§4 矩阵的运算

§4.1 矩阵的加法与数乘

定义 设 $A, B \in \mathbb{R}^{m\times n}$. $\alpha \in \mathbb{R}$

$$A = (a_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots n}} \qquad B = (b_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots, n}}$$

矩阵 A 与 B 的和. 记为 $A+B$

定义为 $m\times n$ 阶矩阵 $C = (c_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots n}}$

其中 $c_{ij} = a_{ij} + b_{ij}$

α 与 A 的数乘. 记为 αA, 定义为

$m\times n$ 阶矩阵 $C = (c_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots m}}$

$$c_{ij} = \alpha a_{ij}$$

例 设 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

计算 $C = -A + 2B$

得 $C = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} + \begin{pmatrix} 10 & 12 \\ 14 & 16 \end{pmatrix}$

$= \begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix}$.

注: 由实数的运算规律可知:

矩阵的加法满足

交换律. 结合律.

$$A + O_{m\times n} = A$$

$$A + (-A) = O_{m\times n}$$

矩阵的数乘满足 $\forall \alpha, \beta \in \mathbb{R}$. $A \in \mathbb{R}^{m\times n}$

$(\alpha\beta) A = \alpha(\beta A) = \beta(\alpha A)$

和对 $\forall A, B \in \mathbb{R}^{m\times n}$

$$\alpha(A+B) = \alpha A + \alpha B$$

$$(\alpha+\beta) A = \alpha A + \beta A.$$

命题 4.1 设 $\alpha \in \mathbb{R}$, $A, B \in \mathbb{R}^{m\times n}$

则 (i) $(A+B)^t = A^t + B^t$

(ii) $(\alpha A)^t = \alpha A^t$

证 (i). 设 $A = (a_{ij})_{\substack{i=1,\cdots,m\\ j=1,\cdots n}}$, $B = (b_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots n}}$

则 $A^t = (a'_{ji})_{\substack{j=1,\cdots n\\ i=1,\cdots m}}$ $B^t = (b'_{ji})_{\substack{j=1,\cdots n\\ i=1,\cdots m}}$

$a'_{ji} = a_{ij}$ $b'_{ji} = b_{ij}$

$A^t+B^t=(a'_{ji}+b'_{ji})_{\substack{j=1\cdots n\\ i=1\cdots m}}$ 第j行第i列的元素是 $a_{ij}+b_{ij}$

$A+B=(a_{ij}+b_{ij})_{\substack{i=1\cdots n\\ j=1\cdots m}}$ $A+B$

由转置的定义. $(A+B)^t=A^t+B^t$

(ii) 类似. $\square$

注

$A=\begin{pmatrix}\vec{A}_1\\ \vdots\\ \vec{A}_m\end{pmatrix}$, $B=\begin{pmatrix}\vec{B}_1\\ \vdots\\ \vec{B}_m\end{pmatrix}$

$=(\vec{A}^{(1)},\cdots\vec{A}^{(n)})$ $=(\vec{B}^{(1)},\cdots\vec{B}^{(n)})$

$A+B=\begin{pmatrix}\vec{A}_1+\vec{B}_1\\ \vdots\\ \vec{A}_m+\vec{B}_m\end{pmatrix}=(\vec{A}^{(1)}+\vec{B}^{(1)},\cdots,\vec{A}^{(n)}+\vec{B}^{(n)})$

类似地 $\forall\alpha\in\mathbb{R}$

$\alpha A=\begin{pmatrix}\alpha\vec{A}_1\\ \vdots\\ \alpha\vec{A}_m\end{pmatrix}=(\alpha\vec{A}^{(1)},\cdots\alpha\vec{A}^{(n)})$

定义: 设 $\varphi,\psi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$

$\varphi+\psi:\ \mathbb{R}^n\longrightarrow\mathbb{R}^m$
$\vec{x}\longmapsto\varphi(\vec{x})+\psi(\vec{x})$

设 $\alpha\in\mathbb{R}$

$\alpha\varphi:\ \mathbb{R}^n\longrightarrow\mathbb{R}^m$
$\vec{x}\longmapsto\alpha\varphi(\vec{x})$

验证: $\varphi+\psi$, $\alpha\varphi$ ~~都是线性映射~~

验证: $\varphi+\psi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$

$\forall\vec{u},\vec{v}\in\mathbb{R}^n$. $\alpha,\beta\in\mathbb{R}$

$(\varphi+\psi)(\alpha\vec{u}+\beta\vec{v})$

$=\varphi(\alpha\vec{u}+\beta\vec{v})+\psi(\alpha\vec{u}+\beta\vec{v})$ [映射加法定义]

$=\alpha\varphi(\vec{u})+\beta\varphi(\vec{v})+\alpha\psi(\vec{u})+\beta\psi(\vec{v})$ [线性映射定义]

$=\alpha(\varphi+\psi)(\vec{u})+\beta(\varphi+\psi)(\vec{v})$ [映射加法定义]

于是 $\varphi+\psi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$

同理可验证 $\alpha\varphi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$

问题：设 $\varphi, \psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, $\alpha \in \mathbb{R}$

设 φ, ψ 的矩阵分别为 A, B.

$\varphi+\psi$, $\alpha\varphi$ 的矩阵是什么？

设 C 是 $\varphi+\psi$ 的矩阵

$$C = \left((\varphi+\psi)(\vec{e}^{(1)}), \cdots (\varphi+\psi)(\vec{e}^{(n)})\right)$$

$$= \left(\varphi(\vec{e}^{(1)})+\psi(\vec{e}^{(1)}), \cdots, \varphi(\vec{e}^{(n)})+\psi(\vec{e}^{(n)})\right)$$

$$= \left(\vec{A}^{(1)}+\vec{B}^{(1)}, \cdots, \vec{A}^{(n)}+\vec{B}^{(n)}\right)$$

$$= A+B$$

同理可证：$\alpha\varphi$ 的矩阵是 αA.

注：设 $A \in \mathbb{R}^{m\times n}$, $\varphi_A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ 是以 A 为矩阵的线性映射

则 $\varphi_A + \varphi_B = \varphi_{A+B}$

$$\alpha\varphi_A = \varphi_{\alpha A}$$

§4.2 矩阵的乘法

设 ~~$\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$~~,

设 $\varphi \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^m)$, ~~其~~

$\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\psi"] \arrow[dr, dashed, "\varphi\circ\psi"'] & \mathbb{R}^s \arrow[d, "\varphi"] \\
 & \mathbb{R}^m
\end{tikzcd}$$

验证 $\varphi\circ\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

设 $\alpha, \beta \in \mathbb{R}$.
$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\varphi\circ\psi(\alpha\vec{u}+\beta\vec{v}) = \varphi(\psi(\alpha\vec{u}+\beta\vec{v}))$$

$$= \varphi(\alpha\psi(\vec{u})+\beta\psi(\vec{v}))$$

$$= \alpha\varphi(\psi(\vec{u}))+\beta\varphi(\psi(\vec{v})) = \alpha\varphi\circ\psi(\vec{u})+\beta\varphi\circ\psi(\vec{v})$$

于是 $\varphi\circ\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

问题 设 φ 的矩阵是 $A \in \mathbb{R}^{m\times s}$

ψ ⋯⋯ $B \in \mathbb{R}^{s\times n}$

$\varphi\circ\psi$ 是矩阵是什么？

设 $\vec{e}^{(1)},\dots,\vec{e}^{(n)}$ 是 $\mathbb{R}^n$ 的标准基

$\vec{\xi}^{(1)},\dots,\vec{\xi}^{(s)}$ 是 $\mathbb{R}^s$ 的标准基

$\vec{\varepsilon}^{(1)},\dots,\vec{\varepsilon}^{(m)}$ 是 $\mathbb{R}^m$ 的 - - -

~~$\varphi\circ\psi$~~ 设 $C\in\mathbb{R}^{m\times n}$ 是 $\varphi\circ\psi$ 的矩阵表示，

$$C=\left(\varphi\circ\psi(\vec{e}^{(1)}),\ \dots,\ \varphi\circ\psi(\vec{e}^{(n)})\right)$$

$$=\left(\varphi(\psi(\vec{e}^{(1)})),\ \dots,\ \varphi(\psi(\vec{e}^{(n)}))\right)$$

$$=\left(\varphi(\vec{B}^{(1)}),\ \dots\ \varphi(\vec{B}^{(n)})\right)$$

设 $B=(b_{kj})_{\substack{k=1,\dots s\\ j=1,\dots n}}$，

$$C=\left(\varphi\begin{pmatrix}b_{11}\\ b_{21}\\ \vdots\\ b_{s1}\end{pmatrix},\ \dots\ \varphi\begin{pmatrix}b_{1n}\\ b_{2n}\\ \vdots\\ b_{sn}\end{pmatrix}\right)$$

$$=\left(\varphi\Big(\sum_{k=1}^{s}b_{k1}\vec{\xi}_k\Big),\ \dots,\ \varphi\Big(\sum_{k=1}^{s}b_{kn}\vec{\xi}_k\Big)\right)$$

$$=\left(\sum_{k=1}^{s}b_{k1}\varphi(\vec{\xi}_k),\ \dots,\ \sum_{k=1}^{s}b_{kn}\varphi(\vec{\xi}_k)\right)$$

$$=\left(\sum_{k=1}^{s}b_{k1}\vec{A}^{(k)},\ \dots,\ \sum_{k=1}^{s}b_{kn}\vec{A}^{(k)}\right)$$

C 中第 j 列是 $\sum_{k=1}^{s}b_{kj}\vec{A}^{(k)}$，$j=1,\dots n$

设 $\vec{A}^{(k)}=\begin{pmatrix}a_{1k}\\ \vdots\\ a_{mk}\end{pmatrix}$，$k=1,2,\dots s$

则 C 中第 i 行第 j 列的元素

$$c_{ij}=\sum_{k=1}^{s}b_{kj}a_{ik}=\sum_{k=1}^{s}a_{ik}b_{kj}$$

定义：设 $A\in\mathbb{R}^{m\times s}$，$B\in\mathbb{R}^{s\times n}$

A 与 B 乘积 C 是 $m\times n$ 阶矩阵

令 $A=(a_{ik})_{\substack{i=1,\dots m\\ k=1,\dots s}}$　$B=(b_{kj})_{\substack{k=1,\dots s\\ j=1,\dots n}}$

$C=(c_{ij})_{\substack{i=1,\dots m\\ j=1,\dots n}}$

则 $c_{ij}=\sum_{k=1}^{s}a_{ik}b_{kj}$　$\begin{matrix}i=1,\dots m\\ j=1,\dots n.\end{matrix}$

记 C 为 AB.

~~解~~.

注：只有当 A 的列数 = B 的行数时
AB 才有定义

注 设 $A, B \in \mathbb{R}^{n\times n}$.
$AB = BA$ 不一定成立

例：设 $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$，$B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$$AB=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$BA=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

注 $A \neq O_{m\times s}$, $B\neq O_{s\times n}$.
AB 可能是 $O_{m\times n}$

例 设 $A=(\alpha_1,\cdots\alpha_n)_{1\times n}$　$B=\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}_{n\times 1}$

计算 AB　BA

解：
AB 是 1×1 的矩阵 $\alpha_1\beta_1+\cdots+\alpha_n\beta_n$
BA 是 $n\times n$ 的矩阵

$$\begin{pmatrix}\alpha_1\beta_1 & \alpha_2\beta_1 & \cdots & \alpha_n\beta_1\\ \alpha_1\beta_2 & \alpha_2\beta_2 & & \alpha_n\beta_2\\ \vdots & \vdots & & \vdots\\ \alpha_1\beta_n & \alpha_n\beta_n & & \alpha_n\beta_n\end{pmatrix}.$$

矩阵乘法的向量版

设 $A=(a_{ik})_{\substack{i=1,\cdots m\\k=1,\cdots s}}$　$B=(b_{kj})_{\substack{k=1,\cdots s\\j=1\cdots m}}$

$$C=AB=(c_{ij})_{\substack{i=1,\cdots m\\j=1\cdots n}}$$

$$c_{ij}=\sum_{k=1}^{s}a_{ik}b_{kj}=(a_{i1},\cdots a_{is})\begin{pmatrix}b_{1j}\\ \vdots\\ b_{sj}\end{pmatrix}$$

$$=\vec{A_i}\ \vec{B}^{(j)}$$

$$C=\begin{pmatrix}\vec{A}_1\vec{B}^{(1)}, & \vec{A}_1\vec{B}^{(2)}, & \cdots & \vec{A}_1\vec{B}^{(n)}\\ \vec{A}_2\vec{B}^{(1)}, & \vec{A}_2\vec{B}^{(2)}, & \cdots & \vec{A}_2\vec{B}^{(n)}\\ \vec{A}_m\vec{B}^{(1)} & \vec{A}_m\vec{B}^{(2)} & & \vec{A}_m\vec{B}^{(n)}\end{pmatrix}$$

$$=\left(\vec{A}_i\vec{B}^{(j)}\right)_{\substack{i=1,2\cdots m\\j=1\cdots n}}.$$

例 设 $A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$ ~~B~~ 计算

AA^t

$$AA^t=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$$

$$=\begin{pmatrix}14&32\\32&77\end{pmatrix}$$

例 设 $A=(a_{ij})_{m\times n}$. $\vec{x}=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$, $\vec{y}=(y_1,\cdots y_m)$

计算 $A\vec{x}$ 和 $\vec{y}A$

$$A\vec{x}=A\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\left(\vec{A}_i\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\right)_{i=1,\cdots m}$$

$$=\begin{pmatrix}a_{11}x_1+\cdots+a_{1n}x_n\\\cdots\\a_{m1}x_1+\cdots+a_{mn}x_n\end{pmatrix}$$

$$\vec{y}\cdot A=\left((y_1\cdots y_m)\vec{A}^{(j)}\right)_{j=1,\cdots n}=$$

$$(y_1a_{11}+\cdots+y_ma_{m1},\ \cdots\ y_1a_{1n}+\cdots+y_ma_{mn})$$

注:

$$(L)\quad\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\\cdots\\a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m\end{cases}$$

设 $A=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\&\cdots&&\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}$

向量版 $x_1\vec{A}^{(1)}+\cdots+x_n\vec{A}^{(n)}=\vec{b}$. $\vec{b}=\begin{pmatrix}b_1\\\vdots\\b_n\end{pmatrix}$

矩阵版 $A\vec{x}=\vec{b}$ $\vec{x}=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$

齐次 $A\vec{x}=\vec{0}$

设 φ 的矩阵表示为 A

$$\varphi(\vec{x})=x_1\varphi(\vec{e}^{(1)})+\cdots+x_n\varphi(\vec{e}^{(n)})$$

$$=x_1\vec{A}^{(1)}+\cdots+x_n\vec{A}^{(n)}=A\vec{x}$$

~~设~~ 记号化简. 设 $A\in\mathbb{R}^{m\times n}$. $\vec{x}=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$

(L) $A\vec{x}=\vec{b}$. ~~其~~ $\vec{b}=\begin{pmatrix}b_1\\\vdots\\b_m\end{pmatrix}$

(H) $A\vec{x}=\vec{0}_m$

$\varphi_A(\vec{x})=A\vec{x}$

例：设 $A\in\mathbb{R}^{m\times s}$，$B^{s\times n}$

证明 $AB = \underbrace{\begin{pmatrix}\vec{A}_1 B\\ \vdots \\ \vec{A}_m B\end{pmatrix}}_{\text{行向量版}} = \underbrace{(A\vec{B}^{(1)}, \cdots A\vec{B}^{(n)})}_{\text{列向量版}}$

设 $C = \begin{pmatrix}\vec{A}_1 B\\ \vdots\\ \vec{A}_m B\end{pmatrix} = (c_{ij})_{m\times n}$

$c_{ij} = \vec{A}_i B$ 中的第j个元素

$= \vec{A}_i \vec{B}^{(j)} = AB$ 中位于i行j列的元素

设 $D = (A\vec{B}^{(1)}, \cdots A\vec{B}^{(n)}) = (d_{ij})_{m\times n}$

$d_{ij} = A\vec{B}^{(j)}$ 中位于第i行的元素

$= \vec{A}_i \vec{B}^{(j)} = AB$ 中位于i行j列的元素

例：记号：设 $M = \begin{pmatrix}\lambda_1 & & O\\ & \lambda_2 & \\ O & & \ddots\ \lambda_n\end{pmatrix}_{n\times n}$，$\lambda_i\in\mathbb{R}$

称为 n 阶对角矩阵. 记为

$\mathrm{diag}(\lambda_1, \cdots \lambda_n)$. 当 $\lambda_1 = \cdots = \lambda_n =: \lambda$ 时

$\mathrm{diag}(\lambda_1, \cdots \lambda_n)$ 简记为 $\mathrm{diag}_n(\lambda)$

例 设 $A\in\mathbb{R}^{m\times n}$ 计算

$\mathrm{diag}(\lambda_1, \cdots \lambda_m) A$ 和 ~~diag~~ $A\,\mathrm{diag}(\lambda_1, \cdots \lambda_n)$

解：$\mathrm{diag}(\lambda_1, \cdots \lambda_m) A$ 的第i行是

$\underbrace{(0\cdots 0\ \lambda_i\ 0\cdots 0)}_{\vec{v}_i} A = (\vec{v}_i \vec{A}^{(1)}, \cdots \vec{v}_i \vec{A}^{(n)})$

$= (\lambda_i a_{i1}, \cdots \lambda_i a_{in})$

$= \lambda_i \vec{A}_i$

$\Rightarrow \mathrm{diag}(\lambda_1, \cdots \lambda_m) A = \begin{pmatrix}\lambda_1 \vec{A}_1\\ \vdots\\ \lambda_m \vec{A}_m\end{pmatrix}$

$A\,\mathrm{diag}(\lambda_1, \cdots \lambda_n)$ 的第j列

$= A\cdot(\lambda_j \vec{e}^{(j)}) = \begin{pmatrix}\vec{A}_1(\lambda_j \vec{e}^{(j)})\\ \vdots\\ \vec{A}_m(\lambda_j)\vec{e}^{(j)}\end{pmatrix}$

$= \lambda_j \begin{pmatrix}\vec{A}_1 \vec{e}^{(j)}\\ \vdots\\ \vec{A}_m \vec{e}^{(j)}\end{pmatrix} = \lambda_j \begin{pmatrix}a_{1j}\\ \vdots\\ a_{mj}\end{pmatrix} = \lambda_j \vec{A}^{(j)}$

$\Rightarrow A\,\mathrm{diag}(\lambda_1, \cdots \lambda_n) = (\lambda_1 \vec{A}^{(1)}, \cdots \lambda_n \vec{A}^{(n)})$

注：设 $A\in\mathbb{R}^{m\times n}$　$E_mA=A$　$AE_n=A$.

§4.3 矩阵乘法的规律.

I. 结合律：设 $A\in\mathbb{R}^{m\times s}$, $B\in\mathbb{R}^{s\times k}$, $C\in\mathbb{R}^{k\times n}$

$$(AB)C=A(BC)$$

验证：设 φ_A, φ_B, φ_C 是以 A, B, C 为矩阵的线性映射.

由映射的结合律可知

$$(\varphi_A\circ\varphi_B)\circ\varphi_C=\varphi_A\circ(\varphi_B\circ\varphi_C)$$

由矩阵乘法定义

$$(\varphi_A\circ\varphi_B)\circ\varphi_C=\varphi_{AB}\circ\varphi_C=\varphi_{(AB)\circ C}$$

$$\varphi_A\circ(\varphi_B\circ\varphi_C)=\varphi_A\circ\varphi_{BC}=\varphi_{A\circ(BC)}$$

由定理3.2　$(AB)\circ C=A(BC)$

II. 与加法的分配律

设 $A\in\mathbb{R}^{m\times s}$, $B, C\in\mathbb{R}^{s\times n}$.

则　$A(B+C)=AB+AC$

验证：φ_A, φ_B, φ_C 是以 A, B, C 为矩阵表示的线性映射.

$\forall\vec{x}\in\mathbb{R}^n$

$$\varphi_A\circ(\varphi_B+\varphi_C)(\vec{x})=\varphi_A((\varphi_B+\varphi_C)(\vec{x}))$$

$$=\varphi_A(\varphi_B(\vec{x})+\varphi_C(\vec{x}))=\varphi_A\circ\varphi_B(\vec{x})+\varphi_A\circ\varphi_C(\vec{x})$$

于是　$\varphi_A\circ(\varphi_B+\varphi_C)=\varphi_A\circ\varphi_B+\varphi_A\circ\varphi_C$

$$\varphi_A\circ(\varphi_B+\varphi_C)=\varphi_A\circ(\varphi_{B+C})=\varphi_{A(B+C)}$$

$$\varphi_A\circ\varphi_B+\varphi_A\circ\varphi_C=\varphi_{AB}+\varphi_{AC}=\varphi_{AB+AC}$$

由定理3.2　$A(B+C)=AB+AC$.

同理　设 $A, B\in\mathbb{R}^{m\times s}$　$C\in\mathbb{R}^{s\times n}$

$$(A+B)C=AC+BC.$$

III 与数乘的分配. 设 $\alpha\in\mathbb{R}$. $A\in\mathbb{R}^{m\times s}$, $B\in\mathbb{R}^{s\times n}$

$$A(\alpha B)=(\alpha A)B=\alpha(AB)$$

验证：略.

IV 不成立的规律 设 $A\in\mathbb{R}^{m\times s}$, $B\in\mathbb{R}^{s\times n}$

AB 有意义 BA 只有当 $n=m$ 才有意义

设 $A, B\in\mathbb{R}^{n\times n}$

AB 不一定等于 BA

设 $A\in\mathbb{R}^{m\times s}$, $B, C\in\mathbb{R}^{s\times n}$

$$AB=AC \qquad A\neq O_{m\times s}$$

不能推出 $B=C$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\neq\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

左右消去律不成立.

命题4.2. 设 $A\in\mathbb{R}^{m\times s}$, $B\in\mathbb{R}^{s\times n}$

则 $(AB)^t=B^tA^t$

证: 设 $A=(a_{ik})_{\substack{i=1,\dots,m\\k=1,\dots,s}}$, $A^t=(a'_{ki})_{\substack{k=1\cdots s\\i=1\cdots m}}$

$B=(b_{kj})_{\substack{k=1\cdots s\\j=1,\cdots n}}$ $B^t=(b'_{jk})_{\substack{j=1\cdots n\\k=1\cdots s}}$

且 $a'_{ki}=a_{ik}$, $b'_{jk}=b_{kj}$

证: $C=AB=(c_{ij})_{\substack{i=1,\cdots m\\j=1,\cdots n}}$

则 $C^t=(c'_{ji})_{\substack{j=1,\cdots,n\\i=1,\cdots m}}$ $c'_{ji}=c_{ij}$

设 $D=B^tA^t=(d_{ji})_{\substack{j=1,\cdots m\\i=1,\cdots n}}$

$\cancel{B^t=(b_{jk})_{\substack{j=1,\cdots n\\k=1\cdots s}} \quad A^t=(a_{ki})_{\substack{k=1,\cdots s\\i=1,\cdots m}}}$

$\cancel{d_{ji}=\sum_{k=1}^{s}b_{jk}a_{ki}}$

$$c_{ij}=\sum_{k=1}^{s}a_{ik}b_{kj}$$

$$d_{ji}=\sum_{k=1}^{s}b'_{jk}\cdot a'_{ki}=\sum_{k=1}^{s}a_{ik}b_{kj}$$

$$=c_{ij}=c'_{ji} \Rightarrow D=C^t. \quad \square$$

§4.3 矩阵秩的不等式
(The 4th contact)

定理 4.1 设 $A\in\mathbb{R}^{m\times s}$, $B\in\mathbb{R}^{s\times n}$

则 (i) $\operatorname{rank}(AB)\leq\min(\operatorname{rank}(A),\operatorname{rank}(B))$

(ii) $\operatorname{rank}(AB)\geq\operatorname{rank}(A)+\operatorname{rank}(B)-s$

证: 法1 利用线性映射

回忆: 设 ψ 线性映射, M 是其矩阵表示

$$\operatorname{im}\psi = V_c(M) \Rightarrow \dim(\operatorname{im}(\psi)) = \operatorname{rank}(M).$$

设 φ_A, φ_B 是以 A, B 为矩阵的线性映射

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_B"] \arrow[dr, "\varphi_A\circ\varphi_B = \varphi_{AB}"'] & \mathbb{R}^s \arrow[d, "\varphi_A"] \\ & \mathbb{R}^m \end{tikzcd}$$

令 $I_A = \operatorname{im}(\varphi_A)$

$I_B = \operatorname{im}(\varphi_B)$

$I_{AB} = \operatorname{im}(\varphi_{AB})$

则 (ii) 等价于

$$\dim I_A + \dim I_B - s \leq \dim I_{AB} \leq \min(\dim I_A, \dim I_B)$$

令 $K_A = \cancel{\dim}\ \ker(\varphi_A)$

$I_B, K_A \subset \mathbb{R}^s$ 是子空间. 于是 $I_B \cap K_A$ 也是 $\mathbb{R}^s$ 的子空间. 我们有 $I_B \cap K_A \subset I_B \subset \mathbb{R}^s$

设 $\vec{v}_1 \ldots \vec{v}_d$ 是 $I_B \cap K_A$ 的一组基.

由基扩充定理

$\vec{v}_1, \ldots \vec{v}_d, \vec{v}_{d+1}, \ldots \vec{v}_k$ 是 I_B 的一组基

$\vec{v}_1, \ldots \vec{v}_d, \vec{v}_{d+1} \ldots \vec{v}_k\ \vec{v}_{k+1}, \ldots \vec{v}_s$ 是 $\mathbb{R}^s$ 的一组基

注意到 $I_{AB} = \varphi_A \circ \varphi_B(\mathbb{R}^n) = \varphi_A(I_B)$. (*)

断言1 $\varphi_A(\vec{v}_{d+1}), \ldots \varphi_A(\vec{v}_k)$ 是 I_{AB} 的一组基

断言1 的证明 $\because \vec{v}_{d+1}, \ldots \vec{v}_k \in I_B$ 且 $I_{AB} = \varphi_A(I_B)$

$\therefore \varphi_A(\vec{v}_{d+1}), \ldots \varphi_A(\vec{v}_k) \in I_{AB}$

若 $\cancel{\exists}\ \alpha_{d+1}, \ldots \alpha_k \in \mathbb{R}$ 使得

$$\alpha_{d+1}\varphi_A(\vec{v}_{d+1}) + \cdots + \alpha_k\varphi_A(\vec{v}_k) = \vec{0}_m$$

则 $\varphi_A(\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_k\vec{v}_k) = \vec{0}_m$

于是 $\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_k\vec{v}_k \in K_A \cap I_B$

由此 $\exists\ \alpha_1 \ldots \alpha_d \in \mathbb{R}$

$$\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_k\vec{v}_k = \alpha_1\vec{v}_1 + \cdots + \alpha_d\vec{v}_d$$

$$(-\alpha_1)\vec{v}_1 + \cdots + (-\alpha_d)\vec{v}_d + \alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_k\vec{v}_k = \vec{0}_s$$

$\because \vec{v}_1, \ldots \vec{v}_k$ 线性无关

$\therefore \alpha_{d+1} = \cdots = \alpha_k = 0$

于是 $\varphi_A(\vec{v}_{d+1}), \ldots \varphi_A(\vec{v}_k)$ 线性无关

设 $\vec{w} \in I_{AB}$ 则 $\exists\ \vec{v} \in I_B$ 使得

$$\varphi_A(\vec{v}) = \vec{w}$$

$\exists\ \beta_1, \ldots \beta_k \in \mathbb{R}$

$$\vec{v} = \beta_1\vec{v}_1 + \cdots + \beta_d\vec{v}_d + \beta_{d+1}\vec{v}_{d+1} + \cdots + \beta_k\vec{v}_k$$

$$\vec{w} = \varphi_A(\vec{v}) = \beta_{d+1}\varphi_A(\vec{v}_{d+1}) + \cdots + \beta_k\varphi_A(\vec{v}_k).$$

即：$I_{AB} = \langle \varphi_A(\vec{v}_{d+1}), \cdots \varphi_A(\vec{v}_k) \rangle$

综上所述，断言1成立．从而 $\dim I_{AB} = k-d$

断言2. $k-d \le \dim I_A \le s-d$

断言2的证明：

$$I_B \subset \mathbb{R}^s$$

$$I_{AB} = \varphi_A(I_B) \subset \varphi_A(\mathbb{R}^s) = I_A$$

$$k-d = \dim I_{AB} \le \dim I_A$$

$$I_A = \langle \varphi_A(\vec{v}_1), \cdots \varphi_A(\vec{v}_d), \varphi_A(\vec{v}_{d+1}), \cdots \varphi_A(\vec{v}_s) \rangle$$

$$= \langle \varphi_A(\vec{v}_{d+1}), \cdots \varphi_A(\vec{v}_s) \rangle$$

$$\dim I_A \le s-d$$

由断言1 $\quad \dim I_{AB} = k-d \le k = \dim I_B$

由断言2 $\quad \dim I_{AB} \le \dim I_A$

$$\Rightarrow \quad \dim I_{AB} \le \min(\dim I_A, \dim I_B)$$

由断言2,1 $\quad \dim I_A + \dim I_B \le s-\cancel{d}+k = s+(k-d)$

$$= s + \dim I_{AB}$$

$$\Rightarrow \quad \dim I_{AB} \ge \dim I_A + \dim I_B - s$$

$\square$

注：(i) 的纯矩阵证明

设 $A = (a_{ik})_{\substack{i=1,\cdots,m \\ k=1,\cdots,s}} \quad B = (b_{kj})_{\substack{k=1,\cdots,s \\ j=1,\cdots,n}}$

设 $P = AB$

$$P = (A\vec{B}^{(1)}, \cdots, A\vec{B}^{(n)})$$

$$\vec{P}^{(j)} = A\vec{B}^{(j)} = b_{1j}\vec{A}^{(1)} + \cdots + b_{sj}\vec{A}^{(s)} \in V_c(A), \; j=1\cdots n$$

$$\Rightarrow \quad V_c(P) \subset V_c(A) \Rightarrow \operatorname{rank}(P) \le \operatorname{rank}(A)$$

$$P = \begin{pmatrix} \vec{A}_1 B \\ \vdots \\ \vec{A}_m B \end{pmatrix} \quad \vec{P}_i = \vec{A}_i B = b_{i1}\vec{B}_1 + \cdots + b_{is}\vec{B}_s$$

$$\in V_r(A), \; i=1\cdots m$$

$$V_r(P) \subset V_r(B) \Rightarrow \operatorname{rank}(P) \le \operatorname{rank}(B)$$

所以 (i) 成立

注 (ii) 也有纯矩阵的证明．

例：设 $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times n}$

如果 $AB = O_{m\times n}$

证明 $\quad \operatorname{rank}(A) + \operatorname{rank}(B) \le s$

证法1　利用定理 4.1 (ii)

$$\mathrm{rank}(AB) \geq \mathrm{rank}(A) + \mathrm{rank}(B) - s$$

(Sylvester 不等式)

$$0 \geq \mathrm{rank}(A) + \mathrm{rank}(B) - s$$

$$\Rightarrow \mathrm{rank}(A) + \mathrm{rank}(B) \leq s.$$

证法2　$AB = O_{m\times n}$

$$\Rightarrow A\vec{B}^{(1)} = \vec{O}_m, \ \cdots \ A\vec{B}^{(n)} = \vec{O}_m$$

设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}$

则 $\vec{B}^{(1)}, \cdots \vec{B}^{(n)} \in \mathrm{sol}(A\vec{x} = \vec{O}_m)$

$$V_c(B) = \langle \vec{B}^{(1)}, \cdots \vec{B}^{(n)} \rangle \subset \mathrm{sol}(A\vec{x} = \vec{O}_m)$$

$$\mathrm{rank}(B) = \dim V_c(B) \leq \dim \mathrm{sol}(A\vec{x} = \vec{O}_m) = s - \mathrm{rank}(A)$$

(定理3.1)

$$\mathrm{rank}(A) + \mathrm{rank}(B) \leq s$$

§5 方阵

§5.1 方阵的代数运算

记 $\mathbb{R}^{n\times n}$ 为 $M_n(\mathbb{R})$

设 $A, B, C \in M_n(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$.

加法:
$$A + B = B + A$$
$$A + (B + C) = (A + B) + C$$
$$A + O_{n\times n} = A$$
$$A + (-A) = O_{n\times n}$$

数乘
$$\alpha(\beta A) = \beta(\alpha A) = (\alpha\beta)A$$

乘法
$$A(BC) = (AB)C$$
$$AE_n = E_n A = A$$

分配律
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$$A(\alpha B) = (\alpha A)B = \alpha(AB)$$
$$\alpha(A+B) = \alpha A + \alpha B$$

为此我们称 $M_n(\mathbb{R})$ 是一个代数