1. $V=\langle\vec{e_1},\cdots\vec{e_4}\rangle$. $\vec{\varepsilon_1}=\vec{e_1}$, $\vec{\varepsilon_2}=-\vec{e_2}$, $\vec{\varepsilon_3}=2\vec{e_3}+3\vec{e_4}$. $\vec{\varepsilon_4}=4\vec{e_3}+\vec{e_4}$

(i) $\mathrm{char}(F)=0$ 证明 $\vec{\varepsilon_1},\cdots\vec{\varepsilon_4}$ 构成基. 并求矩阵 $A,B\in M_4(F)$ s.t.

$$(\vec{\varepsilon_1},\cdots\vec{\varepsilon_4})=(\vec{e_1}\cdots\vec{e_4})A,\quad (\vec{e_1}\cdots\vec{e_4})=(\vec{\varepsilon_1}\cdots\vec{\varepsilon_4})B.$$

(ii) $\mathrm{char}(F)=?$ $\vec{\varepsilon_1}\cdots\vec{\varepsilon_4}$ 不是基时.

解 (i). $(\vec{\varepsilon_1}\cdots\vec{\varepsilon_4})=(e_1\cdots\vec{e_4})\underbrace{\begin{pmatrix}1&&&\\&-1&&\\&&2&4\\&&3&1\end{pmatrix}}_{=A}$ $|A|=+10$ $\because \mathrm{char}(F)=0$

$\therefore |A|\neq 0$

$\because \mathrm{rank}\,A=4$ $\therefore$ $\vec{\varepsilon_1}\cdots\vec{\varepsilon_4}$ 构成基.

$$B=A^{-1}=\begin{pmatrix}+1&&&\\&-1&&\\&&-\frac{1}{10}&\frac{2}{5}\\&&\frac{3}{10}&-\frac{1}{5}\end{pmatrix}$$

(ii) $\because |A|=10$ $\therefore$ $\mathrm{char}(F)=2$ 或 5 时 $|A|=0$ $\Rightarrow$ A不满秩 $\Rightarrow$ 不构成基.

2. $q(x_1x_2x_3)=-3x_3^2+4x_1x_2$

(i) $A=\begin{pmatrix}0&2&0\\2&0&0\\0&0&-3\end{pmatrix}$

(ii) 矩阵法: $(A\ \ E)=\begin{pmatrix}0&2&0&1&0&0\\2&0&0&0&1&0\\0&0&-3&0&0&1\end{pmatrix}\xrightarrow[(ii)+(i)]{(2)+(1)}\begin{pmatrix}4&2&0&1&1&0\\2&0&0&0&1&0\\0&0&-3&0&0&1\end{pmatrix}$

$\xrightarrow[-\frac{1}{2}(i)+(ii)]{-\frac{1}{2}(1)+(2)}\begin{pmatrix}4&0&0&1&1&0\\0&-1&0&-\frac{1}{2}&\frac{1}{2}&0\\0&0&-3&0&0&1\end{pmatrix}$ 令 $P=\begin{pmatrix}1&-\frac{1}{2}&0\\1&\frac{1}{2}&0\\0&0&1\end{pmatrix}$ 则 $P^tAP=\begin{pmatrix}4&&\\&-1&\\&&-3\end{pmatrix}$

第2方法 $q=-3x_3^2+4x_1x_2$ 令 $x_1=y_1+y_2$, $x_2=y_1-y_2$ $x_3=y_3$

$=-3y_3^2+4y_1^2-4y_2^2$ $\vec{x}=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&1\end{pmatrix}\vec{y}$ 则令 $P=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&1\end{pmatrix}$ $P^tAP=\begin{pmatrix}4&&\\&-4&\\&&-3\end{pmatrix}$

(iii) 正惯性指数 1 负惯性指数 2.

3. $A=\begin{pmatrix} 1 & 4 & \cdots & 4 \\ 4 & 1 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & \cdots & 1 \end{pmatrix} \in M_{2n}(\mathbb{R})$

(i) $\Delta_k = \begin{vmatrix} 1 & 4 & \cdots & 4 \\ \vdots & \ddots & & \vdots \\ 4 & \cdots & & 1 \end{vmatrix} = \begin{vmatrix} 4(k-1)+1 & 4 & \cdots & 4 \\ \vdots & 1 & \ddots & \vdots \\ 4(k-1)+1 & 4 & \cdots & 1 \end{vmatrix} = 4(k-1)+1 \begin{vmatrix} 1 & 4 & \cdots & 4 \\ 1 & 1 & \ddots & \vdots \\ 1 & 4 & \cdots & 1 \end{vmatrix}$

$= (4(k-1)+1)\cdot \begin{vmatrix} 1 & 0 & & \\ 1 & -3 & \ddots & \\ \vdots & \vdots & & \\ 1 & 0 & & -3 \end{vmatrix} = (4k-3)(-3)^{k-1}$

$\frac{\Delta_{k+1}}{\Delta_k} = \frac{4k+1}{4k-3}\cdot(-3) < 0 \quad (\forall\ k \geq 1) \qquad \Delta_1 = 1 > 0 \qquad \Delta_0 = 1 > 0$

由Jacobi方法 $A \sim_c \begin{pmatrix} \frac{\Delta_1}{\Delta_0} & & & \\ & \frac{\Delta_2}{\Delta_1} & & \\ & & \ddots & \\ & & & \frac{\Delta_{2n}}{\Delta_{2n-1}} \end{pmatrix}$ ∴签名为 $(1, 2n-1)$

(ii) 当 $n=1$ 时 A 的签名为 $(1,1)$ 与 $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ 合同

当 $n>1$ 时 A 的签名为 $(1, 2n-1) \neq (n, n)$ ∴不与 $\begin{pmatrix} E_n & \\ & -E_n \end{pmatrix}$ 合同

4. $\phi: M_n(\cancel{\Phi}\mathbb{Q}) \longrightarrow M_n(\mathbb{Q})$
$\quad A \longmapsto A - A^t$

(i) ϕ 是线性算子
(ii) 求 $\operatorname{rank}\phi$ (iii) $\ker\phi$ 与 $\operatorname{im}\phi$ 是否是直和.

Pf: (i) $\forall \alpha, \beta \in \mathbb{Q}$. $A, B \in M_n(\mathbb{Q})$ $\phi(\alpha A + \beta B) = (\alpha A + \beta B) - (\alpha A + \beta B)^t$

$= \alpha(A - A^t) + \beta(B - B^t)$

$= \alpha\phi(A) + \beta\phi(B) \Rightarrow$ 线性

(ii) $\operatorname{rank}\phi = \dim(\operatorname{im}\phi)$. $\forall A \in M_n(\mathbb{Q})$ $\phi(A)$ 是斜对称矩阵 $((A-A^t)^t = A^t - A = -(A-A^t))$

$\forall$ 斜对称矩阵 B $(B^t = -B)$ $\exists A = \frac{B}{2}$ s.t. $\phi(A) = \frac{B}{2} - (\frac{B}{2})^t = \frac{B}{2} + \frac{B}{2} = B$

$\therefore \operatorname{im}\phi = AM_n(\mathbb{Q})$ (斜对称矩阵空间) $\therefore \dim(\operatorname{im}\phi) = \frac{n(n-1)}{2}$ $\therefore \operatorname{rank}\phi = \frac{n(n-1)}{2}$

(iii). $\forall B \in \ker\phi \cap \operatorname{im}\phi$ $\exists A \in M_n(\cancel{\Phi}\mathbb{Q})$ s.t. $B = \phi(A) = A - A^t \Rightarrow B$ 斜对称.

且 $\phi(B) = \cancel{(A-A^t) - (A-A^t)^t = A - A^t - A^t}\ B - B^t = 0 \Rightarrow B = B^t \Rightarrow B$ 对称.

$\therefore B = O \Rightarrow$ 是直和.

5. (i) $\dim(U_1+U_2+U_3) \leq \dim(U_1)+\dim(U_2)+\dim(U_3)$

$$\dim(U_1+U_2+U_3) = \dim((U_1+U_2)) + \dim U_3 - \dim((U_1+U_2)\cap U_3)$$

$$= \dim U_1 + \dim U_2 - \dim(U_1\cap U_2) + \dim U_3 - \dim((U_1+U_2)\cap U_3)$$

$$= \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1\cap U_2) - \dim((U_1+U_2)\cap U_3)$$

$$\leqslant \dim U_1 + \dim U_2 + \dim U_3$$

(ii) $\dim((U_1+U_2)\cap U_3) + \dim(U_1\cap U_2) = \dim((U_1+U_3)\cap U_2) + \dim(U_1\cap U_3)$

$\dim(U_1+U_2+U_3) = \dim(U_1+U_3+U_2) = \dim U_1 + \dim U_3 + \dim U_2 - \dim(U_1\cap U_3) - \dim((U_1+U_3)\cap U_2)$

$\therefore \dim(U_1\cap U_3) + \dim((U_1+U_3)\cap U_2) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1+U_2+U_3)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad \dim(U_1\cap U_2) + \dim((U_1+U_2)\cap U_3)$ □

6. $\mathcal{A} \in \mathcal{L}(V)$ 满足 $\mathcal{A}^2 = -\mathcal{A}$ 证明：

(i) $\ker\mathcal{A} + \operatorname{im}\mathcal{A} = V$

(ii) $\dim(\ker(\mathcal{A}+\mathcal{E})) = \operatorname{rank}\mathcal{A}$

Pf. (ii) 设 $f(t) \in \mathbb{Q}[t]$ 满足 $f(t) = t^2+t = (t+1)t$

$\because \gcd(t+1, t) = 1$ 且 $f(\mathcal{A}) = 0$

$\therefore \ker(\mathcal{A}+\mathcal{E}) \oplus \ker\mathcal{A} = V$

$$\Rightarrow \dim(\ker(\mathcal{A}+\mathcal{E})) = \dim V - \dim(\ker\mathcal{A})$$

$$= \dim(\operatorname{im}\mathcal{A})$$

$$= \operatorname{rank}\mathcal{A}.$$

(i) $\because \mathcal{A}^2 = -\mathcal{A}$ $\therefore \operatorname{rank}\mathcal{A}^2 = \cancel{\operatorname{rank}\mathcal{A}}\ \operatorname{rank}(-\mathcal{A}) = \operatorname{rank}\mathcal{A}$

由核像分解 $\ker\mathcal{A} \oplus \operatorname{im}\mathcal{A} = V \Rightarrow \ker\mathcal{A} + \operatorname{im}\mathcal{A} = V$ □

7. $A, B \in SM_n(\mathbb{R})$ 正定

(i) A^2 与 $A+B$ 均正定

(ii) 举例说明 AB 不一定对称

(iii) 证明 AB 相似于正定矩阵.

Pf (i) $\because$ A 正定 $\therefore$ $\exists$ 可逆矩阵 $P \in GL_n(\mathbb{R})$ s.t. $A = P^t P$

则 $A^2 = (P^t P)\cdot(P^t P)$ 令 $Q = P^t P$ $\because$ P 可逆 $\therefore$ Q 可逆 且 $A^2 = Q^t \cdot Q$

$\therefore$ A^2 正定

$\because$ A, B 正定 $\therefore$ 对 $\forall \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$ $\vec{x}^t A \vec{x} > 0$, $\vec{x}^t B \vec{x} > 0$

$\therefore$ $\vec{x}^t (A+B) \vec{x} = \vec{x}^t A \vec{x} + \vec{x}^t B \vec{x} > 0$ $\Rightarrow$ $A+B$ 正定.

(ii) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ $\Rightarrow$ $A \cdot B = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}$ 不对称.

(iii) $\because$ A, B 正定 $\therefore$ $\exists$ 可逆矩阵 $P_1, P_2 \in GL_n(\mathbb{R})$ s.t. $A = P_1^t P_1$ $B = P_2^t P_2$

则 $(P_1^t)^{-1} \cdot (AB) \cdot P_1^t = \underline{(P_1^t)^{-1} \cdot P_1^t} P_1 \cdot P_2^t \cdot P_2 \cdot P_1^t = P_1 P_2^t P_2 P_1^t = (P_2 P_1^t)^t \cdot (P_2 P_1^t)$

令 $Q = P_2 P_1^t$ $\because$ $P_1, P_2 \in GL_n(\mathbb{R})$ $\therefore$ $Q \in GL_n(\mathbb{R})$ $\therefore$ $Q^t Q$ 是正定矩阵

即 $\exists$ 可逆矩阵 P_1^t s.t. $(P_1^t)^{-1} \cdot (AB) \cdot P_1^t = Q^t \cdot Q$ $\Rightarrow$ AB 相似于正定矩阵.

8. $A \in SM_n(\mathbb{R})$ 正定. 定义 $q:$
$$\mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \det \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix}$$

(i) 证明, q 是 $\mathbb{R}^n$ 上二次型

(ii) 证明: q 负定

Pf. $q(-\vec{x}) = \det \begin{pmatrix} A & -\vec{x} \\ -\vec{x}^t & 0 \end{pmatrix} = -\det \begin{pmatrix} A & \vec{x} \\ -\vec{x}^t & 0 \end{pmatrix} = \det \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} = q(\vec{x})$.

设 $f(x, y) = \frac{1}{2}[q(\vec{x}+\vec{y}) - q(\vec{x}) - q(\vec{y})]$

则 $f(x,y) = \frac{1}{2}\left[\det\begin{pmatrix} A & (\vec{x}+\vec{y}) \\ (\vec{x}+\vec{y})^t & 0 \end{pmatrix} - \det\begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} - \det\begin{pmatrix} A & \vec{y} \\ \vec{y}^t & 0 \end{pmatrix}\right]$

$$= \frac{1}{2}\left[\det\begin{pmatrix} A & \vec{x} \\ \vec{x}^t+\vec{y}^t & 0 \end{pmatrix} + \det\begin{pmatrix} A & \vec{y} \\ \vec{x}^t+\vec{y}^t & 0 \end{pmatrix} - \det\begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} - \det\begin{pmatrix} A & \vec{y} \\ \vec{y}^t & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\det\begin{pmatrix} A & \vec{x} \\ \vec{y}^t & 0 \end{pmatrix} + \det\begin{pmatrix} A & \vec{y} \\ \vec{x}^t & 0 \end{pmatrix}\right]$$

$\forall\ \alpha, \beta \in \mathbb{R}$. $\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2 \in \mathbb{R}^n$ 有：

$$f(\alpha\vec{x}_1+\beta\vec{x}_2, \vec{y}_1) = \frac{1}{2}\left[\det\begin{pmatrix} A & \alpha\vec{x}_1+\beta\vec{x}_2 \\ \vec{y}_1^t & 0 \end{pmatrix} + \det\begin{pmatrix} A & \vec{y}_1 \\ \alpha\vec{x}_1^t+\beta\vec{x}_2^t & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\alpha\det\begin{pmatrix} A & \vec{x}_1 \\ \vec{y}_1^t & 0 \end{pmatrix} + \beta\det\begin{pmatrix} A & \vec{x}_2 \\ \vec{y}_1^t & 0 \end{pmatrix} + \alpha\det\begin{pmatrix} A & \vec{y}_1 \\ \vec{x}_1^t & 0 \end{pmatrix} + \beta\begin{pmatrix} A & \vec{y}_1 \\ \vec{x}_2^t & 0 \end{pmatrix}\right]$$

$$= \alpha\frac{1}{2}\left[\det\begin{pmatrix} A & \vec{x}_1 \\ \vec{y}_1^t & 0 \end{pmatrix} + \det\begin{pmatrix} A & \vec{y}_1 \\ \vec{x}_1^t & 0 \end{pmatrix}\right] + \beta\cdot\frac{1}{2}\left[\det\begin{pmatrix} A & \vec{x}_2 \\ \vec{y}_1^t & 0 \end{pmatrix} + \det\begin{pmatrix} A & \vec{y}_1 \\ \vec{x}_2^t & 0 \end{pmatrix}\right]$$

$$= \alpha f(\vec{x}_1, \vec{y}) + \beta f(\vec{x}_2, \vec{y}_1)$$

同理可证 $f(\vec{x}_1, \alpha\vec{y}_1+\beta\vec{y}_2) = \alpha f(\vec{x}_1, \vec{y}_1) + \beta f(\vec{x}_1, \vec{y}_2)$

∴ f 是 $\mathbb{R}^n$ 上的双线性型(对称)　∴ q 是二次型.

[另法] $\det\begin{pmatrix} A & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ x_1 \cdots x_n & 0 \end{pmatrix} = \det\begin{pmatrix} \vec{A}^{(1)} & \cdots & \vec{A}^{(n)} & \vec{x} \\ x_1 & \cdots & x_n & 0 \end{pmatrix} \xlongequal{\text{按最后一行展开}} (-1)^{n+2} x_1 \det(\vec{A}^{(2)} \cdots \vec{A}^{(n)}\, \vec{x})$

$+ \cdots + (-1)^{2n+1} x_n \det(\vec{A}^{(1)} \cdots \vec{A}^{(n-1)}\, \vec{x}) = \sum_{i=1}^{n} (-1)^{n+1+i} x_i \cdot \det(\vec{A}^{(1)} \cdots \vec{A}^{(i-1)}\, \vec{A}^{(i+1)} \cdots \vec{A}^{(n)}\, \vec{x})$ $\cancel{\sum_{i=1} (-1)^{n+1+i} x_i \det}$

∵ $\det(\vec{A}^{(1)} \cdots \vec{A}^{(i-1)}\, \vec{A}^{(i+1)} \cdots \vec{A}^{(n)}\, \vec{x}) \xlongequal{\text{按最后一列展开}} (-1)^{n+1} x_1 \cdot M_{1i} + \cdots + (-1)^{2n} x_n \cdot M_{ni}$

∴ $\det\begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} = \sum_{i=1}^{n} (-1)^{n+1+i} x_i \cdot \left(\sum_{j=1}^{n} (-1)^{n+j} x_j M_{ji}\right)$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} (-1)^{i+j+1} M_{ji} \cdot x_i x_j$ 是二次齐次多项式

(其中 M_{ji} 为 A 去掉 j 行 i 列的行列式 → 余子式)　∴ q 是二次型.

(ii) $\because$ A 正定 $\therefore$ ∃ 可逆矩阵 $P \in GL_n(\mathbb{R})$ s.t. $P^tAP = E_n$

令 $Q = \begin{pmatrix} P & \vec{o} \\ \vec{o}^t & 1 \end{pmatrix}$ 且 $B = Q^t \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} Q$

则 $B = \begin{pmatrix} P^t & \\ & 1 \end{pmatrix} \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} \begin{pmatrix} P & \\ & 1 \end{pmatrix} = \begin{pmatrix} E_n & P^t\vec{x} \\ \vec{x}^tP & 0 \end{pmatrix}$

令 $\vec{y} = P^t\vec{x}$ 则 $|B| = \begin{vmatrix} E_n & \vec{y} \\ \vec{y}^t & 0 \end{vmatrix} = \begin{vmatrix} 1 & & & y_1 \\ & \ddots & & \vdots \\ & & 1 & y_n \\ y_1 & \cdots & y_n & 0 \end{vmatrix} = \begin{vmatrix} 1 & & & y_1 \\ & \ddots & & \vdots \\ & & 1 & y_n \\ 0 & \cdots & 0 & -y_1^2 - \cdots - y_n^2 \end{vmatrix}$

$= -y_1^2 - \cdots - y_n^2$

而 $|B| = |Q^t| \cdot q(\vec{x}) \cdot |Q| = |Q|^2 \cdot q(\vec{x})$

$\therefore$ 对 $\forall\ \vec{x} \in \mathbb{R}^n \setminus \{\vec{o}\}$ $q(\vec{x}) = \dfrac{|B|}{|Q|^2} < 0$ ($\because \vec{x} \neq \vec{o}$ $\therefore \vec{y} \neq \vec{o}$)

$\therefore$ q 负定.

另法 (i) $\begin{pmatrix} E_n & \vec{o} \\ -\vec{x}^t A^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} = \begin{pmatrix} A & \vec{x} \\ 0 & -\vec{x}^t A^{-1} \vec{x} \end{pmatrix}$

$\therefore q(\vec{x}) = \det \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} = \det\left[\begin{pmatrix} E_n & \vec{o} \\ -\vec{x}^t A^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} A & \vec{x} \\ \vec{x}^t & 0 \end{pmatrix} \right]$

$= \det \begin{pmatrix} A & \vec{x} \\ 0 & -\vec{x}^t A^{-1} \vec{x} \end{pmatrix} = -|A| \cdot (\vec{x}^t A^{-1} \vec{x})$

$\therefore$ q 为二次型 且 $\because$ A 正定 $\therefore$ A^{-1} 正定 且 $-|A| < 0$

(ii) $\therefore$ q 为负定二次型.