第十次作业

1. (教 P_{83}. T_{12}).

$$(A\ \ E_5)=\begin{pmatrix} 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 8 & 6 & 4 & 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 6 & 9 & 6 & 3 & 0 & 0 & 1 & 0 & 0 \\ 2 & 4 & 6 & 8 & 4 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 2 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 3 & 4 & 2 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow[\substack{-r_1+r_4,\\ -r_1+r_5}]{\substack{(-5)r_1+r_2,\\ (-2)r_1+r_3,}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 \\ 0 & -6 & -12 & -18 & -24 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & -3 & -6 & -9 & 0 & \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -4 & 0 & 0 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & \frac{1}{2} & -1 \end{pmatrix}$$

$$\xrightarrow[\substack{(-\frac{1}{2})r_4\\ (-\frac{1}{3})r_5}]{\substack{(-\frac{1}{6})r_2,\\ -(\frac{1}{3})r_3,}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & -\frac{1}{6} & 0 & 0 & 0 & -\frac{5}{6} \\ 0 & 0 & 1 & 2 & 3 & 0 & -\frac{1}{6} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \xrightarrow[r_3+r_1]{(-2)r_2+r_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & -\frac{1}{6} & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 1 & 2 & 3 & 0 & -\frac{1}{6} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{(-2)r_3+r_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & -\frac{1}{6} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \xrightarrow[r_5+r_3]{(-2)r_4+r_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\xrightarrow{-r_5+r_4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

法2：$(A\ \ E_5)\xrightarrow[i=2,3,4,5]{-r_i+r_{i+1}}$

$$\begin{pmatrix} 1 & -4 & -3 & -2 & -1 & 1 & -1 & & & \\ 1 & +2 & -3 & -2 & -1 & 0 & 1 & -1 & & \\ 1 & 2 & 3 & -2 & -1 & & & 1 & -1 & \\ 1 & 2 & 3 & 4 & -1 & & & & 1 & -1 \\ 1 & 2 & 3 & 4 & 5 & & & & & 1 \end{pmatrix}$$

易错：只能做初等行变换.

$$\xrightarrow[i=4,3,2,1]{-r_i+r_{i+1}} \begin{pmatrix} 1 & -4 & -3 & -2 & -1 & 1 & -1 & & & \\ 0 & 6 & 0 & 0 & 0 & -1 & 2 & -1 & & \\ 0 & 0 & 6 & 0 & 0 & 0 & -1 & 2 & -1 & \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow[\substack{4r_2+3r_3\\ +2r_4+r_5}]{\frac{1}{6}r_i,} (E\ \ A^{-1}).$$

第十次作业.

1.(陈P83. 12).

$$(A\ \ E_5)=\begin{pmatrix}5&4&3&2&1&1&0&0&0&0\\4&8&6&4&2&0&1&0&0&0\\3&6&9&6&3&0&0&1&0&0\\2&4&6&8&4&0&0&0&1&0\\1&2&3&4&5&0&0&0&0&1\end{pmatrix}$$

最后一行换到第一行. 依次乘(-5), (-4), (-3), (-2) 加到第二、三、四、五行.

$$\longrightarrow\begin{pmatrix}1&2&3&4&5&0&0&0&0&1\\0&-6&-12&-18&-24&1&&&&-5\\0&0&-6&-12&-18&&1&&&-4\\0&0&0&-6&-12&&&1&&-3\\0&0&0&0&-6&&&&1&-2\end{pmatrix}\xrightarrow[i=2,\dots,5]{r_i/(-6)}\begin{pmatrix}1&2&3&4&5&0&0&0&0&1\\0&1&2&3&4&-\frac16&0&0&0&\frac56\\0&0&1&2&3&0&-\frac16&0&0&\frac23\\0&0&0&1&2&0&0&-\frac16&0&\frac12\\0&0&0&0&1&0&0&0&-\frac16&\frac13\end{pmatrix}$$

$$\xrightarrow{-2r_5+r_4}\begin{pmatrix}1&2&3&4&5&0&0&0&0&1\\&1&2&3&4&-\frac16&0&0&0&\frac56\\&&1&2&3&0&-\frac16&0&0&\frac23\\&&&1&0&0&0&-\frac16&\frac13&-\frac16\\&&&&1&0&0&0&-\frac16&\frac13\end{pmatrix}\xrightarrow{-3r_5-2r_4+r_3}\begin{pmatrix}1&2&3&4&5&0&0&0&0&1\\&1&2&3&4&-\frac16&0&0&0&\frac56\\&&1&0&0&0&-\frac16&\frac13&-\frac16&0\\&&&1&0&0&0&-\frac16&\frac13&-\frac16\\&&&&1&0&0&0&-\frac16&\frac13\end{pmatrix}$$

$$\xrightarrow{-4r_5-3r_4-2r_3+r_2}\begin{pmatrix}1&2&3&4&5&0&0&0&0&1\\&1&0&0&0&-\frac16&\frac13&-\frac16&0&0\\&&1&0&0&0&-\frac16&\frac13&-\frac16&0\\&&&1&0&0&0&-\frac16&\frac13&-\frac16\\&&&&1&0&0&0&-\frac16&\frac13\end{pmatrix}\xrightarrow[-2r_2+r_1]{-5r_5-4r_4-3r_3}\begin{pmatrix}1&0&0&0&0&\frac13&-\frac16&0&0&0\\&1&0&0&0&-\frac16&\frac13&-\frac16&0&0\\&&1&0&0&0&-\frac16&\frac13&-\frac16&0\\&&&1&0&0&0&-\frac16&\frac13&-\frac16\\&&&&1&0&0&0&-\frac16&\frac13\end{pmatrix}$$

$$(F\ \ E_4)=\begin{pmatrix}2&3&2&1&1&0&0&0\\3&6&4&2&0&1&0&0\\4&8&6&3&0&0&1&0\\2&4&3&2&0&0&0&1\end{pmatrix}\xrightarrow[-2r_4+r_3]{\substack{-r_1+r_4\\-2r_1+r_3}}\begin{pmatrix}2&3&2&1&1&0&0&0\\3&6&4&2&0&1&0&0\\0&0&0&-1&0&0&1&-2\\0&1&1&1&-1&0&0&1\end{pmatrix}.$$

$$\xrightarrow[r_2-\frac32 r_1]{\overleftrightarrow{r_3,r_4}}\begin{pmatrix}2&3&2&1&1&0&0&0\\0&\frac32&1&\frac12&-\frac32&1&0&0\\0&1&1&1&-1&0&0&1\\0&0&0&-1&0&0&1&-2\end{pmatrix}\xrightarrow[-r_4]{r_2-\frac32 r_3}\begin{pmatrix}2&3&2&1&1&0&0&0\\0&0&-\frac12&-1&0&1&0&-\frac32\\0&1&1&1&-1&0&0&1\\0&0&0&1&0&0&-1&2\end{pmatrix}$$

$$\xrightarrow{-2r_2,\ \overleftrightarrow{r_2,r_3}}\begin{pmatrix}2&3&2&1&1&0&0&0\\0&1&1&1&-1&0&0&1\\0&0&1&2&0&-2&0&3\\0&0&0&1&0&0&-1&2\end{pmatrix}\xrightarrow[\substack{-r_4+r_2\\-r_4+r_1}]{-2r_4+r_3}\begin{pmatrix}2&3&2&0&1&0&1&-2\\0&1&1&0&-1&0&1&-1\\0&0&1&0&0&-2&2&-1\\0&0&0&1&0&0&-1&2\end{pmatrix}$$

$$\xrightarrow{-r_3+r_2}\left(\begin{array}{cccc|cccc} 2 & 3 & 2 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 2 \end{array}\right) \xrightarrow[\frac{1}{2}r_1]{-3r_2-2r_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 2 \end{array}\right)$$

2. 柯 P83. T13. 验证 $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc\neq 0 \Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

特别地，$ad-bc=1 \Rightarrow A^{-1}=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

若 $ad-bc=0$，A^{-1} 存在吗？

未讨论是否为0的
不要直接除.

证明：$ad-bc\neq 0$，则 $\frac{1}{ad-bc}$ 有意义.

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & ad-bc \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

若 $ad-bc=1$，则 $A^{-1}=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

若 $ad-bc=0$，则 $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\xrightarrow[b\cdot r_2]{d\cdot r_1}\begin{pmatrix} ad & bd \\ bc & db \end{pmatrix}\xrightarrow{-r_1+r_2}\begin{pmatrix} ad & bd \\ 0 & 0 \end{pmatrix}$

即 $\mathrm{rank}(A)=1<2$. 不可逆.

3. P83. T14. 证明任意矩阵 $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 满足 $A^2=(a+d)A-(ad-bc)E$. (23).

(即 A 是 $x^2-(a+d)x-(ad-bc)=0$ 的一个"根").

证：左 $=A^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix}$.

右 $=(a+d)A-(ad-bc)E=\begin{pmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{pmatrix}-\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+bc & d^2+bc \end{pmatrix}$.

$\therefore$ 左 $=$ 右. $\therefore$ (23) 成立.

4. P85. T15. 若 $ad-bc\neq 0$，用关系(23)求 A^{-1}.

解：$A^2+(a+d)A=(bc-ad)E$. ⟵ E前系数不为0时，A才可逆！

若 $bc-ad\neq 0$，则 $A\left(\frac{1}{bc-ad}A+\frac{a+d}{bc-ad}E\right)=E$

则 $A^{-1}=\frac{1}{bc-ad}A+\frac{a+d}{bc-ad}E=\frac{1}{bc-ad}\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$

5. P83. T16. 证明若 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^m = 0$，则 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0$.

证：由 $A^m = 0$ 知 $ad = bc$.

否则，若 $ad-bc \neq 0$，由 T2 知 A 可逆，则 $0 = A^m = \overbrace{A \cdot A \cdots A}^{m个}$ 可逆. $(\rightarrow\leftarrow)$.

由 T2 知，$A^2 = (a+d)A - (ad-bc)E \Rightarrow A^2 = (a+d)A$.

① $m \geq 3$. 则 $0 = A^m = A^{m-2} \cdot A^2 = A^{m-2} \cdot (a+d)A = (a+d) \cdot A^{m-3} \cdot A^2 = (a+d)^2 A^{m-4} \cdot A^2 = (a+d)^2 A^{m-2}$

$= (a+d)^3 \cdot A^{m-5} \cdot A^2 = (a+d)^4 A^{m-4} = \cdots = (a+d)^{m-2} A^2$.

数　矩阵乘法无消去律

$\Rightarrow a+d = 0 \Rightarrow A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+bc & ad+d^2 \end{pmatrix} \overset{ad=bc}{=\!=} \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} = 0$.

或 $A^2 = 0$.

方2：$A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu\lambda_1 & \mu\lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}(\lambda_1\ \lambda_2)$.

$0 = A^m = \begin{pmatrix} 1 \\ \mu \end{pmatrix}\underbrace{(\lambda_1\ \lambda_2)\begin{pmatrix} 1 \\ \mu \end{pmatrix}}\underbrace{(\lambda_1\ \lambda_2)\cdots\begin{pmatrix} 1 \\ \mu \end{pmatrix}}(\lambda_1\ \lambda_2)$

$= \begin{pmatrix} 1 \\ \mu \end{pmatrix}(\lambda_1\ \lambda_2) \cdot (\lambda_1 + \mu\lambda_2)^{m-1}$

$= (\lambda_1 + \mu\lambda_2)^{m-1} A \Rightarrow A = 0$ or $\lambda_1 = -\mu\lambda_2$

② $m \leq 2$. 则 $A^2 = 0$.　　因 $r(A^2) = r(A^3) =$

6. 设 $A, B \in M_n(\mathbb{R})$ 是 n 阶方阵，E_n 是 n 阶单位阵，计算

$$M = \begin{pmatrix} O_n & E_n \\ E_n & O_n \end{pmatrix}\begin{pmatrix} A & O_n \\ O_n & B \end{pmatrix}\begin{pmatrix} O_n & E_n \\ E_n & O_n \end{pmatrix}$$

并证明 $\operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} B$.

证：$M = \begin{pmatrix} O_n & E_n \\ E_n & O_n \end{pmatrix}\begin{pmatrix} A & O_n \\ O_n & B \end{pmatrix}\begin{pmatrix} O_n & E_n \\ E_n & O_n \end{pmatrix} = \begin{pmatrix} O_n & B \\ A & O_n \end{pmatrix}\begin{pmatrix} O_n & E_n \\ E_n & O_n \end{pmatrix} = \begin{pmatrix} B & O_n \\ O_n & A \end{pmatrix}$.

$\operatorname{rank} M = \operatorname{rank}\begin{pmatrix} B & O_n \\ O_n & A \end{pmatrix} = \operatorname{rank} A + \operatorname{rank} B$　　$(\operatorname{rank} M = \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{rank} A + \operatorname{rank} B)$.

§1. 矩阵总结
(线性映射).

$\mathbb{R}^{m\times n} \overset{1:1}{\longleftrightarrow} \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$A \longmapsto \varphi_A : [\vec{x} \longmapsto A\vec{x}]$.

$A_\varphi = (\varphi(e_1), \cdots, \varphi(e_n)) \longleftarrow\!\shortmid\ \varphi$

1. 矩阵 $A \in \mathbb{R}^{m\times n} \longleftrightarrow \varphi_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$A\vec{x} = \vec{0},\ \vec{x} \in \mathbb{R}^n. \iff \vec{x} \in \ker(\varphi_A)$.

$A\vec{x} = \vec{b},\ \vec{x}, \vec{b} \in \mathbb{R}^n. \iff \vec{b} \in \operatorname{im}(\varphi_A)$　~~当 m=n 且 A 可逆时，$\vec{x} = A^{-1}b$.~~

. $\ker(\varphi_A) = V_A$. 解空间　$\operatorname{im}(\varphi_A) = V_c(A)$.

$\dim\ker(\varphi_A) = \dim V_A$.　　$\dim(\operatorname{im}\varphi_A) = \dim V_c(A) = \operatorname{rank} A$

$\Rightarrow \operatorname{rank}(A) + \dim V_A = n$.　$\dim(\ker\varphi_A) + \dim(\operatorname{im}\varphi_A) = n$.

2. φ_A 满射 $\Longleftrightarrow$ $\mathrm{rank}(A)=m$ 行满秩. $\Longleftrightarrow$ $\forall \vec{b}\in\mathbb{R}^m$，$A\vec{x}=\vec{b}$ 有解.

$\Longleftrightarrow$ $\mathrm{rank}(A\,\vdots\,\vec{b})=\mathrm{rank}(A)=m$.

当 $m=n$ 时 且 A 可逆，$A\vec{x}=\vec{b}\Rightarrow \vec{x}=A^{-1}\vec{b}$. 解存在且唯一.

定理（满秩分解公式）. 若一个 $m\times n$ 阶矩阵 A 的秩为 r，则存在 $m\times r$ 阶列满秩矩阵 B 和 $r\times n$ 阶行满秩矩阵 C，使得 $A=BC$.

证：$\because \mathrm{rank}(A)=r$. $\therefore$ $\exists$ 可逆方阵 $P\in M_m(\mathbb{R})$，$Q\in M_n(\mathbb{R})$.

st. $PAQ=\begin{pmatrix}E_r&0\\0&0\end{pmatrix}\Rightarrow A=P^{-1}\begin{pmatrix}E_r&0\\0&0\end{pmatrix}Q^{-1}$.

$\Rightarrow A=P^{-1}\begin{pmatrix}E_r\\ \underset{(m-r)\times r}{0}\end{pmatrix}(E_r\ \underset{r\times(n-r)}{0})Q^{-1}$. 令 $\overset{m\times r}{B}=\overset{m\times m}{P^{-1}}\overset{m\times r}{\begin{pmatrix}E_r\\0\end{pmatrix}}$，$C=\overset{r\times n}{(E_r\ 0)}\overset{n\times n}{Q^{-1}}$.

下证 $\mathrm{rank}(B)=\mathrm{rank}(C)=r$. 以 B 为例，证 B 列满秩.

由于 $\mathrm{rank}(P^{-1})=m$，$\therefore r=\mathrm{rank}(P^{-1})+\mathrm{rank}\begin{pmatrix}E_r\\0\end{pmatrix}-m\le \underset{\mathrm{rank}\left(P^{-1}\begin{pmatrix}E_r\\0\end{pmatrix}\right)}{\underset{\|}{\mathrm{rank}(B)}}\le \mathrm{rank}\begin{pmatrix}E_r\\0\end{pmatrix}=r$

$\therefore \mathrm{rank}(B)=r$. $\square$.

§2. 行列式.

行列式来源：线性方程组求解.

(1680s-1710s). 发明：莱布尼茨 (Leibniz)(德) 和关孝和 (日).

1750. 明确阐述：克莱姆 (Cramer) (瑞士) $\longrightarrow$ 定义和展开法则

系统化：贝祖 (Bézout) (法) $\to$ 利用行列式判定齐次非零解存在.

(1770s) 连贯逻辑阐述：Vandenonde $\longrightarrow$ 行列式理论分离出成为独立理论.

1. 线性映射 $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$.

线性函数 $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}$.

m 重线性函数 $\varphi:\underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{m\text{个}}\longrightarrow\mathbb{R}$

$(\vec{x}_1,\cdots,\vec{x}_m)\longmapsto\varphi(\vec{x}_1,\cdots,\vec{x}_m)$.

满足 $\forall i\in\{1,\cdots,m\}$. $\vec{x}_i,\vec{x}_i'\in\mathbb{R}^n$，$\alpha,\beta\in\mathbb{R}$.

有 $\varphi(\vec{x}_1,\cdots,\alpha\vec{x}_i+\beta\vec{x}_i',\cdots,\vec{x}_m)=\alpha\varphi(\vec{x}_1,\cdots,\vec{x}_i,\cdots,\vec{x}_m)+\beta\varphi(\vec{x}_1,\cdots,\vec{x}_i',\cdots,\vec{x}_m)$.

斜对称函数：$\varphi: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m个} \longrightarrow \mathbb{R}$

满足 $\forall i,j \in \{1,\cdots,m\}$，且 $i \neq j$，……，

$$\varphi(\vec{x}_1,\cdots,\vec{x}_i,\cdots,\vec{x}_j,\cdots,\vec{x}_m) = -\varphi(\vec{x}_1,\cdots,\vec{x}_j,\cdots,\vec{x}_i,\cdots,\vec{x}_m).$$

2. Def（行列式）. $det: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n个} \longrightarrow \mathbb{R}$　n重线性斜对称函数，

且满足 $det(\vec{e}_1,\cdots,\vec{e}_n)=1$. 则称 det 为行列式

$\forall A \in M_n(\mathbb{R})$，记 $det(A) = det(\vec{A}^{(1)},\cdots,\vec{A}^{(n)}) := |A|$.

· $|A| = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.　共 $n!$ 项.

（每个求和项：每行每列各选一个元的乘积）.

例1. $A = \begin{pmatrix} & & & a_{1,n} \\ & & a_{2,n-1} & \\ & \iddots & & \\ a_{n,1} & & & \end{pmatrix}$　求 $det(A)$.

解：$|A| = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$.

当 $(i,j) \neq (k, n+1-k)$，$k=1,2,\cdots,n$ 时，$a_{ij}=0$.

∴ $|A|$ 求和项中，不为 0 的只有一个：$\varepsilon_\sigma a_{n,1} a_{n-1,2} \cdots, a_{1,n}$.

其中 $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n-1 & \cdots & & 1 \end{pmatrix}$　$\varepsilon_\sigma = (-1)^{\frac{n(n-1)}{2}}$.　（据 P_{45}. T_4）.

∴ $|A| = (-1)^{\frac{n(n-1)}{2}} a_{n1} a_{n-1,2} \cdots a_{1,n}$.　□.

方2：交换列，变为上三角形.

3. 行列式性质.　$A \in M_n(\mathbb{R})$

1). $det(E) = 1$.

2). $det(A) = det(A^t)$.

3). 若 A 中两列或两行相同

或有一行（一列）为 $\vec{0}$，则 $det(A)=0$.

4). 与初等变换关系. $\det(F_{ij}(A)) = -\det(A)$. ∵列变换同样.

$$\det(F_{ij}(\lambda)A) = \det(A).$$

$$\det(F_i(\lambda)\cdot A) = \lambda\det(A) \Rightarrow \det(\lambda A) = \lambda^n\det(A).$$

5). A为三角方阵. 则 $\det(A) =$ 对角线元素之积.

· 计算行列式方法之一, 用初等变换化为上三角阵.

· $A_{ij} = (-1)^{i+j}M_{ij}$, $M_{ij} = \det$(A去掉第i行第j列的矩阵).

6). $\det(A) = \underset{\forall i\in\{1,\cdots,n\}}{a_{i1}A_{i1} + \cdots + a_{in}A_{in}} = \underset{\forall j\in\{1,\cdots,n\}}{a_{1j}A_{1j} + \cdots + a_{nj}A_{nj}}$ 按一行或一列展开.

例: 求 $\begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} \xlongequal[-\alpha_1 r_1 + r_2]{-\alpha_1 r_2 + r_3} \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha_2-\alpha_1 & \alpha_3-\alpha_1 \\ 0 & \alpha_2(\alpha_2-\alpha_1) & \alpha_3(\alpha_3-\alpha_1) \end{vmatrix} = 1\cdot\begin{vmatrix} \alpha_2-\alpha_1 & \alpha_3-\alpha_1 \\ \alpha_2(\alpha_2-\alpha_1) & \alpha_3(\alpha_3-\alpha_1) \end{vmatrix}$

$$= (\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\begin{vmatrix} 1 & 1 \\ \alpha_2 & \alpha_3 \end{vmatrix} = (\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2).$$

· 范德蒙行列式:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & & & \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n}(\alpha_j - \alpha_i)$$

例2. 计算 $\begin{vmatrix} -2 & 3 & 1 \\ 503 & 201 & 298 \\ 5 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 3 & 1 \\ 500+3 & 200+1 & 300-2 \\ 5 & 2 & 3 \end{vmatrix}$

$$= \begin{vmatrix} -2 & 3 & 1 \\ 500 & 200 & 300 \\ 5 & 2 & 3 \end{vmatrix} + \begin{vmatrix} -2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & 3 \end{vmatrix} \xlongequal{A^1 \leftrightarrow A^3} -\begin{vmatrix} 1 & 3 & -2 \\ -2 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} \xlongequal[-3r_1+r_3]{2r_1+r_2} -\begin{vmatrix} 1 & 3 & -2 \\ 0 & 7 & -1 \\ 0 & -7 & 11 \end{vmatrix}$$

$$\xlongequal{r_2+r_3} -\begin{vmatrix} 1 & 3 & -2 \\ 0 & 7 & -1 \\ 0 & 0 & 10 \end{vmatrix} = -70.$$

例4. 一个n阶行列式, 若它的元素满足 $a_{ij} = -a_{ji}$, $i,j = 1,2,\cdots,n$.

则称为反(斜)对称行列式. 证明奇数阶反对称行列式为0.

证明: $i=j$, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$, $\forall i\in\{1,2,\cdots,n\}$.

$i\ne j$, $a_{ij} = -a_{ji}$

即 $A^t=-A$ （反过来，若 $A^t=-A$ 则称 A 反对称）.

$\therefore \det(A^t)=\det(-A)=(-1)^n\det(A)\xlongequal{n奇}-\det(A)$

又 $\det(A^t)=\det(A)$ $\quad\therefore \det(A)=-\det(A)$ $\quad\therefore \det(A)=0$.

2). 证明偶数阶反对称矩阵 A，若 $B=(a_{ij}+b)_{n\times n}$，则 $|A|=|B|$.

证：
$$\det(B)=\begin{vmatrix} b & a_{12}+b & a_{13}+b & \cdots & a_{1n}+b \\ -a_{12}+b & b & a_{23}+b & \cdots & a_{2n}+b \\ -a_{13}+b & -a_{23}+b & b & & a_{3n}+b \\ \vdots & & & \ddots & \vdots \\ -a_{1n}+b & -a_{2n}+b & -a_{3n}+b & \cdots & b \end{vmatrix} \xlongequal{\text{加一行,一列}} \begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & b & a_{12}+b & & a_{1n}+b \\ 0 & -a_{12}+b & b & & \\ \vdots & \vdots & & \ddots & \\ 0 & -a_{1n}+b & & & b \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n+1]{-r_1+r_i} \begin{vmatrix} 1+0 & 0+b & 0+b & \cdots & 0+b \\ -1 & 0 & a_{12} & \cdots & a_{1n} \\ -1 & -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ -1 & -a_{1n} & -a_{2n} & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & a_{12} & \cdots & a_{1n} \\ -1 & -a_{12} & 0 & & a_{2n} \\ \vdots & \vdots & & \ddots & \\ -1 & -a_{1n} & & & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & b & \cdots & b \\ -1 & 0 & a_{12} & \cdots & a_{1n} \\ -1 & -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & -a_{1n} & & \cdots & 0 \end{vmatrix}$$

$$=\det(A)+b\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & a_{12} & \cdots & a_{1n} \\ -1 & -a_{12} & & \ddots & \\ \vdots & \vdots & & & \\ -1 & -a_{1n} & & & 0 \end{vmatrix}=\det(A).$$

例3. 设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$，$\det(A)\triangleq d$.

则 $a_{k1}A_{i1}+\cdots+a_{kn}A_{in}=\begin{cases} d, & k=i \\ 0, & k\neq i \end{cases}$

$a_{1l}A_{1j}+\cdots+a_{nl}A_{nj}=\begin{cases} d, & l=j \\ 0, & l\neq j. \end{cases}$

证：若 $k=i$，则 $d=\sum_{s=1}^{n}a_{is}A_{is}$ 按第 i 行展开求行列式.

若 $k\neq i$，（设 $k<i$）
$$\sum_{s=1}^{n}a_{ks}A_{is} \underset{\text{按第}i\text{行展开}}{=} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ \leftarrow 第k行 \\ \\ \leftarrow 第i行 \\ \\ \\ \end{matrix} =0.$$

（2行一样）.