更正：引理8.5的证明已重写，见讲义

$$P_0=\begin{pmatrix} E_n-A & O \\ O & E_n-B \end{pmatrix} \xrightarrow{\text{初等行}} \begin{pmatrix} E_n-A & O \\ E_n-A & E_n-B \end{pmatrix}=P_1$$

$$\xrightarrow[\text{右乘}\begin{pmatrix} E_n & B \\ O & E_n \end{pmatrix}]{\text{初等列}} \begin{pmatrix} E_n-A & (E_n-A)B \\ E_n-A & (E_n-A)B+E_n-B \end{pmatrix}$$

$$=\begin{pmatrix} E_n-A & (E_n-A)B \\ E_n-A & E_n-AB \end{pmatrix}=P_2$$

$\operatorname{rank}(P_0) \geq \operatorname{rank}(P_2) \geq \operatorname{rank}(E_n-AB)$

$\operatorname{rank}(P_0) = \operatorname{rank}(E_n-A)+\operatorname{rank}(E_n-B)$. $\blacksquare$

第二章 行列式

回忆：$\sigma: \{1,\cdots,n\} \to \{1,\cdots,n\}$ 双射

$i \mapsto \sigma(i)$

σ 称为置换。S_n 是所有置换的集合。$|S_n|=n!$ ε_σ 是 σ 的符号

设 $X=\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n$ ①

X 的行列式定义为

$$\det(X)=\sum_{\sigma\in S_n} \varepsilon_\sigma x_{\sigma(1),1}\cdots x_{\sigma(n),n}.$$

$\det(X)$ 也记作 $|X|$.

§0 斜对称多重线性函数

例：$f: \mathbb{R}^n \to \mathbb{R}$，线性函数

设 $\vec{x}=x_1\vec{e}^{(1)}+\cdots+x_n\vec{e}^{(n)}$

$f(\vec{x})=f(x_1\vec{e}^{(1)}+\cdots+x_n\vec{e}^{(n)})$

$=f(\vec{e}^{(1)})x_1+\cdots+f(\vec{e}^{(n)})x_n$

令 $a_i=f(\vec{e}^{(i)})$

$f(\vec{x})=a_1x_1+\cdots+\overset{a_n}{\cancel{f(\vec{e}^{(n)})}}x_n$

f 由 f 在 $\vec{e}^{(1)},\cdots,\vec{e}^{(n)}$ 下的像确定

定义：设 $f: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m} \longrightarrow \mathbb{R}$

称为 $\mathbb{R}^n$ 上 m 重线性函数，如果

$\forall i \in \{1, \cdots, m\}$，$\vec{y} \in \mathbb{R}^n$，$\alpha, \beta \in \mathbb{R}$

$f(\vec{x}_1, \cdots \vec{x}_{i-1}, \alpha \vec{x}_i + \beta \vec{y}, \vec{x}_{i+1}, \cdots \vec{x}_n)$

$= \alpha f(\vec{x}_1, \cdots \vec{x}_{i-1}, \vec{x}_i, \vec{x}_{i+1}, \cdots \vec{x}_n) +$

$\beta f(\vec{x}_1, \cdots \vec{x}_{i-1}, \vec{y}, \vec{x}_{i+1}, \cdots, \vec{x}_n)$

~~对称 于是~~

例：$\det_2 \quad \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(\vec{x}_1, \vec{x}_2) \mapsto \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

其中 $\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$，$\vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

则 $\det_2$ 是 $\mathbb{R}^2$ 上的二重线性函数

验证：设 $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\det_2(\alpha \vec{x}_1 + \beta \vec{y}, \vec{x}_2) = \begin{vmatrix} \alpha x_{11} + \beta y_1 & x_{12} \\ \alpha x_{21} + \beta y_2 & x_{22} \end{vmatrix}$

②

$= (\alpha x_{11} + \beta y_1) x_{22} - (\alpha x_{21} + \beta y_2) x_{12}$

$= \alpha (x_{11} x_{22} - x_{21} x_{12}) + \beta (y_1 x_{22} - y_2 x_{12})$

$= \alpha \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} + \beta \begin{vmatrix} y_1 & x_{12} \\ y_2 & x_{22} \end{vmatrix} = \alpha \det_2(\vec{x}_1, \vec{x}_2)$

$+ \beta \det_2(\vec{y}, \vec{x}_2)$

同理可证：$\det_2(\vec{x}_1, \alpha \vec{x}_2 + \beta \vec{y})$

$= \alpha \det_2(\vec{x}_1, \vec{x}_2) + \beta \det_2(\vec{x}_1, \vec{y})$

例，设 $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ 是二重线性的

展开 $f(\vec{x} + \vec{z}, \vec{y} + 2\vec{w})$

$= f(\vec{x}, \vec{y} + 2\vec{w}) + f(\vec{z}, \vec{y} + 2\vec{w})$

$= f(\vec{x}, \vec{y}) + f(\vec{x}, 2\vec{w}) + f(\vec{z}, \vec{y}) + f(\vec{z}, 2\vec{w})$

$= f(\vec{x}, \vec{y}) + 2 f(\vec{x}, \vec{w}) + f(\vec{z}, \vec{y}) + 2 f(\vec{z} + \vec{w})$

多重线性函数的表示

设 $f: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m} \longrightarrow \mathbb{R}$ 是 m 重

线性函数

$$\vec{x}_j = \sum_{i=1}^{n} x_{ij} \vec{e}^{(i)}, \quad j = 1, \cdots, m$$

$$f(\vec{x}_1, \cdots, \vec{x}_m) = f\left(\sum_{i_1=1}^{n} x_{i_1 1} \vec{e}^{(i_1)}, \cdots, \sum_{i_m=1}^{n} x_{i_m m} \vec{e}^{(i_m)}\right)$$

$$= \sum_{i_1=1}^{n} x_{i_1 1} f\left(\vec{e}^{(i_1)}, \sum_{i_2=1}^{n} x_{i_2 2} \vec{e}^{(i_2)}, \cdots, \sum_{i_m=1}^{n} x_{i_m m} \vec{e}^{(i_m)}\right)$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \underbrace{f(\vec{e}^{(i_1)}, \vec{e}^{(i_2)}, \cdots, \vec{e}^{(i_m)}) \, x_{i_1 1} x_{i_2 2} \cdots x_{i_m m}}$$

令 $a_{i_1, i_2, \cdots, i_m} = f(\vec{e}^{(i_1)}, \vec{e}^{(i_2)}, \cdots, \vec{e}^{(i_m)})$

则 $$f(\vec{x}_1, \cdots, \vec{x}_m) = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} a_{i_1 \cdots i_m} x_{i_1 1} \cdots x_{i_m m} \quad (*)$$

定义：设 $f: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m} \longrightarrow \mathbb{R}$ 是 $\mathbb{R}^n$ 上 m 重线性函数，如果 $\forall i, j \in \{1, \cdots, m\}$，让

$$f(\vec{x}_1, \cdots \vec{x}_i, \cdots \vec{x}_j, \cdots \vec{x}_m)$$

$$= - f(\vec{x}_1, \cdots, \underset{i}{\vec{x}_j}, \cdots \underset{j}{\vec{x}_i}, \cdots \vec{x}_m)$$

则称 f 是全斜对称的

例：验证 $\det_2$ 是 $\mathbb{R}^2$ 上二重线性函数

设 $\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

$$\det_2(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

$$\det_2(\vec{x}_2, \vec{x}_1) = \begin{vmatrix} x_{12} & x_{11} \\ x_{22} & x_{21} \end{vmatrix}$$

$$= x_{12} x_{21} - x_{11} x_{22} = - \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

例 设 f 是 $\mathbb{R}^n$ 上 m 重斜对称线性函数

证明：$f(\vec{x}_1,\cdots,\underset{i}{\vec{u}},\cdots\underset{j}{\vec{u}},\cdots\vec{x}_m)=0$ $i\neq j$

证：交换第 i 个和第 j 个变量得

$$f(\vec{x}_1,\cdots\underset{i}{\vec{u}},\cdots\underset{j}{\vec{u}},\cdots\vec{x}_m)=-f(\vec{x}_1,\cdots\underset{i}{\vec{u}}\cdots\underset{j}{\vec{u}},\cdots\vec{x}_m)$$

$\Rightarrow$ $2f(\vec{x}_1,\cdots,\vec{u},\cdots\vec{u},\cdots\vec{x}_m)=0$

$\because 2\neq 0.$ $f(\vec{x}_1,\cdots\vec{u},\cdots\vec{u},\cdots\vec{x}_m)=0.$

注：$m>1.$ $f(\vec{0},\vec{x}_2,\cdots\vec{x}_m)=0$

问题：设 $f:\underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{n}\longrightarrow\mathbb{R}$

n 重、斜对称，线性，展开 f

由 $(*)$

$$f(\vec{x}_1,\cdots\vec{x}_n)=\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}a_{i_1,\cdots i_n}x_{i_1,1}\cdots x_{i_n,n}$$

其中 $\vec{x}_j=\begin{pmatrix}x_{1j}\\ \vdots\\ x_{nj}\end{pmatrix}$，$j=1,\cdots n$

$$a_{i_1,\cdots i_n}=f(\vec{e}_{i_1},\cdots,\vec{e}_{i_n}).$$

因为 f 斜对称 所以

如果 $i_1,\cdots i_n$ 中有两个相同

则 $a_{i_1,\cdots i_n}=0$

于是 只要考虑 $i_1,\cdots i_n$ 两两不同的 $a_{i_1\cdots i_n}$.

此时 ~~$i_1,\cdots i_n$~~ $\begin{pmatrix}1&2&\cdots&n\\ i_1&i_2&\cdots&i_n\end{pmatrix}\in S_n$

即 $i_1=\sigma(1)$，$i_2=\sigma(2)$，$\cdots$ $i_n=\sigma(n)$，~~且其中~~

$\sigma\in\begin{pmatrix}1,2&\cdots&n\\ i_1&\cdots&i_n\end{pmatrix}$

于是

$$(**)\quad f(\vec{x}_1,\cdots\vec{x}_n)=\sum_{\sigma\in S_n}a_{\sigma(1),\cdots\sigma(n)}x_{\sigma(1),1,}\cdots x_{\sigma(n),n}$$

引理 0.1 利用以上符号

$$a_{\sigma(1),\cdots,\sigma(n)}=\varepsilon_\sigma\,a_{12\cdots n}.$$

证：设 $\sigma=\tau_1\cdots\tau_k$，其中 $\tau_1,\cdots\tau_k$ 是对换

$k=0$. σ 是恒同映射. 引理显然成立

$k=1$. 设 $\sigma=(i\,j)$

$$a_{\sigma(1),\cdots,\sigma(n)}=\cancel{f(\vec{e}_{\sigma(1)}}\,f(\vec{e}^{(\sigma(1))},\cdots,\vec{e}^{(\sigma(n))})$$

$$=f(\vec{e}^{(1)},\cdots\vec{e}^{(j)},\cdots\vec{e}^{(i)},\cdots,\vec{e}^{(n)})$$

$$=-f(\vec{e}^{(1)},\cdots\vec{e}^{(i)},\cdots\vec{e}^{(j)},\cdots\vec{e}^{(n)})$$

引理成立

$$=-a_{1,2,\cdots n}$$

设 σ 为 $k-1$ 个对换之积时引理成立.

令 $\pi=\tau_2\cdots\tau_k$

$$\cancel{a_{\sigma(1),\cdots\sigma(n)}=f(\vec{e}_{\sigma(1)},\cdots,\vec{e}_{\sigma(n)})}$$

$$\cancel{=f(\vec{e}_{\tau_1\circ\pi(1}}$$

$$a_{\sigma(1),\cdots\sigma(n)}=f(\vec{e}^{(\sigma(1))},\cdots\vec{e}^{(\sigma(n))})\quad[\text{定义}]$$

$$=f(\vec{e}^{[\tau_1(\pi(1))]},\cdots,\vec{e}^{[\tau_1(\pi(n))]})\quad[\sigma\text{定义}]$$

$$=-f(\vec{e}^{[\pi(1)]},\cdots,\vec{e}^{[\pi(n)]})\quad[\text{对换一次}]$$

$$=-\cancel{(-)}\,\varepsilon_\pi\,a_{1,\cdots n}=\varepsilon_\sigma\,a_{1,2,\cdots n}$$

[归纳假设] ☐

由 (**) 和引理 0.1

$$f(\vec{x}_1,\cdots,\vec{x}_n)=\lambda\sum_{\sigma\in S_n}\varepsilon_\sigma\,x_{\sigma(1),1},\cdots x_{\sigma(n),n}$$

其中 $\lambda\in a_{1,2,\cdots n}\in\mathbb{R}$

定义：设 $\vec{x}_1,\cdots\vec{x}_n\in\mathbb{R}^n$

$X=(\vec{x}_1,\cdots\vec{x}_n)\in M_n$ 的行列式

是 $\sum_{\sigma\in S_n}\varepsilon_\sigma\,x_{\sigma(1),1},\cdots x_{\sigma(n),n}$

记为 $\det(X)$ 或 $|X|$.

注：由上述推导可知，如果 f 是 $\mathbb{R}^n$ 上 n 重斜对称、线性函数，则

$$f(\vec{x}_1,\dots,\vec{x}_n)=\lambda \det\left((\vec{x}_1,\dots\vec{x}_n)\right)$$

其中 $\lambda=f(\vec{e}^{(1)},\dots,\vec{e}^{(n)})$.

特别地 $\det(\vec{x}_1,\dots,\vec{x}_n)$ 是 $\mathbb{R}^n$ 上 n 重斜对称线性函数

§1 行列式的基本性质

定理1.1. 设 $A\in M_n$，则 $\det(A)=\det(A^t)$

证：设 $A=(a_{ij})$　$A^t=(a'_{ij})$

其中 $a'_{ij}=a_{ji}$，$i,j\in\{1,\dots,n\}$

$$\det(A^t)=\sum_{\sigma\in S_n}\varepsilon_\sigma a'_{\sigma(1),1}\cdots a'_{\sigma(n),n}$$

$$=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}$$

$$=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma^{-1}\sigma(1),\sigma(1)}\cdots a_{\sigma^{-1}\sigma(n),\sigma(n)}$$

$$\sigma=\begin{pmatrix}1 & 2, & \cdots & n\\ \sigma(1) & \sigma(2), & \cdots, & \sigma(n)\end{pmatrix}$$

$$\det(A^t)=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma^{-1}(1),1}\cdots a_{\sigma^{-1}(n),n}$$

$\because\ \varepsilon_\sigma=\varepsilon_{\sigma^{-1}}$,　$S_n=\{\sigma^{-1}\mid\sigma\in S_n\}$

$$\det(A^t)=\sum_{\sigma^{-1}\in S_n}\varepsilon_{\sigma^{-1}} a_{\sigma^{-1}(1),1}\cdots a_{\sigma^{-1}(n),n}$$

$$=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma(1),1}\cdots a_{\sigma(n),n}$$

$$=\det(A)\qquad\blacksquare$$

例 设 $A=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$

$$A^t=\begin{pmatrix}x_{11} & x_{21}\\ x_{12} & x_{22}\end{pmatrix}$$

$$\det(A^t)=x_{11}x_{22}-x_{12}x_{21}=\det(A)$$

推论1.1 行列式 det(A) 关于行向量也是 n 重的线性斜对称~~线性~~函数。

证：设 $i\in\{1,\cdots,n\}$，$\vec{x}=(x_1,\cdots,x_n)$，$\vec{y}=(y_1,\cdots,y_n)$，$\alpha,\beta\in\mathbb{R}$。$\vec{A_i}=\alpha\vec{x}+\beta\vec{y}$

$$\det(A)=\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ \alpha\vec{x}+\beta\vec{y}\\ \vdots\\ \vec{A_n}\end{pmatrix}=\det(A^t)$$

$$=\det(\vec{A_1}^t,\cdots,\vec{A_{i-1}}^t,\alpha\vec{x}^t+\beta\vec{y}^t,\vec{A_{i+1}}^t,\cdots,\vec{A_n}^t)$$

$$=\alpha\det(\vec{A_1}^t,\cdots,\vec{A_{i-1}}^t,\vec{x}^t,\vec{A_{i+1}}^t,\cdots,\vec{A_n})+$$
$$\beta\det(\vec{A_1}^t,\cdots,\vec{A_{i-1}}^t,\vec{y}^t,\vec{A_{i+1}}^t,\cdots,\vec{A_n})$$

$$=\alpha\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{A_n}\end{pmatrix}+\beta\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{y}\\ \vdots\\ \vec{A_n}\end{pmatrix}$$

类似地，可证：

$$\det\begin{array}{c}\begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{A_j}\\ \vdots\\ \vec{A_i}\\ \vdots\\ \vec{A_n}\end{pmatrix}\end{array}\begin{array}{l}\\ \\ \cdots i\\ \\ \cdots j\\ \\ \\ \end{array}=-\det(A)\quad\square$$

D1 行列式关于行向量或列向量是多重线性的

推论1.2 设 $A\in M_n$。

(i) 如果 A 的某个行(列)向量是零向量，则 $\det(A)=0$

(ii) $\forall\alpha\in\mathbb{R}$ $|\alpha A|=\alpha^n|A|$

证：(i) 设 $i\in\{1,\cdots,n\}$，$\vec{A_i}=(\underbrace{0,\cdots,0}_{n})=:\vec{O}_{1\times n}$

$$\det(A)=\det\begin{array}{c}\begin{pmatrix}\vec{A_1}\\ \vdots\\ O_{1\times n}\\ \vdots\\ \vec{A_n}\end{pmatrix}\end{array}\begin{array}{l}\\ \\ \cdots i\\ \\ \\ \end{array}=\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ O_{1\times n}+O_{1\times n}\\ \vdots\\ \vec{A_n}\end{pmatrix}=\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ O_{1\times n}\\ \vdots\\ \vec{A_n}\end{pmatrix}+\det\begin{pmatrix}\vec{A_1}\\ \vdots\\ O_{1\times n}\\ \vdots\\ \vec{A_n}\end{pmatrix}$$

$=\det(A)+\det(A)$. 于是 $\det(A)=0$

(ii) $$|\alpha A|=\det(\alpha\vec{A}^{(1)},\alpha\vec{A}^{(2)},\cdots,\alpha\vec{A}^{(n)})$$
$$=\alpha\det(\vec{A}^{(1)},\alpha\vec{A}^{(2)},\cdots,\alpha\vec{A}^{(n)})$$
$$=\alpha^2\det(\vec{A}^{(1)},\vec{A}^{(2)},\alpha\vec{A}^{(3)},\cdots,\alpha\vec{A}^{(n)})$$
$$\cdots$$
$$=\alpha^n\det(\vec{A}^{(1)},\vec{A}^{(2)},\cdots,\vec{A}^{(n)})$$
$$=\alpha^n|A|.$$

设 $\vec{X}_1, \cdots, \vec{X}_n \in \mathbb{R}^n$, $\vec{X}_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix}$, $j = 1 \cdots n$

$$\det(\vec{X}_1, \cdots, \vec{X}_n) = \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1}, \cdots x_{\sigma(n),n}$$

验证 $\det: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R}$ 是多重线性函数

设 $\vec{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\det(\vec{X}_1, \cdots \vec{X}_j + \vec{Y}, \cdots \vec{X}_n)$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1}, \cdots (x_{\sigma(j),j} + y_{\sigma(j)}) \cdots x_{\sigma(n),n}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1} \cdots x_{\sigma(j),j} \cdots x_{\sigma(n),n}$$

$$+ \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1} \cdots y_{\sigma(j)} \cdots x_{\sigma(n),n}$$

$$= \det(\vec{X}_1, \cdots \vec{X}_j, \cdots, \vec{X}_n) + \det(\vec{X}_1, \cdots \underset{j}{\vec{Y}}, \cdots \vec{X}_n)$$

类似可证: $\det(\vec{X}_1, \cdots \alpha \vec{X}_j \cdots \vec{X}_n) = \alpha \det(\vec{X}_1, \cdots \vec{X}_j, \vec{X}_n)$

设 $X = (x_{ij})_{\substack{i=1\cdots n \\ j=1\cdots n}}$

$Y = (y_{ij})$ 由 X 交换第 i 列和第 j 列得到的 $(i \neq j)$

令 $\tau = (i\,j)$ 则 $\vec{Y}^{(k)} = \vec{X}^{(\tau(k))}$

$$\det(Y) = \sum_{\sigma \in S_n} \varepsilon_\sigma y_{\sigma(1),1} \cdots y_{\sigma(n),n}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),\tau(1)} \cdots x_{\sigma(n),\tau(n)}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma\tau^{-1}(\tau(1)),\tau(1)} \cdots x_{\sigma\tau^{-1}\tau(n),\tau(n)}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma\tau^{-1}(1),1} \cdots x_{\sigma\tau^{-1}(n),n}$$

$$= -\sum_{\sigma \in S_n} \varepsilon_{\sigma\tau^{-1}} x_{\sigma\tau^{-1}(1),1} \cdots x_{\sigma\tau^{-1}(n),n}$$

$$= -\sum_{\pi \in S_n} \varepsilon_\pi x_{\pi(1),1}, \cdots x_{\pi(n),n}$$

$$= -\det(X) \quad \boxed{}$$

D2 斜对称.

推论1.3. 设 $A\in M_n$. 如果 A 中两行或两列相同，则 $\det(A)=0$.

证: 设 $i,j\in\{1,\cdots,n\}$, $i\neq j$. $\vec{A}_i=\vec{A}_j$

$$\det(A)=\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_i\\ \vdots\\ \vec{A}_j\\ \vdots\\ \vec{A}_n\end{pmatrix}=-\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_j\\ \vdots\\ \vec{A}_i\\ \vdots\\ \vec{A}_n\end{pmatrix}\begin{matrix}\\ \\ \cdots i\\ \\ \cdots j\\ \\ \\ \end{matrix}=-\det(A)$$

$$2\det(A)=0 \quad \because 2\neq 0 \quad \therefore \det(A)=0$$

推论1.4 设 $A\in M_n$. 如果 $\mathrm{rank}(A)<n$, 则 $\det(A)=0$

证: 设 $\mathrm{rank}(A)<n$. 则 $\vec{A}^{(1)},\cdots\vec{A}^{(n)}$ 线性相关.

不妨设 $\vec{A}^{(1)}=\alpha_2\vec{A}^{(2)}+\cdots+\alpha_n\vec{A}^{(n)}$, $\alpha_2,\cdots\alpha_n\in\mathbb{R}$

$$\det(A)=\det\left(\alpha_2\vec{A}^{(2)}+\cdots+\alpha_n\vec{A}^{(n)},\ \vec{A}^{(2)},\cdots,\vec{A}^{(n)}\right)$$

$$=\sum_{i=2}^{n}\alpha_i\det\left(\vec{A}^{(i)},\ \vec{A}^{(2)},\cdots,\vec{A}^{(n)}\right)=0 \quad ∎$$

D3 初等变换下的行列式

设 $A\in M_n$, $i,j\in\{1,\cdots,n\}$, $i\neq j$

则 (I) $|E_{ij}^{(n)}A|=|AE_{ij}^{(n)}|=-|A|$

(II) $|E_{ij}^{(n)}(\lambda)A|=|AE_{ij}^{(n)}(\lambda)|=|A|$

(III) $|E_i^{(n)}(\lambda)A|=|AE_i^{(n)}(\lambda)|=\lambda|A|$

验证: (I) 即推论1.3. (III) 多重线性

(II) $$\det\left(E_{ij}^{(n)}(\lambda)A\right)=\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_i+\lambda\vec{A}_j\\ \vec{A}_j\\ \vdots\\ \vec{A}_n\end{pmatrix}$$

$$=\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_i\\ \vdots\\ \vec{A}_j\\ \vdots\\ \vec{A}_n\end{pmatrix}+\lambda\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_j\\ \vdots\\ \vec{A}_j\\ \vdots\\ \vec{A}_n\end{pmatrix}=\det\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vdots\\ \vec{A}_n\end{pmatrix}=\det(A).$$

D4 正规化 $|E_n|=1$

$$\det(E_n)=\det\left(\vec{e}^{(1)},\cdots\vec{e}^{(n)}\right)=1.$$

D_5 上(下)三角形矩阵的行列式

设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn}\end{pmatrix}$ 或 $A=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$

则 $\det(A)=a_{11}a_{22}\cdots a_{nn}$.

验证：考虑上三角形矩阵 A

$$|A|=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma(1),1}\cdots a_{\sigma(n),n}$$

若 σ 不是恒同映射，则 $\exists i\in\{1,\dots,n\}$，使得

$$\sigma(i)\neq i \qquad (*)$$

设 i 是最小正整数满足 $(*)$

则 $\sigma(1)=1,\dots\sigma(i-1)=i-1$. 于是 $\sigma(i)>i$

由此可知. $a_{\sigma(i),i}=0 \Rightarrow$

$$a_{\sigma(1),1}\cdots a_{\sigma(i),i}\cdots a_{\sigma(n),n}=0$$

$|A|=\varepsilon_e a_{11}a_{22}\cdots a_{nn}$. 　$e\in S_n$ 恒同

$=a_{11}a_{22}\cdots a_{nn}$

命题1.1 设 $A\in M_n$. 则

$$|A|=0 \Longleftrightarrow \operatorname{rank}(A)<n$$

证："$\Leftarrow$" 推论1.3

"$\Rightarrow$" 由

设 $\operatorname{rank}(A)=r$. 由定理6.1（第二章）

通过初等行变换和列变换

$$A\longrightarrow \underbrace{\begin{pmatrix} E_r & 0 \\ 0 & 0\end{pmatrix}}_{B}$$

由行列式在初等变换下的性质

$\det(B)=0$ 于是 $r<n$. $\boxed{}$

设 $\det(E_n)=1$.

例：计算 $\Delta=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1\end{vmatrix}$ 的值

$$\Delta=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2\end{vmatrix}=1\cdot(-2)(-2)(-2)=-8.$$

例：计算 $\Delta = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$$\Delta = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

例 $\Delta = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & a \end{vmatrix}_{n\times n}$

把第2列，…第n列都加到第一列上

$$\Delta = \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \cdots & b \\ & & \cdots & & \\ a+(n-1)b & b & b & \cdots & a \end{vmatrix}$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ & & \cdots & & \\ 1 & b & b & \cdots & a \end{vmatrix}$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ & 0 & & \ddots & \\ & & & & a-b \end{vmatrix}$$

$$= (a+(n-1)b)\,(a-b)^{n-1}.$$

§2 行列式进一步性质

§2.1 按一行或一列展开

定义：设 $A\in M_n$，$i,j\in\{1,\cdots,n\}$ 去掉 A 的第 i 行和第 j 列得到的 $n-1$ 阶方阵的行列式 ~~记为~~ 称为 $|A|$ 关于 i 行 j 列的余式 记为 M_{ij}. 而 $(-1)^{i+j}M_{ij}$ 称为 $|A|$ 关于 i 行 j 列的代数余子式，记为 A_{ij}

例 $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$ $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

引理2.1 设 $A \in M_n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,n-1} & 0 \\ a_{21} & a_{22} & \cdots & a_{2\,n-1} & 0 \\ & & \cdots & & \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

则 $\det(A) = a_{nn} A_{nn}$

证: $$|A| = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n-1),1}\, a_{\sigma(n),n}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(n)=n}} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n-1),1}\, a_{n,n}$$

$$= \sum_{\tau \in S_{n-1}} \varepsilon_\tau a_{\tau(1),1} \cdots a_{\tau(n-1),n-1} a_{nn}$$

$$= a_{nn} \sum_{\tau \in S_{n-1}} \varepsilon_\tau a_{\tau(1),1} \cdots a_{\tau(n-1),n-1}$$

$$= a_{nn} M_{nn}$$

$$= a_{nn} A_{nn}.$$ ▨

引理2.2 设 $A \in M_n$. $\vec{A}^{(j)} = a_{ij} \vec{e}^{(i)}$

则 $|A| = a_{ij} A_{ij}$

证: 由假设可知

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ & & & \cdots & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ & & & \cdots & & & \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = (-1)^{n-j+n-i} \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} & 0 \\ & & \cdots & & & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} & 0 \\ & & \cdots & & & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} & 0 \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} & a_{ij} \end{pmatrix}$$

$$= (-1)^{i+j} M_{ij}\, a_{ij} \qquad \text{(引理2.1)}$$

$$= a_{ij} A_{ij}.$$

定理2.1 设 $A=(a_{ij})_{n\times n}$ 则

(i) $|A| = \sum_{i=1}^{n} a_{ij} A_{ij}$ （按第j行展开）

(ii) $|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$ （按第j列展开）

证：(i)

$$|A| = \det\left(\vec{A}^{(1)}, \cdots, \vec{A}^{(j-1)}, \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}, \vec{A}^{(j+1)}, \cdots, \vec{A}^{(n)}\right)$$

$$= \det\left(\vec{A}^{(1)}, \cdots, \vec{A}^{(j-1)}, \sum_{i=1}^{n} a_{ij}\vec{e}^{(i)}, \vec{A}^{(j+1)}, \cdots, \vec{A}^{(n)}\right)$$

$$= \sum_{i=1}^{n} a_{ij} \det\left(\vec{A}^{(1)}, \cdots, \vec{A}^{(j-1)}\ a_{ij}\vec{e}^{(i)}, \vec{A}^{(j+1)}, \cdots \vec{A}^{(n)}\right)$$

$$= \sum_{i=1}^{n} a_{ij} A_{ij} \qquad \text{（引理2.2）}$$

(ii) 类似 或 转置 $\square$

例：展开 $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ 按第一列展开

$$\Delta = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}.$$

例. 计算 $D = \begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}_{5\times 5}$

$$= -2\begin{vmatrix} 5 & 3 & -1 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & -1 & 4 \\ 0 & 2 & 3 & 5 \end{vmatrix} = -10\begin{vmatrix} -2 & 3 & 1 \\ -4 & -1 & 4 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 20\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \\ -1 & 3 & 5 \end{vmatrix} = 20\begin{vmatrix} 1 & 3 & 1 \\ 0 & -7 & 2 \\ 0 & 6 & 6 \end{vmatrix}$$

$$= 20\begin{vmatrix} -7 & 2 \\ 6 & 6 \end{vmatrix} = +120\begin{vmatrix} -7 & 2 \\ 1 & 1 \end{vmatrix} = -1080 \quad \square$$

例 Vandermonde 行列式 $n \geq 2$

$$V_n(x_1,\cdots,x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ & \cdots & & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

展开 $V_n(x_1,\cdots,x_n)$.

$n=2$ $\quad \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$

$n=3$ $\quad \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2-x_1 & x_3-x_1 \\ x_1^2 & x_2^2-x_1^2 & x_3^2-x_1^2 \end{vmatrix}$

$$= \begin{vmatrix} x_2-x_1, & x_3-x_1 \\ (x_2-x_1)(x_2+x_1), & (x_3-x_1)(x_3+x_1) \end{vmatrix}$$

$$= (x_2-x_1)(x_3-x_1)\begin{vmatrix} 1 & 1 \\ x_2+x_1 & x_3+x_1 \end{vmatrix}$$

$$= (x_2-x_1)(x_3-x_1)(x_3-x_2)$$

用第$n-1$行 $\times\, x_1$ 去减第n行

$$V_n(x_1,\cdots,x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ & \cdots & & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ 0 & x_2^{n-1}-x_2^{n-2}x_1 & \cdots & x_n^{n-1}-x_n^{n-2}x_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ & \cdots & & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ 0 & x_2^{n-2}(x_2-x_1) & \cdots & x_n^{n-2}(x_n-x_1) \end{vmatrix}$$

$=$ 用第$n-2$行乘以x_1去减$n-1$行

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ & \cdots & & \\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3} \\ 0 & x_2^{n-3}(x_2-x_1) & \cdots & x_n^{n-3}(x_n-x_1) \\ \vdots & \vdots & & \vdots \\ 0 & x_2^{n-2}(x_2-x_1) & & x_n^{n-2}(x_n-x_1) \end{vmatrix}$$

用第i行
乘以x_1去减i+1行
~~$i=1,2,\dots,n-1$~~
$i=n-1, n-2, \dots, 1$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2-x_1 & \cdots & x_n-x_1 \\ 0 & x_2(x_2-x_1) & \cdots & x_n(x_n-x_1) \\ & \cdots & & \\ 0 & x_2^{n-3}(x_2-x_1) & & x_n^{n-3}(x_n-x_1) \\ 0 & x_2^{n-2}(x_2-x_1), & \cdots & x_n^{n-2}(x_n-x_1) \end{vmatrix}$$

$$=(x_2-x_1)\cdots(x_n-x_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ & \cdots & & \\ x_2^{n-3} & x_3^{n-3} & \cdots & x_n^{n-3} \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$=\left[\prod_{2\le k\le n}(x_k-x_1)\right] V_{n-1}(x_2, x_3, \dots x_n) \quad (*)$$

断言: $V_n = \prod_{1\le i<j\le n}(x_j-x_i)$

$n=2$ ✓

设$n-1$时断言成立

则 $V_{n-1}(x_2, x_3, \dots x_n) = \prod_{2\le i<j\le n}(x_j-x_i)$

由(*)可得

$$V_n(x_1,\dots,x_n) = \left(\prod_{2\le k\le n}(x_k-x_1)\right)\left(\prod_{2\le i<j\le n}(x_j-x_i)\right)$$

$$= \prod_{1\le i<j\le n}(x_j-x_i) \quad \square$$