第7次作业

1.(柯 P_{81}. 1) 下述映射哪些是线性映射?

a) $[x_1, \cdots x_n] \xmapsto{\varphi} [x_n, \cdots x_1]$

$$\forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \alpha, \beta \in \mathbb{R} \quad \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha x_n + \beta y_n \\ \vdots \\ \alpha x_1 + \beta y_1 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_n \\ \vdots \\ x_1 \end{pmatrix} + \beta \begin{pmatrix} y_n \\ \vdots \\ y_1 \end{pmatrix}$$ 即 $\varphi(\alpha \vec{x} + \beta \vec{y}) = \alpha \varphi(\vec{x}) + \beta \varphi(\vec{y}) \Rightarrow$ 线性映射.

$$\varphi(\vec{e}_i) = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow (n-i+1) \\ \\ \\ \\ \end{array}$$ ∴ 矩阵为 $$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

b) $[x_1, \cdots x_n] \xmapsto{\varphi} [x_1, x_2^2 \cdots x_n^n]$

$$\forall \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n. \quad \alpha, \beta \in \mathbb{R}.$$

$$\varphi(\alpha \vec{x} + \beta \vec{y}) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ (\alpha x_2 + \beta y_2)^2 \\ \vdots \\ (\alpha x_n + \beta y_n)^n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha^2 x_2^2 \\ \vdots \\ \alpha^n x_n^n \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 2\alpha\beta x_2 y_2 \\ \vdots \\ \sum_{k=1}^{n-1} \binom{n}{k} (\alpha x_n)^k (\beta y_n)^{n-k} \end{pmatrix}}_{=\vec{z}} + \begin{pmatrix} \beta y_1 \\ \beta^2 y_2^2 \\ \vdots \\ \beta^n y_n^n \end{pmatrix}$$

$\neq \cancel{\alpha\varphi(\vec{x}) + \beta\varphi(\vec{y}) + \vec{z}}$ $\alpha\varphi(\vec{x}) + \beta\varphi(\vec{y})$ $\cancel{\because \vec{z} \neq \vec{0}}$ ∴ 不是线性映射.

c) $[x_1, \cdots x_n] \xmapsto{\varphi} [x_1, x_1 + x_2 \cdots x_1 + x_2 + \cdots + x_n]$

$$\forall \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \alpha, \beta \in \mathbb{R}.$$

$$\varphi(\alpha \vec{x} + \beta \vec{y}) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_1 + \beta y_1 + \cdots + \alpha x_n + \beta y_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y \\ \alpha(x_1 + x_2) + \beta(y_1 + y_2) \\ \vdots \\ \alpha(x_1 + \cdots + x_n) + \beta(y_1 + \cdots + y_n) \end{pmatrix} = \alpha \varphi(\vec{x}) + \beta \varphi(\vec{y})$$

∴ 为线性映射. $$\varphi(\vec{e}_i) = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{array} \begin{array}{l} \} \text{行} \\ \\ \leftarrow (i) \\ \\ \\ \\ \end{array}$$ ∴ 矩阵为 $$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

注:每个分量均是线性 即为线性映射.

2.(1)

$$A=\begin{pmatrix}1&3&5&-4&0\\1&3&2&-2&1\\1&-2&3&-1&-1\\1&2&1&-1&1\end{pmatrix}\to\begin{pmatrix}1&2&1&-1&1\\0&1&4&-3&-1\\0&0&-3&2&1\\0&-4&2&0&-2\end{pmatrix}\to\begin{pmatrix}1&2&1&-1&1\\0&1&4&-3&-1\\0&0&-3&2&1\\0&0&18&-12&-6\end{pmatrix}\to$$

$$\begin{pmatrix}1&2&1&-1&1\\0&1&4&-3&-1\\0&0&-3&2&1\\0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&2&4&-3&0\\0&1&1&-1&0\\0&0&-3&2&1\end{pmatrix}\to\begin{pmatrix}1&0&2&-1&0\\0&1&1&-1&0\\0&0&-3&2&1\end{pmatrix}$$

$\therefore \begin{cases}x_1=-2x_3+x_4\\x_2=-x_3+x_4\\x_5=3x_3-2x_4\end{cases}$ ⇒ 解空间基底为 $\left\{\begin{pmatrix}-2\\-1\\1\\0\\3\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\\-2\end{pmatrix}\right\}$ $\dim(V_{H}(A))=2.$ $(5-3)$

(2) 非齐次线性方程组增广矩阵 $B=(A\,\vdots\,b)=\left(\begin{array}{ccccc|c}1&3&5&-4&0&2\\1&3&2&-2&1&1\\1&-2&3&-1&-1&3\\1&2&1&-1&1&1\end{array}\right)$

$B\xrightarrow{\text{类似(1)}}\left(\begin{array}{ccccc|c}1&0&2&-1&0&2\\0&1&1&-1&0&0\\0&0&-3&2&1&-1\end{array}\right)$ 且相容

则解为 $\begin{pmatrix}2\\0\\0\\0\\-1\end{pmatrix}+\begin{pmatrix}-2\\-1\\1\\0\\3\end{pmatrix}\alpha+\begin{pmatrix}1\\1\\0\\1\\-2\end{pmatrix}\beta$ $(\alpha,\beta\in\mathbb{R})$. (特解+通解)

3. $A\in\mathbb{R}^{5\times7}$ $\overset{V_A}{(V_H(A))}=\langle\vec{v}_1,\vec{v}_2,\vec{v}_3\rangle$ 3个基底. $\Rightarrow \dim V_A=3 \Rightarrow \operatorname{rank}A=7-3=4.$

$\because 0\le\operatorname{rank}A\le5$ $\therefore 2\le 7-\operatorname{rank}A\le7$ $\Rightarrow 2\le\dim V_A\le7$ $\Rightarrow \dim V_A\ne1$

4. $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ 线性映射，ϕ 单 $\Rightarrow m\ge n$.

证. ϕ 单 $\Rightarrow \ker\phi=\{\vec 0\}\Rightarrow \dim(\operatorname{Im}\phi)=n$ 又$\because$ ~~di~~ $\operatorname{im}\phi\subseteq\mathbb{R}^m$ $\therefore\dim(\operatorname{Im}\phi)\le m$

即 $n\le m$.

[反证法] 假设 $m<n$ $\because$ ϕ 的矩阵 $A\in\mathbb{R}^{m\times n}$ $\therefore$ 以 A 为系数矩阵的齐次线性方程组一定有非平凡解 (未元个数 $n>$ 方程个数 $m\ge$ 阶 $\operatorname{rank}A$)

即 $\exists\ \vec{x}\in\mathbb{R}^n\setminus\{\vec 0\}$ s.t. $\phi(\vec x)=\vec 0$ 显然 $\phi(\vec 0)=\vec 0$ $\therefore$ ϕ 不是单射 $\rightarrow\leftarrow$. □.

5. (1) 设 $A \in \mathbb{R}^{m\times n}$ A增加一行 则秩或加1或不变.

(2) 设 $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank} A = r$. 则 $\forall$ s行组成一矩阵B, $\operatorname{rank} B \geq r+s-m$.

Pf (1). 考虑 A 的行空间 $V_r = \langle \vec{v}_1, \cdots \vec{v}_m \rangle \subseteq \mathbb{R}^n$ 则 $\operatorname{rank} A = \dim \langle \vec{v}_1, \cdots \vec{v}_m \rangle$

A增加一行 即行增加一行向量 $\vec{v}_{m+1}$ 则要么 $\vec{v}_{m+1} \in V_r$ 则 $\operatorname{rank} A = \dim V_r$ 不变.

要么 $\vec{v}_{m+1} \notin V_r$ 即 $\vec{v}_{m+1}$ 与 $\{\vec{v}_1, \cdots \vec{v}_m\}$ 线性无关 则 $\dim \langle \vec{v}_1, \cdots \vec{v}_{m+1} \rangle = \operatorname{rank} A + 1$.

(2). $B \in \mathbb{R}^{s\times n}$ 由 (1) B增加一行 rank B 不变或+1.

则 B 增加 $(m-s)$行 后得到 A $\Rightarrow$ $\operatorname{rank} B \leq \operatorname{rank} A \leq \operatorname{rank} B + m - s$

$\therefore$ $\operatorname{rank} B \geq r+s-m$ (略证).

6. (柯 P61.4) $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}$ $B = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}$

试用平面上 n条直线所成集合的几何性质 给出 $\operatorname{rank} A = \operatorname{rank} B$ 的充要条件.

解: 考虑 $A^T = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \vdots & \\ \alpha_n & \beta_n \end{pmatrix}$ $B^T = \left(\begin{array}{cc|c} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots \\ \alpha_n & \beta_n & \gamma_n \end{array}\right)$

$\uparrow$ 二元齐次线性方程组系数矩阵 $\uparrow$ 非齐次增广矩阵.

非齐次线性方程组相容 $\iff$ $\operatorname{rank} A^T = \underline{\operatorname{rank} A = \operatorname{rank} B} = \operatorname{rank} B^T$.

$\Updownarrow$

$\exists (x, y) \in \mathbb{R}^2$ s.t. $\alpha_i x + \beta_i y = \gamma_i$ $(i = 1, 2 \cdots n)$ 均成立.

$\Updownarrow$

平面上 n条直线 $\alpha_i x + \beta_i y = \gamma_i$ $(i = 1, 2 \cdots n)$ 有交点.

线性映射与矩阵运算.

Thm1 $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathbb{R}^{m\times n}$ 是双射.

$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \{\varphi: \mathbb{R}^n \to \mathbb{R}^m \mid \varphi$ 为线性映射$\}$，$\mathbb{R}^{m\times n} = \{m\times n$ 阶实矩阵$\}$

Pf. 给定 $\varphi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$，取 $\mathbb{R}^n$ 标准基 $\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$行 $(i=1,2,\cdots n)$

令 $\vec{A}^{(i)} = \varphi(\vec{e}_i) \in \mathbb{R}^m$，$A = (\vec{A}^{(1)} \ \cdots \ \vec{A}^{(n)}) \in \mathbb{R}^{m\times n}$

则定义 $\Phi(\varphi) = A$ （称 A 为 φ 的矩阵）

给定 $A \in \mathbb{R}^{m\times n}$ 令 $\vec{A}^{(1)}, \cdots \vec{A}^{(n)}$ 为 A 的全部列向量 则 $\vec{A}^{(i)} \in \mathbb{R}^m$ $(i=1,2\cdots n)$

构造映射 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 显然 φ 为线性映射.

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$ 且满足 $\varphi(\vec{e}_i) = \vec{A}^{(i)}$ $(i=1,2\cdots n)$

则定义 $\Psi(A) = \varphi$

下面考虑 $\Psi\circ\Phi: \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

对 $\forall\ \varphi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ $\Phi(\varphi) = A = (\varphi(\vec{e}_1) \ \cdots \ \varphi(\vec{e}_n))$

设 $\Psi(A) = \psi$ 即则 $\forall \vec{x} \in \mathbb{R}^n$ $\psi(\vec{x}) = x_1\varphi(\vec{e}_1) + \cdots + x_n\varphi(\vec{e}_n) = \varphi(\vec{x}) \Rightarrow \psi = \varphi$

$\therefore \Psi(\Phi(\varphi)) = \Psi\circ\Phi(\varphi) = \varphi \Rightarrow \Psi\circ\Phi = \mathrm{id}_{\mathrm{Hom}}$

再考虑 $\Phi\circ\Psi: \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$

对 $\forall\ A \in \mathbb{R}^{m\times n}$ $\Psi(A) = \varphi: \vec{x} \mapsto x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$

$\because \varphi(\vec{e}_i) = \vec{A}^{(i)}$ $(i=1,2\cdots n)$ $\therefore \Phi(\varphi) = A$ 即 $\Phi\circ\Psi(A) = A \Rightarrow \Phi\circ\Psi = \mathrm{id}_{\mathbb{R}^{m\times n}}$

综上 $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ 到 $\mathbb{R}^{m\times n}$ 为一个一一映射 □

Thm2 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 为线性映射 A 为 φ 的矩阵.

则 $\dim(\ker\varphi) + \dim(\mathrm{im}\,\varphi) = n$

$\dim(\ker\varphi) = \dim(V_A)$ $(V_A = \ker\varphi)$

$\dim(\mathrm{im}\,\varphi) = \dim(V_c(A)) = \mathrm{rank}\,A$ $(V_c(A) = \mathrm{im}\,\varphi)$

例：如何计算线性映射 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 的 $\ker\varphi$ 和 $\operatorname{Im}\varphi$ 的基底.

基本过程：1) 求 φ 的矩阵 $A \in \mathbb{R}^{m\times n}$

2) 通过初等行变换将 A 转化为阶梯形. 求 V_A 的一组基 ($\ker\varphi$)

3) 通过列初等列变换得到 $V_c(A)$ 的一组基 ($\operatorname{im}\varphi$)

设 $\varphi: \mathbb{R}^5 \longrightarrow \mathbb{R}^4$　　求 $\ker\varphi$ 和 $\operatorname{im}\varphi$ 的一组基.

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} \longmapsto \begin{pmatrix} x_1+x_2+x_3+x_4+x_5\\ 3x_1+2x_2+x_3+x_4-3x_5\\ x_2+2x_3+2x_4+6x_5\\ 5x_1+4x_2+3x_3+3x_4-x_5 \end{pmatrix}$$

解：φ 的矩阵 $A = \begin{pmatrix} 1&1&1&1&1\\ 3&2&1&1&-3\\ 0&1&2&2&6\\ 5&4&3&3&-1 \end{pmatrix}$

行变换：$A \longrightarrow \begin{pmatrix} 1&1&1&1&1\\ 0&-1&-2&-2&-6\\ 0&1&2&2&6\\ 0&-1&-2&-2&-6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&1&1&1&1\\ 0&1&2&2&6\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&-1&-1&-5\\ 0&1&2&2&6\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$

即 $\begin{cases} x_1 = x_3+x_4+5x_5\\ x_2 = -2x_3-2x_4-6x_5 \end{cases}$　得到 V_A 一组基：$\left\{ \begin{pmatrix} 1\\-2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\-2\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 5\\-6\\0\\0\\1 \end{pmatrix} \right\}$

$\therefore \ker\varphi = \left\langle \begin{pmatrix} 1\\-2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\-2\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 5\\-6\\0\\0\\1 \end{pmatrix} \right\rangle$

列变换：$A \rightarrow \begin{pmatrix} 1&0&0&0&0\\ 3&-1&-2&-2&-6\\ 0&1&2&2&6\\ 5&-1&-2&-2&-6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&0&0&0\\ 3&-1&0&0&0\\ 0&1&0&0&0\\ 5&-1&0&0&0 \end{pmatrix}$

$\therefore V_c(A) = \operatorname{im}\varphi = \left\langle \begin{pmatrix} 1\\3\\0\\5 \end{pmatrix}, \begin{pmatrix} 0\\-1\\1\\-1 \end{pmatrix} \right\rangle$.

(求 $\operatorname{im}\varphi$ 的另一种方法：找 $\ker\varphi$ 的直和补：$\left\langle \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix} = \vec{v}_1, \begin{pmatrix} 0\\1\\0\\0\\0 \end{pmatrix} = \vec{v}_2 \right\rangle$　则 $\varphi(\vec{v}_1), \varphi(\vec{v}_2)$ 构成 $\operatorname{im}\varphi$ 的基

$\therefore \operatorname{im}\varphi = \left\langle \begin{pmatrix} 1\\3\\0\\5 \end{pmatrix}, \begin{pmatrix} 1\\2\\1\\4 \end{pmatrix} \right\rangle$)

注：设 $\{\vec{v}_1, \cdots \vec{v}_r\}$ 构成 $\ker\varphi$ 的直和补的一组基 则 $\{\varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r)\}$ 为 $\mathrm{im}\varphi$ 的基.

Pf. 设 $V = \langle \vec{v}_1, \cdots \vec{v}_r \rangle$ 则 $\ker\varphi \oplus V = \mathbb{R}^n$

线性无关：设 $\alpha_1, \cdots \alpha_r \in \mathbb{R}$ st $\alpha_1\varphi(\vec{v}_1) + \cdots + \alpha_r\varphi(\vec{v}_r) = \vec{0}$ ($\in \mathbb{R}^m$)

$\because$ φ是线性映射 $\therefore$ $\varphi(\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r) = \vec{0} \Rightarrow \alpha\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \in \ker\varphi$.

又 $\because$ $\alpha\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \in V$ 且 $V \cap \ker\varphi = \{\vec{0}\}$ $\therefore$ $\alpha\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0}$.

且 $\vec{v}_1, \cdots \vec{v}_r$ 线性无关（基） $\therefore$ $\alpha_1 = \cdots = \alpha_r = 0$ $\therefore$ $\varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r)$无关.

$\mathrm{im}\varphi = \langle \varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r) \rangle$：显然 $\varphi(\vec{v}_i) \in \mathrm{im}\varphi$ ($i = 1, 2 \cdots r$)

$\therefore$ $\langle \varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r) \rangle \subseteq \mathrm{im}\varphi$ 且 $\dim(\mathrm{im}\varphi) = n - \dim(\ker\varphi) = \dim V = r$

$\dim \langle \varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r) \rangle = r$ $\therefore$ $\mathrm{im}\varphi = \langle \varphi(\vec{v}_1), \cdots \varphi(\vec{v}_r) \rangle$ □.

矩阵乘法.

设 $A \in \mathbb{R}^{m\times s}$ $B \in \mathbb{R}^{s\times n}$ 则 $\varphi_A \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^m)$, $\varphi_B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$.

定义 $A\cdot B$ 为复合映射 $\varphi_A \circ \varphi_B$ 对的矩阵.（即 $\varphi_A \circ \varphi_B = \varphi_{AB}$）.

则 $A\cdot B \in \mathbb{R}^{m\times n}$ 且若 $A = (a_{ij})_{m\times s}$, $B = (b_{ij})_{s\times n}$ 则 $A\cdot B = (c_{ij})_{m\times n}$ 满足：

$$c_{ij} = \sum_{k=1}^{s} \cancel{b_{ik}}\, a_{ik}\cdot b_{kj} \quad (i = 1, 2 \cdots m,\ j = 1, 2 \cdots n)$$

（矩阵乘法即是"一行乘一列" $c_{ij} = (a_{i1} \cdots a_{is}) \cdot \begin{pmatrix} b_{1j} \\ \vdots \\ b_{sj} \end{pmatrix}$,

Prop 1) 结合律： $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times k}$ $C \in \mathbb{R}^{k\times n}$

$(AB)\cdot C = A\cdot(B\cdot C)$ （映射复合满足结合律）

2) 分配律 $A, B \in \mathbb{R}^{m\times s}$, $C \in \mathbb{R}^{s\times n}$

$(A+B)\cdot C = A\cdot C + B\cdot C$

$A, B \in \mathbb{R}^{m\times s}$. $C \in \mathbb{R}^{k\times m}$

$C\cdot(A+B) = C\cdot A + C\cdot B$

3) 数乘交换 $A \in \mathbb{R}^{m\times s}$. $B \in \mathbb{R}^{s\times n}$ $\alpha(A\cdot B) = A\cdot(\alpha B)$

4) 转置 ： $(AB)^t = B^t \cdot A^t$

eg. $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

5). 交换律不成立. 消去律不成立 ！

ie. $A\cdot B$ 不一定有意义 且一般即使有意义 $AB \neq BA$. $A\cdot C = B\cdot C \not\Rightarrow A = B$