第九周作业

1. $a,b,c\in\mathbb{R}$. $m,n\in\mathbb{Z}^+$ 求矩阵:

(1) $\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}^m$ (2) $\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix}^m_{n\times n}$

解: (1) 设 $A=\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}$ 则 $\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}^m=(E_3+A)^m=\sum_{k=0}^{m}\binom{m}{k}E^{m-k}\cdot A^k$

且 $A^0=E_3$. $A^1=A$. $A^2=\begin{pmatrix}0 & 0 & ab\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ $A^k=O_3$ $(k\geqslant 3)$

$\therefore$ 原式 $=A^0+\binom{m}{1}A^1+\binom{m}{2}A^2=\begin{pmatrix}1 & ma & mc+\frac{m(m-1)}{2}ab\\ 0 & 1 & mb\\ 0 & 0 & 1\end{pmatrix}$ (也可用数归法)

(2) 设 $J_n=\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix}_{n\times n}$ 即求 J_n^m

直观推测 $J_n^m=\begin{cases}\begin{pmatrix}\overbrace{0 \cdots 0}^{m个} & 1 & & 0\\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix}\Big\}m个 \quad \begin{pmatrix}O_{(n-m)\times m} & E_{n-m}\\ O_{m\times m} & O_{m\times(n-m)}\end{pmatrix} & (m<n)\\ O_n & (m\geqslant n)\end{cases}$

数学归纳法证明:

设 $m<n$: $m=1$ 时 $J_n^1=J_n$ 成立.

假设上述等式在 $m-1$ 时成立. 即 $J_n^{m-1}=\begin{pmatrix}\overbrace{0 \cdots 0}^{m-1} & 1 & & \\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix}\Big\}m-1$

考虑 $J_n^m=J_n\cdot J_n^{m-1}=\begin{pmatrix}0 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix}\cdot\begin{pmatrix}\overbrace{0 \cdots 0}^{m-1} & 1 & & \\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix}=\begin{pmatrix}\overbrace{0 \cdots 0}^{m} & 1 & & \\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix}\Big\}m$

$\therefore J_n^m=\begin{pmatrix}\overbrace{0 \cdots 0}^{m个} & 1 & & \\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix}\Big\}m个$ 对 $m<n$ 均成立.

$\therefore J_n^{n-1}=\begin{pmatrix}\overbrace{0 \cdots 0}^{n-1} & 1\\ & 0\\ & \vdots\\ & 0\end{pmatrix}\Big\}n-1$ 则 $J_n^n=\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix}\begin{pmatrix}0 \cdots 0 & 1\\ & 0\\ & \vdots\\ & 0\end{pmatrix}=O_n$

$\therefore J_n^m=O_n$ 对 $m\geqslant n$ 均成立.

2.（柯 P82 5） $H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ 求 $H^t \cdot H$

解： $H^t \cdot H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 4E_4.$

3（柯 P82. 9） 设 $A, B, C \in M_n(\mathbb{R})$ $ABC = O$ 则 $\operatorname{rank} A + \operatorname{rank} B + \operatorname{rank} C \leq 2n.$

Pf.（利用 Sylvester 不等式：$\forall\ M \in \mathbb{R}^{m\times s}$, $N \in \mathbb{R}^{s\times n}$ $\operatorname{rank} \overset{M}{A} + \operatorname{rank} \overset{N}{B} - s \leq \operatorname{rank} \overset{MN}{\cancel{AB}}$ ）

$\operatorname{rank}(A\cdot B) + \operatorname{rank} C - n \leq \operatorname{rank}(A\cdot B\cdot C) = 0$ （$\because AB \in \mathbb{R}^{m\times n}$. $C \in \mathbb{R}^{m\times n}$ ）

$\operatorname{rank} A + \operatorname{rank} B - n \leq \operatorname{rank} A\cdot B$

将上述两式相加得 $\operatorname{rank} AB + \operatorname{rank} C - n + \operatorname{rank} A + \operatorname{rank} B - n \leq \operatorname{rank} AB$

$\Rightarrow \operatorname{rank} A + \operatorname{rank} B + \operatorname{rank} C \leq 2n$

（另法：$\varphi_A : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_{BC} = \varphi_B \circ \varphi_C$ $I_{BC} = \operatorname{im} \varphi_{BC}$. $K_A = \ker \varphi_A$. $\because ABC = 0 \Rightarrow \varphi_A \circ \varphi_{BC} = 0 \Rightarrow I_{BC} \subseteq K_A$. $\therefore \dim(I_{BC}) \leq \dim(K_A) \Rightarrow \operatorname{rank} BC \leq n - \operatorname{rank} A$ 又 $\operatorname{rank} BC \geq \operatorname{rank} B + \operatorname{rank} C - n$ $\therefore$ ____ ）

4.（柯 P82.10） 求 $A = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{pmatrix}$ 的秩

（if $\operatorname{rank} B = 1$, $\operatorname{rank} C = 1$ then $\exists\ x_i \neq 0$ $y_j \neq 0$ $\Rightarrow A \neq O \Rightarrow \operatorname{rank} A = 1$ ）

if $\operatorname{rank} B = 0$ or $\operatorname{rank} C = 0$ then $A = O \Rightarrow \operatorname{rank} A = 1$.

解： $A = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot (y_1 \cdots y_n)$ 令 $B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{n\times 1}$ $C = (y_1 \cdots y_n) \in \mathbb{R}^{1\times n}$

则 $\operatorname{rank} B = \begin{cases} 0 & (x_1 = \cdots = x_n = 0) \\ 1 & \text{otherwise} \end{cases}$ $\operatorname{rank} C = \begin{cases} 0 & (y_1 = \cdots = y_n = 0) \\ 1 & \text{otherwise} \end{cases}$

$A = B\cdot C \Rightarrow \operatorname{rank} A \leq \min(\operatorname{rank} B, \operatorname{rank} C) \Rightarrow \operatorname{rank} A = \begin{cases} 0 & \text{otherwise} \\ 1 & \exists\ x_i \cdot y_j \neq 0\ (i, j \in \{1, \cdots, n\}) \end{cases}$

5（柯 P82. 11） $A = (a_{ij})$ 非退化对称矩阵 $\Rightarrow A^{-1}$ 也对称.

尽量不要操作矩阵内元素.
学会直接操作矩阵.

Pf. A 为对称矩阵 ie $a_{ij} = a_{ji}$ $\therefore A^t = A$

$\because \cancel{A\cdot A^t}$ $A\cdot A^{-1} = E$ $\therefore (A\cdot A^{-1})^t = (A^{-1})^t \cdot A^t = (A^{-1})^t \cdot A = E$

$\therefore (A^{-1})^t$ 也为 A 的逆. 由于非退化矩阵逆的唯一性 $\therefore (A^{-1})^t = A^{-1}$

$\therefore A^{-1}$ 也为对称矩阵. $((A^{-1})^t = (A^t)^{-1} = A^{-1})$

6. $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{p\times q}$ $C \in \mathbb{R}^{m\times q}$ $D \in \mathbb{R}^{p\times n}$, $(m,n,p,q \in \mathbb{Z}^+)$ 证明：

(1) $\operatorname{rank}\begin{pmatrix} A & O_{m\times q} \\ O_{p\times n} & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} O_{m\times q} & A \\ B & O_{p\times n} \end{pmatrix} = \operatorname{rank} A + \operatorname{rank} B$

(2) $\operatorname{rank}\begin{pmatrix} A & C \\ O_{p\times n} & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} C & A \\ B & O_{p\times n} \end{pmatrix} \geqslant \operatorname{rank} A + \operatorname{rank} B$

(3) $\operatorname{rank}\begin{pmatrix} A & O_{m\times q} \\ D & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} O_{m\times q} & A \\ B & D \end{pmatrix} \geqslant \operatorname{rank} A + \operatorname{rank} B$

解 (1) 设 $M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$. 不妨设 $\{\vec{A}^{(1)}, \cdots \vec{A}^{(r)}\}$ 构成 $V_c(A)$ 一组基. $\{\vec{B}^{(1)}, \cdots \vec{B}^{(s)}\}$ 构成 $V_c(B)$ 基.

设 $V_A = \langle \begin{pmatrix} \vec{A}^{(1)} \\ \vec{0} \end{pmatrix}, \cdots \begin{pmatrix} \vec{A}^{(r)} \\ \vec{0} \end{pmatrix} \rangle$ $V_B = \langle \begin{pmatrix} \vec{0} \\ \vec{B}^{(1)} \end{pmatrix}, \cdots \begin{pmatrix} \vec{0} \\ \vec{B}^{(s)} \end{pmatrix} \rangle$

下面证 $V_c(M) = V_A \oplus V_B$

对 $\forall\ \vec{v} \in V_c(M)$ $\vec{v} = \sum_{i=1}^{n} \alpha_i \begin{pmatrix} \vec{A}^{(i)} \\ \vec{0} \end{pmatrix} + \sum_{j=1}^{q} \beta_j \begin{pmatrix} \vec{0} \\ \vec{B}^{(j)} \end{pmatrix} = \sum_{i=1}^{r} \alpha_i' \begin{pmatrix} \vec{A}^{(i)} \\ \vec{0} \end{pmatrix} + \sum_{j=1}^{s} \beta_j' \begin{pmatrix} \vec{0} \\ \vec{B}^{(j)} \end{pmatrix}$

$\therefore V_c(M) \subseteq V_A + V_B$ 显然 $V_A + V_B \subseteq V_c(M)$ $\therefore V_c(M) = V_A + V_B$.

$\forall\ \vec{v} \in V_A \cap V_B$ $\vec{v} = \sum_{i=1}^{r} \alpha_i \begin{pmatrix} \vec{A}^{(i)} \\ \vec{0} \end{pmatrix} = \sum_{j=1}^{s} \beta_j \begin{pmatrix} \vec{0} \\ \vec{B}^{(j)} \end{pmatrix} \Rightarrow \sum_{i=1}^{r} \alpha_i \vec{A}^{(i)} = \vec{0}_m,\ \sum_{j=1}^{s} \beta_j \vec{B}^{(j)} = \vec{0}_p$

$\because \vec{A}^{(1)}, \cdots \vec{A}^{(r)}$ 线性无关 $\therefore \alpha_1 = \cdots = \alpha_r = 0$ 同理 $\beta_1 = \cdots = \beta_s = 0$ $\therefore \vec{v} = \vec{0}_{m+p}$.

$\therefore V_c(M) = V_A \oplus V_B$ 易证 $\begin{pmatrix} \vec{A}^{(1)} \\ \vec{0} \end{pmatrix}, \cdots \begin{pmatrix} \vec{A}^{(r)} \\ \vec{0} \end{pmatrix}$ 线性无关 $\therefore \dim V_A = r$ 同理 $\dim V_B = s$.

$\therefore \operatorname{rank} M = \dim V_c(M) = \dim V_A + \dim V_B = r + s = \operatorname{rank} A + \operatorname{rank} B$.

类似地 可证明 $\operatorname{rank}\begin{pmatrix} O & A \\ B & O \end{pmatrix} = \operatorname{rank} A + \operatorname{rank} B$.

(2). 设 $M_1 = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$, $M_2 = \begin{pmatrix} C & A \\ B & O \end{pmatrix}$ 则 $V_c(M_1) = \langle \begin{pmatrix} \vec{A}^{(1)} \\ \vec{0} \end{pmatrix}, \cdots \begin{pmatrix} \vec{A}^{(n)} \\ \vec{0} \end{pmatrix}, \begin{pmatrix} \vec{C}^{(1)} \\ \vec{B}^{(1)} \end{pmatrix}, \cdots \begin{pmatrix} \vec{C}^{(q)} \\ \vec{B}^{(q)} \end{pmatrix} \rangle$

$= V_c(M_2)$ $\therefore \operatorname{rank} M_1 = \operatorname{rank} M_2$.

不妨设 ~~$\begin{pmatrix} \vec{A}^{(1)} \\ \vec{0} \end{pmatrix}, \cdots \vec{A}^{(r)}$~~ $\vec{A}^{(1)}, \cdots \vec{A}^{(r)}$ 构成 $V_c(A)$ 一组基. $\vec{B}^{(1)}, \cdots \vec{B}^{(s)}$ 构成 $V_c(B)$ 一组基.

令 $V_A = \langle \begin{pmatrix} \vec{A}^{(1)} \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} \vec{A}^{(r)} \\ 0 \end{pmatrix} \rangle$ $V_B = \langle \begin{pmatrix} \vec{C}^{(1)} \\ \vec{B}^{(1)} \end{pmatrix}, \cdots \begin{pmatrix} \vec{C}^{(s)} \\ \vec{B}^{(s)} \end{pmatrix} \rangle$

设 $\beta_1 \begin{pmatrix} \vec{C}^{(1)} \\ \vec{B}^{(1)} \end{pmatrix} + \cdots + \beta_s \begin{pmatrix} \vec{C}^{(s)} \\ \vec{B}^{(s)} \end{pmatrix} = \vec{0}_{m+p} \Rightarrow \beta_1 \vec{B}^{(1)} + \cdots + \beta_s \vec{B}^{(s)} = \vec{0}_p \Rightarrow \beta_1 = \cdots = \beta_s = 0$

$\therefore \dim V_A = r$ $\dim V_B = s$ 且 $V_A \cap V_B = \vec{0}$ 显然 $V_A \oplus V_B \subseteq V_c(M_1)$

$\therefore \operatorname{rank} M_1 = \dim V_c(M_1) \geqslant \dim V_A + \dim V_B = \operatorname{rank} A + \operatorname{rank} B$

(3) 类似 (2) 证明. □.

矩阵相抵

Lemma1（打洞引理）

设 $A \in \mathbb{R}^{m\times n}$　$\exists\ P \in M_m(\mathbb{R})$，$Q \in M_n(\mathbb{R})$　s.t.

(i)　P = 若干m阶初等矩阵之积.

(ii)　Q = 若干 n阶 初等矩阵之积.

(iii)　$PAQ = \begin{pmatrix} E_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{pmatrix}$　其中 $r = \operatorname{rank} A$.

（左乘行变换，右乘列变换）

Thm1　$A \in M_n(\mathbb{R})$ 可逆 $\Longleftrightarrow$ A = 若干 n阶初等矩阵之积 $\Longleftrightarrow$ $\operatorname{rank} A = n$

Def1　$A, B \in \mathbb{R}^{m\times n}$　如果 $\exists$ 可逆矩阵 $P \in M_m(\mathbb{R})$，$Q \in M_n(\mathbb{R})$ s.t.

$PAQ = B$　则称 A 和 B 初等等价（相抵）　记为 $A \sim_e B$

注：相抵是等价关系.

Cor1 (1) 设 $A \in \mathbb{R}^{m\times n}$　$\operatorname{rank} A = r$　则　$A \sim_e \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$

(2)　$A, B \in \mathbb{R}^{m\times n}$　$A \sim_e B \Longleftrightarrow \operatorname{rank} A = \operatorname{rank} B$

(3).　$\mathbb{R}^{m\times n}/\sim_e = \left\{ \overline{O_{m\times n}},\ \overline{\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}} \;\middle|\; r = 1, 2, \cdots \min(m,n) \right\}$

矩阵求逆.

给定 $A \in M_n(\mathbb{R})$ 可逆　求 A^{-1}

(i)　设 $B = (A\ \ E_n)$

(ii)　$B \xrightarrow{\text{初等行变换}} (E_n\ \ C)$

则　$A^{-1} = C$.

矩阵分块.

设 $A \in \mathbb{R}^{m\times s}$　$B \in \mathbb{R}^{s\times n}$

$A = (A_{ik})_{\substack{i=1,\cdots p \\ k=1,\cdots h}}$　$A_{ik} \in \mathbb{R}^{m_i \times s_k}$

$B = (B_{kj})_{\substack{k=1,\cdots h \\ j=1,\cdots q}}$　$B_{kj} \in \mathbb{R}^{s_k \times n_j}$

其中 $m_1 + \cdots + m_p = m$，$s_1 + \cdots + s_h = s$

$n_1 + \cdots + n_q = n$

则 $AB = (C_{ij})_{\substack{i=1,\cdots p \\ j=1,\cdots q}}$　$C_{ij} = \sum_{k=1}^{h} A_{ik} \cdot B_{kj}$

例1. 设 $A=\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}_{n\times n}$ $B=\begin{pmatrix}0&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{pmatrix}_{n\times n}$ 求 A^{-1} 和 B^{-1}. $(n\geq 2)$

解: $(A\ E_n)=\left(\begin{array}{cccc|cccc}1&1&\cdots&1&1&0&\cdots&0\\0&1&\cdots&1&0&1&\cdots&0\\\vdots&\vdots&&\vdots&\vdots&\vdots&&\vdots\\0&0&\cdots&1&0&0&\cdots&1\end{array}\right)\xrightarrow[i=2,3\cdots n]{-(i)+(i-1)}\left(\begin{array}{cccc|cccc}1&0&\cdots&0&1&-1&\cdots&0\\0&1&\cdots&0&0&1&-1\cdots&0\\\vdots&&\ddots&\vdots&\vdots&&\ddots&-1\\0&\cdots&&1&0&\cdots&&1\end{array}\right)$

$\therefore A^{-1}=\begin{pmatrix}1&-1&&\\&1&\ddots&\\&&\ddots&-1\\&&&1\end{pmatrix}$

另法] $J_n^m=\begin{pmatrix}\overbrace{0\cdots0}^{m}&1&\ddots&\\&&&1\\&&&0\\&&&\vdots\\&&&0\end{pmatrix}$ $\Big\}m$ $\therefore A=E_n+J_n+J_n^2+\cdots+J_n^{n-1}$

$\because (E_n-J_n)(E_n+J_n+J_n^2+\cdots+J_n^{n-1})=E_n-J_n^n=E_n.$

$\therefore A^{-1}=E_n-J_n=\begin{pmatrix}1&-1&&\\&1&\ddots&\\&&\ddots&-1\\&&&1\end{pmatrix}$

$(B\ E_n)=\left(\begin{array}{cccc|cccc}0&1&\cdots&1&1&0&\cdots&0\\1&0&\cdots&1&0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&0&0&0&\cdots&1\end{array}\right)\xrightarrow[\text{第一行乘}\frac{1}{n-1}]{\text{各行均加到第一行}}\left(\begin{array}{cccc|cccc}1&1&\cdots&1&\frac{1}{n-1}&\frac{1}{n-1}&\cdots&\frac{1}{n-1}\\1&0&\cdots&1&0&1&\cdots&0\\\vdots&\vdots&&\vdots&\vdots&\vdots&&\vdots\\1&1&\cdots&0&0&0&\cdots&1\end{array}\right)$

$\xrightarrow[i=2,\cdots n]{-(1)+(i),\,-(i)}\left(\begin{array}{cccc|cccc}1&1&\cdots&1&\frac{1}{n-1}&\frac{1}{n-1}&\cdots&\frac{1}{n-1}\\0&1&\cdots&0&\frac{1}{n-1}&-\frac{n-2}{n-1}&\cdots&\frac{1}{n-1}\\\vdots&\vdots&\ddots&\vdots&&&&\\0&0&\cdots&1&\frac{1}{n-1}&\frac{1}{n-1}&\cdots&-\frac{n-2}{n-1}\end{array}\right)\xrightarrow[i=2,\cdots n]{-(i)+(1)}\left(\begin{array}{ccc|cccc}1&&&-\frac{n-2}{n-1}&\frac{1}{n-1}&\cdots&\frac{1}{n-1}\\&\ddots&&\frac{1}{n-1}&-\frac{n-2}{n-1}&\cdots&\frac{1}{n-1}\\&&&\vdots&\vdots&\ddots&\vdots\\&&1&\frac{1}{n-1}&\frac{1}{n-1}&\cdots&-\frac{n-2}{n-1}\end{array}\right)$

$\therefore B^{-1}=\begin{pmatrix}-\frac{n-2}{n-1}&&\frac{1}{n-1}\\&\ddots&\\\frac{1}{n-1}&&-\frac{n-2}{n-1}\end{pmatrix}$

另法]. 令 $N=B+E_n=\begin{pmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&&\vdots\\1&1&\cdots&1\end{pmatrix}$ $N^2=nN$.

设 $(N-E_n)(\lambda N-E_n)=\lambda N^2-(\lambda+1)N+E_n=(n\lambda-\lambda-1)N+E_n$

令 $\lambda=\frac{1}{n-1}$ 则 $(n-1)\lambda-1=0\Rightarrow \overset{B}{\overset{\|}{(N-E_n)}}\cdot\overset{B^{-1}}{\overset{\|}{\left(\frac{1}{n-1}N-E_n\right)}}=E_n$

例2. 设 $A=\begin{pmatrix}2&5\\1&3\end{pmatrix}$, $B=\begin{pmatrix}4&-1\\2&1\end{pmatrix}$ 求 $X, Y\in M_2(\mathbb{R})$ s.t. $AX=B$, $YA=B$.

解：[方法一] 求 A^{-1}：$\begin{pmatrix}2&5&1&0\\1&3&0&1\end{pmatrix}\xrightarrow{(1)\leftrightarrow(2)}\begin{pmatrix}1&3&0&1\\2&5&1&0\end{pmatrix}\xrightarrow{-2(1)+(2)}\begin{pmatrix}1&3&0&1\\0&-1&1&-2\end{pmatrix}\xrightarrow[-(2)]{3(2)+(1)}$

$\begin{pmatrix}1&0&3&-5\\0&1&-1&2\end{pmatrix}\Rightarrow A^{-1}=\begin{pmatrix}3&-5\\-1&2\end{pmatrix}$. 则 $X=A^{-1}B=\begin{pmatrix}2&-8\\0&3\end{pmatrix}$.

$Y=BA^{-1}=\begin{pmatrix}4&-1\\2&1\end{pmatrix}\begin{pmatrix}3&-5\\-1&2\end{pmatrix}=\begin{pmatrix}13&-22\\5&-8\end{pmatrix}$.

[方法二] $(A\ B)=\begin{pmatrix}2&5&4&-1\\1&3&2&1\end{pmatrix}\xrightarrow{(1)\leftrightarrow(2)}\begin{pmatrix}1&3&2&1\\2&5&4&-1\end{pmatrix}\xrightarrow{-2(1)+(2)}\begin{pmatrix}1&3&2&1\\0&-1&0&-3\end{pmatrix}\xrightarrow[-(2)]{3(2)+(1)}$

$\begin{pmatrix}1&0&2&-8\\0&1&0&3\end{pmatrix}\Rightarrow X=A^{-1}B=\begin{pmatrix}2&-8\\0&3\end{pmatrix}$

对方程 $YA=B$ 两边转置 $A^t\cdot Y^t=B^t$

$(A^t\ B^t)=\begin{pmatrix}2&1&4&2\\5&3&-1&1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&13&5\\0&1&-22&-8\end{pmatrix}\Rightarrow Y^t=(A^t)^{-1}\cdot B^t=\begin{pmatrix}13&5\\-22&-8\end{pmatrix}$

$\therefore Y=\begin{pmatrix}13&-22\\5&-8\end{pmatrix}$

例3. 设 $A\in M_m(\mathbb{R})$, $B\in M_n(\mathbb{R})$ 均可逆. $C\in\mathbb{R}^{m\times n}$ 则

(1) $\begin{pmatrix}E_m&C_{m\times n}\\O_{n\times m}&E_n\end{pmatrix}^{-1}=\begin{pmatrix}E_m&-C\\O&E_n\end{pmatrix}$, $\begin{pmatrix}E_n&O\\C&E_m\end{pmatrix}^{-1}=\begin{pmatrix}E_n&O\\-C&E_m\end{pmatrix}$

(2) $\begin{pmatrix}A&C\\O&B\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1}&-A^{-1}CB^{-1}\\O&B^{-1}\end{pmatrix}$

Pf (1). 注意到 $\forall D\in\mathbb{R}^{m\times n}$ 有 $\begin{pmatrix}E_m&C\\0&E_n\end{pmatrix}\cdot\begin{pmatrix}E_m&D\\0&E_n\end{pmatrix}=\begin{pmatrix}E_m&C+D\\0&E_n\end{pmatrix}$

则 $\begin{pmatrix}E_m&C\\0&E_n\end{pmatrix}\cdot\begin{pmatrix}E_m&-C\\0&E_n\end{pmatrix}=\begin{pmatrix}E_m&\\&E_n\end{pmatrix}=E_{m+n}$ 同理 $\begin{pmatrix}E_m&-C\\0&E_n\end{pmatrix}\cdot\begin{pmatrix}E_m&C\\0&E_n\end{pmatrix}=E_{m+n}$.

$\therefore\begin{pmatrix}E_m&C\\0&E_n\end{pmatrix}^{-1}=\begin{pmatrix}E_m&-C\\0&E_n\end{pmatrix}$ 类似地 $\begin{pmatrix}E_n&0\\C&E_m\end{pmatrix}^{-1}=\begin{pmatrix}E_n&0\\-C&E_m\end{pmatrix}$.

(2). $\begin{pmatrix}A^{-1}&0\\0&B^{-1}\end{pmatrix}\begin{pmatrix}A&C\\0&B\end{pmatrix}=\begin{pmatrix}E_m&A^{-1}C\\0&E_n\end{pmatrix}$ 由(1) $\left(\begin{pmatrix}E_m&-A^{-1}C\\0&E_n\end{pmatrix}\cdot\begin{pmatrix}A^{-1}&\\&B^{-1}\end{pmatrix}\right)\cdot\begin{pmatrix}A&C\\0&B\end{pmatrix}=E_{m+n}$

$\therefore\begin{pmatrix}A&C\\0&B\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1}&-A^{-1}CB^{-1}\\0&B^{-1}\end{pmatrix}$ (同理 $\begin{pmatrix}A&C\\0&B\end{pmatrix}\cdot\begin{pmatrix}A^{-1}&\\&B^{-1}\end{pmatrix}\cdot\begin{pmatrix}E_m&-CB^{-1}\\&E_n\end{pmatrix}=E_{m+n}$)

例 设 $A,B\in M_n(\mathbb{R})$ 交换 即 $AB=AB$ 求证 $\operatorname{rank}(A+B)+\operatorname{rank}AB\le \operatorname{rank}A+\operatorname{rank}B$

Pf. 设线性映射 $\varphi_A:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\vec{x}\longmapsto A\vec{x}$, $\varphi_B:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\vec{x}\longmapsto B\vec{x}$ $\qquad \varphi_{AB}=\varphi_A\circ\varphi_B$, $\varphi_{A+B}=\varphi_A+\varphi_B$

设 $\mathbb{R}^n$ 子空间 $K_A=\ker\varphi_A$, $K_B=\ker\varphi_B$. $K_{AB}=\ker\varphi_{AB}$. $K_{A+B}=\ker(\varphi_{A+B})=\ker(\varphi_A+\varphi_B)$.

首先证明 $K_A+K_B\subseteq K_{AB}$ 如下:

$\forall\ \vec{x}\in K_A+K_B$ 则 $\exists\ \vec{x}_A\in K_A,\ \vec{x}_B\in K_B$ s.t. $\vec{x}=\vec{x}_A+\vec{x}_B$

则 $\varphi_{AB}(\vec{x})=\varphi_{AB}(\vec{x}_A)+\varphi_{AB}(\vec{x}_B)=\varphi_A\circ\varphi_B(\vec{x}_A)+\varphi_A\circ\varphi_B(\vec{x}_B)$

$\because AB=BA$ $\therefore \varphi_A\circ\varphi_B=\varphi_B\circ\varphi_A$ $\therefore \varphi_{AB}(\vec{x})=\varphi_B\circ\overbrace{\varphi_A(\vec{x}_A)}^{=\vec{0}}+\varphi_A\circ\overbrace{\varphi_B(\vec{x}_B)}^{=\vec{0}}$

$\therefore \varphi_{AB}(\vec{x})=\vec{0}$ $\therefore \vec{x}\in K_{AB}$ 则 $K_A+K_B\subseteq K_{AB}$

$\therefore \dim(K_A+K_B)\le\dim(K_{AB})$

又 $\because \dim(K_A+K_B)=\dim K_A+\dim K_B-\dim(K_A\cap K_B)$ (维数公式)

再证 $K_A\cap K_B\subseteq K_{A+B}$

$\forall\ \vec{x}\in K_A\cap K_B$ 即 $\vec{x}\in K_A$ 且 $\vec{x}\in K_B$ $\therefore \varphi_A(\vec{x})=\vec{0}$, $\varphi_B(\vec{x})=\vec{0}$

则 $\varphi_{A+B}(\vec{x})=(\varphi_A+\varphi_B)(\vec{x})=\varphi_A(\vec{x})+\varphi_B(\vec{x})=\vec{0}+\vec{0}=\vec{0}$ $\therefore \vec{x}\in\ker(\varphi_{A+B})$

即 $K_A\cap K_B\subseteq K_{A+B}$ $\therefore \dim(K_A\cap K_B)\le\dim K_{A+B}$

$\therefore \cancel{\dim(K_A+K_B)}\ \dim(K_{AB})\geq\dim(K_A+K_B)\geq\dim K_A+\dim K_B-\dim K_{A+B}$

$\therefore \underbrace{\dim(K_{AB})}_{n-\operatorname{rank}AB}+\underbrace{\dim(K_{A+B})}_{n-\operatorname{rank}(A+B)}\geq\underbrace{\dim\cancel{A}K_A}_{n-\operatorname{rank}A}+\underbrace{\dim\cancel{B}K_B}_{n-\operatorname{rank}B}$

$\therefore \operatorname{rank}A+\operatorname{rank}B\geq\operatorname{rank}AB+\operatorname{rank}(A+B)$ $\square$.