bases for the rational space curve \( F(s, t) \), and let \( A \) be a point on \( F(s, t) \). Then for each parameter pair \((s_0, t_0)\) corresponding to the point \( A \), \( F(s_0, t_0) = A \), \( \text{mult}_F(s_0, t_0) = k \) if and only if \((s_0, t_0)\) is a root of \( \gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) \) of multiplicity \( k \).

Proof. By Theorem 5.3, since the \( \mu \)-basis elements are moving planes that follow the space curve \( F(s, t) \), \((s_0, t_0)\) must be a root of \( \gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) \) of multiplicity at least \( k \).

On the other hand, since by Proposition 6.2 every moving plane \( L(s, t) \) that follows the curve \( F(s, t) \) is a combination of \( p(s, t), q(s, t), r(s, t) \), following Corollary 5.5 that \( \deg(\gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A)) \) can be no greater than \( k \).

Let \( L(s, t) \) be a moving plane that follows the rational space curve and let \( L(s_0, t_0) \) be a generic plane through the point \( F(s_0, t_0) \). Again write the moving plane as a combination of the \( \mu \)-basis elements: \( L(s, t) = \alpha(s, t)p(s, t) + \beta(s, t)q(s, t) + \gamma(s, t)r(s, t) \). Since \((s_0, t_0)\) is a root of \( \gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) \) of multiplicity \( k \), it follows that \((s_0, t_0)\) is a root of \( L(s, t) \cdot A \) of multiplicity at least \( k \). Thus \( \text{mult}_F(s_0, t_0) \geq k \). If \( \text{mult}_F(s_0, t_0) = k + 1 \), then by the first part of the proof, \((s_0, t_0)\) must be a root of \( \gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) \) of multiplicity \( k + 1 \), contrary to assumption.

Corollary 6.5. Suppose that \( p(s, t), q(s, t) \) and \( r(s, t) \) are \( \mu \)-bases of the rational space curve \( F(s, t) \), and let \( A \) be a singular point on \( F(s, t) \). Let \((s_i, t_i)\) be all the different parameters corresponding to the point \( A \) and let \( \text{mult}_F(s_i, t_i) = k_i, i = 0, 1, \ldots, l - 1 \). Then \( \gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) = \prod_{i=0}^{l-1} (s_i - t_i)^{k_i} \).

Proof. This result follows directly from Theorem 6.4.

Corollary 6.6. Suppose that \( p(s, t), q(s, t) \) and \( r(s, t) \) are \( \mu \)-bases for the rational space curve \( F(s, t) \), and let \( A \) be a singular point on \( F(s, t) \). Then \( \deg(\gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A)) = \text{order}_F(A) \), and the roots of this \( \gcd \) are the parameters with proper multiplicity corresponding to the point \( A \) on the curve \( F(s, t) \).

Proof. This result follows directly from Theorem 6.4 and Corollary 6.5.

Corollary 6.7. Let \( p, q \) and \( r \) be \( \mu \)-bases with degrees \( \mu_1 \leq \mu_2 \leq \mu_3 \) for the rational space curve \( F(s, t) \). Then on the curve \( F(s, t) \):

1. There are no singular points of order \( k > \mu_3 \).
2. There are no singular points of order \( \mu_2 < k < \mu_3 \).
3. If there is a singular point of order \( \mu_1 < k < \mu_2 \), then \( p \cdot A = 0 \) and \( q \cdot A \neq 0 \).
4. All the singular points of order \( k > \mu_1 \) are collinear.

**Proof.** Suppose that \( A \) is a singular point of order \( k \) on the rational space curve \( F(s, t) \). By Corollary 6.6, \( \deg(p(s, t) \cdot A, q(s, t) \cdot A) = k \). Therefore:
1. Obviously \( k \leq \deg(r(s, t) \cdot A) = \mu_3 \).
2. If \( \mu_2 < k < \mu_3 \), then \( \deg(p \cdot A) \leq \mu_1 < k \) and \( \deg(q \cdot A) \leq \mu_2 < k \). Hence by Corollary 5.7, \( p \cdot A = q \cdot A = 0 \), and \( k = \deg\gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) = \mu_3 \), contradicting the assumption that \( \mu_2 < k < \mu_3 \). Therefore, there is no singularity of order \( \mu_2 < k < \mu_3 \).
3. Similarly, if there is a singular point of order \( \mu_1 < k < \mu_2 \), then by Corollary 5.7, \( p \cdot A = 0 \). If we also have \( q \cdot A = 0 \), then \( k = \deg\gcd(p(s, t) \cdot A, q(s, t) \cdot A, r(s, t) \cdot A) = \mu_3 \), contradicting the assumption that \( \mu_1 < k < \mu_2 \).
4. Suppose that there were three non-collinear singular points of order \( k > \mu_1 \), and let \( \Pi \) be the plane determined by these three non-collinear points. By part 3 of this corollary, any singular point of order \( k > \mu_1 \) is an axial point of \( p(s, t) \). Thus these three singular points must lie on every plane \( p(s, t) \), so \( p(s, t) = \Pi \) for all \( s, t \). Hence since \( p(s, t) \) follows the curve \( F(s, t) \), the curve \( F(s, t) \) must be planar, contrary to assumption. Therefore, all the singular points of order \( k > \mu_1 \) are collinear. \( \square \)

**Corollary 6.8.** Suppose \( p(s, t), q(s, t) \), and \( r(s, t) \) are a \( \mu \)-basis of degrees \( (1, 1, d - 2) \) for the rational space curve \( F(s, t) \). Then
1. \( F(s, t) \) has no singularities if and only if the axes of \( p \) and \( q \) do not intersect.
2. \( F(s, t) \) has exactly one singular point \( A \) if and only if the axes of \( p \) and \( q \) intersect at the point \( A \). Moreover, in this case \( \ord_{\Gamma}(A) = d - 2 \).

**Proof.** First, since \( \mu_1 = \mu_2 = 1 \), by Corollary 2.3, \( p(s, t) \) and \( q(s, t) \) are both axial moving planes. Therefore:
1. By Corollary 6.7, the space curve \( F(s, t) \) has no singularities of order \( 1 < k < d - 2 \). Moreover, the space curve \( F(s, t) \) has no singularity of order \( d - 2 \) if and only if there does not exist a point \( A \) such that \( p \cdot A = q \cdot A = 0 \), which means the axes of \( p \) and \( q \) do not intersect.
2. By Corollary 5.7, the rational space curve \( F(s, t) \) has a singular point \( A \) if and only if \( p(s, t) \cdot A = q(s, t) \cdot A = 0 \). This is equivalent to saying that the axes of \( p(s, t) \) and \( q(s, t) \) intersect at the point \( A \). In this case, \( \ord_{\Gamma}(A) = \deg\gcd(p \cdot A, q \cdot A, r \cdot A) = d - 2 \). \( \square \)

In addition to the extreme case where the degrees of two of the elements of the \( \mu \)-basis are one, we can also say something about the balanced case where all the elements of the \( \mu \)-basis have the same degree. We begin with two simple lemmas.

**Lemma 6.9.** Four singular points of order \( n \) on a rational space curve of degree \( 3n \) cannot lie on a plane. Similarly, three singular points of order \( n \) on a rational space curve of degree \( 3n \) cannot lie on a line.

**Proof.** If four singular points of order \( n \) on a rational space curve of degree \( 3n \) were to lie on a plane, then this plane would intersect the rational space curve of degree \( 3n \) at four singular points of total multiplicity at least \( 4n \), which is impossible. Similarly, three singular points of order \( n \) on a rational space curve of degree \( 3n \) cannot lie on a line. \( \square \)

**Lemma 6.10.** Suppose \( p(s, t), q(s, t) \), and \( r(s, t) \) are a \( \mu \)-basis of degrees \( (n, n, n) \) for the rational space curve \( F(s, t) \). If the curve has 4 singular points of order \( n \), then two elements of the \( \mu \)-basis can be chosen as axial moving planes that follow the curve with axes through two disjoint pairs of the singular points.

**Proof.** Let \( A, B, C, D \) be four singular points of order \( n \) on the curve. By Theorem 3.3, there are two moving planes \( L_1, L_2 \) of degree \( n = 3n - n - n \) with axes \( AB \) and \( CD \) that follow the curve. Moreover, \( L_1, L_2 \) are linearly independent, otherwise, they would have the same axis, contradicting Lemma 6.9. Since \( p(s, t), q(s, t) \), and \( r(s, t) \) are a \( \mu \)-basis, \( L_i = a_i p + b_i q + c_i r \), for some constants, \( a_i, b_i, c_i, i = 1, 2 \), with \( (a_1, b_1, c_1) \neq (k_1, b_2, c_2) \), otherwise, \( L_1 = kL_2 \). Without loss of generality, we can assume that \( (a_1, b_1) \neq (a_2, b_2) \). Then
\[
\begin{align*}
p &= \frac{b_2 L_1 - b_1 L_2 - (b_2 c_1 - b_1 c_2) r}{a_1 b_2 - a_2 b_1}, \\
q &= \frac{a_2 L_1 - a_1 L_2 - (a_2 c_1 - a_1 c_2) r}{a_2 b_1 - a_1 b_2},
\end{align*}
\]
and
\[
[L_1, L_2, r] = (a_1 b_2 - a_2 b_1)[p, q, r].
\]
Thus, \( L_1, L_2, r \) is a \( \mu \)-basis. \( \square \)
Theorem 6.11. Suppose $p(s, t), q(s, t)$ and $r(s, t)$ are a $\mu$-basis of degrees $(n, n, n)$ with $n \geq 2$ for a rational space curve $C$ with a parametrization $F(s, t)$ as in (1). Then the curve $C$ can have at most 4 singular points of order $n$.

Proof. Suppose that the curve $C$ has four singular points of order $n$. By Lemma 6.9, we can assume without loss of generality that these four singular points are $A = (1, 0, 0, 0), B = (0, 1, 0, 0), C = (0, 0, 1, 0), D = (0, 0, 0, 1)$. Moreover by Lemma 6.10, we can find a $\mu$-basis $p, q, r$ where $p$ has axis $\overline{AB}$ and $q$ has axis $\overline{CD}$. Therefore

$$p = (0, 0, p_2, p_3), \quad q = (q_0, q_1, 0, 0),$$

where $p_2, p_3, q_0, q_1$ are homogeneous polynomials in $s, t$ of degree $n$.

Since $A, B, C, D$ are singular points of order $n$, by Corollary 6.6 we must have

$$r \cdot A = c_0q \cdot A, \quad r \cdot B = c_1q \cdot B, \quad r \cdot C = c_2p \cdot C, \quad r \cdot D = c_3p \cdot D,$$

for some constants $c_i, i = 0, 1, 2, 3$. Therefore, $r = (c_0q_0, c_1q_1, c_2p_2, c_3p_3)$.

Moreover, since the curve $F(s, t)$ is the outer product of $p, q, r$.

$$f_0 = (c_2 - c_3)q_1p_2p_3, \quad f_1 = -(c_2 - c_3)q_0p_2p_3, \quad f_2 = (c_1 - c_0)q_0q_1p_3, \quad f_3 = -(c_1 - c_0)q_0q_1p_2. \quad (8)$$

Therefore $c_0 \neq c_1, c_2 \neq c_3$, and $\gcd(p_2, p_3) = \gcd(q_0, q_1) = 1$, since the curve $F(s, t)$ is a non-planar curve and $\gcd(f_0, f_1, f_2, f_3) = 1$.

Suppose there were five singular points of order $n$, and let the fifth singular point be $E = (e_0, e_1, e_2, e_3)$. Since by Lemma 6.9 $E$ cannot lie on $\overline{AB}$ or $\overline{CD}$, both $(e_2, e_3) \neq (0, 0)$ and $(e_0, e_1) \neq (0, 0)$. Without loss of generality, we can assume that $e_0, e_2 \neq 0$.

Let $F(s_0, t_0) = E = (e_0, e_1, e_2, e_3)$. Since $e_0, e_2 \neq 0$, it follows by Eq. (8) that

$$q_0(s_0, t_0) \neq 0, \quad q_1(s_0, t_0) \neq 0, \quad p_2(s_0, t_0) \neq 0, \quad p_3(s_0, t_0) \neq 0,$$

thus $e_1, e_3 \neq 0$.

Since $E$ is a singular point of order $n$, by Corollary 6.6 there are constants $k_1, k_2$ such that

$$q \cdot E = k_1p \cdot E, \quad r \cdot E = k_2p \cdot E.$$

Moreover by Lemma 6.9 $E$ cannot lie on the axes of $p$ or $q$, so $q \cdot E \neq 0$, and $p \cdot E \neq 0$; thus $k_1 \neq 0$. Solving these linear equations for $q_0, q_1$ yields

$$q_0 = \frac{(k_1c_1 - k_2 + c_2)e_2p_2 + (k_1c_1 - k_2 + c_3)e_3p_3}{(c_1 - c_0)e_0}, \quad \quad (9)$$

$$q_1 = \frac{(-k_1c_0 + k_2 - c_2)e_2p_2 + (-k_1c_0 + k_2 - c_3)e_3p_3}{(c_1 - c_0)e_1}. \quad \quad (10)$$

Substituting the formulas for $q_0, q_1$ in Eqs. (9) and (10) into Eq. (8), we get four homogeneous expressions of degree 3 in the variables $p_2, p_3$—that is

$$f_i(s, t) = g_i(u, v), \quad \text{where} \quad u = p_2(s, t), \quad v = p_3(s, t), \quad i = 0, 1, 2, 3,$$

and $g_i$ are homogeneous polynomials in $u, v$ of degree 3. Hence the space curve of degree 3n is a reparametrized cubic space curve and thus the parametrization is not 1–1, contradicting the assumption that $F$ is a generic 1–1 map. Therefore, a rational space curve of degree 3n with a $\mu$-basis of degrees $(n, n, n)$ can have at most 4 singular points of order $n$.

We can now make a general statement about the singularities of the highest possible order on a rational space curve.

Theorem 6.12. Suppose that $p(s, t), q(s, t), r(s, t)$ are a $\mu$-basis of degrees $(\mu_1, \mu_2, \mu_3)$ with $\mu_1 \leq \mu_2 \leq \mu_3$ and $\mu_3 \geq 2$ for a rational space curve $C$ with a parametrization $F(s, t)$ as in (1). Then the curve $C$ can have no singularity of order greater than $\mu_3$.

Moreover, the number of singularities of order $\mu_3$ is bounded as follows:

1. If $\mu_3 > \mu_2$, then the curve $C$ has at most one singularity of order $\mu_3$.
2. If $\mu_3 = \mu_2$ and $\mu_1 < \mu_3$, then the curve $C$ has at most two singularities of order $\mu_3$.
3. If $\mu_1 = \mu_2 = \mu_3$, then the curve $C$ has at most four singularities of order $\mu_3$.

Proof. By Corollary 6.7, the curve $C$ can have no singularities of order greater than $\mu_3$. Moreover,

1. If $\mu_3 > \mu_2$, then the curve $C$ can have at most one singularity of order $\mu_3$. Otherwise, if the curve $C$ had singularities of order $\mu_3$ at two points $P, Q$, then by Corollary 6.6, we would have

$$p(s, t) \cdot P = p(s, t) \cdot Q = 0, \quad q(s, t) \cdot P = q(s, t) \cdot Q = 0.$$
Thus the line $\overline{PQ}$ would be an axis of both $p(s, t)$ and $q(s, t)$. Therefore since $p(s, t)$ and $q(s, t)$ both follow the parametrization $F(s, t)$, the moving planes $p(s, t)$ and $q(s, t)$ would coincide, contradicting the fact that $p(s, t), q(s, t)$, and $r(s, t)$ form a $\mu$-basis. This result also follows by Theorem 3.3, since if there were two singularities of order $\mu_3$, then by Theorem 3.3 there would be a moving plane of degree less than $\mu_1$ that follows the curve, which is impossible.

2. If $\mu_3 = \mu_2$ and $\mu_1 < \mu_3$, then the curve $C$ can have at most two singularities of order $\mu_3$. Otherwise, if the curve $C$ had three or more singularities of order $\mu_3$, then there would be a plane $\Pi$ containing at least three singularities of order $\mu_3$, so this plane would intersect the curve $C$ at least three points with multiplicity $3\mu_3 > \mu_1 + \mu_2 + \mu_3 = \deg(F)$, which is impossible.

3. If $\mu_1 = \mu_2 = \mu_3$, then by Theorem 6.11, the curve $C$ can have at most four singularities of order $\mu_3$. \qed

7. Computing singularities for low degree rational space curves

In this section, we investigate methods to compute all the singularities for rational space curves $C$ given by parametrizations $F(s, t)$ as in Eq. (1) of low degree. First, we review some basic cases: if $\deg(F) = 0$, then $F(s, t)$ is a point; if $\deg(F) = 1$, then $F(s, t)$ is a line; if $\deg(F) = 2$, then $F(s, t)$ is a conic, thus a planar curve; if $\deg(F) = 3$, then $F(s, t)$ is a rational cubic space curve, and we have already seen in Corollary 3.6 that a rational cubic space curve has no singularities. Therefore, we begin our discussion with rational space curves of degree 4.

Since the degrees of the elements of a $\mu$-basis are unique and sum to the degree of the curve, from now on we will use the degrees $(\mu_1, \mu_2, \mu_3)$ of a $\mu$-basis $p, q, r$ to denote the type of a rational space curve $F(s, t)$, and we shall assume that the $\mu$-basis elements $p, q, r$ are arranged so that $\mu_1 \leq \mu_2 \leq \mu_3$.

7.1. Singularities of rational space curves of degree 4

Theorem 7.1. Suppose $F(s, t)$ is a rational space curve of degree 4. Then

1. $F(s, t)$ has no singularities if and only if the axes of the $\mu$-basis elements $p$ and $q$ do not intersect.
2. $F(s, t)$ has exactly one singular point $A$ if and only if the axes of the $\mu$-basis elements $p$ and $q$ intersect at the point $A$. Moreover, in this case, $\text{order}(A) = 2$.

Proof. A degree 4 rational space curve must be of the type $(1, 1, 2)$. Therefore this result follows directly by Corollary 6.8. \qed

Based on Theorem 7.1 and Lemma 2.2, we now present an algorithm for finding the singular point of a rational quartic space curve. Before we present this algorithm, recall from Section 2 that $M_p$ and $M_q$ denote the coefficient matrices of the moving planes $p$ and $q$.

Algorithm 1 (Search for the singularity of a rational quartic space curve).

1. Using the algorithm in Song and Goldman (2009), calculate a $\mu$-basis $p, q, r$ for the rational quartic space curve.
2. Determine the rank of the matrix $[M_p \quad M_q]^T$.
3. If $\text{rank}[M_p \quad M_q]^T = 4$, then the rational quartic space curve has no singularity; otherwise, if $\text{rank}[M_p \quad M_q]^T = 3$, then solve the equation $[M_p \quad M_q]^T x = 0$. The root corresponds to the singular point.

Note that Algorithm 1 is also valid for any curve of type $(1, 1, d - 2)$.

Example 7.2 (A rational quartic space curve with one singularity of order 2). Consider the rational quartic space curve given by $F(s, t) = (s^4 - 5s^3t + s^2t^2 + 21st^3 - 17t^4, ~ s^4 + 8s^3t + 17s^2t^2 - 2st^3 - 24t^4, ~ s^4 + s^3t - 2s^2t^2 + t^4, ~ t^4)$.

Using the algorithm in Song and Goldman (2009), we compute a $\mu$-basis for $F(s, t)$:

$p = (-180t - 109s, ~ 135t - 87s, ~ 196s, ~ 180t - 87s)$,
$q = (4s, ~ -3s, ~ 45t - s, ~ -45t - 3s)$,
$r = (135t^2 + 29s^2 + 90st, ~ 12s^2, ~ -41s^2, ~ 2295t^2 - 1203s^2 - 1305st)$.

The axis line of $p$ is given by the intersection of the two planes

\[ \begin{cases} -784x_0 - 87x_1 + 784x_2 = 0, \\ 225x_0 - 196x_2 - 29x_3 = 0, \end{cases} \]

and the axis line of $q$ is given by the intersection of the two planes

\[ \begin{cases} x_2 - x_3 = 0, \\ 4x_0 - 3x_1 - 4x_3 = 0. \end{cases} \]
These two lines (four planes) intersect at the point \( A = (1, 0, 1, 1) \). Thus by Theorem 7.1, \( A \) is a singular point of \( F(s, t) \) of order 2. We can also apply Corollary 6.6 to show that the two parameters corresponding to the point \( A \) are \((1, 1)\) and \((-2, 1)\).

More directly, by Algorithm 1, the block matrix

\[
\begin{bmatrix}
M_p \\
M_q
\end{bmatrix} = \begin{bmatrix}
-109 & -87 & 196 & -87 \\
-180 & 135 & 0 & 180 \\
4 & -3 & -1 & -3 \\
0 & 0 & 45 & -45
\end{bmatrix}
\]

has rank 3, and up to a constant multiple, \( A = (1, 0, 1, 1) \) is the only solution of the equation \([M_p \ M_q]^T \cdot x = 0\). Therefore, \( A = (1, 0, 1, 1) \) is a singular point of order 2.

7.2. Singularities of rational space curves of degree 5

There are two types of rational quintic space curves: type \((1, 1, 3)\) and type \((1, 2, 2)\). Below we shall treat each case in turn.

7.2.1. Rational quintic space curves of type \((1,1,3)\)

**Theorem 7.3.** Suppose that \( F(s, t) \) is a rational space curve of type \((1,1,3)\). Then

1. \( F(s, t) \) has no singularities if and only if the axes of the \( \mu \)-basis elements \( p \) and \( q \) do not intersect.
2. \( F(s, t) \) has exactly one singular point \( A \) if and only if the axes of the \( \mu \)-basis elements \( p \) and \( q \) intersect at the point \( A \). Moreover, in this case, \( \text{ord}_F(A) = 3 \).

**Proof.** The result follows directly by Corollary 6.8. \( \square \)

Algorithm 1 can be used to search for the singularity of a rational quintic space curve of type \((1,1,3)\).

**Example 7.4** (A rational quintic space curve with one singularity of order 3). Consider the rational quintic space curve given by

\[
F(s, t) = (s^3 t + s^2 t^2 - 2 s^2 t^3 + t^5, s^5 + 5 s^4 t + 6 s^3 t^2 - 4 s^2 t^3 - 8 s t^4, s^4 t - 3 s^2 t^3 + 2 s t^4 + t^5, t^5).
\]

Using the algorithm in Song and Goldman (2009), we compute a \( \mu \)-basis for \( F(s, t) \):

\[
p = (s - t, 0, -s, t),
\]

\[
q = (9 s, -t, -8 s - 4 t, 4 t - s),
\]

\[
r = (11 s^3, -4 t^3 - 3 s^2 t + 4 s t^2, -8 s^3, 13 s^3 + 16 s^2 t - 32 s t^2).
\]

The axis line of \( p \) is given by the intersection of the two planes

\[
\begin{cases}
x_0 - x_3 = 0, \\
x_2 - x_3 = 0,
\end{cases}
\]

and the axis line of \( q \) is given by the intersection of the two planes

\[
\begin{cases}
9 x_0 - 8 x_2 - x_3 = 0, \\
36 x_0 - x_1 - 36 x_2 = 0.
\end{cases}
\]

These two lines intersect at the point \( A = (1, 0, 1, 1) \). Thus by Theorem 7.3, \( A \) is a singular point of \( F(s, t) \) of order 3. We can also apply Corollary 6.6 to show that the three parameters corresponding to the point \( A \) are \((1, 1), (-2, 1) \) and \((0, 1)\).

More directly, by Algorithm 1, the block matrix

\[
\begin{bmatrix}
M_p \\
M_q
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
9 & 0 & -8 & -1 \\
0 & -1 & -4 & 4
\end{bmatrix}
\]

has rank 3, and up to a constant multiple, \( A = (1, 0, 1, 1) \) is the only solution for the equation \([M_p \ M_q]^T \cdot x = 0\). Therefore, \( A = (1, 0, 1, 1) \) is a singular point of order 3.
7.2.2. Rational quintic space curves of type (1,2,2)

**Theorem 7.5.** Suppose that $F(s, t)$ is a rational space curve of type (1,2,2) with $\mu$-basis $p, q, r$. Then the only singular points of $F(s, t)$ are of order 2 and there can be at most two such singular points. Moreover, a point $A$ is a singular point of $F(s, t)$ if and only if the following two conditions are satisfied:

1. $A$ is on the axis of $p$, i.e. $p \cdot A = 0$.
2. There are constants $k_1, k_2$ not both zero such that $k_1 q \cdot A + k_2 r \cdot A = 0$.

Notice that if $k_2 = 0$, then $A$ lies on the intersection of the axes of $p$ and $q$. Similarly, if $k_1 = 0$, then $A$ lies on the intersection of the axes of $p$ and $r$.

**Proof.** It follows directly from Corollary 6.6 that the only singular points of $F(s, t)$ are of order 2, and the singular points must lie on the axis of $p$. Furthermore, there can be at most two such singular points. For if there were three singular points of order 2, then every plane through the axis of $p$ would intersect the quintic space curve in at least three points of total multiplicity at least 6, which is impossible.

Moreover, by Corollary 6.6, a point $A$ is a singular point of $F(s, t)$ of order 2 if and only if $p \cdot A = 0$ and $\deg((\gcd(q \cdot A, r \cdot A)) = 2$. The latter condition is equivalent to saying that there are constants $k_1, k_2$ not both zero such that $k_1 q \cdot A + k_2 r \cdot A = 0$.

Now, $k_2 = 0$ if and only if $p \cdot A = q \cdot A = 0$, that is, if and only if $A$ is the intersection point of the axes of $p$ and $q$. Similarly, $k_1 = 0$ if and only if $p \cdot A = r \cdot A = 0$, that is, if and only if $A$ is the intersection point of the axes of $p$ and $r$. □

**Algorithm 2** (Search for the singularities of a rational quintic space curve of type (1,2,2)).

1. Using the algorithm in Song and Goldman (2009), calculate a $\mu$-basis $p, q, r$ for the rational quintic space curve.
2. Determine the ranks of the matrices $[M_p \ M_q]^T$, and $[M_p \ M_r]^T$.
3. If rank$[M_p \ M_q]^T < 4$, then solve the equation $[M_p \ M_q]^T x = 0$. The root corresponds to a singular point of order 2.
4. If rank$[M_p \ M_r]^T < 4$, then solve the equation $[M_p \ M_r]^T x = 0$. The root corresponds to a singular point of order 2.
5. If two singularities have not yet been found, then
   (a) Find two points $A, B$ on the axis of $p$ by solving the equation $M_p x = 0$.
   (b) Solve the equation $M_q (a A + b B) = k M_r (a A + b B)$ for the variables $a, b, k$ with $k \neq 0$. The points $a A + b B$, if they exist, correspond to singular points of order 2.

**Example 7.6** (A rational quintic space curve with two singularities of order 2). Consider the rational quintic space curve given by $F(s, t) = (s^5, s^2 t^2, s^2 t^3, t^5)$.

We shall apply Algorithm 2 to find the singular points of $F(s, t)$. Using the Algorithm in Song and Goldman (2009), we compute a $\mu$-basis for $F(s, t)$: $p = (0, t, -s, 0), \ q = (t^2, -s^2, 0, 0), \ r = (0, 0, t^2, -s^2)$.

Now it is easy to check that rank$[M_p \ M_q]^T = 3$. Thus there is one solution to each of the following equations

$$[M_p \ M_q]^T x = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x = 0.$$ 

namely, $A = [0, 0, 0, 1]^T$, and $B = [1, 0, 0, 0]^T$. Therefore, $A = (0, 0, 0, 1)$ and $B = (1, 0, 0, 0)$ are two singular points of order 2, and there can be no other singular points on $F(s, t)$.

7.3. Singularities of rational space curves of degree 6

There are three types of rational sextic space curves: type (1,1,4), type (1,2,3) and type (2,2,2). Below we shall treat each case in turn.

7.3.1. Rational sextic space curves of type (1,1,4)

**Theorem 7.7.** Suppose $F(s, t)$ is a rational space curve of type (1,1,4). Then

1. $F(s, t)$ has no singularities if and only if the axes of the $\mu$-basis elements $p$ and $q$ do not intersect.
2. $F(s, t)$ has exactly one singular point $A$ if and only if the axes of the $\mu$-basis elements $p$ and $q$ intersect at the point $A$. Moreover, in this case, order$_F(A) = 4$.
Proof. The result follows directly by Corollary 6.8. □

Algorithm 1 can be used to search for the singularity of a rational sextic space curve of type (1,1,4).

7.3.2. Rational sextic space curves of type (1,2,3)

**Theorem 7.8.** Suppose that \( F(s, t) \) is a rational space curve of type (1,2,3) with \( \mu \)-basis \( p, q, r \). Then the singular points of \( F(s, t) \) are of order 2 or 3, and there can be at most two singular points of which at most one can be of order 3. Moreover, a point \( A \) is a singular point of \( F(s, t) \) if and only if the following two conditions are satisfied:

1. \( A \) is on the axis of \( p \), i.e. \( p \cdot A = 0 \).
2. There are constant \( k_1, k_2, k_3 \) not all zero such that \((k_1 s + k_2 t)q \cdot A = k_3 r \cdot A\).

Notice that if \( k_1 = k_2 = 0 \), then \( A \) is of order 2, and lies on the intersection of the axes of \( p \) and \( r \). If \((k_1, k_2) \neq (0, 0) \) and \( k_3 \neq 0 \), then \( A \) is of order 2, and neither the axes of \( p \) and \( q \) nor the axes of \( p \) and \( r \) intersect. If \( k_3 = 0 \), then \( A \) is of order 3, and lies on the intersection of the axes of \( p \) and \( q \).

Proof. It follows directly from Corollary 6.6 that the only singular points of \( F(s, t) \) are of order 2 or 3, and all the singular points must lie on the axis of \( p \). Furthermore, there can be at most two singular points of which at most one can be of order 3. For if there were three singular points of order 2 or two singular points of order 3, then due to these points every plane through the axis of \( p \) would intersect the sextic curve \( F(s, t) \) in points with a total multiplicity of at least 6. But every plane in the moving plane \( p \) must pass through the axis of \( p \). Therefore since the moving plane \( p \) follows the curve \( F(s, t) \), most of the planes in \( p \) would necessarily intersect this rational sextic curve with a total multiplicity of at least 7, which is impossible.

Moreover, by Corollary 6.6, a point \( A \) is a singular point on \( F(s, t) \) if and only if \( p \cdot A = 0 \) and

\[
(k_1 s + k_2 t)q \cdot A = k_3 r \cdot A, \quad \text{for some choices of } k_1, k_2, k_3.
\]

Note, if \( k_1 = k_2 = 0 \) and \( k_3 \neq 0 \), then by Eq. (11), \( r \cdot A = 0 \). Thus \( A \) is a singular point of order 2. When \((k_1, k_2) \neq (0, 0) \) and \( k_3 \neq 0 \), then by Eq. (11), \( q \cdot A \neq 0 \) or \( r \cdot A \neq 0 \), and \(\gcd(q \cdot A, r \cdot A)\) is, up to a constant multiple, \( q \cdot A \). Thus \( A \) is a singular point of order 2. If \((k_1, k_2) \neq (0, 0) \), and \( k_3 = 0 \), then by Eq. (11), \( q \cdot A = 0 \). Thus \( A \) is a singular point of order 3. □

**Algorithm 3** (Search for the singularities of a rational sextic space curve of type (1,2,3)).

1. Using the algorithm in Song and Goldman (2009), calculate a \( \mu \)-basis \( p, q, r \) for the rational sextic space curve.
2. Find two points \( A, B \) on the axis of \( p \) by solving the equation \( \text{M}_{p}\text{x} = 0 \).
3. Solve the equation \( M_{(k_1 s + k_2 t)q}(aA + bB) = k_3 M_{r}(aA + bB) \) for the variables \( a, b, k_1, k_2, k_3 \). If \( k_3 = 0 \), then the point \( aA + bB \) corresponds to a singular point of order 3; otherwise, the point \( aA + bB \) corresponds to a singular point of order 2.

**Example 7.9** (A rational sextic space curve with two singularities: one of order 2 and one of order 3). Consider the rational sextic space curve given by

\[
F(s, t) = (s^6, s^4 t^2, s^3 (s - t) t^2, t^6).
\]

We shall apply Algorithm 3 to find the singular points of \( F(s, t) \). Using the algorithm in Song and Goldman (2009), we compute a \( \mu \)-basis for \( F(s, t) \): \( p = (0, -s + t, s, 0) \), \( q = (-t^2, s^2, 0, 0) \), \( r = (0, -t^3, t^3, s^3) \).

The equation

\[
\text{M}_{p} \text{x} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{x} = 0
\]

has two distinct solutions \( A = [1, 0, 0, 0]^T \), \( B = [0, 0, 0, 1]^T \), and \( M_{(k_1 s + k_2 t)q}(aA + bB) = k_3 M_{r}(aA + bB) \) yields

\[
[0 \ 0 \ -ak_1 \ -k_2 b]^T = [k_3 b \ 0 \ 0 \ 0]^T.
\]

Equality holds if \( k_3 = a = 0 \), \( k_1, k_2, b \neq 0 \); or \( k_1 = k_2 = b = 0 \), \( k_3, a \neq 0 \). The former corresponds to the singular point \( aA + bB = B = (0, 0, 0, 1) \), the latter corresponds to the singular point \( aA + bB = A = (1, 0, 0, 0) \), and \( \text{order}_{F}(A) = 2 \), \( \text{order}_{F}(B) = 3 \).

7.3.3. Rational sextic space curves of type (2,2,2)

**Theorem 7.10.** Suppose that \( F(s, t) \) is a rational space curve of type (2,2,2) with \( \mu \)-basis \( p, q, r \). Then the singular points of \( F(s, t) \) are all of order 2. Moreover, \( F(s, t) \) has at most 4 singular points, and a point \( A \) is a singular point of \( F(s, t) \) if and only if one of the following three conditions holds:

1. The point \( A \) is the common axis point of two of the \( \mu \)-basis elements \( p, q, r \).
2. The point $A$ is on the axis of one of the $\mu$-basis elements $p, q, r$ (suppose $p$), and $q \cdot A = kr \cdot A$ for some non-zero constant $k$.
3. The point $A$ is not on the axis of $p, q, r$, and $p \cdot A = k_1q \cdot A = k_2r \cdot A$ for some non-zero constants $k_1$ and $k_2$.

**Proof.** It follows by Corollary 6.7 and Theorem 6.11 that the order of each singular point must be 2 and there are at most 4 singular points of order 2. We will now prove that a point $A$ is a singular point of $F(s, t)$ if and only if one of the following three conditions holds.

1. Suppose that the point $A$ is the common axis of $p$ and $q$, so that $p \cdot A = q \cdot A = 0$. Then $\gcd(p \cdot A, q \cdot A, r \cdot A) = r \cdot A$. Since $\deg(r \cdot A) = 2$, the point $A$ is a singular point of order 2.
2. Suppose that the point $A$ is on the axis of $p$, but is not on the axis of $q$ or $r$. Then $p \cdot A = 0, q \cdot A \neq 0, r \cdot A \neq 0$. Thus $\gcd(p \cdot A, q - A, r \cdot A) = \gcd(q \cdot A, r \cdot A)$. Therefore, the point $A$ is a singular point if and only if $q \cdot A = kr \cdot A$ for some non-zero constant $k$.
3. Suppose that the point $A$ is not on the axis of $p, q, r$. Then $p \cdot A \neq 0, q \cdot A \neq 0, r \cdot A \neq 0$. Thus, by Corollary 6.6, the point $A$ is a singular point if and only if $p \cdot A = k_1q \cdot A = k_2r \cdot A$ for some non-zero constants $k_1$ and $k_2$. □

**Algorithm 4 (Search for the singularities of a rational sextic space curve of type (2,2,2)).**

1. Using the algorithm in Song and Goldman (2009), calculate a $\mu$-basis $p, q, r$ for the rational sextic space curve.
2. Determine the ranks of the matrices $[M_p]_{M_q}, [M_p]_{M_r}$ and $[M_q]_{M_r}$.
3. If $\operatorname{rank}[M_p]_{M_q} \neq 4$, then solve the equation $[M_p]_{M_q}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
4. If $\operatorname{rank}[M_p]_{M_r} \neq 4$, then solve the equation $[M_p]_{M_r}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
5. If $\operatorname{rank}[M_q]_{M_r} \neq 4$, then solve the equation $[M_q]_{M_r}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
6. If there is a non-zero value $k$ such that $\operatorname{rank}[M_p]_{M_q+kr} \neq 4$, then solve the equation $[M_p]_{M_q+kr}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
7. If there is a non-zero value $k'$ such that $\operatorname{rank}[M_q]_{M_p+k'r} \neq 4$, then solve the equation $[M_q]_{M_p+k'r}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
8. If there is a non-zero value $k''$ such that $\operatorname{rank}[M_r]_{M_p+k''q} \neq 4$, then solve the equation $[M_r]_{M_p+k''q}^T \cdot x = 0$. The root corresponds to a singular point of order 2.
9. If there exist non-zero constants $k_1, k_2$, then solve the equation $[M_{p+k_1q}]_{M_{p+k_2r}}^T \cdot x = 0$. The root corresponds to a singular point of order 2.

**Example 7.11 (A rational sextic space curve with four singularities of order 2).** Consider the rational sextic space curve given by

$$ F = (s^2(t^2(s - t)^2), s^2(t^2(s - t)^2), t^2(s - t)^2(s - 2t)^2, s^2(s - t)^2(s - 2t)^2). $$

We shall apply Algorithm 4 to find the singular points of $F(s, t)$. Using the algorithm in Song and Goldman (2009), we compute a $\mu$-basis for $F(s, t)$:

$$ p = (s^2 - 4st + 4t^2, 0, 0, -t^2), $$
$$ q = (-s^2 + 4st - 4t^2, s^2 - 2st + t^2, 0, 0), $$
$$ r = (-s^2 + 4st - 4t^2, s^2 - 2st + t^2, s^2, 0). $$

Once again it is easy to check that

$$ \operatorname{rank}[M_p]_{M_q}^T = 3, \quad \operatorname{rank}[M_q]_{M_r}^T = 3, \quad \operatorname{rank}[M_p]_{M_r}^T = 4. $$

There is one solution to each of the equations $[M_p]_{M_q}^T \cdot x = 0$ and $[M_q]_{M_r}^T \cdot x = 0$, namely, $A = (0, 0, 1, 0)$ and $B = (0, 0, 0, 1)$. Thus the points $A = (0, 0, 1, 0)$ and $B = (0, 0, 0, 1)$ are singular points of order 2.

If $k = -1$, then $\operatorname{rank}[M_p]_{M_{q+kr}}^T < 4$, and the equation $[M_p]_{M_{q+kr}}^T \cdot x = 0$ has the solution $C = (0, 1, 0, 0)$. Thus $C = (0, 1, 0, 0)$ is a singular point of order 2.

If $k_1 = k_2 = 1$, then $\operatorname{rank}[M_{p+k_1q}]_{M_{p+k_2r}}^T < 4$, and the equation

$$ [M_{p+k_1q}]_{M_{p+k_2r}}^T \cdot x = 0 $$

has the solution $D = (1, 0, 0, 0)$. Thus $D = (1, 0, 0, 0)$ is a singular point of order 2.

Therefore, we conclude there are four singular points $A = (0, 0, 1, 0), B = (0, 0, 0, 1), C = (0, 1, 0, 0)$ and $D = (1, 0, 0, 0)$ of order 2 and the curve has no other singular points.

**8. Conclusion**

Previous work by Song et al. (2007) shows that there is a one-to-one correspondence between the singularities of rational planar curves and the axial moving lines that follow these curves. The axes of the moving lines are located at the singularities of the rational curves. Moreover, the orders of the singularities and the degrees of the associated axial moving
Table 1

<table>
<thead>
<tr>
<th>Curve types</th>
<th>Maximum number of singularities</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>no singular points</td>
</tr>
<tr>
<td>(1, 1, 2)</td>
<td>1 double point</td>
</tr>
<tr>
<td>(1, 1, 3)</td>
<td>1 triple point</td>
</tr>
<tr>
<td>(1, 2, 2)</td>
<td>2 double points</td>
</tr>
<tr>
<td>(1, 1, 4)</td>
<td>1 singular point of order 4</td>
</tr>
<tr>
<td>(1, 2, 3)</td>
<td>2 double points, or 1 triple point and 1 double point</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>4 double points</td>
</tr>
<tr>
<td>(1, 1, d – 2)</td>
<td>1 singular point of order d – 2</td>
</tr>
<tr>
<td>(h, n, n)</td>
<td>4 singular points of order n</td>
</tr>
<tr>
<td>(μ₁, μ₂, μ₃) and μ₃ &gt; μ₂</td>
<td>1 singular point of order μ₃ (as well as other possible singular points of lower order)</td>
</tr>
<tr>
<td>(μ₁, μ₂, μ₃) and μ₃ = μ₂, μ₁ &lt; μ₃</td>
<td>2 singular point of order μ₃ (as well as other possible singular points of lower order)</td>
</tr>
</tbody>
</table>

Motivated by these results for rational planar curves, we derived correspondences between the singularities of rational space curves and the axial moving planes that follow these curves. We also showed how to employ μ-bases for the moving planes that follow rational space curves to compute all the singularities of rational space curves of low degree. The methods we employ are simple and straightforward, and are based solely on elementary techniques from linear algebra. Similar techniques can be applied to detect the singularities and determine the order of the singularities of higher degree rational space curves, though the higher the degree the more the cases that need to be addressed.

We close with Table 1 summarizing our results for the maximal number and the order of all the possible singularities on rational space curves of degree less than or equal to six. We also provide tight bounds on the number of possible singularities of highest order for rational space curves of arbitrary degree. In our future work, we hope to provide bounds as well on the number of singularities of lower order.

Acknowledgements

This work was partially supported by NSF grant CCR-020331, by the 111 Project of China grant b07033, by the NSF of China grant 10671192, and by the 100 Talent Project sponsored by CAS of China. We would like to thank Laurent Busé, Carlos D’Andrea, Brendan Hassett, Bill Hoffman and Augusto Nobile for their helpful conversations and suggestions. The referees also made useful suggestions for improving our paper.

This article grew out of some ideas initially discussed during the mini-workshop on Surface Modeling and Syzygies at Mathematisches Forschungsinstitut Oberwolfach, Germany. We would like to thank the organizers of the mini-workshop and the hosting organizations for their support and hospitality.

References


