中国科学院大学

课程编号：B01GB001Y-B02

课程名称：线性代数 I-B（期中）

试题专用纸

任课教师：李子明

注意事项：

1.考试时间为 120 分钟，考试方式 闭 卷；

2.全部答案写在答题纸上；

3.考试结束后，请将本试卷和答题纸、草稿纸一并交回。

1. (15分) 计算齐次线性方程组

$$\begin{cases} x_1 - x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - x_4 &= 0 \end{cases}$$

解空间的维数和一组基.

2. (15分) 确定下列置换的阶数并判定其奇偶性:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 6 & 10 & 7 & 4 & 5 & 9 & 2 & 1 \end{pmatrix}.$$

3. (15分)

(i) 计算 $\gcd(49,35), \operatorname{lcm}(49,35)$ 和一对整数 m,n 使得

$$49m + 35n = \gcd(49,35).$$

(ii) 设:

$$S = \{(a,b) \,|\, a,b \in \mathbb{Z} \text{ 且 } 49a = 35b\}.$$

请问 S 是不是 $\mathbb{R}^{1\times 2}$ 的子空间? 并说明理由.

4. (15分) 设 $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 是线性映射，向量 $\vec{u}, \vec{v} \in \mathbb{R}^n$. 如果 $\vec{u} - \vec{v} \in \ker(\phi)$, 则称 $\vec{u}$ 和 $\vec{v}$ 是关于 ϕ 等价的，记为 $\vec{u} \sim_\phi \vec{v}$.

(i) 证明: $\sim_\phi$ 是等价关系.

(ii) 证明: 当 $m < n$ 时, 存在 $\vec{u}, \vec{v} \in \mathbb{R}^n$, 使得 $\vec{u} \neq \vec{v}$ 且 $\vec{u} \sim_\phi \vec{v}$.

5. (10分) 设 $\vec{x}_1, \ldots, \vec{x}_k$ 是 $\mathbb{R}^n$ 中的向量, 其中 $k > 2$. 令

$$\vec{y}_1 = \vec{x}_1 + \vec{x}_2,\ \vec{y}_2 = \vec{x}_2 + \vec{x}_3,\ \ldots,\ \vec{y}_{k-1} = \vec{x}_{k-1} + \vec{x}_k,\ \vec{y}_k = \vec{x}_k + \vec{x}_1.$$

(i) 证明: 如果 $\vec{x}_1,\ldots,\vec{x}_k$ 线性相关, 则 $\vec{y}_1,\ldots,\vec{y}_k$ 也线性相关.

(ii) 当 $\vec{x}_1,\ldots,\vec{x}_k$ 线性无关时, $\vec{y}_1,\ldots,\vec{y}_k$ 一定线性无关吗? 如果是, 请证明; 否则, 请举出反例.

6. (10分) 设 $A_n=(a_{ij})$, 其中 $a_{ij}=\max(i,j)$, $i=1,\ldots,n$, $j=1,\ldots,n$.

(i) 计算 A_2 和 A_3 的秩.

(ii) 计算 A_n 的秩.

7. (10分) 设: $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ 是线性映射, V 是 $\mathbb{R}^m$ 的子空间, $U=\phi^{-1}(V)$.

(i) 证明: U 是 $\mathbb{R}^n$ 的子空间.

(ii) 证明: 如果 ϕ 是满射, 则 $\dim U\geq\dim V$.

(iii) 举例说明当 ϕ 不是满射时不等式 $\dim U\geq\dim V$ 可能不成立.

8. (10分)

(i) 设 U 是 $\mathbb{R}^n$ 的 $n-1$ 维子空间. 证明存在实系数的齐次线性方程使得 U 是该方程的解空间.

(ii) 设 V 是 $\mathbb{R}^n$ 的 d 维子空间, 其中 $0\leq d<n-1$. 证明: 存在 $\mathbb{R}^n$ 中 $n-1$ 维的子空间 $U_1,\ldots,U_{n-d}$ 使得

$$V=\bigcap_{i=1}^{n-d}U_i.$$

"一"期中考试

1. $\dim V_H = 1 \qquad V_H = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

2. $\pi = \underbrace{(1\,3\,6\,4\,10)}_{5}\underbrace{(2\,8\,9)}_{3}\underbrace{(5\,7)}_{2}$

$\mathrm{ord}(\pi) = \mathrm{lcm}(5,3,2) = 30$， $\varepsilon_\pi = (-1)^{4+2+1} = -1$

奇置换

3. $\underbrace{(-2)}_{m} \, 49 + \underbrace{3}_{n} \cdot 35 = 7 = \gcd(49,35)$

$m = -2+5k, \quad n = 3-7k \quad (k \in \mathbb{Z})$

$\mathrm{lcm}(49,35) = \dfrac{49 \times 35}{7} = 245$

设 $S \subset \mathbb{Z} \times \mathbb{Z}$ 　 $\sqrt{2}S \not\subset \mathbb{Z}\times\mathbb{Z}$，$S$ 不是 $\mathbb{R}$ 上的子空间

4. 等价关系验证 ✓

(5) (i) 法1（利用引理X）

设 $V = \langle \vec{x}_1, \dots, \vec{x}_k \rangle$. 因为 $\vec{x}_1, \dots, \vec{x}_k$ 线性相关 ①

所以 $\dim V < k$. 因为 $\vec{y}_1, \dots, \vec{y}_k \in V$

所以 $\vec{y}_1, \dots, \vec{y}_k$ 线性相关.

法2（利用线性组合引理）

因为 $\vec{x}_1, \dots, \vec{x}_k$ 线性相关. 所以不妨设

$\vec{x}_k \in \langle \vec{x}_1, \dots, \vec{x}_{k-1} \rangle$

因为 $\vec{y}_1, \dots, \vec{y}_k \in \langle \vec{x}_1, \dots, \vec{x}_k \rangle$

所以 $\vec{y}_1, \dots, \vec{y}_k \in \langle \vec{x}_1, \dots, \vec{x}_{k-1} \rangle$

即 $\vec{y}_1, \dots, \vec{y}_k$ 都是 $\vec{x}_1, \dots, \vec{x}_{k-1}$ 的线性组合

于是 $\vec{y}_1, \dots, \vec{y}_k$ 线性相关（线性组合引理）

(ii) 法1 当 $k=4$ 时

$\underbrace{(\vec{x}_1 + \vec{x}_2)}_{\vec{y}_1} - \underbrace{(\vec{x}_2 + \vec{x}_3)}_{\vec{y}_2} + \underbrace{(\vec{x}_3 + \vec{x}_4)}_{\vec{y}_3} - \underbrace{(\vec{x}_4 + \vec{x}_1)}_{\vec{y}_4} = \vec{0}$

$\Rightarrow \vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4$ 线性相关

证2 设 $\alpha_1,\dots,\alpha_k \in \mathbb{R}$ 使得

$$\alpha_1\vec{y}_1+\dots+\alpha_k\vec{y}_k=\vec{0}$$

$$\Longleftrightarrow \alpha_1(\vec{x}_1+\vec{x}_2)+\alpha_2(\vec{x}_2+\vec{x}_3)+\dots+\alpha_k(\vec{x}_k+\vec{x}_1)=\vec{0}$$

$$\Longleftrightarrow (\alpha_1+\alpha_k)\vec{x}_1+(\alpha_1+\alpha_2)\vec{x}_2+\dots+(\alpha_{k-1}+\alpha_k)\vec{x}_k=\vec{0}$$

$\because$ $\vec{x}_1,\dots,\vec{x}_k$ 线性无关

$\therefore$ (H) $\begin{cases}\alpha_1+\alpha_k=0\\ \alpha_1+\alpha_2=0\\ \vdots\\ \alpha_{k-1}+\alpha_k=0\end{cases}$

当 k 为偶数时 (H) 有非零解

$\Downarrow$

$\vec{y}_1,\dots,\vec{y}_k$ 线性相关

5(i) 的~~典型~~错误(I)

(i) $\alpha_1\vec{y}_1+\dots+\alpha_k\vec{y}_k=\vec{0}$

$$\Longleftrightarrow \alpha_1(\vec{x}_1+\vec{x}_2)+\alpha_2(\vec{x}_2+\vec{x}_3)+\dots+\alpha_k(\vec{x}_k+\vec{x}_1)=\vec{0}$$

$$\Longleftrightarrow (\alpha_1+\alpha_k)\vec{x}_1+(\alpha_1+\alpha_2)\vec{x}_2+\dots+(\alpha_{k-1}+\alpha_k)\vec{x}_k=\vec{0}$$

$\because$ $\vec{x}_1,\dots,\vec{x}_k$ 线性相关

$\alpha_1+\alpha_k$, $\alpha_1+\alpha_2$, $\dots$ $\alpha_{k-1}+\alpha_k$ 不全为零

当!

$\Rightarrow$ $\alpha_1,\dots,\alpha_k$ 不全为零

$\Rightarrow$ $\vec{y}_1,\dots,\vec{y}_k$ 线性相关

②

~~$\exists\ \alpha_1,\dots,\alpha_k\in\mathbb{R}$~~

(II) $\because$ $\vec{x}_1,\dots,\vec{x}_k$ 线性相关

$\therefore$ $\exists\ \alpha_1,\dots,\alpha_k\in\mathbb{R}$. ~~使得~~ 不全为零

使得 $\alpha_1\vec{x}_1+\dots+\alpha_k\vec{x}_k=\vec{0}$

设 $\beta_1,\dots,\beta_k\in\mathbb{R}$ 使得

$$\beta_1\vec{y}_1+\dots+\beta_k\vec{y}_k=\vec{0}$$

则 $(\beta_1+\beta_k)\vec{x}_1+\dots+(\beta_1+\beta_2)\vec{x}_2+\dots+(\beta_{k-1}+\beta_k)\vec{x}_k=\vec{0}$

$\boxed{\text{于是}}$ $\begin{cases}\beta_1+\beta_k=\alpha_1\\ \beta_1+\beta_2=\alpha_2\\ \vdots\\ \beta_{k-1}+\beta_k=\alpha_k\end{cases}$ $\because$ $\alpha_1,\dots,\alpha_k$ 不全为零

$\therefore$ $\beta_1,\dots,\beta_k$ - - -

错!

$\Rightarrow$ $\vec{y}_1,\dots,\vec{y}_k$ 线性相关

挽救:

① 考虑方程组 (L) $\begin{cases}\beta_1+\beta_k=\alpha_1\\ \beta_1+\beta_2=\alpha_2\\ \vdots\\ \beta_{k-1}+\beta_k=\alpha_k\end{cases}$

② 证明 (L) 相容

③ 推出 $\vec{x}_1,\dots,\vec{x}_k$ 线性相关

注 (L) 变元个数 = 方程个数 $\not\Rightarrow$ (L) 有解.

⑥ "="

$$A_n \xrightarrow[i=1,2,\dots,n-1]{r_i - r_{i+1}} \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ & & \ddots & & & & \\ & * & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ n & n & \cdots & \cdots & \cdots & n & n \end{pmatrix}$$

$\Rightarrow \operatorname{rank}(A_n) = n.$

⑦ (i) $U = \varphi^{-1}(V) = \{\vec{x} \in \mathbb{R}^n \mid \varphi(\vec{x}) \in V\}$

设 $\alpha_1, \alpha_2 \in \mathbb{R}$，$\vec{u}_1, \vec{u}_2 \in U$．则存在 $\vec{v}_1, \vec{v}_2 \in V$ 使得 $\varphi(\vec{u}_1) = \vec{v}_1$，$\varphi(\vec{u}_2) = \vec{v}_2$

$$\Rightarrow \underbrace{\varphi(\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2) = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2}_{\varphi\text{是线性的}} \underbrace{\in V}_{V\text{是子空间}}$$

$\Rightarrow \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 \in U$

$\Rightarrow U$ 是子空间

错误　把 φ^{-1} 看成映射．　$\vec{u}_1 = \varphi^{-1}(\vec{v}_1)$, $\vec{u}_2 = \varphi^{-1}(\vec{v}_2)$

$\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 = \alpha_1\varphi^{-1}(\vec{v}_1) + \alpha_2\varphi^{-1}(\vec{v}_2) = \varphi^{-1}(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) \in U$

(ii) 设 $\vec{v}_1, \dots, \vec{v}_d$ 是 V 的基　③

∵ φ 满 ∴ $\exists\, \vec{u}_1, \dots, \vec{u}_d \in U$ 使得

$\varphi(\vec{u}_i) = \vec{v}_i$，$i=1,\dots,d$

∵ $\vec{v}_1, \dots, \vec{v}_d$ 线性~~相关~~无关

∴ $\vec{u}_1, \dots, \vec{u}_d$ 线性无关　(命题3.1 (iii))

$\Rightarrow \dim U \geq d = \dim V$.

(iii) $\varphi: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$

设 $V = \mathbb{R}^2$, $U = \varphi^{-1}(V) = \mathbb{R}$.　$\dim\mathbb{R} < \dim\mathbb{R}^2$.

错误．$\varphi: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$　$\varphi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

φ 不是线性映射．

⑧. 设 V 的一组基是 $\vec{v}_1, \dots, \vec{v}_{n-1}$

令 $\vec{v}_j = \begin{pmatrix} v_{1j} \\ \vdots \\ v_{nj} \end{pmatrix}$，$j=1,2,\dots,n-1$

考虑方程组

$\{ a_1 v_{1j} + \cdots + a_n v_{nj} = 0$，$j=1,\dots,n-1$

其中 $a_1, \dots, a_n$ 是未知数

该方程组有n-1个方程. n个未知数.

所以有非零解 $\alpha=(\alpha_1, \alpha_2, \cdots \alpha_n)^t$.

令 则 $\vec{v}_1, \cdots \vec{v}_{n-1}$ 是

(H) $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ 的解

设 (H) 的解空间为 V_α. 则

$\dim V_\alpha \leq n-1$ （$\because \alpha_1 \cdots \alpha_n$ 不全为零）

$\vec{v}_1, \cdots \vec{v}_{n-1} \in V_\alpha \Rightarrow V \subset V_\alpha$.

$\Rightarrow V = V_\alpha$. □

(ii) 设 $\vec{v}_1, \cdots, \vec{v}_d$ 是 V 的一组基

$$\vec{v}_j = \begin{pmatrix} v_{1j} \\ \vdots \\ v_{nj} \end{pmatrix} \quad j=1,2,\cdots d$$

考虑方程组

$$a_1 v_{1j} + \cdots + a_n v_{nj} = 0,$$

$j=1,2,\cdots d$, $a_1, \cdots a_n$ 是未知数

即

$$(H) \quad \underbrace{\begin{pmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{21} & v_{22} & \cdots & v_{n2} \\ & & \cdots & \\ v_{d1} & v_{2d} & \cdots & v_{nd} \end{pmatrix}}_{B} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

则 $B^t = (\vec{v}_1, \cdots \vec{v}_d) \Rightarrow \mathrm{rank}(B^t) = d \Rightarrow \mathrm{rank}(B) = d$ ④

$\Rightarrow \dim V_B = n-d$ （定理2.4）

设 $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \cdots \begin{pmatrix} a_{1,n-d} \\ \vdots \\ a_{n,n-d} \end{pmatrix}$ 是 V_B 的一组基

考虑方程组

~~$a_{11}x_1, a_{21}, \cdots a_{1,n-d}$~~

$$(H) \begin{cases} a_{11}x_1 + a_{21}x_1 + \cdots + a_{n1}x_n = 0 \\ \\ a_{1,n-d}x_1 + a_{n-d,2}x_2 + \cdots + a_{n-d,n}x_n = 0 \end{cases}$$

则 $V \subset V_H$ 且 $\dim V_H = n-d$

$\Rightarrow V = V_H = \bigcap_{i=1}^{n-d} V_i$. 其中 V_i

是 $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0$ 的解空间 □

另法(ii) 设 $\vec{v}_1, \cdots \vec{v}_d$ 是 V 的一组基

由基扩充定理.

$\vec{v}_1, \cdots \vec{v}_d, \vec{v}_{d+1}, \cdots \vec{v}_n$

是 V 的一组基.

设 $V_i = \langle \vec{v}_1, \cdots \vec{v}_d, \vec{v}_{d+1}, \cdots \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots \vec{v}_n \rangle$

$i = d+1, \cdots n$

则 $\dim V_i = n-1$. 令 $U = \bigcap_{i=d+1}^{n} V_i$

$\because V \subset V_i,\ i = d+1, \cdots n \quad \therefore V \subset U$.

反之 设 $\vec{u} \in U$. 则 $\exists \alpha_1, \cdots \alpha_d, \alpha_{d+1}, \cdots \alpha_n \in \mathbb{R}$

使得 $\vec{u} = \alpha_1 \vec{v}_1 + \cdots + \alpha_d \vec{v}_d + \alpha_{d+1} \vec{v}_{d+1} + \cdots + \alpha_i \vec{v}_i + \cdots + \alpha_n \vec{v}_n$

$\because \vec{u} \in V_i,\ \vec{u} = \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_d + \beta_{d+1} \vec{v}_{d+1} + \cdots + 0 \cdot \vec{v}_i + \cdots + \beta_n \vec{v}_n$

其中 $\beta_i \in \mathbb{R}$.

$\Rightarrow (\alpha_1 - \beta_1)\vec{v}_1 + \cdots (\alpha_d - \beta_d)\vec{v}_d + (\alpha_{d+1} - \beta_{d+1})\vec{v}_{d+1} + (\alpha_i - 0)\vec{v}_i$

$+ (\alpha_n - \beta_n)\vec{v}_n = \vec{0}$

$\Rightarrow \alpha_i = 0, \quad i = d, d+1, \cdots n$

$\Rightarrow \vec{u} = \alpha_1 \vec{v}_1 + \cdots + \alpha_d \vec{v}_d \Rightarrow \vec{u} \in V$

$\Rightarrow U = V$ ▨

作业

1. $\varphi: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \end{pmatrix}$$

① 验证 φ 线性. ② 求 A_φ ③ 求 $\ker(\varphi)$ 和 $\mathrm{im}(\varphi)$ 的基.

解 ① 因为像的每个坐标都是线性函数 所以 φ 是线性的 (命题3.2)

②

$$A_\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

~~rank A_φ~~

③ $\ker(\varphi) = V_{A_\varphi}$

考虑方程组 $A_\varphi \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow V_{A_\varphi} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \ker(\varphi) = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

$\operatorname{im}(\varphi) = V_c(A_\varphi) \Rightarrow \dim \operatorname{im}(\varphi) = 2$

（定理3.1）

$\operatorname{im}(\varphi) = \langle \vec{A}^{(1)}, \vec{A}^{(2)} \rangle = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \rangle$ ☒

矩阵代数，中心元，可逆元

设 $A, B \in M_n(\mathbb{R})$ 一般而言 $AB \neq BA$

定义：设 $A = (a_{ij})_{n\times n}$. A的迹定义为

$a_{11} + a_{22} + \cdots + a_{nn}$

记为 $\operatorname{tr}(A)$. tr … trace

例：设 $\alpha, \beta \in \mathbb{R}$. $A, B \in M_n(\mathbb{R})$

证明 $\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B)$.

证：设 $A = (a_{ij})$, $B = (b_{ij})$

$\operatorname{tr}(\alpha A + \beta B) = \operatorname{tr}((\alpha a_{ij} + \beta b_{ij}))$

$= \sum_{i=1}^{n} (\alpha a_{ii} + \beta b_{ii}) = \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii}$

$= \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B)$.

命题：设 $A, B \in M_n(\mathbb{R})$ 证明 ⑥

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

证明：设 $C = (c_{ij}) = AB$

$D = (d_{ij}) = BA$

则 $c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$. $d_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}$

$\operatorname{tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{k=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$

$\operatorname{tr}(D) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki} b_{ik}$

$\Rightarrow \operatorname{tr}(C) = \operatorname{tr}(D)$

注：tr 是交换不变量

注：rank 不是交换不变量

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\operatorname{rank}(AB) = 1$, $\operatorname{rank}(BA) = 0$

例：证明不存在 $A,B \in M_n(\mathbb{R})$. 使得

$$AB-BA=E$$

证：设 $\exists A,B \in M_n(\mathbb{R})$ 使得上式成立

则 $tr(AB-BA)=tr(E)=n$

$tr(AB)-tr(BA)=n$

$\overset{\shortparallel}{0}$ $\rightarrow\leftarrow$. ☒

回忆：$M_n(\mathbb{R})$ 消去律不成立

但如果 A 可逆，， $B, C \in \cancel{M_n(\mathbb{R})}\ \mathbb{R}^{m\times n}$

$$AB=AC \Rightarrow B=C$$

验证：$A^{-1}(AB)=A^{-1}(AC)$

$$\Rightarrow (A^{-1}A)B=(A^{-1}A)C \Rightarrow B=C.$$

同样，如果 $A \in M_n(\mathbb{R})$

$$BA=CA \Rightarrow B=C.$$

也可以利用满秩证明

例：设 $A \in M_n(\mathbb{R})$ 可逆

则 A^t 可逆且 $(A^t)^{-1}=(A^{-1})^t$

证：$(A^{-1})^t A^t=(AA^{-1})^t=E^t=E$

由推论5.2 $(A^{-1})^t=(A^t)^{-1}$. ☒

例：设 $A \in M_n(\mathbb{R})$ 且 $E+A$ 可逆

证明 $(E+A)^{-1}(E-A)=(E-A)(E+A)^{-1}$

证：由"消去律"，只要证

$$(E+A)\left[(E+A)^{-1}(E-A)\right]=(E+A)\left[(E-A)(E+A)^{-1}\right]$$

左 $=E-A$ 右 $=(E+A)(E-A)(E+A)^{-1}$

$(E+A)(E-A)=E^2-A^2$ $(\because EA=AE)$

$$=(E-A)(E+A)$$

右 $=E-A$

左 $=$ 右 ☒

定义：设 $A \in M_n(\mathbb{R})$. 如果 $k \in \mathbb{Z}^+$ 使得 $A^k = O$，则称 A 是幂零元

例 $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

例：设 $A \in M_n(\mathbb{R})$ 幂零. 证明 $E-A$ 可逆

证：设 $A^k = O$

$k=1$. $A = O$ $\quad E - A = E$ 可逆

$k=2$ $A^2 = O$ $\quad E = E - A^2 = (E-A)(E+A)$

$(\because AE = EA)$

$\Rightarrow E-A$ 可逆

$k=3$ $A^3 = O$ $\quad E = E - A^3 = (E-A)(E+A+A^2)$

$\Rightarrow E-A$ 可逆

$E = E - A^k = (E-A)(E+A+\cdots+A^{k-1})$

$\Rightarrow E-A$ 可逆.

例：设 $A \in \mathbb{R}^{m \times s}$, $B \in \mathbb{R}^{s \times n}$

证明：如果 $AB = O_{m\times n}$. 则

$\mathrm{rank}(A) + \mathrm{rank}(B) \leq s$

证：法1 (Sylvester's inequality)

$O = \mathrm{rank}(AB) \leq \mathrm{rank}(A) + \mathrm{rank}(B) - s$

$\Rightarrow \mathrm{rank}(A) + \mathrm{rank}(B) \leq s$

法2 $\ker + \mathrm{im}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_B"] \arrow[dr, "\varphi_{AB}"'] & \mathbb{R}^s \arrow[d, "\varphi_A"] \\ & \mathbb{R}^m \end{tikzcd}$$

$\varphi_{AB} = O$

$\forall \vec{x} \in \mathbb{R}^n$

$\varphi_A(\varphi_B(\vec{x})) = \vec{0}_m$

$\Rightarrow \mathrm{im}(\varphi_B) \subset \ker(\varphi_A)$

$\Rightarrow \mathrm{rank}\, B \leq \dim \ker(\varphi_A)$

$= s - \mathrm{rank}(A)$

$\Rightarrow \mathrm{rank}(A) + \mathrm{rank}(B) \leq s$ $\square$

期中试题

1. $\begin{cases} x_1 - x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \end{cases}$ 求解空间一组基和维数.

解：系数矩阵为 $\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \rightarrow$

$\begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

$\therefore \begin{cases} x_1 = x_4 \\ x_2 = -x_4 \\ x_3 = x_4 \\ x_4 = x_4 \end{cases}$ $\therefore \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ 为解空间一组基. 解空间维数为 1.

2. 求置换 $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 6 & 10 & 7 & 4 & 5 & 9 & 2 & 1 \end{pmatrix}$ 阶数和符号.

解：$\sigma = (1\ 3\ 6\ 4\ 10)(2\ 8\ 9)(5\ 7)$

$\therefore$ 阶为 $\mathrm{lcm}(5,3,2) = 30$

符号为 $(-1)^{5-1+3-1+2-1} = -1$ $\therefore$ σ 为奇置换.

3. i) 计算 $\gcd(49,35)$ $\mathrm{lcm}(49,35)$ 和 $m,n \in \mathbb{Z}$ s.t. $49m + 35n = \gcd(49,35)$.

ii) $S = \{(a,b) \in \mathbb{Z}^2 \mid 49a = 35b\}$ 是否为 $\mathbb{R}^2$ 子空间?

解 i). $49 = 1\cdot 35 + 14$
$35 = 2\cdot 14 + 7$
$14 = 2\cdot 7 + 0$
$\Rightarrow$ $\gcd(49,35) = 7$, $\mathrm{lcm}(49,35) = \dfrac{49\cdot 35}{7} = 245$

$m_2 = m_0 - q_1 m_1 = 1 - 0 = 1$ $\quad n_2 = n_0 - q_1 n_1 = 0 - 1 = -1$

$m_3 = m_1 - q_2 m_2 = 0 - 2 = -2$ $\quad n_3 = n_1 - q_2 n_2 = 1 - 2\cdot(-1) = 3$

$\therefore (-2)\cdot 49 + 3\cdot 35 = 7 = \gcd(49,35)$ $\quad -2\cdot 7 + 3\cdot 5 = 1$

即 $\begin{cases} m = -2 \\ n = 3 \end{cases}$ 满足 $49m + 35n = \gcd(49,35)$. $\quad (-2+5k)\cdot 7 + (3-7k)\cdot 5 = 1$

ii) 不是 S 对数乘不封闭.

4. $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性映射. $\vec{u}, \vec{v} \in \mathbb{R}^n$. 如果 $\vec{u}-\vec{v} \in \ker\phi$ 称记 $\vec{u} \sim_\phi \vec{v}$.

i) $\sim_\phi$ 是等价关系.

ii) $m<n$ 时 $\exists\ \vec{u}, \vec{v} \in \mathbb{R}^n$ s.t. $\vec{u} \neq \vec{v}$ 且 $\vec{u} \sim_\phi \vec{v}$.

Pf i) 反身性: $\forall\ \vec{u} \in \mathbb{R}^n$ $\vec{u}-\vec{u} = \vec{0}$ $\therefore \phi(\vec{0}) = \vec{0}$ $\Rightarrow \vec{u}-\vec{u} \in \ker\phi$ $\therefore \vec{u} \sim_\phi \vec{u}$.

~~对~~ 对称性: 如果 $\vec{u}, \vec{v} \in \mathbb{R}^n$ 满足 $\vec{u} \sim_\phi \vec{v}$ 则 $\vec{u}-\vec{v} \in \ker\phi \Rightarrow \phi(\vec{u}-\vec{v}) = \vec{0}$

$\Rightarrow \phi(\vec{u}) = \phi(\vec{v}) \Rightarrow \phi(\vec{v}-\vec{u}) = \vec{0} \Rightarrow \vec{v}-\vec{u} \in \ker\phi \Rightarrow \vec{v} \sim_\phi \vec{u}$.

~~对~~ 传递性: 如果 $\vec{u} \sim_\phi \vec{v}$, $\vec{v} \sim_\phi \vec{w}$. 即 $\vec{u}-\vec{v} \in \ker\phi$. $\vec{v}-\vec{w} \in \ker\phi$.

则 $\phi(\vec{u}-\vec{w}) = \phi(\vec{u}-\vec{v}) + \phi(\vec{v}-\vec{w}) = \vec{0}+\vec{0} = \vec{0}$

$\Rightarrow \vec{u}-\vec{w} \in \ker\phi \Rightarrow \vec{u} \sim_\phi \vec{w}$

ii). 设线性映射 ϕ 对应的矩阵为 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$

如果 $m<n$ 则以 A 为系数矩阵的 n 元线性齐次方程组一定有非平凡解.

不妨设一组非平凡解为 $\vec{w} \in \mathbb{R}^n$ 则对 $\forall\ \vec{u} \in \mathbb{R}^n$ $\vec{u}+\vec{w} \sim_\phi \vec{u}$ 且 $\vec{u}+\vec{w} \neq \vec{u}$.

($\because \phi(\vec{u}+\vec{w}-\vec{u}) = \phi(\vec{w}) = A\cdot\vec{w} = \vec{0} \Rightarrow \vec{u}+\vec{w}-\vec{u} \in \ker\phi \Rightarrow \vec{u}+\vec{w} \sim_\phi \vec{u}$).

即 $\exists\ \vec{u} \in \mathbb{R}^n$. $\vec{v} = \vec{u}+\vec{w} \in \mathbb{R}^n$ s.t. $\vec{u} \neq \vec{v}$ 且 $\vec{u} \sim_\phi \vec{v}$

方法二: $\because n = \dim(\mathbb{R}^n) = \dim(\operatorname{im}\phi) + \dim(\ker\phi)$ $\dim(\operatorname{im}\phi) \leq \dim(\mathbb{R}^m) = m$ $\therefore \dim(\ker\phi) \geq n-m$. $\therefore \ker\phi$ 非平凡. $\Rightarrow \exists\ \vec{w} \neq \vec{0} \in \ker\phi$.

5. 设 $\vec{v}, \vec{x}_1 \cdots \vec{x}_k \in \mathbb{R}^n$, $k>1$ 且 $\vec{x}_1, \cdots \vec{x}_k$ 线性无关.

如果 $\vec{v} = \sum_{i=1}^{k} \alpha_i \vec{x}_i$ 且 $\alpha_k \neq 0$ 则 $\vec{v}, \vec{x}_1, \cdots \vec{x}_{k-1}$ 线性无关.

Pf. 设 $\beta_0, \beta_1, \cdots \beta_{k-1} \in \mathbb{R}$ s.t. $\beta_0 \vec{v} + \sum_{i=1}^{k-1} \beta_i \vec{x}_i = \vec{0}$

则 $\beta_0 \cdot \sum_{i=1}^{k} \alpha_i \vec{x}_i + \sum_{i=1}^{k-1} \beta_i \vec{x}_i = \vec{0} \Rightarrow \sum_{i=1}^{k-1} (\beta_0\alpha_i + \beta_i)\vec{x}_i + \beta_0 \alpha_k \vec{x}_k = \vec{0}$

$\because \vec{x}_1, \cdots \vec{x}_k$ 线性无关 则 $\begin{cases} \beta_0 \alpha_k = 0 \\ \beta_0 \alpha_i + \beta_i = 0 \quad (i=1,\cdots k-1) \end{cases} \xRightarrow{\alpha_k \neq 0} \begin{cases} \beta_0 = 0 \\ \beta_i = 0 \quad (i=1,\cdots k-1) \end{cases}$

$\therefore \vec{v}, \vec{x}_1, \cdots \vec{x}_{k-1}$ 线性无关.

6. $A=(a_{ij})_{n\times n}$ 为 n阶方阵 且 $a_{ij}=\max(i,j)=\begin{cases} j & (i\leq j)\\ i & (i>j)\end{cases}$ $(i,j\in\{1,2\cdots n\})$

1) 计算 rank A   2) 判断由 A (对应) 关于标准基的线性映射 φ 是否为双射

解：
1) $A=\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n\\ 2 & 2 & 3 & \cdots & n-1 & n\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ n-1 & n-1 & n-1 & \cdots & n-1 & n\\ n & n & n & \cdots & n & n\end{pmatrix} \xrightarrow[i=n,n-1,\cdots 2]{-(i)+(i-1)} \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0\\ -1 & -1 & 0 & \cdots & 0 & 0\\ -1 & -1 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -1 & -1 & -1 & \cdots & 0 & 0\\ -1 & -1 & -1 & \cdots & -1 & 0\\ n & n & n & \cdots & n & n\end{pmatrix}$

$\Rightarrow$ rank $A=n$

2). $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $[\vec{e}_i \longmapsto \vec{A}^{(i)}]$ $(i=1,2\cdots n$, $\vec{e}_i=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$ ←第i行, $\vec{A}^{(i)}$ 为 A 第i列)

~~$\vec{v} \longmapsto A\vec{v}$~~

∵ rank $A=n$ ∴ $\vec{A}^{(1)},\cdots \vec{A}^{(n)}$ 线性无关 $\Rightarrow$ $\vec{A}^{(1)},\cdots \vec{A}^{(n)}$ 可构成 $\mathbb{R}^n$ 一组基.

设 $\vec{v}=\sum_{i=1}^{n}x_i\vec{e}_i\in\mathbb{R}^n$ s.t. $\varphi(\vec{v})=\vec{0}$ 即 $\sum_{i=1}^{n}x_i\vec{A}^{(i)}=\vec{0}$ $\Rightarrow x_1=\cdots=x_n=0$

即 $\ker\varphi=\{\vec{0}\}$ ∴ φ 为单射.

对 $\forall\ \vec{w}\in\mathbb{R}^n$ 由于 $\vec{A}^{(1)},\cdots \vec{A}^{(n)}$ 构成 $\mathbb{R}^n$ 一组基，$\exists!\ y_1,\cdots y_n\in\mathbb{R}$ s.t.

$\vec{w}=\sum_{i=1}^{n}y_i\vec{A}^{(i)}$ 令 $\vec{v}=\sum_{i=1}^{n}y_i\vec{e}_i$ 则 $\varphi(\vec{v})=\vec{w}$ 即∃原像 $\Rightarrow$ φ 为满射.

∴ φ 是双射.

5. $\vec{x}_1,\cdots \vec{x}_k\in\mathbb{R}^n$ $(k>2)$ $\vec{y}_1=\vec{x}_1+\vec{x}_2$, $\vec{y}_2=\vec{x}_2+\vec{x}_3$, $\cdots$ $\vec{y}_k=\vec{x}_k+\vec{x}_1$.

(1) $\vec{x}_1,\cdots \vec{x}_k$ 线性相关 ∴ $\dim\overbrace{\langle \vec{x}_1,\cdots \vec{x}_k\rangle}^{=V}<k$.

且 $\vec{y}_i\in V$ ∴ $\dim\langle\vec{y}_1,\cdots \vec{y}_k\rangle\leq\dim V<k$ $\Rightarrow$ $\vec{y}_1,\cdots \vec{y}_k$ 线性相关.

(2). 不是 eg. $k=4$ $\vec{y}_1-\vec{y}_2+\vec{y}_3-\vec{y}_4=\vec{x}_1+\vec{x}_2-\vec{x}_2-\vec{x}_3+\vec{x}_3+\vec{x}_4-\vec{x}_4-\vec{x}_1=\vec{0}$

线性相关.

7. $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性映射. $V \subseteq \mathbb{R}^m$ 子空间. $U = \phi^{-1}(V)$ 证明.

(1) $U \subseteq \mathbb{R}^n$ 为子空间 (2) $\dim U \geq \dim(V \cap \operatorname{im}\phi)$

Pf. (1) $\forall\ \vec{u}_1, \vec{u}_2 \in U$. $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$. $\phi(\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2) = \alpha_1\phi(\vec{u}_1) + \alpha_2\phi(\vec{u}_2)$

$\because U = \phi^{-1}(V) := \{\vec{u} \in \mathbb{R}^n \mid \phi(\vec{u}) \in V\}$ $\therefore \phi(\vec{u}_1) \in V.\ \phi(\vec{u}_2) \in V$.

由于 V 是线性子空间 $\therefore \alpha_1\phi(\vec{u}_1) + \alpha_2\phi(\vec{u}_2) \in V$ $\therefore \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 \in \phi^{-1}(V)$.

(2) 设 $\varphi = \phi|_U$. 则 $\varphi: U \to V$ 为线性映射 ($\because \phi$ 是线性映射且 U, V 为子空间)

$\therefore \dim U = \dim(\operatorname{im}\varphi) + \dim(\ker\varphi)$

下面证: $\operatorname{im}\varphi = V \cap \operatorname{im}\phi$

$\forall\ \vec{y} \in \operatorname{im}\varphi$ 即 $\exists\ \vec{x} \in \cancel{\mathbb{R}^n}\ U$ s.t. $\vec{y} = \phi(\vec{x}) \Rightarrow \vec{y} \in \operatorname{im}\phi$ 且 $\vec{y} \in V$

$\therefore \vec{y} \in \operatorname{im}\phi \cap V$ $\forall\ \vec{y} \in \operatorname{im}\phi \cap V \Rightarrow \vec{y} \in \operatorname{im}\phi$ 且 $\vec{y} \in V \Rightarrow \exists\ \vec{x} \in \phi^{-1}(V)$

s.t. $\vec{y} = \phi(\vec{x})$ 即 $\exists\ \vec{x} \in U$ s.t. $\vec{y} = \phi(\vec{x}) = \varphi(\vec{x}) \Rightarrow \vec{y} \in \operatorname{im}\varphi$.

$\therefore \operatorname{im}\varphi = V \cap \operatorname{im}\phi$ $\therefore \dim(\operatorname{im}\varphi) = \dim(V \cap \operatorname{im}\phi)$. 且 $\dim(\ker\varphi) \geq 0$

$\therefore \dim U \geq \dim(V \cap \operatorname{im}\phi)$.

8. (i) $U \subseteq \mathbb{R}^n$ 为 $n-1$ 维子空间. 证明 $\exists$ ~~非平凡的线性方程~~ $a_1x_1 + \cdots + a_nx_n = 0$ $(*)$ s.t. U 是以方程的解空间.

(ii). $V \subseteq \mathbb{R}^n$ d 维子空间. $(0 \leq d < n-1)$ 证明 $\exists\ \mathbb{R}^n$ 中 $n-1$ 维子空间 $U_1, \cdots U_{n-d}$ s.t.

$$V = \bigcap_{i=1}^{n-d} U_i.$$

Pf. 设 $\left\{\vec{\beta}_i = \begin{pmatrix} b_{i1} \\ \vdots \\ b_{in} \end{pmatrix} \in \mathbb{R}^n \,\middle|\, i = 1, 2 \cdots n-1\right\}$ 为 U 的一组基. 如果 U 是

$a_1x_1 + \cdots + a_nx_n = 0$ 的解空间 则 $\sum_{j=1}^{n} a_j b_{ij} = 0$ $(i = 1, \cdots n-1)$

$\Rightarrow B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{(n-1)\times 1}$ $\Rightarrow \because \operatorname{rank} B = n-1 \therefore$ 有非平凡解 $\overset{s.t.}{\Rightarrow} U$ 为 $(*)$ 解空间

~~(ii)~~.

(ii) 设 $\{\vec{v}_i = \begin{pmatrix} c_{i1} \\ \vdots \\ c_{in} \end{pmatrix} \in \mathbb{R}^n \mid i=1,2\cdots d\}$ 为 V 的一组基. 令 $C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{d1} & \cdots & c_{dn} \end{pmatrix}_{d\times n}$

考虑 $C\cdot\vec{Y} = \vec{0}$ 的解空间. 首先该解空间为 $(n-d)$ 维 ($\because \operatorname{rank} C = d$)

则可找到该解空间的一组基材 $\{\vec{y}_j = \begin{pmatrix} y_{j1} \\ \vdots \\ y_{jn} \end{pmatrix} \in \mathbb{R}^n \mid j=1,2\cdots n-d\}$

则 $\sum_{k=1}^{n} c_{ik}\cdot y_{jk} = \vec{0}$ 对 $i=1,2\cdots d$, $j=1,2\cdots n-d$ 均成立.

$$\Rightarrow \begin{cases} y_{11}c_{i1} + y_{12}c_{i2} + \cdots + y_{1n}c_{in} = 0 \\ \vdots \\ y_{n-d,1}c_{i1} + y_{n-d,2}c_{i2} + \cdots + y_{n-d,n}c_{in} = 0 \end{cases} \quad \text{对 } i=1,2\cdots d \text{ 成立.}$$

$$\Rightarrow \vec{v}_i = \begin{pmatrix} c_{i1} \\ \vdots \\ c_{in} \end{pmatrix} \text{ 为 ~~齐次~~ 线性方程组 } \begin{cases} y_{11}x_1 + y_{12}x_2 + \cdots + y_{1n}x_n = 0 \\ \vdots \\ y_{n-d,1}x_1 + y_{n-d,2}x_2 + \cdots + y_{n-d,n}x_n = 0 \end{cases} (**) \text{ 的解.}$$
$(i=1,2\cdots d)$

即 V 是 $(**)$ 的解空间. 而对于每个方程 $y_{j1}x_1 + \cdots + y_{jn}x_n = 0$ $(j=1,\cdots n-d)$

其解空间为设为 U_j $\because \vec{y}_j \neq \vec{0}$ $\therefore \dim U_j = n-1$

则 $(**)$ 的解空间也可看成每个方程的公共解空间 即 $\bigcap_{j=1}^{d-n} U_j$

$\therefore V = \bigcap_{j=1}^{d-n} U_j$ □.

7. $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性. $V \subseteq \mathbb{R}^m$. $U = \phi^{-1}(V)$

i) $U \subseteq \mathbb{R}^n$ 子空间. ii) ϕ 满 $\Rightarrow \dim U \geq \dim V$. iii) 举例 不满足 $\dim U \geq \dim V$ 不成立.

Pf. i) $U = \phi^{-1}(V) = \{\vec{x} \in \mathbb{R}^n \mid \phi(\vec{x}) \in V\} \subseteq \mathbb{R}^n$.

$\forall \vec{x}, \vec{y} \in U$. $\alpha, \beta \in \mathbb{R}$. $\phi(\alpha\vec{x} + \beta\vec{y}) = \alpha\phi(\vec{x}) + \beta\phi(\vec{y})$

$\because \vec{x}, \vec{y} \in U$ 则 $\exists$ $\vec{u}, \vec{v} \in V$ s.t. $\phi(\vec{x}) = \vec{u}$. $\phi(\vec{y}) = \vec{v}$

$\therefore \phi(\alpha\vec{x} + \beta\vec{y}) \in V \Rightarrow \alpha\vec{x} + \beta\vec{y} \in \phi^{-1}(V) = U$.

ii) ~~ϕ满$\Rightarrow$~~ 设 $\{\vec{v}_1, \cdots \vec{v}_k\}$ 构成 V 的一组基. 由 ϕ 满 $\Rightarrow$ $\exists\, \vec{u}_i \in U$ s.t. $\phi(\vec{u}_i) = \vec{v}_i$.

$\because \vec{v}_1, \cdots \vec{v}_k$ 线性无关 $\Rightarrow$ $\vec{u}_1, \cdots \vec{u}_k$ 线性无关 且 $\langle \vec{u}_1 \cdots \vec{u}_k \rangle \subseteq U$ $\therefore \dim U \geq \dim V$

iii) 取入 $m > n$ 令 $V = \mathbb{R}^m$ 即可.

第8次作业.

1. $\varphi: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1+x_2 \\ x_1-x_3 \\ x_2+x_3 \\ x_1+2x_2+x_3 \end{pmatrix}$$

(1) φ 线性

(2) 求矩阵

(3) 求 $\ker\varphi$. $\mathrm{im}\varphi$ 一基.

解 (1) $\forall \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$. $\alpha, \beta \in \mathbb{R}$. $\varphi\left(\alpha\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} \alpha x_1+\beta y_1 \\ \alpha x_2+\beta y_2 \\ \alpha x_3+\beta y_3 \end{pmatrix}\right) = \begin{pmatrix} \alpha x_1+\beta y_1+\alpha x_2+\beta y_2 \\ \alpha x_1+\beta y_1-\alpha x_3-\beta y_3 \\ \alpha x_2+\beta y_2+\alpha x_3+\beta y_3 \\ \alpha x_1+\beta y_1+2\alpha x_2+2\beta y_2+\cdots \end{pmatrix}$

$$= \alpha\begin{pmatrix} x_1+x_2 \\ x_1-x_3 \\ x_2+x_3 \\ x_1+2x_2+x_3 \end{pmatrix} + \beta\begin{pmatrix} y_1+y_2 \\ y_1-y_3 \\ y_2+y_3 \\ y_1+2y_2+y_3 \end{pmatrix} = \alpha\varphi(\vec{x})+\beta\varphi(\vec{y}) \Rightarrow \text{线性映射}.$$

(2). $\varphi\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. $\varphi\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$. $\varphi\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ 矩阵为 $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ 设 $A\vec{x} = \vec{0}$ 则解空间 $V_A = \ker\varphi$.

$$A \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \xrightarrow{\cancel{x_1 = -x_2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow V_A = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{im}\varphi = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \mathrm{im}\varphi = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

错误点 ① $\because \dim(\mathrm{im}\varphi) = 2$ $(= \mathrm{rank}A)$

$\therefore \mathrm{im}\varphi = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ 没道理.

② $\mathrm{im}\varphi = \langle A^{(1)}, A^{(2)}, A^{(3)} \rangle$ 不能做初行变换再算.

2. (1) $\begin{pmatrix} -1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & -3 \\ \cancel{6}2 & 5 \end{pmatrix}$

(2) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\,\cancel{\cos}\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

(3) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$ 数学归纳法证明.

注 $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ 相当于逆时针旋转 α 角. $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto A\vec{x}$

3. $A=(a_{ij})_{m\times n}$ $\quad J_m=\begin{pmatrix}0&1&&\\&0&1&\\&&\ddots&\ddots\\&&&1\\&&&0\end{pmatrix}_{m\times m}$ $\quad J_n=\begin{pmatrix}0&1&&\\&0&1&\\&&\ddots&\ddots\\&&&1\\&&&0\end{pmatrix}_{n\times n}$

$$J_m\cdot A=\begin{pmatrix}a_{21}&a_{22}&\cdots&a_{2n}\\a_{31}&a_{32}&\cdots&a_{3n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\\0&0&&0\end{pmatrix}\qquad AJ_n=\begin{pmatrix}0&a_{11}&a_{12}&\cdots&a_{1\,n-1}\\0&a_{21}&a_{22}&\cdots&a_{2\,n-1}\\\vdots&\vdots&\vdots&&\vdots\\0&a_{m1}&a_{m2}&\cdots&a_{m,n-1}\end{pmatrix}$$

注：左乘行变换 右乘列变换.

初等矩阵（初等变换矩阵）

1. 互换两行（列）：

$$A_1=\begin{array}{c}\begin{pmatrix}1&&&&\\&\ddots&&&\\&&0&1&\\&&&0&\\&&&&\ddots\\&&&&&1\end{pmatrix}\begin{array}{l}\\\\\leftarrow i行\\\leftarrow j行\\\\\\\end{array}\\ \qquad\quad\uparrow i列\ \ \uparrow j列\end{array}$$

2. 某行（列）乘λ（$\lambda\neq0$）：

$$A_2=\begin{array}{c}\begin{pmatrix}1&&&\\&\ddots&&\\&&\lambda&\\&&&\ddots\\&&&&1\end{pmatrix}\begin{array}{l}\\\\\leftarrow i行\\\\\\\end{array}\\ \uparrow i列\end{array}$$

3. 第i行乘λ加到j行：
(j列) (i列)

$$A_3=\begin{array}{c}\begin{pmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&\lambda&1&\\&&&&\ddots\\&&&&&1\end{pmatrix}\begin{array}{l}\\\\\leftarrow i行\\\leftarrow j行\\\\\\\end{array}\\ \qquad\uparrow i列\ \ \uparrow j列\end{array}$$

<u>Thm</u> 对矩阵B作初等行变换 相当于对B左乘若干初等矩阵.
(列) (右)

注：初等变换不改变矩阵的秩

$\therefore$ rank(AB) = rank(BA) = rank B，其中A是若干初等矩阵的乘积.