§2.4 关于秩和维数的等式与不等式

(The first contact)

命题2.1 设 $A\in\mathbb{R}^{m\times n}$ 则

$$\operatorname{rank}(A)\leqslant\min(m,n)$$

证: $V_r(A)\subset\mathbb{R}^{1\times n}\Rightarrow\dim V_r(A)\leqslant n$

$\Rightarrow\operatorname{rank}(A)\leqslant n$

类似 $V_c(A)\subset\mathbb{R}^{m}\Rightarrow\dim V_c(A)\leqslant m$

$\Rightarrow\operatorname{rank}(A)\leqslant m$ ▨

定理3.1

定义: 设 $A=\begin{pmatrix}a_{11}&a_{12},\cdots&a_{1n}\\a_{21}&a_{22}&\cdots a_{2n}\\ \cdots&\cdots&\cdots\\a_{m1}&a_{m2}&\cdots a_{mn}\end{pmatrix}_{m\times n}$

A的转置是 $\begin{pmatrix}a_{11}&a_{21}&\cdots a_{m1}\\a_{12}&a_{22}&\cdots a_{m2}\\ \vdots&\vdots&\vdots\\a_{1n}&a_{2n}&a_{mn}\end{pmatrix}_{n\times m}$

记为 A^t (transpose)

命题2.2 $\operatorname{rank}(A)=\operatorname{rank}(A^t)$

证: 由转置的定义可知

$$\dim V_r(A)=\dim V_c(A^t)$$

于是 $\operatorname{rank}(A)=\operatorname{rank}(A^t)$ (定理3.1)

例(p61. 习题4)

回忆习题课. 设 $U,V\subset\mathbb{R}^n$ 是子空间

则 $\dim(U+V)+\dim(U\cap V)=\dim U+\dim V$.

§3 线性映射

§3.1 定义和性质

定义: 设 $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ 是映射.

如果 $\forall\alpha,\beta\in\mathbb{R}$, $\vec{u},\vec{v}\in\mathbb{R}^n$

$$\varphi(\alpha\vec{u}+\beta\vec{v})=\alpha\varphi(\vec{u})+\beta\varphi(\vec{v})$$

则称 φ 是线性映射

注1: $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ 是线性的

$\Longleftrightarrow$ (i) $\forall\vec{u},\vec{v}\in\mathbb{R}^n$.

$\varphi(\vec{u}+\vec{v})=\varphi(\vec{u})+\varphi(\vec{v})$

(ii) $\forall\alpha\in\mathbb{R}$. $\vec{u}\in\mathbb{R}^n$

$\varphi(\alpha\vec{u})=\alpha\varphi(\vec{u})$.

验证："⟹" 取 $\alpha=\beta=1$ 得 (i)，取 $\beta=0$ 得 (ii)

"⟸" $\varphi(\alpha\vec{u}+\beta\vec{v}) = \varphi(\alpha\vec{u})+\varphi(\beta\vec{v})$ （∵ (i)）

$= \alpha\varphi(\vec{u})+\beta\varphi(\vec{v})$ （∵ (ii)）.

注(ii) 设 $\vec{0}_n \in \mathbb{R}^n$，$\vec{0}_m \in \mathbb{R}^m$ 是零向量

如果 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的，则

$$\varphi(\vec{0}_n) = \vec{0}_m$$

验证： $\varphi(\vec{0}_n) = \varphi(\vec{0}_n+\vec{0}_n) = \varphi(\vec{0}_n)+\varphi(\vec{0}_n)$

$\Rightarrow \varphi(\vec{0}_n) = \vec{0}_m$

命题3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的

$\alpha_1,\dots,\alpha_k \in \mathbb{R}$，$\vec{v}_1,\dots,\vec{v}_k \in \mathbb{R}^n$

则 (i) $\varphi(\alpha_1\vec{v}_1+\dots+\alpha_k\vec{v}_k) = \alpha_1\varphi(\vec{v}_1)+\dots+\alpha_k\varphi(\vec{v}_k)$

(ii) 如果 $\vec{v}_1,\dots,\vec{v}_k$ 线性相关，则 $\varphi(\vec{v}_1),\dots,\varphi(\vec{v}_k)$ 也线性相关

(iii) 如果 $\varphi(\vec{v}_1),\dots,\varphi(\vec{v}_k)$ 线性无关，则 $\vec{v}_1,\dots,\vec{v}_k$ 也线性无关

证：(i) 对 k 归纳.

$k=1$ $\varphi(\alpha_1\vec{v}_1) = \alpha_1\varphi(\vec{v}_1)$ （注1(ii)）

设 $k-1$ 时结论成立，当 k 时

$\varphi(\alpha_1\vec{v}_1+\dots+\alpha_{k-1}\vec{v}_{k-1}+\alpha_k\vec{v}_k)$

$= \varphi(\alpha_1\vec{v}_1+\dots+\alpha_{k-1}\vec{v}_{k-1})+\varphi(\alpha_k\vec{v}_k)$ （注1）

$= \underbrace{\alpha_1\varphi(\vec{v}_1)+\dots+\alpha_{k-1}\varphi(\vec{v}_{k-1})}_{\text{归纳假设}} + \overbrace{\alpha_k\varphi(\vec{v}_k)}^{\text{注1}}$.

(ii) 存在 $\alpha_1,\dots,\alpha_k \in \mathbb{R}$，不全为零

使得 $\alpha_1\vec{v}_1+\dots+\alpha_k\vec{v}_k = \vec{0}_n$

$\varphi(\alpha_1\vec{v}_1+\dots+\alpha_k\vec{v}_k) = \varphi(\vec{0}_n) = \vec{0}_m$

$\parallel$

$\alpha_1\varphi(\vec{v}_1)+\dots+\alpha_k\varphi(\vec{v}_k)$ （结论1）

(iii) 是 (ii) 的逆否命题.

例：零映射. $\varphi_0: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \longmapsto \vec{0}_m$

设 $\vec{x}, \vec{y} \in \mathbb{R}^n$，$\alpha, \beta \in \mathbb{R}$

$\varphi_0(\alpha\vec{x}+\beta\vec{y}) = \vec{0}_m$

$\alpha\varphi_0(\vec{x})+\beta\varphi_0(\vec{y}) = \alpha\cdot\vec{0}_m+\beta\cdot\vec{0}_m = \vec{0}_m$

于是 $\varphi_0(\alpha\vec{x}+\beta\vec{y}) = \alpha\varphi_0(\vec{x})+\beta\varphi_0(\vec{y})$

φ_0 是线性的

恒同映射 $\varphi_I : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto \vec{x}$

$\forall \alpha,\beta \in \mathbb{R}$, $\vec{x},\vec{y} \in \mathbb{R}^n$

$\varphi_I(\alpha\vec{x}+\beta\vec{y}) = \alpha\vec{x}+\beta\vec{y}$

$\alpha\varphi_I(\vec{x})+\beta\varphi_I(\vec{y}) = \alpha\vec{x}+\beta\vec{y}$

于是 $\varphi_I(\alpha\vec{x}+\beta\vec{y}) = \alpha\varphi_I(\vec{x})+\beta\varphi_I(\vec{y})$.

φ_I 是线性的

线性函数：设 $\alpha_1\cdots\alpha_n \in \mathbb{R}$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R},\quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n$$

设 $\vec{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y}=\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. $\lambda \in \mathbb{R}$

$$f(\vec{x}+\vec{y}) = f\left(\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}\right) = \alpha_1(x_1+y_1)+\cdots+\alpha_n(x_n+y_n)$$

$$= (\alpha_1x_1+\cdots+\alpha_nx_n)+(\alpha_1y_1+\cdots+\alpha_ny_n)$$

$$= f(\vec{x})+f(\vec{y})$$

$$f(\lambda\vec{x}) = f\left(\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}\right) = \alpha_1(\lambda x_1)+\cdots+\alpha_n(\lambda x_n)$$

$$= \lambda(\alpha_1x_1+\cdots+\alpha_nx_n) = \lambda f(\vec{x}).$$

f 是线性的

~~引理 3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}$~~

~~如果 $f_1,\cdots f_n$ 都是 $\mathbb{R}^n \to \mathbb{R}$ 的线性函数~~

~~则 φ 是线性映射.~~

~~证：设 $\vec{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y}=\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. $\lambda \in \mathbb{R}$~~

$\varphi(\vec{x}+\vec{y}) =$

引理 3.1. 设 $f_1,\cdots f_m$ 是 $\mathbb{R}^n \to \mathbb{R}$ 线性函数,

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m,\quad \vec{x} \longmapsto \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$$

则 φ 是线性映射

证：设 $\alpha,\beta\in\mathbb{R}$, $\vec{x},\vec{y}\in\mathbb{R}^n$

$$\varphi(\alpha\vec{x}+\beta\vec{y})=\begin{pmatrix}f_1(\alpha\vec{x}+\beta\vec{y})\\ \vdots\\ f_m(\alpha\vec{x}+\beta\vec{y})\end{pmatrix}=\begin{pmatrix}\alpha f_1(\vec{x})+\beta f_1(\vec{y})\\ \vdots\\ \alpha f_m(\vec{x})+\beta f_m(\vec{y})\end{pmatrix}$$

$$=\alpha\begin{pmatrix}f_1(\vec{x})\\ \vdots\\ f_m(\vec{x})\end{pmatrix}+\beta\begin{pmatrix}f_1(\vec{y})\\ \vdots\\ f_m(\vec{y})\end{pmatrix}=\alpha\varphi(\vec{x})+\beta\varphi(\vec{y})\quad\boxed{}$$

例：嵌入 ~~e~~ 设 $n\leq m$

$$\varphi_e:\ \mathbb{R}^n\longrightarrow\mathbb{R}^m$$
$$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\longmapsto\begin{pmatrix}x_1\\ \vdots\\ x_n\\ 0\\ \vdots\\ 0\end{pmatrix}\Big\}m-n$$

是线性映射

例 投射 设 $n\geq m$

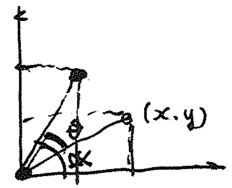
$$\varphi_p:\ \mathbb{R}^n\longrightarrow\mathbb{R}^m$$
$$\begin{pmatrix}x_1\\ x_m\\ x_n\end{pmatrix}\longmapsto\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}$$

是线性映射

例. 旋转 $\varphi_r:\ \mathbb{R}^2\longrightarrow\mathbb{R}^2$

固定 $\theta\in[0,2\pi)$ $\quad\begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}x\cos\theta-y\sin\theta\\ x\sin\theta+y\cos\theta\end{pmatrix}$

线性映射



设 $\rho=\sqrt{x^2+y^2}$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\rho\cos\alpha\\ \rho\sin\alpha\end{pmatrix}\longrightarrow\begin{pmatrix}\rho\cos(\alpha+\theta)\\ \rho\sin(\alpha+\theta)\end{pmatrix}$$

$$=\begin{pmatrix}\rho\cos\alpha\cos\theta-\rho\sin\alpha\sin\theta\\ \rho\sin\alpha\cos\theta+\rho\cos\alpha\sin\theta\end{pmatrix}=\begin{pmatrix}x\cos\theta-y\sin\theta\\ x\sin\theta+y\cos\theta\end{pmatrix}$$

例 平移 设 $\vec{v}\in\mathbb{R}^n\setminus\{\vec{0}_n\}$

$$\varphi:\ \mathbb{R}^n\longrightarrow\mathbb{R}^n\qquad \varphi(\vec{0}_n)=\vec{v}\neq\vec{0}_n$$
$$\vec{x}\longmapsto\vec{x}+\vec{v}$$

φ 不是线性映射

例 抛物线： $f:\ \mathbb{R}\to\mathbb{R}$
$$x\longmapsto x^2$$

$$f(x+y)=x^2+2xy+y^2$$

$$f(x)+f(y)=x^2+y^2$$

$\because 2\neq 0.\quad f(x+y)\neq f(x)+f(y)$

f 不是线性的

命题3.2 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的

(i) 设 $V \subset \mathbb{R}^n$ 是子空间

$$\varphi(V) = \{\varphi(\vec{v}) \in \mathbb{R}^m \mid \vec{v} \in V\}$$

是 $\mathbb{R}^m$ 的子空间且 $\dim V \geq \dim \varphi(V)$

(ii) 设 $W \subset \mathbb{R}^m$ 是子空间

$$\varphi^{-1}(W) = \{\vec{v} \in \mathbb{R}^n \mid \varphi(\vec{v}) \in W\}$$

是 $\mathbb{R}^n$ 的子空间.

证: (i) 设 $\alpha_1, \alpha_2 \in \mathbb{R}$. $\vec{w}_1, \vec{w}_2 \in \varphi(V)$

则 $\exists\ \vec{v}_1, \vec{v}_2 \in V$, $\vec{w}_1 = \varphi(\vec{v}_1)$, $\vec{w}_2 = \varphi(\vec{v}_2)$

$$\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 = \alpha_1\varphi(\vec{v}_1) + \alpha_2\varphi(\vec{v}_2)$$
$$= \varphi(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2)$$

$\because$ V 是子空间 $\therefore$ $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \in V$

$\therefore$ $\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 \in \varphi(V)$.

设 $d = \dim \varphi(V)$. 则 $\varphi(V)$ 有一组基 $\vec{w}_1, \dots \vec{w}_d$

于是 $\exists\ \vec{v}_1, \dots \vec{v}_d \in V$ 使得 $\vec{w}_i = \varphi(\vec{v}_i)$, $i = 1, \dots d$.

由~~命题~~ 3.1 (iii) $\vec{v}_1, \dots \vec{v}_d$ 线性无关

于是 $\dim V \geq d$

(ii) 设 $\alpha_1, \alpha_2 \in \mathbb{R}$. $\vec{v}_1, \vec{v}_2 \in \varphi^{-1}(W)$

则 $\varphi(\vec{v}_1), \varphi(\vec{v}_2) \in W$

$$\alpha_1\varphi(\vec{v}_1) + \alpha_2\varphi(\vec{v}_2) \in W$$
$$\parallel$$
$$\varphi(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2)$$

于是 $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \in \varphi^{-1}(W)$ ☐

§3.2 核与像.

定义: 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的

记 $\varphi(\mathbb{R}^n)$ 为 $\mathrm{im}(\varphi)$ 称为 φ 的像空间

记 $\varphi^{-1}(\{\vec{0}_m\})$ 为 $\ker(\varphi)$ 称为 φ 的核空间

注: $\ker(\varphi) \subset \mathbb{R}^n$. $\varphi(\mathbb{R}^n) \subset \mathbb{R}^m$

它们分别是生活在不同空间中的子空间.

例. φ_0 $\ker(\varphi_0) = \mathbb{R}^n$ $\quad \operatorname{im}(\varphi_0) = \{\vec{0}_m\}$

$\ker(\varphi_I) = \{\vec{0}_n\}$ $\quad \operatorname{im}(\varphi_I) = \mathbb{R}^n$

$\ker(\varphi_e) = \{\vec{0}_n\}$ $\quad \operatorname{im}(\varphi_e) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \middle| \; x_1 \ldots x_n \in \mathbb{R} \right\}$ ($m-n$)

$\ker(\varphi_p) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \middle| \; x_{m+1}, \ldots x_n \in \mathbb{R} \right\}$

$\operatorname{im}(\varphi_p) = \mathbb{R}^m$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \alpha_1 x_1 + \cdots + \alpha_n x_n$

$\operatorname{im}(f) = \begin{cases} \{0\} & 当 \alpha_1 = \cdots = \alpha_n = 0 \\ \mathbb{R} & 当 \alpha_1, \ldots \alpha_n 不全为零 \end{cases}$

$\ker(f) = \operatorname{sol}(\alpha_1 x_1 + \cdots + \alpha_n x_n = 0)$

$\operatorname{im}(\varphi_r) = \mathbb{R}^2$ $\quad \ker(\varphi_r) = \{\vec{0}_2\}$

命题3.3 设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性

则 (i) φ 是满射 $\Longleftrightarrow \operatorname{im}(\varphi) = \mathbb{R}^m$

(ii) φ 是单射 $\Longleftrightarrow \ker(\varphi) = \{\vec{0}_n\}$

证: (i) 由满射的定义

(ii) "$\Rightarrow$" 因为 φ 线性, 所以 $\varphi(\vec{0}_n) = \vec{0}_m$

因为 φ 单射. 所以 $\varphi^{-1}(\{\vec{0}_m\}) = \{\vec{0}_n\}$

$\ker(\varphi) = \varphi^{-1}(\{\vec{0}_m\})$.

"$\Leftarrow$" 设 $\vec{u}, \vec{v} \in \mathbb{R}^n$ 且 $\varphi(\vec{u}) = \varphi(\vec{v})$

则 $\varphi(\vec{u}) - \varphi(\vec{v}) = \vec{0}_m$

$\varphi(\vec{u} - \vec{v}) = \vec{0}_m \Longrightarrow \vec{u} - \vec{v} \in \ker(\varphi)$

$\because \ker(\varphi) = \{\vec{0}_n\}$ $\quad \therefore \vec{u} - \vec{v} = \vec{0}_n \Rightarrow \vec{u} = \vec{v}$

φ 是单射 $\boxtimes$

例 验证 φ_r 是单射

设 $\varphi_r\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 则

$\begin{cases} x\cos\theta + y\sin\theta = 0 & \cdots ① \\ -x\sin\theta + y\cos\theta = 0 & \cdots ② \end{cases}$

$① \times \sin\theta + ② \times \cos\theta \Longrightarrow y = 0 \Longrightarrow x = 0$.

于是 φ_r 是单射.

定理3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射

则 $\dim(\ker(\varphi)) + \dim(\mathrm{im}(\varphi)) = n$

(The second contact)

证: ~~如果 φ 是单射~~. 设 $\vec{v_1}, \dots \vec{v_d}$ 是 $\ker(\varphi)$ 的一组基. 由基扩充定理. $\mathbb{R}^n$ 有基底 $\vec{v_1}, \dots \vec{v_d}, \vec{v_{d+1}}, \dots \vec{v_n}$

注意 $\ker(\varphi) = \{\vec{0_n}\}$ 时. 取 $\mathbb{R}^n$ 中任意基底即可.

断言1 $\varphi(\vec{v_{d+1}}), \dots \varphi(\vec{v_n})$ 线性无关

~~该~~断言1之证明 设 $\alpha_{d+1}, \dots, \alpha_n \in \mathbb{R}$

使得

$$\alpha_{d+1}\varphi(\vec{v_{d+1}}) + \dots + \alpha_n \varphi(\vec{v_n}) = \vec{0_m}$$

则 $\varphi(\alpha_{d+1}\vec{v_{d+1}} + \dots + \alpha_n \vec{v_n}) = \vec{0_m}$

于是 $\alpha_{d+1}\vec{v_{d+1}} + \dots + \alpha_n \vec{v_n} \in \ker(\varphi)$

因为 $\vec{v_1}, \dots \vec{v_d}$ 是 $\ker(\varphi)$ 的基

所以 $\exists$ ~~$\alpha_{d+1}, \dots \alpha_n \in$~~ $\alpha_1, \dots \alpha_d \in \mathbb{R}$

使得

$$\alpha_1\vec{v_1} + \dots + \alpha_d \vec{v_d} = \alpha_{d+1}\vec{v_{d+1}} + \dots + \alpha_n \vec{v_n}$$

$$\alpha_1\vec{v_1} + \dots + \alpha_d \vec{v_d} + (-\alpha_{d+1})\vec{v_{d+1}} + \dots + (-\alpha_n)\vec{v_n} = \vec{0_n}$$

$$\Rightarrow \alpha_1 = \dots = \alpha_d = \alpha_{d+1} = \dots = \alpha_n = 0$$

断言1成立

断言2 $\varphi(\vec{v_{d+1}}), \dots \varphi(\vec{v_n})$ 是 $\mathrm{im}(\varphi)$ 的一组基

断言2之证明 由断言1. 只要证

$$\mathrm{im}(\varphi) = \langle \varphi(\vec{v_{d+1}}), \dots \varphi(\vec{v_n}) \rangle$$ 即可

显然 $\langle \varphi(\vec{v_{d+1}}), \dots \varphi(\vec{v_n}) \rangle \subset \mathrm{im}(\varphi)$

反之设 $\vec{w} \in \mathrm{im}(\varphi)$

$\exists \vec{u} \in V$ 使得 $\varphi(\vec{u}) = \vec{w}$

设 $\vec{u} = \beta_1\vec{v_1} + \dots + \beta_d\vec{v_d} + \beta_{d+1}\vec{v_{d+1}} + \dots + \beta_n\vec{v_n}$

$(\beta_1, \dots \beta_n \in \mathbb{R})$

$$\varphi(\vec{u}) = \beta_1\varphi(\vec{v_1}) + \dots + \beta_d\varphi(\vec{v_d}) + \beta_{d+1}\varphi(\vec{v_{d+1}}) + \dots + \beta_n\varphi(\vec{v_n})$$

$$= \beta_{d+1}\varphi(\vec{v_{d+1}}) + \dots + \beta_n\varphi(\vec{v_n}) \quad (\because \vec{v_1}, \dots \vec{v_d} \in \ker(\varphi))$$

$$\Rightarrow \vec{w} \in \langle \varphi(\vec{v_{d+1}}), \dots \varphi(\vec{v_n}) \rangle$$

断言2成立

由此可知 $\dim \operatorname{im}(\varphi) = n-d$ ☒

~~推理~~

推论3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性

(i) φ 是满射 $\Rightarrow n \geq m$

(ii) φ 是单射 $\Rightarrow n \leq m$

(iii) φ 是双射 $\Rightarrow n = m$

证: (i) φ 是满射 $\Rightarrow \dim \operatorname{im}(\varphi) = m$

由 $\dim \ker(\varphi) + m = n$

得出 $m \leq n$

(ii) φ 是单射 $\underset{\text{命题3.3}}{\Rightarrow} \ker(\varphi) = \{\vec{0}_n\}$

$\Rightarrow \dim \ker(\varphi) = 0$

$\Rightarrow \dim \operatorname{im}(\varphi) = n$ (定理3.2)

又因为 $\operatorname{im}(\varphi) \subset \mathbb{R}^m \Rightarrow$ ~~dim~~ $n \leq m$

(iii) 由 (i)(ii) 直接可得

推论3.2 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ 线性

则以下三个结论等价

(i) φ 是双射 (ii) φ 是单射 (iii) φ 是满射

证 (i) $\Rightarrow$ (ii) 显然.

(ii) $\Rightarrow$ (iii) φ 是单射 $\Rightarrow \ker(\varphi) = \{\vec{0}_n\}$

$\underset{\text{定理3.2}}{\Rightarrow} \dim \operatorname{im}(\varphi) = n \underset{\text{命题1.4}}{\Rightarrow}$ ~~dim~~ $\operatorname{im}(\varphi) = \mathbb{R}^n$

(iii) $\Rightarrow$ (i) $\operatorname{im}(\varphi) = \mathbb{R}^n \Rightarrow$ ~~dim~~ $\dim(\operatorname{im}(\varphi)) = n$

$\Rightarrow \ker(\varphi) = \{\vec{0}_n\}$

$\underset{\text{定理3.2}}{\Longrightarrow}$ ~~dim~~ $\dim \ker(\varphi) = 0 \Rightarrow \ker(\varphi) = \{\vec{0}_n\}$

$\underset{\text{命题3.3}}{\Longrightarrow} \varphi$ 单射 $\Rightarrow \varphi$ 是双射.

例: φ_r 是满射

§3.3 线性映射的矩阵表示

$\mathbb{R}^n$ 标准基 $\vec{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}_{n-1}, \vec{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}_{n-2} \cdots \vec{e}^{(n)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \}_{n-1}$

$\mathbb{R}^m$ 标准基 $\vec{\varepsilon}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}_{m-1}, \vec{\varepsilon}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}_{m-2}$

$\cdots \vec{\varepsilon}^{(m)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \}_{m-1}$

设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的

设 $\varphi(\vec{e}^{(j)}) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ $j=1,2,\cdots n$

定义：$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

$= (\varphi(\vec{e}^{(1)}), \varphi(\vec{e}^{(2)}), \cdots \varphi(\vec{e}^{(n)}))$

称为 φ 在标准基下的矩阵表示

注：$\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 其矩阵表示 $A \in \mathbb{R}^{m\times n}$

注：在本章中只考虑标准基．于是 φ 在标准基下的矩阵表示 简称为 φ 的矩阵（表示）

引理 3.2 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性的，

$A \in \mathbb{R}^{m\times n}$ 是 φ 的矩阵

则 $\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = x_1 \vec{A}^{(1)} + \cdots + x_n \vec{A}^{(n)}$

证：令 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

则 $\vec{x} = x_1 \vec{e}^{(1)} + \cdots + x_n \vec{e}^{(n)}$

$\varphi(\vec{x}) = x_1 \varphi(\vec{e}^{(1)}) + \cdots + x_n \varphi(\vec{e}^{(n)})$

$= x_1 \vec{A}^{(1)} + \cdots + x_n \vec{A}^{(n)}$

例：φ_0 的矩阵表示是 $O_{m\times n}$

φ_I 的矩阵是 $(\vec{e}^{(1)}, \cdots \vec{e}^{(n)}) = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}$

φ_e 的矩阵是 $\begin{pmatrix} \vec{e}^{(1)} & \cdots & \vec{e}^{(n)} \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{pmatrix} \}m-n$

$= \begin{pmatrix} E_n \\ O_{(m-n)\times n} \end{pmatrix}$

φ_p 的矩阵

$m\left\{\begin{pmatrix} 1 & 0 & & 0 & \cdots & 0 \\ 0 & 0 & & 0 & & \vdots \\ 0 & 0 & \ddots & 0 & & \vdots \\ 0 & 0 & & 1 & & 0 \end{pmatrix}\right.$

$= (E_m, O_{m\times(n-m)})$

f: $(\alpha_1, \alpha_2, \cdots \alpha_n)$

φ_r: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

记号 $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ 是从 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的所有线性映射的集合.

定理 3.2 $\Phi: \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}^{m\times n}$

$\varphi \longmapsto \varphi$ 的矩阵

是双射

证: 先证 Φ 是满射.

设 $A \in \mathbb{R}^{m\times n}$ 定义:

$$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$$

令 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

$$\varphi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_n + \cdots + a_{mn}x_n \end{pmatrix}$$

由引理 3.1, $\varphi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$$\Phi(\varphi) = (\varphi(\vec{e}^{(1)}), \cdots \varphi(\vec{e}^{(n)})) = (\vec{A}^{(1)}, \cdots \vec{A}^{(n)}) = A$$

于是 Φ 是满射

设 ~~$\Phi, \Psi \in \mathbb{R}^{m\times n}$~~ $\varphi, \psi \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ (10)

且 $\Phi(\varphi) = \Phi(\psi) = A$

设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ 由引理 3.2

$$\varphi(\vec{x}) = x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$$

$$\psi(\vec{x}) = x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$$

于是 $\varphi = \psi$. ∎

记号 $\varphi_A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, $A \in \mathbb{R}^{m\times n}$

则 A 代表 φ_A 的矩阵.

应用 1. 确定线性映射的核与像

命题 3.4 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射, $A \in \mathbb{R}^{m\times n}$ 是其矩阵

则 (i) $\mathrm{im}(\varphi) = V_c(A)$

(ii) $\ker(\varphi)$ 是齐次线性方程组

$$x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)} = \vec{0}_m$$

的解空间.

证: (i)

$$\operatorname{im}(\varphi)=\left\{\varphi\left(\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\right)\,\middle|\, x_1,\dots x_n\in\mathbb{R}\right\}$$

$$=\left\{x_1\vec{A}^{(1)}+\dots+x_n\vec{A}^{(n)}\,\middle|\,x_1,\dots,x_n\in\mathbb{R}\right\}$$

$$=\langle\vec{A}^{(1)},\dots,\vec{A}^{(n)}\rangle=V_c(A).$$

(ii) $$\ker(\varphi)=\left\{\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\,\middle|\,\varphi\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\vec{O}_m\right\}$$

$$=\left\{\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\,\middle|\,x_1\vec{A}^{(1)}+\dots+x_n\vec{A}^{(n)}=\vec{O}_m\right\}\quad\boxtimes$$

例 设 $A=\begin{pmatrix}0&1&2\\3&4&5\end{pmatrix}$ 是线性映射

$\varphi:\ \mathbb{R}^3\to\mathbb{R}^2$ 的矩阵

计算 $\ker(\varphi)$ 和 $\operatorname{im}(\varphi)$ 的基底

解: 考虑方程组

$$x_1\begin{pmatrix}0\\3\end{pmatrix}+x_2\begin{pmatrix}1\\4\end{pmatrix}+x_3\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

即 $$\begin{cases}3x_1+4x_2+5x_3=0\\ \qquad\quad x_2+2x_3=0\end{cases}$$

$$\begin{cases}x_2=-2x_3\\ x_1=\frac{1}{3}(3x_3)=x_3\end{cases}$$

通解 $$\begin{cases}x_1=t\\x_2=-2t\\x_3=t\end{cases}$$

$$\ker(\varphi)=\left\langle\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rangle$$

$\operatorname{rank}(A)=2.\qquad \operatorname{im}(\varphi)=\langle\underbrace{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}}_{\text{基底}}\rangle$

应用II 确定齐次线性方程组解空间的维数

定理3.3 设

$$(H):\begin{cases}a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=0\\a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=0\\ \qquad\cdots\\a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=0\end{cases}$$

设 $A=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\&\cdots&\\a_{m1}&\cdots&a_{mn}\end{pmatrix}$ V_H 是 (H) 的解空间

则 $\dim V_H+\operatorname{rank}(A)=n$

(The third contact).

证：由命题 3.4

$\ker(\varphi) = V_H$， $\operatorname{im}(\varphi) = V_c(A)$

$\dim\ker(\varphi) = \dim V_H$， $\dim\operatorname{im}(\varphi) = \operatorname{rank}(A)$

由定理3.1

$\dim V_H + \operatorname{rank}(A) = n$ ▨

例：计算

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_4 = 0 \end{cases}$$ 的维数

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

$\operatorname{rank}(A) = 2 \Rightarrow$ 解空间维数 $= 4 - 2 = 2$

补充

例 P61. 习题4 设

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}$$

试用平面上 n 条直线的几何性质给出 ⑫

$\operatorname{rank}(A) = \operatorname{rank}(B)$

的条件.

证：由命题2.2

$\operatorname{rank}(A) = \operatorname{rank}(A^t)$

$\operatorname{rank}(B) = \operatorname{rank}(B^t)$

设 l_i：$\alpha_i x + \beta_i y = \gamma_i$ 的直线

$$(L) \begin{cases} \alpha_1 x + \beta_1 y = \gamma_1 \\ \vdots \\ \alpha_n x + \beta_n y = \gamma_n \end{cases}$$

则 A^t 是 (L) 的系数矩阵，B^t 是 (L) 的增广矩阵

$\operatorname{rank}(A^t) = \operatorname{rank}(B^t) \iff$ (L) 有共同解

即 $l_1, \ldots l_n$ 有公共交点，