第十二次作业.

1. (P_{102}. T8). $A, B \in M_n(\mathbb{R})$. 证明 $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B| \cdot |A-B|$.

证: $$\begin{pmatrix} E & E \\ 0 & E \end{pmatrix}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} E & -E \\ 0 & E \end{pmatrix} = \begin{pmatrix} A+B & B+A \\ B & A \end{pmatrix}\begin{pmatrix} E & -E \\ 0 & E \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ B & A-B \end{pmatrix}$$

$$\therefore \left|\begin{pmatrix} E & E \\ 0 & E \end{pmatrix}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} E & -E \\ 0 & E \end{pmatrix}\right| = \left|\begin{pmatrix} E & E \\ 0 & E \end{pmatrix}\right| \cdot \left|\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right| \cdot \left|\begin{pmatrix} E & -E \\ 0 & E \end{pmatrix}\right| = \begin{vmatrix} A+B & \\ B & A-B \end{vmatrix}$$

$$\therefore \begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & \\ B & A-B \end{vmatrix} = |A+B| \cdot |A-B|.$$

2. 设 $X \in \mathbb{R}^{n\times k}$, $Y \in \mathbb{R}^{k\times n}$. 证明 $|E_n + XY| = |E_k + YX|$.

证: 方1. $$\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k+YX & Y+YXY \\ X & E_n+XY \end{pmatrix} = \begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n+XY \end{pmatrix}.$$

从想乘行列式入手, 找矩阵分别乘和等矩阵得到共同的矩阵

取行列式: $$\begin{vmatrix} E_k+YX & 0 \\ X & E_n \end{vmatrix} \cdot \begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix} = \begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix} \begin{vmatrix} E_k & 0 \\ X & E_n+XY \end{vmatrix}$$

即 $|E_k+YX| \cdot |E_n| = |E_k| \cdot |E_n+XY| \Rightarrow |E_n+XY| = |E_k+YX|$.

方2. $$\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & E_n+XY \end{pmatrix} \Rightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_n+XY|. \quad ①.$$

$$\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix} = \begin{pmatrix} E_k+YX & 0 \\ 0 & E_n \end{pmatrix} \Rightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_k+YX|. \quad ②.$$

同一矩阵通过不同初等变换得到想化det对应的矩阵

由 ①. ② 知 $|E_n+XY| = |E_k+YX|$

☑.

推论1. 设 $X \in \mathbb{R}^{n\times k}$, $Y \in \mathbb{R}^{k\times n}$,

$\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}$ 可逆 $\Leftrightarrow$ E_n+XY 可逆 $\Leftrightarrow$ E_k+YX 可逆.

推论2. (Sylvester 等式). $\text{rank}(E_n+XY)+k = \text{rank}(E_k+YX)+n$.

3. (P_{108}. T2). 用 rank A 表示 rank $A^{\vee}$. $A \in M_n(\mathbb{R})$.

解: $AA^{\vee} = |A| \cdot E$.

1) 若 $|A| \neq 0$; 即: $\text{rank}(A) = n$, ~~则 $|A| \neq 0$~~

$\therefore |A| \cdot |A^{\vee}| = |A|^n \cdot |E| = |A|^n$, $\therefore |A^{\vee}| = |A|^{n-1} \neq 0$

$\therefore \text{rank}(A^{\vee}) = n = \text{rank}(A)$.

(实际上, $A^{-1} = \frac{1}{|A|} A^{\vee}$ $\therefore \text{rank}(A^{\vee}) = \text{rank}(A)$.).

2). 若 $|A|=0$，则 $AA^{\vee}=0$，则 $0=\mathrm{rank}(AA^{\vee})\geq \mathrm{rank}(A)+\mathrm{rank}(A^{\vee})-n$

2.1). $\mathrm{rank}(A)=n-1$，则存在一个 A 的 r 阶子式 $\neq 0$，设 $A_{ij}\neq 0$. $1\leq i,j\leq n$.

$\therefore A^{\vee}\neq 0$，$\therefore$ $1\leq \mathrm{rank}(A^{\vee})\leq n-\mathrm{rank}(A)=1 \Rightarrow \mathrm{rank}(A^{\vee})=1$

2.2). $\mathrm{rank}(A)<n-1$，则 $A_{ij}=0$ 对 $\forall\, 1\leq i\leq n,\ 1\leq j\leq n$ 成立.

$\therefore A^{\vee}=0$，$\therefore \mathrm{rank}(A^{\vee})=0$.

4. $A\in\mathbb{R}^{(n-1)\times n}$, $\mathrm{rank}(A)=n-1$. 证明 $AX=0$ 的基础解系由 $X^{0}=[D_1,-D_2,\cdots,(-1)^{n-1}D_n]$ 组成，其中 D_i 是 $A=(a_{ij})$ 中去掉第 i 列所得矩阵的行列式. 任意解 $X=\lambda X^{0}$, $\lambda\in\mathbb{R}$

证：$\because \mathrm{rank}(A)=n-1$，$\therefore A$ 有 $n-1$ 个线性无关列向量.

设去掉第 i 列后其余 $n-1$ 列线性无关. ~~则 $|A^{(1)},\cdots,A^{(i-1)},A^{(i+1)},\cdots,A^{(n)}|\neq 0$.~~

则方程组

$$\begin{cases} a_{11}x_1+\cdots+a_{1,i-1}x_{i-1}+a_{1,i+1}x_{i+1}+\cdots+a_{1n}x_n=-a_{1i}x_i \\ \cdots \\ a_{n-1,1}x_1+\cdots+a_{n-1,i-1}x_{i-1}+a_{n-1,i+1}x_{i+1}+\cdots+a_{n-1,n}x_n=-a_{n-1,i}x_i \end{cases} \quad (*)$$

在固定 $x_i=-1$ 时，解存在且唯一. 记这个唯一解为 $[\alpha_1,\cdots,\alpha_{i-1},\alpha_{i+1},\cdots,\alpha_n]$.

设 $\vec{\alpha}=[\alpha_1,\cdots,\alpha_{i-1},-1,\alpha_{i+1},\cdots,\alpha_n]$. 则 $A\vec{\alpha}=0$. 即 $\vec{\alpha}\in V_A$.

设 $(*)$ 的系数矩阵为 $\tilde{A}$，则 $|\tilde{A}|\neq 0$，由 Cramer Thm 知

$1\leq j\leq i-1$，$$\alpha_j=\frac{|A^{(1)},\cdots,\underset{\text{第}j\text{列}}{A^{(i)}},\cdots,A^{(i-1)},A^{(i+1)},\cdots,A^{(n)}|}{|\tilde{A}|}$$

$i+1\leq j\leq n$，$$\alpha_j=\frac{|A^{(1)},\cdots,A^{(i-1)},A^{(i+1)},\cdots,\overset{\text{第}j-1\text{列}}{A^{(i)}},\cdots,A^{(n)}|}{|\tilde{A}|}$$

令 $X^{0}=(-1)^{i}\cdot|\tilde{A}|\cdot\vec{\alpha}\triangleq[X_1^{0},\cdots,X_{i-1}^{0},X_{i+1}^{0},\cdots,X_n^{0}]$，则 $X^{0}\in V_A$.

$1\leq j\leq i-1$，$$X_j^{0}=(-1)^{i}\cdot|A^{(1)},\cdots,\underset{\uparrow\text{第}j\text{列}}{A^{(i)}},\cdots,\underset{\uparrow\text{第}i-1\text{列}}{A^{(i-1)}},A^{(i+1)},\cdots A^{(n)}|$$

$$=(-1)^{i}\cdot(-1)^{i-j-1}|A^{(1)},\cdots,A^{(i-1)},A^{(i)},A^{(i+1)},\cdots,A^{(n)}|=(-1)^{j-1}D_j$$

$(-1)^{-j-1}=(-1)^{j+1}=(-1)^{j-1}$

$i+1\leq j\leq n$，$$X_j^{0}=(-1)^{i}\cdot|A^{(1)},\cdots,A^{(i-1)},\underset{\uparrow\text{第}i\text{列}}{A^{(i+1)}},\cdots,\underset{\uparrow j-1\text{列}}{A^{(i)}},\cdots,A^{(n)}|$$

$$=(-1)^{i}\cdot(-1)^{j-i-1}\cdot|A^{(1)},\cdots,A^{(i-1)},A^{(i)},A^{(i+1)},\cdots,A^{(n)}|=(-1)^{j-1}D_j$$

$j=i$，$X_i^{0}=(-1)^{i-1}|\tilde{A}|=(-1)^{i-1}D_i$ 由于 $|\tilde{A}|\neq 0$，$\therefore$ ~~X_i^{0}~~ $\neq 0$ $\therefore X^{0}\neq\vec{0}$

又 $\dim(V_A)=n-\mathrm{rank}(A)=1$，$\therefore V_A=\langle X^{0}\rangle$ ◻.

方2：证 $AX^{0}=0$ 且 $X^{0}\neq\vec{0}$ 即可. 即 $\forall\, 1\leq i\leq n$，均有 $\sum_{j=1}^{n}(-1)^{j+1}a_{ij}D_j=0$.

考虑 $A_i=\begin{pmatrix} a_{i1}\cdots a_{in} \\ a_{11}\cdots a_{1n} \\ \cdots \end{pmatrix}$ 则 $0=|A_i|=\sum_{j=1}^{n}(-1)^{j+1}a_{ij}D_j=\sum_{j=1}^{n}(-1)^{j+1}a_{ij}D_j$ ✓

5. 证明：若 $A,B,C,D \in M_n(\mathbb{R})$，$|A| \neq 0$，则 $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD-ACA^{-1}B|$

此外，验证 $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{cases} |AD-CB|, & 若 AC=CA \\ |DA-CB|, & 若 AB=BA. \end{cases}$ $= |A|\cdot|D-CA^{-1}B|.$

证：$\because$ $|A| \neq 0$，$\therefore$ A 可逆.

$$\begin{pmatrix} A^{-1} & \\ & A \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & A^{-1}B \\ AC & AD \end{pmatrix}, \quad \begin{pmatrix} E & 0 \\ -AC & E \end{pmatrix}\begin{pmatrix} E & A^{-1}B \\ AC & AD \end{pmatrix} = \begin{pmatrix} E & A^{-1}B \\ 0 & AD-ACA^{-1}B \end{pmatrix}.$$

$$\therefore \begin{vmatrix} E & 0 \\ -AC & E \end{vmatrix} \cdot \begin{vmatrix} A^{-1} & \\ & A \end{vmatrix} \cdot \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} E & A^{-1}B \\ 0 & AD-ACA^{-1}B \end{vmatrix}$$

即 $|E|\cdot|E|\cdot|A^{-1}|\cdot|A|\cdot\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |E|\cdot|AD-ACA^{-1}B|. \Rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD-ACA^{-1}B|.$

又 $|AD-ACA^{-1}B| = |A(D-CA^{-1}B)| = |A|\cdot|D-CA^{-1}B|.$

若 $AC=CA$，则 $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD-ACA^{-1}B| = |AD-C(AA^{-1})B| = |AD-CB|.$

若 $AB=BA$，则 $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\cdot|D-CA^{-1}B| = |D-CA^{-1}B|\cdot|A| = |DA-CA^{-1}BA| = |DA-CB|$

6. 1) 设 $A=(a_{ij}) \in M_n(\mathbb{R})$，证：若 $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$，$i=1,2,\cdots,n$. 则 $|A| \neq 0$.

证：反证法. 假设 $|A|=0$，则 $Ax=0$ 存在非零解，

设 x^0 为 $Ax=0$ 的一个非零解，且 $|x_k^0| = \max_{1\leq i\leq n}\{|x_1^0|,\cdots,|x_n^0|\}$

则 $|x_k^0|>0$，且 $|x_k^0| \geq |x_i^0|$，$\forall\ i \leq n$.

取方程组第 k 行：$a_{kk}x_k^0 + \sum_{j\neq k} a_{kj}x_j^0 = 0$

$\therefore |a_{kk}||x_k^0| = \left|\sum_{j\neq k} a_{kj}x_j^0\right| \leq \sum_{j\neq k}|a_{kj}||x_j^0| \leq \sum_{j\neq k}|a_{kj}|\cdot|x_k^0| = |x_k^0|\cdot\sum_{j\neq k}|a_{kj}|$

但 $|x_k^0| \neq 0$，$\therefore |a_{kk}| \leq \sum_{j\neq k}|a_{kj}|$ $(\rightarrow\leftarrow)$.

$\therefore |A| \neq 0$. $\square$.

2). 设 $A=(a_{ij}) \in M_n(\mathbb{R})$. 证：若 $a_{ii} > \sum_{j\neq i}|a_{ij}|$，$i=1,2,\cdots,n$. 则 $|A|>0$.

证：对 n 归纳：$n=1$. √.

假设 $n-1$ 时结论成立.

n 时. $\because$ $a_{11}>0$，$\therefore$ 由高斯消去法

$$|A| = \left|\begin{matrix} a_{11} & a_{12} & \cdots & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\left(\frac{a_{21}}{a_{11}}\right)a_{12} & & a_{23}-\left(\frac{a_{21}}{a_{11}}\right)a_{13} & \cdots & a_{2n}-\left(\frac{a_{21}}{a_{11}}\right)\cdot a_{1n} \\ 0 & a_{32}-\left(\frac{a_{31}}{a_{11}}\right)a_{12} & & a_{33}-\left(\frac{a_{31}}{a_{11}}\right)a_{13} & \cdots & a_{3n}-\left(\frac{a_{31}}{a_{11}}\right)a_{1n} \\ \cdots & & & & & \\ 0 & a_{n2}-\left(\frac{a_{n1}}{a_{11}}\right)a_{12} & & a_{n3}-\left(\frac{a_{n1}}{a_{11}}\right)a_{13} & \cdots & a_{nn}-\left(\frac{a_{n1}}{a_{11}}\right)a_{1n} \end{matrix}\right|_{n\times n} \quad \begin{matrix} -A_1\left(\frac{a_{21}}{a_{11}}\right)+A_2 \\ -A_1\cdot\left(\frac{a_{31}}{a_{11}}\right)+A_3 \end{matrix}$$

$$= a_{11}\cdot a_{11}^{-(n-1)} \left|\begin{matrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ & \cdots & & \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{matrix}\right|_{(n-1)\times(n-1)}$$

设 B 为上述 $n-1$ 阶行列式所对应的矩阵. $B=(b_{ij})\in M_{n-1}(\mathbb{R})$

由归纳假设. 只要证：$\forall\ i\in\{1,\cdots,n-1\}$, $b_{ii} > \sum_{j\neq i}|b_{ij}|$. 即可.

$$b_{ii} = \begin{vmatrix} a_{11} & a_{1,i+1} \\ a_{i+1,1} & a_{i+1,i+1} \end{vmatrix} = a_{11}\cdot a_{i+1,i+1} - a_{1,i+1}\,a_{i+1,1}$$

$$b_{ij} = \begin{vmatrix} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{vmatrix} = a_{11}\cdot a_{i+1,j+1} - a_{1,j+1}\,a_{i+1,1} \quad , \ i\neq j$$

$$\begin{aligned} b_{ii} - \sum_{j\neq i}|b_{ij}| &= a_{11}\cdot a_{i+1,i+1} - a_{1,i+1}\,a_{i+1,1} - \left|a_{11}\cdot\sum_{\substack{j\neq i\\ j=1}}^{n-1} a_{i+1,j+1} + a_{i+1,1}\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{1,j+1}\right| \\ &= a_{11}\left(a_{i+1,i+1} - \left|\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{i+1,j+1}\right|\right) - a_{i+1,1}\left(a_{1,i+1} + \left|\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{1,j+1}\right|\right). \\ &\geqslant a_{11}\cdot\left(a_{i+1,i+1} - \sum_{\substack{j=1\\ j\neq i}}^{n-1}|a_{i+1,j+1}|\right) - |a_{i+1,1}|\cdot\underbrace{\sum_{j=1}^{n-1}|a_{1,j+1}|}_{< a_{11}} \\ &> a_{11}\cdot\left(a_{i+1,i+1} - \sum_{\substack{j=1\\ j\neq i}}^{n-1}|a_{i+1,j+1}| - |a_{i+1,1}|\right) \\ &= a_{11}\cdot\left(a_{i+1,i+1} - \sum_{\substack{j=1\\ j\neq i+1}}^{n}|a_{i+1,j}|\right) > 0 \end{aligned}$$

由归纳假设知，$|B|>0 \Longrightarrow |A| = a_{11}^{2-n}|B| > 0$ $\square$.

朱

选做 证明

$$B_n(s,t)=\begin{vmatrix}\binom{s}{t} & \binom{s}{t+1} & \cdots & \binom{s}{t+n-1}\\ \binom{s+1}{t} & \binom{s+1}{t+1} & \cdots & \binom{s+1}{t+n-1}\\ \cdots & \cdots & \cdots & \\ \binom{s+n-1}{t} & \binom{s+n-1}{t+1} & \cdots & \binom{s+n-1}{t+n-1}\end{vmatrix}=\frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}.$$

不妨设 $s\geq t$，否则 $B_n(s,t)=0$

证： 第 k 行： $\left[\binom{s+k-1}{t}\quad\binom{s+k-1}{t+1}\quad\cdots\quad\binom{s+k-1}{t+n-1}\right]$

$$=\left[\frac{s+k-1}{t}\cdot\frac{(s+k-2)!}{(t-1)!(s+k-t-1)!},\ \frac{s+k-1}{t+1}\frac{(s+k-2)!}{t!(s+k-t-2)!},\ \cdots,\ \frac{s+k-1}{t+n-1}\frac{(s+k-2)!}{(t+n-2)!(s+k-t-n)!}\right]$$

$$=\left[\frac{s+k-1}{t}\binom{s+k-2}{t-1},\ \frac{s+k-1}{t+1}\binom{s+k-2}{t},\ \cdots,\ \frac{s+k-1}{t+n-1}\binom{s+k-2}{t+n-2}\right]$$

知 $B_n(s,t)$ 中第 k 行可提出公因子 $(s+k-1)$, $k=1,2,\cdots,n$.

第 j 列可提出公因子 $(t+j-1)$, $j=1,2,\cdots,n$.

则
$$B_n(s,t)=\frac{s\cdot(s+1)\cdots(s+n-1)}{t(t+1)\cdots(t+n-1)}\cdot\begin{vmatrix}\binom{s-1}{t-1} & \binom{s-1}{t} & \cdots & \binom{s-1}{t+n-2}\\ \binom{s}{t-1} & \binom{s}{t} & \cdots & \binom{s}{t+n-2}\\ \cdots & & & \\ \binom{s+n-2}{t-1} & \binom{s+n-2}{t} & \cdots & \binom{s+n-2}{t+n-2}\end{vmatrix}$$

$$=\frac{(n+s-1)!}{(s-1)!\,n!}\Big/\left[\frac{(n+t-1)!}{(t-1)!\,n!}\right]\cdot B_n(s-1,t-1)=\frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}}B_n(s-1,t-1)$$

$$=\frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}}\begin{vmatrix}\frac{s-1}{t-1}\binom{s-2}{t-2} & \frac{s-1}{t}\binom{s-2}{t-1} & \cdots & \frac{s-1}{t+n-2}\binom{s-2}{t+n-3}\\ \frac{s}{t-1}\binom{s-1}{t-2} & \frac{s}{t}\binom{s-1}{t-1} & \cdots & \frac{s}{t+n-2}\binom{s-1}{t+n-3}\\ \cdots & & & \\ \frac{s+n-2}{t-1}\binom{s+n-3}{t-2} & \frac{s+n-2}{t}\binom{s+n-3}{t-1} & \cdots & \frac{s+n-2}{t+n-2}\binom{s+n-3}{t+n-3}\end{vmatrix}$$

$$=\frac{\binom{n+s-1}{n}\cdot\binom{n+s-2}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}}B_n(s-2,t-2)=\underset{\uparrow}{\cdots}=\frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}\cdot \underset{\uparrow}{B_n(s-t,0)}.$$

不断从各行,各列提公因子,直到第一列变为1.

$s\geq t$

$$B_n(s-t,0)=\begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-1} \\ 1 & \binom{s-t+1}{1} & \binom{s-t+1}{2} & \cdots & \binom{s-t+1}{n-1} \\ \cdots \\ 1 & \binom{s-t+n-1}{1} & \binom{s-t+n-1}{2} & \cdots & \binom{s-t+n-1}{n-1} \end{vmatrix} \xlongequal[i=n-1,\cdots,1]{-r_i+r_{i+1}} \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-1} \\ 0 & \underset{=1}{\binom{s-t}{0}} & \binom{s-t}{1} & \cdots & \binom{s-t}{n-2} \\ \cdots \\ 0 & 1 & \binom{s-t+n-2}{1} & \cdots & \binom{s-t+n-2}{n-2} \end{vmatrix}_{n\times n}$$

(班长定理 $\binom{s+1}{t}-\binom{s}{t}=\binom{s}{t-1}$).

$$\xlongequal[\text{展开}]{\text{按第1列}} B_{n-1}(s-t,0)=\begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-2} \\ 1 & \binom{s-t+1}{1} & \binom{s-t+1}{2} & \cdots & \binom{s-t+1}{n-2} \\ \cdots \\ 1 & \binom{s-t+n-2}{1} & \binom{s-t+n-2}{2} & \cdots & \binom{s-t+n-2}{n-2} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$\xlongequal[i=n-2,\cdots,1]{-r_i+r_{i+1}} \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-2} \\ 0 & \binom{s-t}{0}=1 & \binom{s-t}{1} & \cdots & \binom{s-t}{n-3} \\ 0 & 1 & \binom{s-t+1}{1} & \cdots & \binom{s-t+1}{n-3} \\ \cdots \\ 0 & 1 & \binom{s-t+n-3}{1} & \cdots & \binom{s-t+n-3}{n-3} \end{vmatrix}_{(n-1)\times(n-1)}. = B_{n-2}(s-t,0)$$

按第1列展开

$$=\cdots=B_2(s-t,0)=\begin{vmatrix} \binom{s-t}{0} & \binom{s-t}{1} \\ \binom{s-t+1}{0} & \binom{s-t+1}{1} \end{vmatrix} \xlongequal{-r_1+r_2} \begin{vmatrix} 1 & \binom{s-t}{1} \\ 0 & \underset{=1}{\binom{s-t}{0}} \end{vmatrix} = 1.$$

$$\therefore B_n(s,t)=\frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}.$$

□.

§1. 行列式的应用

例1. 证明. 1). $(AB)^{\vee} = B^{\vee}A^{\vee}$. 2). $(A^{t})^{\vee} = (A^{\vee})^{t}$. 3). $(\lambda A)^{\vee} = \lambda^{n-1}A^{\vee}$. 4). $(A^{\vee})^{\vee} = |A|^{n-2}A$

证: 1). $\forall\, i,j \in \{1,\cdots,n\}$. 证明 $(AB)^{\vee}$ 的第 i 行 j 列元素与 $B^{\vee}A^{\vee}$ 的对应元素相等.

设 $AB = C$. 则 $\left((AB)^{\vee}\right)_{ij} = (C^{\vee})_{ij} = C_{ji} = (-1)^{i+j}\begin{vmatrix} C_{11} \cdots C_{1,i-1} & C_{1,i+1} \cdots & C_{1,n} \\ \cdots & & \\ C_{j-1,1} \cdots C_{j-1,i-1} & C_{j-1,i+1}, \cdots & C_{j-1,n} \\ C_{j+1,1} \cdots C_{j+1,i-1} & C_{j+1,i+1}, \cdots, & C_{j+1,n} \\ \cdots & & \\ C_{n,1} \cdots C_{n,i-1} & C_{n,i+1}, \cdots, & C_{n,n} \end{vmatrix}$

$$= (-1)^{i+j}\begin{vmatrix} A_1B^{(1)} \cdots & A_1B^{(i-1)}, A_1B^{(i+1)} \cdots & A_1B^{(n)} \\ \cdots & & \\ A_{j-1}B^{(1)} \cdots & A_{j-1}B^{(i-1)}, A_{j-1}B^{(i+1)} \cdots & A_{j-1}B^{(n)} \\ A_{j+1}B^{(1)} \cdots & A_{j+1}B^{(i-1)}, A_{j+1}B^{(i+1)} \cdots & A_{j+1}B^{(n)} \\ \cdots & & \\ A_nB^{(1)} \cdots & A_nB^{(i-1)}, A_nB^{(i+1)}, \cdots & A_nB^{(n)} \end{vmatrix} = (-1)^{i+j}\left|\underset{(n-1)\times n}{\begin{pmatrix} A_1 \\ \vdots \\ A_{j-1} \\ A_{j+1} \\ \vdots \\ A_n \end{pmatrix}}\underset{n\times(n-1)}{\left(B^{(1)}, \cdots, B^{(i-1)}, B^{(i+1)}, \cdots, B^{(n)}\right)}\right|$$

Binet-Cauchy公式

$(A^{\vee})_{ij} = A_{ji}$ $= (-1)^{i+j} \cdot \sum_{k=1}^{n} M^{A}_{jk} \cdot M^{B}_{ki}$, 这里 $\begin{cases} M^{A}_{jk} \text{ 是}A\text{关于}(j,k)\text{的余子式} \\ M^{B}_{ki} \text{ 是 } B \text{ 关于 } (k,i) \text{ 的余子式.} \end{cases}$

$= \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} (B^{\vee})_{ik}(A^{\vee})_{kj} = (B^{\vee}A^{\vee})_{ij}$

2). 令 $B = A^{t}$，则 $(A^{t})^{\vee} = B^{\vee}$. 设 $B^{\vee}$第 i 行 j 列元素为 $(B^{\vee})_{ij}$

则 $(B^{\vee})_{ij} = B_{ji}$ 为$B=A^{t}$关于(j,i)的代数余子式. 即 A_{ij}

$\therefore (B^{\vee})_{ij} = A_{ij} = (A^{\vee})_{ji}$ $\therefore (A^{t})^{\vee} = B^{\vee} = (A^{\vee})^{t}$

3). 令 $B = \lambda A$，则 $B_{ij} = \lambda^{n-1}A_{ij}$ ，$\therefore \lambda A = B^{\vee} = \begin{pmatrix} \lambda^{n-1}A_{11}, \cdots, \lambda^{n-1}A_{n1} \\ \cdots \\ \lambda^{n-1}A_{1n}, \cdots, \lambda^{n-1}A_{nn} \end{pmatrix} = \lambda^{n-1}A^{\vee}$.

4). $n=2$时，设 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 则 $(A^{\vee})^{\vee} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{\vee} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ ✓.

$n \geq 3$. 若$|A| \neq 0$，则 A 可逆. $\therefore (A^{\vee})^{\vee} = (|A|A^{-1})^{\vee} = |A|^{n-1} \cdot |A^{-1}| \cdot (A^{-1})^{-1} = |A|^{n-2} \cdot A$

若$|A| = 0$. 当$\mathrm{rank}(A) = n-1$, $r(A^{\vee}) = 1$, $\therefore (A^{\vee})^{\vee} = 0$ ($A^{\vee}$的r阶子式均为0, $r \geq 2$).

当$\mathrm{rank}(A) < n-1$, $r(A^{\vee}) = 0$, $\therefore A^{\vee} = 0$ $\therefore (A^{\vee})^{\vee} = 0$.

都有 $(A^{\vee})^{\vee} = 0 \cdot A = |A|^{n-2} \cdot A$

综上，$n \geq 2$ 时，$(A^{\vee})^{\vee} = A$.

例2. Recall. $|E_n+\overset{n\times k}{X}\overset{k\times n}{Y}| = |E_k+\overset{k\times n}{Y}\overset{n\times k}{X}|$.

计算
$$\Delta = \begin{vmatrix} 1+a_1b_1 & a_1b_2\cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2\cdots & a_2b_n \\ \vdots & & \\ a_nb_1 & a_nb_2\cdots & 1+a_nb_n \end{vmatrix}$$

解: $$\Delta = \left| E_n + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} (b_1\ b_2\ \cdots\ b_n) \right| = \left| 1 + (b_1\ b_2\cdots b_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right| = 1+\sum_{i=1}^{n} a_ib_i$$

§2. 群的定义.

1. 设 G 是一个非空集合, * 是 G 上的一个二元运算(封闭性), 称 (G,*) 是一个群, 若满足

群 { 幺半群 { 半群 {
- G_0. G 关于 * 封闭: $\forall a,b\in G.\ a*b\in G$
- G_1. 结合律: $\forall a,b,c\in G,\ (a*b)*c = a*(b*c)$
- G_2. 单位元: $\exists e\in G,$ st. $\forall a\in G$. $a*e = e*a = a$.
- G_3. 逆元存在: $\forall a\in G,\ \exists b\in G$, st. $a*b = b*a = e$.

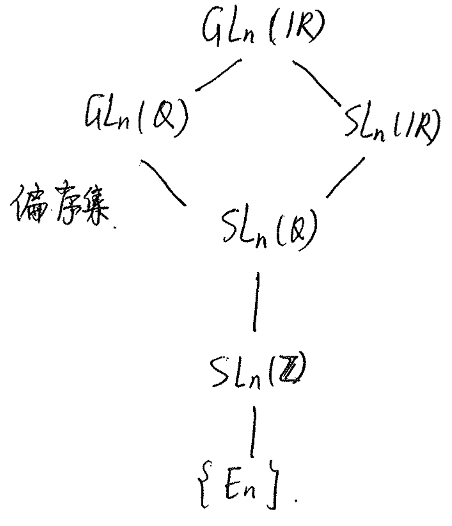
例: 1. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$.

2. $(\mathbb{Q}^*,\times)$, $(\mathbb{R}^*,\times)$, $(\mathbb{C}^*,\times)$. 记 $\mathbb{Q}^*=\mathbb{Q}\setminus\{0\}$.

3. $\mathbb{R}$ 上的 n 阶一般线性群: $GL_n(\mathbb{R}) = \{A\in M_n(\mathbb{R}) \mid |A|\neq 0\}$.

二元运算: 矩阵乘法. 单位元: E_n.

4. $\mathbb{R}$ 上的 n 阶特殊线性群: $SL_n(\mathbb{R}) = \{A\in M_n(\mathbb{R}) \mid |A|=1\}$.



(作业). 验证 $SL_n(\mathbb{Z})$ 是群.

2. 同态映射. $(G,\cdot,e_G)$, $(\overline{G},*,e_{\overline{G}})$ 是两个群.

映射 $\phi: G\longrightarrow \overline{G}$. 满足. 若 $\forall a,b\in G$

$$\phi(a\cdot b) = \phi(a)*\phi(b) \quad (保持群结构)$$

注: 1).
$$\begin{tikzcd} G\times G \arrow[r, "\cdot"] \arrow[d, "\phi"'] & G \arrow[d, "\phi"] \\ \overline{G}\times\overline{G} \arrow[r, "*"] & \overline{G} \end{tikzcd}$$
(a,b). $a\cdot b$; $\phi(a),\phi(b)$. $\phi(a)*\phi(b)$.

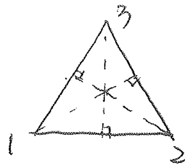
2). $\phi(e_G) = e_{\overline{G}}$

$\forall a\in G$. $\phi(a^{-1}) = \phi(a)^{-1}$.

3. 同构映射: "同态 + 双射".

注: 若 ϕ 同构, 则 ϕ^{-1} 也是同构. (ϕ 双射. 则 ϕ^{-1} 存在. 只需验证 ϕ^{-1} 保持群结构).

例 1. 考虑将等边三角形变到自身的所有变换.



旋转变换. $\begin{cases} \varphi_0 \longleftrightarrow e & (\text{旋转 } 0^\circ). \\ \varphi_1 \longleftrightarrow (132). & (\text{旋转 } 120^\circ) \\ \varphi_3 \longleftrightarrow (123). & (\text{旋转 } 240^\circ). \end{cases}$ 顺时针.

对称变换 $\begin{cases} \psi_1 \longleftrightarrow (23) \\ \psi_2 \longleftrightarrow (13) \\ \psi_3 \longleftrightarrow (12). \end{cases}$

$G = \{\varphi_0, \varphi_1, \varphi_2, \psi_1, \psi_2, \psi_3\}$

是 $\triangle_{123}$ 的变换群.

则 $G \cong S_3$ (3个元素的置换群). $(\psi_1\psi_1 = \psi_2 \longleftrightarrow (132)(23) = (13))$.

P101. 4. (补). $\Delta_n(k_1,x_1;\cdots;k_m,x_m)=\begin{vmatrix} M^n_{k_1}(x_1) \\ M^n_{k_2}(x_2) \\ \vdots \\ M^n_{k_m}(x_m)\end{vmatrix}$. 其中 $x_1,\cdots,x_m$ 是未知量

$k_1,\cdots,k_m\in\mathbb{N}$.

且 $k_1+\cdots+k_m=n$.

$M^n_k(x)$ 是 $k\times n$ 阶矩阵, 形如

$$M^n_k(x)=\begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 0 & 1 & \binom{2}{1}x & \cdots & \binom{n-1}{1}x^{n-2} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}x^{n-3} \\ \cdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{k-1}x^{n-k}\end{pmatrix}$$

证明 $\Delta_n(k_1,x_1;\cdots;k_m,x_m)=\prod\limits_{1\le j<i\le m}(x_i-x_j)^{k_ik_j}$

证: 先以 $\begin{vmatrix}k_1=2\\k_2=3\end{vmatrix}$ 为例, 观察变换过程

$$\Delta_5(2,x_1;3,x_2)=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 0 & 1 & \binom{2}{1}x_1 & \binom{3}{1}x_1^2 & \binom{4}{1}x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2 & \binom{4}{1}x_2^3 \\ 0 & 0 & 1 & \binom{3}{2}x_2 & \binom{4}{2}x_2^2 \end{vmatrix} \xlongequal[i=3,2,1]{-x_1\cdot A^i+A^{i+1}} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2-x_1 & (x_2-x_1)x_2 & (x_2-x_1)x_2^2 & (x_2-x_1)x_2^3 \\ 0 & 1 & \binom{2}{1}x_2-x_1 & [\binom{3}{1}x_2-\binom{2}{1}x_1]x_2 & [\binom{4}{1}x_2-\binom{3}{1}x_1]x_2^2 \\ 0 & 0 & 1 & \binom{3}{2}x_2-x_1 & [\binom{4}{2}x_2-\binom{3}{2}x_1]x_2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ x_2-x_1 & (x_2-x_1)x_2 & (x_2-x_1)x_2^2 & (x_2-x_1)x_2^3 \\ 1 & \binom{2}{1}x_2-x_1 & [\binom{3}{1}x_2-\binom{2}{1}x_1]x_2 & [\binom{4}{1}x_2-\binom{3}{1}x_1]x_2^2 \\ 0 & 1 & \binom{3}{2}x_2-x_1 & [\binom{4}{2}x_2-\binom{3}{2}x_1]x_2 \end{vmatrix} = (x_2-x_1)\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & \binom{2}{1}x_2-x_1 & [\binom{3}{1}x_2-\binom{2}{1}x_1]x_2 & [\binom{4}{1}x_2-\binom{3}{1}x_1]x_2^2 \\ 0 & 1 & \binom{3}{2}x_2-x_1 & [\binom{4}{2}x_2-\binom{3}{2}x_1]x_2 \end{vmatrix}$$

$$\xlongequal{A_3-A_2}(x_2-x_1)\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & x_2-x_1 & (x_2-x_1)\binom{2}{1}x_2 & (x_2-x_1)\binom{3}{1}x_2^2 \\ 0 & 1 & \binom{3}{2}x_2-x_1 & [\binom{4}{2}x_2-\binom{3}{2}x_1]x_2 \end{vmatrix} = (x_2-x_1)^2\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2 \\ 0 & 1 & \binom{3}{2}x_2-x_1 & [\binom{4}{2}x_2-\binom{3}{2}x_1]x_2 \end{vmatrix}$$

$$\boxed{\begin{array}{c}\text{班长定理化简系数.}\\ \binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k-1}.\end{array}}$$

$$\xlongequal{A_4-A_3}(x_2-x_1)^2\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2 \\ 0 & 0 & x_2-x_1 & (x_2-x_1)\binom{3}{2}x_2 \end{vmatrix} = (x_2-x_1)^3\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2 \\ 0 & 0 & 1 & \binom{3}{2}x_2^2 \end{vmatrix} = (x_2-x_1)^3\cdot\begin{vmatrix} M^{5-1}_{2-1}(x_1) \\ M^{5-1}_{3}(x_2) \end{vmatrix}$$

$$= (x_2-x_1)^3 \begin{vmatrix} M_1^4(x_1) \\ M_3^4(x_2) \end{vmatrix} = (x_2-x_1)^3 (x_2-x_1)^3 = (x_2-x_1)^{3\cdot 2}.$$

对一般情况用数学归纳法证:

$n=2$. 则 $k_1=k_2=1$. $\Delta_2(1,x_1;1,x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = (x_2-x_1)^{1\cdot 1} = x_2-x_1$ 成立.

假设, $n-1$ 时, 命题成立.

n. 1). 对 $M_{k_1}^n(x_1)$ 作初等列变换: $M_{k_1}^n(x_1)\cdot F_{n-1,n}(-x_1)\cdot F_{n-2,n-1}(-x_1)\cdots F_{1,2}(-x_1)$,

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x_1 & x_1^2 & \cdots & x_1^{n-3} & x_1^{n-2} \\ 0 & 0 & 1 & \binom{2}{1}x_1 & \cdots & \binom{n-3}{1}x_1^{n-4} & \binom{n-2}{1}x_1^{n-3} \\ & & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{k_1-2}x_1^{n-k_1-1} & \binom{n-2}{k_1-2}x_1^{n-k_1} \end{pmatrix} = \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{k-1,1} & M_{k_1-1}^{n-1}(x_1) \end{pmatrix}$$

$k_1=1$ 时, $M_{k_1-1}^{n-1}(x_1)$ 退化, 则 $M_{k_1}^n(x_1)F_{n-1,n}(-x_1)\cdots F_{1,2}(-x_1) = (1, 0, \cdots, 0)$

(这里 $F_{n-1,n}(-x_1), \cdots, F_{1,2}(-x_1)$ 为初等矩阵, $n\times n$ 阶. $F_{ij}(-x_1)$ 表示第 i 列乘 $(-x_1)$ 加到第 j 列).

2). 对 $M_{k_i}^n(x_i)$ 作对应的初等列变换:

$M_{k_i}^n(x_i)\,F_{n-1,n}(-x_1)\,F_{n-2,n-1}(-x_1)\cdots F_{1,2}(-x_1)$.

$$= \begin{pmatrix} 1 & x_i-x_1 & (x_i-x_1)x_i & (x_i-x_1)x_i^2 & \cdots & (x_i-x_1)x_i^{n-3} & (x_i-x_1)x_i^{n-2} \\ 0 & 1 & \binom{2}{1}x_i-x_1 & \left[\binom{3}{1}x_i-\binom{2}{1}x_1\right]x_i & \cdots & \left[\binom{n-2}{1}x_i-\binom{n-3}{1}x_1\right]x_i^{n-4}, & \left[\binom{n-1}{1}x_i-\binom{n-2}{1}x_1\right]x_i^{n-3} \\ 0 & 0 & 1 & \binom{3}{2}x_i-x_1 & \cdots & \left[\binom{n-2}{2}x_i-\binom{n-3}{2}x_1\right]x_i^{n-5}, & \left[\binom{n-1}{2}x_i-\binom{n-2}{2}x_1\right]x_i^{n-4} \\ & \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_i-1}x_i-\binom{n-3}{k_i-1}x_1\right]x_i^{n-k_i-2}, & \left[\binom{n-1}{k_i-1}x_i-\binom{n-2}{k_i-1}x_1\right]x_i^{n-k_i-1} \end{pmatrix}$$

$$\triangleq \begin{pmatrix} 1 & (x_i-x_1)(1 \quad x_i \quad x_i^2 \cdots x_i^{n-3} \quad x_i^{n-2}) \\ 0 & N(k_i, x_i, x_1) \end{pmatrix}.$$

其中 $N(k_i, x_i, x_1)$ 为 $(k_i-1)\times(n-1)$ 阶矩阵.

3).

$$\Delta_n(k_1,x_1;\cdots,k_m,x_m)=\begin{vmatrix} M^n_{k_1}(x_1)\\ \vdots \\ M^n_{k_m}(x_m)\end{vmatrix} = \left|\begin{pmatrix}M^n_{k_1}(x_1)\\ \vdots\\ M^n_{k_m}(x_m)\end{pmatrix} F_{n-1,n}(-x_1)\,F_{n-2,n-1}(-x_1)\cdots F_{1,2}(-x_1)\right|$$

$$=\begin{vmatrix} \begin{pmatrix}1&0\\0&M^{n-1}_{k_1-1}(x_1)\end{pmatrix}\\ \begin{pmatrix}1&(x_2-x_1)(1,x_2,x_2^2,\cdots,x_2^{n-2})\\0&N(k_2,x_2,x_1)\end{pmatrix}\\ \cdots\\ \begin{pmatrix}1&(x_m-x_1)(1,x_m,x_m^2,\cdots,x_m^{n-2})\\0&N(k_m,x_m,x_1)\end{pmatrix}\end{vmatrix}_{n\times n} \overset{\text{第一行展开}}{=\!=\!=\!=} \begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix}(x_2-x_1)(1,x_2,\cdots,x_2^{n-2})\\ N(k_2,x_2,x_1)\end{pmatrix}\\ \cdots\\ \begin{pmatrix}(x_m-x_1)(1,x_m,\cdots,x_m^{n-2})\\N(k_m,x_m,x_1)\end{pmatrix}\end{vmatrix}_{(n-1)\times(n-1)}$$

$$=(x_2-x_1)(x_3-x_1)\cdots(x_m-x_1)\begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix}1 & x_2 & x_2^2 & \cdots & x_2^{n-2}\\ & & N(k_2,x_2,x_1) & & \end{pmatrix}\\ \cdots\\ \begin{pmatrix}1 & x_m & x_m^2 & \cdots & x_m^{n-2}\\ & & N(k_m,x_m,x_1) & & \end{pmatrix}\end{vmatrix}_{(n-1)\times(n-1)} \qquad (\bigstar).$$

$$=\left[\prod_{i=2}^{m}(x_i-x_1)\right]\begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2}\\ 1 & \binom{2}{1}x_2-x_1 & \left[\binom{3}{1}x_2-\binom{2}{1}x_1\right]x_2 & \cdots & \left[\binom{n-2}{1}x_2-\binom{n-3}{1}x_1\right]x_2^{n-4}, & \left[\binom{n-1}{1}x_2-\binom{n-2}{1}x_1\right]x_2^{n-3}\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \cdots & \left[\binom{n-2}{2}x_2-\binom{n-3}{2}x_1\right]x_2^{n-5}, & \left[\binom{n-1}{2}x_2-\binom{n-2}{2}x_1\right]x_2^{n-4}\\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1}, & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2}\\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2}, & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1}\end{pmatrix}\\ \ast \end{vmatrix}$$

利用班长定理：$\binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k}$ 化简行列式中元素系数.

$\gamma_{k_1+1}-\gamma_{k_1}$ …

并将所得第 k_1+1 行的公因子 (x_2-x_1) 提到行列式外

$$=\left[\prod_{i=2}^{n}(x_i-x_1)\right](x_2-x_1)\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ \begin{pmatrix} 1 & x_2 & x_2^{2} & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\ 0 & 1 & \binom{3}{2}x_2-x_1 & \cdots & \left[\binom{n-2}{2}x_2-\binom{n-3}{2}x_1\right]x_2^{n-5} & \left[\binom{n-1}{2}x_2-\binom{n-2}{2}x_1\right]x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1}, & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2}, & \left[\binom{n-1}{k_2-1}\cancel{x_2}\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1} \end{pmatrix} \\ * \end{vmatrix}$$

$\gamma_{k_1+2}-\gamma_{k_1+1}$

并将所得第 k_1+2 行的公因子 (x_2-x_1) 提到行列式外

$$=\left[\prod_{i=2}^{n}(x_i-x_1)\right](x_2-x_1)^2\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ \begin{pmatrix} 1 & x_2 & x_2^{2} & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2 & \binom{n-2}{1}x_2^{n-3} \\ 0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1} & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2} & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1} \end{pmatrix} \\ * \end{vmatrix}$$

重复上述过程

$= \cdots$

$r_{k_2+k_1-2}-r_{k_2+k_1-3}$

并将所得第 k_2+k_1-2 行中公因子 (x_2-x_1) 提到行列式之外

$$\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-2}\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\ 0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \binom{n-3}{k_2-2}x_2^{n-k_2-1} & \binom{n-2}{k_2-2}x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2} & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1} \end{pmatrix} \\ * \end{vmatrix}$$

$r_{k_2+k_1-1}-r_{k_2+k_1-2}$

并将第 k_2+k_1-1 行公因子 (x_2-x_1) 提到行列式外

$$\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\ 0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \binom{n-3}{k_2-2}x_2^{n-k_2-1} & \binom{n-2}{k_2-2}x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \binom{n-3}{k_2-1}x_2^{n-k_2-2} & \binom{n-2}{k_2-1}x_2^{n-k_2-1} \end{pmatrix} \\ * \end{vmatrix}$$

$$=\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ M_{k_2}^{n-1}(x_2) \\ \begin{pmatrix} 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ & & N(k_3,x_3,x_1) & & \end{pmatrix} \\ \begin{pmatrix} 1 & x_m & x_m^2 & \cdots & x_m^{n-2} \end{pmatrix} \end{vmatrix}$$

类似做法

对 $N(k_i,x_i,x_1)$ 作行变换

$$\xlongequal[i=3,\cdots,m]{\text{作行变换}}\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}\cdots(x_m-x_1)^{k_m-1}\begin{vmatrix} M_{k_1-1}^{n-1}(x_1) \\ M_{k_2}^{n-1}(x_2) \\ M_{k_3}^{n-1}(x_3) \\ \cdots \\ M^{n-1}(x_m) \end{vmatrix}$$

$$= \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i}\right] \cdot \left[\prod_{2 \leq j < i \leq m} (x_i - x_j)^{k_i k_j}\right] \cdot \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i (k_1 - 1)}\right]$$

$$= \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i k_1}\right] \left[\prod_{2 \leq j < i \leq m} (x_i - x_j)^{k_i k_j}\right]$$

$$= \prod_{1 \leq j < i \leq m} (x_i - x_j)^{k_i k_j}$$

□.