第十次作业.

1. 设域 F, $\mathrm{char}(F)=0$. 证明下矩阵不相似.

(1) $\begin{pmatrix}1&1\\1&2\end{pmatrix} \not\sim \begin{pmatrix}1&2\\0&3\end{pmatrix}$. (2) $\begin{pmatrix}0&2&1\\0&1&3\\3&0&0\end{pmatrix} \not\sim \begin{pmatrix}1&0&0\\0&0&2\\0&3&1\end{pmatrix}$.

Pf: (1). $\begin{vmatrix}1&1\\1&2\end{vmatrix}=2-1=1$, $\begin{vmatrix}1&2\\0&3\end{vmatrix}=3$ $1\neq 3$.

∵ 行列式是相似不变量. ∴ 两矩阵不相似.

(2). $tr\begin{pmatrix}0&2&1\\0&1&3\\3&0&0\end{pmatrix}=1$, $tr\begin{pmatrix}1&0&0\\0&0&2\\0&3&1\end{pmatrix}=2$. $1\neq 2$.

∵ 迹是相似不变量. ∴ 两矩阵不相似.

2. 计算 $T\in GL_2(\mathbb{R})$, s.t. $T^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}T=\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

解. 设 $T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, 满足 $T^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}T=\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

即 $\begin{pmatrix}0&1\\0&0\end{pmatrix}T=T\begin{pmatrix}0&0\\1&0\end{pmatrix}$. 即 $\begin{pmatrix}c&d\\0&0\end{pmatrix}=\begin{pmatrix}b&0\\d&0\end{pmatrix}$.

$\Rightarrow c=b$, $d=0$ $\Rightarrow$ $T=\begin{pmatrix}a&b\\b&0\end{pmatrix}$, $b\neq 0$

可验证 $|T|=-b^2$. $T^{-1}=\frac{1}{(-b^2)}\begin{pmatrix}0&-b\\-b&a\end{pmatrix}=\begin{pmatrix}0&\frac{1}{b}\\\frac{1}{b}&-\frac{a}{b^2}\end{pmatrix}$.

且 $T^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}T=\begin{pmatrix}0&0\\0&\frac{1}{b}\end{pmatrix}\begin{pmatrix}a&b\\b&0\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

则 $T=\begin{pmatrix}a&b\\b&0\end{pmatrix}$, $b\neq 0$ 为所求. $(|T|=-b^2\neq 0,\ b\neq 0)$.

3. 设 V 是域 F 上的线性空间. $\vec{v}_1,\vec{v}_2,\vec{v}_3$ 是 V 的一组基. 设 $\mathscr{A}$ 是 V 上的线性算子. 满足 $\mathscr{A}(\vec{v}_1)=\vec{v}_1$, $\mathscr{A}(\vec{v}_2)=\vec{v}_1+\vec{v}_2$, $\mathscr{A}(\vec{v}_3)=\vec{v}_1+\vec{v}_2+\vec{v}_3$

(1). 证明: $\mathscr{A}$ 可逆.

(2). 求 $2\mathscr{A}-\mathscr{A}^{-1}$ 在基 $\vec{v}_1,\vec{v}_2,\vec{v}_3$ 下的矩阵.

(1). Pf: 证明 $\mathscr{A}$ 可逆，即证 $\mathscr{A}$ 在一组基 $\vec{v}_1, \vec{v}_2, \vec{v}_3$ 下的矩阵可逆.

$$(\mathscr{A}(\vec{v}_1), \mathscr{A}(\vec{v}_2), \mathscr{A}(\vec{v}_3)) = (\vec{v}_1, \vec{v}_2, \vec{v}_3)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \triangleq (\vec{v}_1, \vec{v}_2, \vec{v}_3) A.$$

$|A| = 1 \Rightarrow A$ 可逆 $\Rightarrow \mathscr{A}$ 是可逆线性算子.

(2). 由于 $\phi: \mathcal{L}(V) \longrightarrow M_3(F)$, 其中 A 是 $\mathscr{A}$ 在 $\vec{v}_1, \vec{v}_2, \vec{v}_3$ 下矩阵.

$\mathscr{A} \longmapsto A$

则 ϕ 是同构(环同构).

$\because \mathscr{A}$ (A) 可逆. $\therefore \phi(\mathscr{A}^{-1}) = \phi(\mathscr{A})^{-1} = A^{-1}$.

$\phi(2\mathscr{A} - \mathscr{A}^{-1}) = 2\phi(\mathscr{A}) - \phi(\mathscr{A}^{-1}) \doteq 2A - A^{-1}$

而 $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore \phi(2\mathscr{A} - \mathscr{A}^{-1}) = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

即为 $2\mathscr{A} - \mathscr{A}^{-1}$ 在 $\vec{v}_1, \vec{v}_2, \vec{v}_3$ 下的矩阵

4. 设 A 是域 F 上的 n 阶方阵. E 为 n 阶单位阵. 且 $A^2 = 2A + 3E$.

证明: $\mathrm{rank}(A+E) + \mathrm{rank}(A-3E) = n$.

Pf: 设 $f(t) = t^2 - 2t - 3 = (t-3)(t+1) \in F[t]$. 则 $f(A) = 0$.

设 $p(t) = t-3$, $q(t) = t+1$. 则 $\gcd(p, q) = 1$.

$\because$ 设 V 为 F 上 n 维线性空间; $\mathscr{A}: V \longrightarrow V$ 是述为 V 上线性算子.

$\mathcal{L}(V) \cong M_n(F)$ $\qquad \vec{x} \longmapsto A\vec{x}$. 且 A 为 $\mathscr{A}$ 在标准基下的矩阵.

由 $f(\underset{\mathscr{A}}{A}) = 0$ 知 $f(\overset{A}{\mathscr{A}}) = 0$. 由核核定理知,

用核核分解定理需同构到算子上.

$\ker(\mathscr{A} + id) \oplus \ker(\mathscr{A} - 3id) = V$

$\underline{\ker(\overset{\text{矩阵}}{A+E})} \oplus \ker(A-3E) \neq M_n(F)$

无意义.

$\Rightarrow \dim\ker(\mathscr{A}+id) + \dim\ker(\mathscr{A}-3id) = \underbrace{\dim V = n}.$

$\Rightarrow \dim V - \mathrm{rank}(\mathscr{A}+id) + \dim V - \mathrm{rank}(\mathscr{A}-3id) = \dim V.$

$\Rightarrow n = \mathrm{rank}(\mathscr{A}+id) + \mathrm{rank}(\mathscr{A}-3id) = \mathrm{rank}(A+E) + \mathrm{rank}(A-3E).$ ▭

注: $\forall p \in F[t]$. $\mathscr{A} \in \mathcal{L}(V)$ 在某组基下矩阵为 A. 则 $p(\mathscr{A})$ 在该基下矩阵为 $p(A)$

5. 证明：对有限维向量空间 V 上的任意线性算子 $\mathcal{A}$、$\mathcal{B}$

$$\operatorname{rank}\mathcal{A} = \operatorname{rank}\mathcal{B}\mathcal{A} + \dim(\operatorname{Im}\mathcal{A} \cap \ker\mathcal{B}).$$

Pf: 设 $W = \operatorname{Im}\mathcal{A}$. 则 $W \subseteq V$ 是子空间

则 $\mathcal{B}|_W : W \longrightarrow V$ 是 W 到 V 的线性映射.
$\quad\quad x \longmapsto \mathcal{B}(x)$

则有 $\dim W = \dim \operatorname{Im}(\mathcal{B}|_W) + \dim \ker(\mathcal{B}|_W)$

又 $\operatorname{Im}(\mathcal{B}|_W) = \mathcal{B}(W) = \mathcal{B}(\operatorname{Im}\mathcal{A}) = \mathcal{B}(\mathcal{A}(V)) = \operatorname{im}(\mathcal{B}\mathcal{A})$.

$\ker(\mathcal{B}|_W) = \ker\mathcal{B} \cap \operatorname{Im}\mathcal{A}$.

$\therefore \dim(\operatorname{im}\mathcal{A}) = \dim(\operatorname{im}\mathcal{B}\mathcal{A}) + \dim(\ker\mathcal{B} \cap \operatorname{Im}\mathcal{A})$.

$\therefore \operatorname{rank}\mathcal{A} = \operatorname{rank}\mathcal{B}\mathcal{A} + \dim(\ker\mathcal{B} \cap \operatorname{Im}\mathcal{A})$. ☒.

6. 设 $\forall\, \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{L}(V)$. V 有限维线性空间.

证明: $\operatorname{rank}\mathcal{B}\mathcal{A} + \operatorname{rank}\mathcal{A}\mathcal{C} \leq \operatorname{rank}\mathcal{A} + \operatorname{rank}\mathcal{B}\mathcal{A}\mathcal{C}$.

Pf: 由 T5. $\operatorname{rank}(\mathcal{B}\mathcal{A}) = \operatorname{rank}(\mathcal{A}) - \dim(\operatorname{Im}\mathcal{A} \cap \ker\mathcal{B})$.

$$\operatorname{rank}(\mathcal{A}\mathcal{C}) = \operatorname{rank}(\mathcal{B}\mathcal{A}\mathcal{C}) + \dim(\operatorname{Im}\mathcal{A}\mathcal{C} \cap \ker\mathcal{B}).$$

$$\Rightarrow \operatorname{rank}(\mathcal{B}\mathcal{A}) + \operatorname{rank}(\mathcal{A}\mathcal{C}) = \operatorname{rank}(\mathcal{A}) + \operatorname{rank}(\mathcal{B}\mathcal{A}\mathcal{C}) + \dim(\operatorname{Im}\mathcal{A}\mathcal{C} \cap \ker\mathcal{B}) - \dim(\operatorname{Im}\mathcal{A} \cap \ker\mathcal{B}).$$

又 $\operatorname{Im}\mathcal{A}\mathcal{C} = \mathcal{A}(\operatorname{Im}\mathcal{C}) \subseteq \mathcal{A}(V) = \operatorname{Im}\mathcal{A}$

$$\Rightarrow \dim(\operatorname{Im}\mathcal{A}\mathcal{C} \cap \ker\mathcal{B}) \leq \dim(\operatorname{Im}\mathcal{A} \cap \ker\mathcal{B})$$

$$\Rightarrow \operatorname{rank}(\mathcal{B}\mathcal{A}) + \operatorname{rank}(\mathcal{A}\mathcal{C}) \leq \operatorname{rank}(\mathcal{A}) + \operatorname{rank}(\mathcal{B}\mathcal{A}\mathcal{C}).$$ ☒.

· 一般情况（矩阵角度）. $\forall\, A \in F^{m\times n}$, $B \in F^{n\times p}$, $C \in F^{p\times q}$

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B) \leq \operatorname{rank}(ABC).$$

Pf: 方1. 设 $\varphi_A: F^n \to F^m$, $\vec{x} \longmapsto A\vec{x}$ $\quad$ $\varphi_B: F^p \to F^n$, $\vec{x} \longmapsto B\vec{x}$ $\quad$ $\varphi_C: F^q \to F^p$, $\vec{x} \longmapsto C\vec{x}$

记 $I_B = \mathrm{im}\,\varphi_B$; $I_{BC} = \mathrm{im}\,\varphi_{BC} = \mathrm{im}(\varphi_B \circ \varphi_C)$.

定义: $\psi: I_B \longrightarrow F^m$, $\vec{x} \longmapsto A\vec{x}$, $\quad \phi: I_{BC} \longrightarrow F^n$, $\vec{x} \longmapsto A\vec{x}$. $\quad$ 即 $\psi = \varphi_A|_{I_B} = \varphi_A \circ \varphi_B$

$\phi = \varphi_A|_{I_{BC}} = \varphi_A \circ \varphi_{BC} = \varphi_A \circ \varphi_B \circ \varphi_C$

则 $\psi \in \mathrm{Hom}(I_B, F^m)$, $\phi \in \mathrm{Hom}(I_{BC}, F^n)$

$$\Rightarrow \begin{cases} \dim(I_B) = \dim(\mathrm{im}\,\psi) + \dim(\ker\psi). \\ \dim(I_{BC}) = \dim(\mathrm{im}\,\phi) + \dim(\ker\phi) \end{cases} \Rightarrow \begin{cases} \dim\ker\psi = \overset{\mathrm{rank}\,\varphi_B}{\overset{\parallel}{\mathrm{rank}\,B}} - \mathrm{rank}\,\psi \\ \dim\ker\phi = \underset{\mathrm{rank}(\varphi_B\circ\varphi_C).}{\underset{\parallel}{\mathrm{rank}\,BC}} - \mathrm{rank}\,\phi. \end{cases}$$

又 $\mathrm{rank}\,\psi = \dim(\mathrm{im}\,\psi) = \dim(\varphi_A(I_B)) = \dim(\mathrm{im}\,\varphi_A \circ \varphi_B) = \mathrm{rank}(AB).$

$\mathrm{rank}\,\phi = \dim(\mathrm{im}\,\phi) = \dim(\varphi_A(I_{BC})) = \dim(\mathrm{im}\,\varphi_A \circ \varphi_B \circ \varphi_C) = \mathrm{rank}(ABC).$

$\Rightarrow \dim\ker\psi = \mathrm{rank}\,B - \mathrm{rank}\,AB$, $\quad \dim\ker\phi = \mathrm{rank}\,BC - \mathrm{rank}\,ABC.$

$\because I_{BC} \subseteq I_B \quad \therefore \overset{\subseteq I_{BC}}{\ker\phi} \subseteq \overset{\subseteq I_B}{\ker\psi} \quad \therefore \dim\ker\phi \leq \dim\ker\psi$

$\therefore \mathrm{rank}\,BC - \mathrm{rank}\,ABC \leq \mathrm{rank}\,B - \mathrm{rank}\,AB$, 即 $\mathrm{rank}\,AB + \mathrm{rank}\,BC + \mathrm{rank}\,B \leq \mathrm{rank}\,ABC.$

$\square$

方2.

$$\mathrm{rank}\begin{pmatrix} B_{n\times p} & O_{n\times q} \\ O_{m\times p} & ABC \end{pmatrix} = \mathrm{rank}(B) + \mathrm{rank}(ABC).$$

$$\because \begin{pmatrix} E_n & O_{n\times m} \\ -A_{m\times n} & E_m \end{pmatrix} \begin{pmatrix} B_{n\times p} & O_{n\times q} \\ O_{m\times p} & ABC \end{pmatrix} \begin{pmatrix} E_p & C_{p\times q} \\ O_{q\times p} & E_q \end{pmatrix} = \begin{pmatrix} B & BC \\ -AB & 0 \end{pmatrix}$$

$$\therefore \mathrm{rank}\begin{pmatrix} B_{n\times p} & O_{n\times q} \\ O_{m\times p} & ABC \end{pmatrix} = \mathrm{rank}\begin{pmatrix} B & BC \\ -AB & 0 \end{pmatrix} \geq \mathrm{rank}(BC) + \mathrm{rank}(AB).$$

$$\therefore \mathrm{rank}(B) + \mathrm{rank}(ABC) \geq \mathrm{rank}(AB) + \mathrm{rank}(BC)$$

$\square$

7. 证明：对有限维向量空间 V 上的任意线性算子 $\mathscr{A}$ 和 $\forall i\in\mathbb{Z}^+$

$$\dim(\mathrm{im}\mathscr{A}^{i+1}\cap\ker\mathscr{A})=\dim(\ker\mathscr{A}^{i})-\dim(\ker\mathscr{A}^{i+1}).$$

Pf: 由 T5， $\mathrm{rank}\,\mathscr{A}^{i+1}=\mathrm{rank}\,\mathscr{A}\mathscr{A}^{i+1}+\dim(\mathrm{Im}\mathscr{A}^{i+1}\cap\ker\mathscr{A})$.

$\Rightarrow\ \dim(\mathrm{Im}\mathscr{A}^{i+1})=\dim(\mathrm{Im}\mathscr{A}^{i})+\dim(\mathrm{Im}\mathscr{A}^{i+1}\cap\ker\mathscr{A})$

$\Rightarrow\ \dim V-\dim(\ker\mathscr{A}^{i+1})=\dim V-\dim(\ker\mathscr{A}^{i})+\dim(\mathrm{Im}\mathscr{A}^{i+1}\cap\ker\mathscr{A})$

$\Rightarrow\ \dim(\ker\mathscr{A}^{i})-\dim(\ker\mathscr{A}^{i+1})=\dim(\mathrm{Im}\mathscr{A}^{i+1}\cap\ker\mathscr{A})$.

$(\mathscr{A}^{i+1},\ \mathscr{A}^{i}\in\mathcal{L}(V))$. 四.

极小多项式. 设 V 为域 F 上的有限维线性空间. $\vec{e}_1,\cdots,\vec{e}_n$ 为一组基.

1. $\varphi:\ \mathcal{L}(V)\xrightarrow{\sim}M_n(F)$.

$\mathscr{A}\longmapsto A$, A 为 $\mathscr{A}$ 在 $\vec{e}_1,\cdots,\vec{e}_n$ 下的矩阵表示.

$\mathscr{A}:V\to V\ \longleftarrow\!\!\!\!\mid\ A$ 其中 $\vec{x}$ 为任一向量在 $\vec{e}_1,\cdots,\vec{e}_n$ 下坐标.

$\vec{x}\longmapsto A\vec{x}$.

即 $\mathscr{A}:(\vec{e}_1,\cdots,\vec{e}_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\longmapsto(\vec{e}_1,\cdots,\vec{e}_n)A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$.

易证 φ 为线性同构，且为环同构. $\Rightarrow\ \dim\mathcal{L}(V)=\dim M_n(F)=n^2$.

2. 设 $\mathscr{A}\in\mathcal{L}(V)$， $A=\varphi(\mathscr{A})\in M_n(F)$.

$F[\mathscr{A}]=\{\sum_{i=0}^{m}f_i\mathscr{A}^i\mid f_i\in F,\ m\in\mathbb{N}\}=\langle\mathscr{A}^0,\mathscr{A}^1,\mathscr{A}^2,\cdots\rangle_F\subseteq\mathcal{L}(V)$.

$F[A]=\{\sum_{i=0}^{m}f_iA^i\mid f_i\in F,\ m\in\mathbb{N}\}=\langle A^0,A^1,A^2,\cdots\rangle_F\subseteq M_n(F)$.

以 $\mathscr{A}$ 为例. A 类似.

$F[\mathscr{A}]$ 为 $\mathcal{L}(V)$ 上交换环 $(\underbrace{F[\mathscr{A}],+,0}_{\text{线性空间}},\circ,id)$

$\because\ \dim_F F[\mathscr{A}]\le\dim_F\mathcal{L}(V)=n^2$ $\therefore\ id,\mathscr{A},\cdots,\mathscr{A}^{n^2}$ 在 F 上线性相关.

于是 $F[\mathscr{A}]$ 可以有更简单的表示方式.

$\xi_{\mathscr{A}}:\ F[t]\longrightarrow F[\mathscr{A}]$ 是满的环同态

$f(t)\longmapsto f(\mathscr{A})$. (赋值同态)

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3. 定义：设 $\mathscr{A}\in\mathcal{L}(V)$，$f(t)\in F[t]$ 若 $f(\mathscr{A})=0$ 则称 f 是 $\mathscr{A}$ 的零化多项式。

若 $g(t)\in F[t]$ 且为 $\mathscr{A}$ 的零化多项式中次数最低首一的非零多项式，

则称 g 为 $\mathscr{A}$ 的极小多项式，记作 $\mu_{\mathscr{A}}$.

注：i) 若 $f(\mathscr{A})=0$，则 $\mu_{\mathscr{A}}\mid f$.　　ii). $\deg\mu_{\mathscr{A}}\le n^2$.

例. $A=\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\in M_2(F)$ $\alpha_1,\alpha_2\in F$. 求 A 的极小多项式.

解：若 $\deg\mu_A=1$，设 $\underset{(首一)}{f_0E+A}=O_2$. 即 $\begin{pmatrix}f_0+\alpha_1 & 0\\ 0 & f_0+\alpha_2\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

(对角矩阵. 数乘矩阵)

$$\Rightarrow\begin{cases}f_0+\alpha_1=0\\ f_0+\alpha_2=0\end{cases}\Rightarrow\alpha_1=\alpha_2=\alpha.\Rightarrow f_0=-\alpha\Rightarrow\mu_A=t-\alpha.$$

若 $\alpha_1\ne\alpha_2$，$4\ge\deg\mu_A>1$. 设 $\underset{(首一)}{f_0E+f_1A+A^2=O}$.

$$\Rightarrow\begin{cases}f_0+\alpha_1f_1+\alpha_1^2=0\\ f_0+\alpha_2f_1+\alpha_2^2=0\end{cases}\Rightarrow\begin{cases}f_0+\alpha_1f_1=-\alpha_1^2\\ f_0+\alpha_2f_1=-\alpha_2^2\end{cases}$$

因为 $\begin{vmatrix}1&\alpha_1\\1&\alpha_2\end{vmatrix}=\alpha_2-\alpha_1\ne0$，所以上方程组有解且唯一.

由 Cramer 法则，$f_0=\dfrac{\begin{vmatrix}-\alpha_1^2&\alpha_1\\-\alpha_2^2&\alpha_2\end{vmatrix}}{\begin{vmatrix}1&\alpha_1\\1&\alpha_2\end{vmatrix}}$，$f_1=\dfrac{\begin{vmatrix}1&-\alpha_1^2\\1&-\alpha_2^2\end{vmatrix}}{\begin{vmatrix}1&\alpha_1\\1&\alpha_2\end{vmatrix}}$.

$$\Rightarrow\mu_A=t^2+\frac{\begin{vmatrix}1&-\alpha_1^2\\1&-\alpha_2^2\end{vmatrix}}{\begin{vmatrix}1&\alpha_1\\1&\alpha_2\end{vmatrix}}t+\frac{\begin{vmatrix}-\alpha_1^2&\alpha_1\\-\alpha_2^2&\alpha_2\end{vmatrix}}{\begin{vmatrix}1&\alpha_1\\1&\alpha_2\end{vmatrix}}=t^2-(\alpha_1+\alpha_2)t+\alpha_1\alpha_2.$$

$\deg\mu_A=2$.

(方法2). 复习引理：设 $A=\begin{pmatrix}A_1&\\&A_2\end{pmatrix}$，则 $\mu_A=\operatorname{lcm}(\mu_{A_1},\mu_{A_2})$.

$$\mu_A=\operatorname{lcm}(\mu_{\alpha_1},\mu_{\alpha_2})=\operatorname{lcm}(t-\alpha_1,t-\alpha_2)=(t-\alpha_1)(t-\alpha_2)=t^2-(\alpha_1+\alpha_2)t+\alpha_1\alpha_2.$$