第十一次作业.

1. $f(t)=-t^3+4t+1$ $\quad A=\begin{pmatrix}1&2\\1&0\end{pmatrix}$. 求 μ_A 和 $f(A)$.

解: 若 $\deg(\mu_A)=1$. 设 $\mu_A=t+a$. 则 $A+aE_2=0$.

$$\Rightarrow\begin{cases}1+a=0\\2=0\\1=0\\a=0\end{cases}\Rightarrow \text{无解}.\quad \therefore \deg\mu_A>1.$$

若 $\deg(\mu_A)=2$ 设 $\mu_A=t^2+at+b$. 则 $A^2+aA+bE_2=0$.

$$\Rightarrow\begin{pmatrix}3&2\\1&2\end{pmatrix}+\begin{pmatrix}a&2a\\a&0\end{pmatrix}+\begin{pmatrix}b&0\\0&b\end{pmatrix}=0\Rightarrow\begin{cases}3+a+b=0\\2+2a=0\\1+a+0=0\\2+b=0\end{cases}\Rightarrow\begin{cases}a=-1\\b=-2\end{cases}$$

$\therefore \mu_A=t^2-t-2$.

方2. (用特征多项式). $\chi_A=|\lambda E-A|=\begin{vmatrix}\lambda-1&-2\\-1&\lambda\end{vmatrix}=\lambda(\lambda-1)-2=\lambda^2-\lambda-2$
$=(\lambda-2)(\lambda+1)$.

则 $\mu_A\mid\chi_A$ 由于 $A-2E\neq0$ 且 $A+E\neq0$. 则 $\mu_A=\chi_A=\lambda^2-\lambda-2$.

$$f(A)=-A^3+4A+E=-\begin{pmatrix}5&6\\3&2\end{pmatrix}+\begin{pmatrix}4&8\\4&0\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\1&-1\end{pmatrix}.$$

2. $A,B\in M_n(K)$, 若 A、B 有一个非退化. 则 $AB\sim_s BA$.

Pf: 不妨设 A 非退化. 即 A 可逆. 则 $A^{-1}(AB)A=(A^{-1}A)(BA)=BA$.

$\Rightarrow AB\sim_s BA$. 同理可证 B 非退化.

但若 A,B 均退化. $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. $B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. $\overset{\begin{pmatrix}0&1\\0&0\end{pmatrix}}{AB}\not\sim_s\overset{\begin{pmatrix}0&0\\0&0\end{pmatrix}}{BA}$.

3. $\mathcal{A},\mathcal{B}\in\mathcal{L}(V)$. 若 $\mathcal{A}\mathcal{B}=\mathcal{B}\mathcal{A}$. 则对 $\forall\lambda\in K$. $\ker(\lambda\mathcal{E}-\mathcal{A})$ 是 $\mathcal{B}$-子空间.

Pf: $\forall\ \vec{x}\in\ker(\lambda\mathcal{E}-\mathcal{A})$. 则 $(\lambda\mathcal{E}-\mathcal{A})(\vec{x})=\lambda\vec{x}-\mathcal{A}\vec{x}=\vec{0}$.

且 $(\lambda\mathcal{E}-\mathcal{A})(\mathcal{B}\vec{x})=\lambda\mathcal{B}\vec{x}-\mathcal{A}\mathcal{B}\vec{x}=\lambda\mathcal{B}\vec{x}-\mathcal{B}\mathcal{A}\vec{x}=\mathcal{B}(\lambda\vec{x}-\mathcal{A}\vec{x})=\vec{0}$

$\therefore \mathscr{B}\vec{x} \in \ker(\lambda\mathcal{E}-\mathscr{A})$. $\therefore \ker(\lambda\mathcal{E}-\mathscr{A})$ 是 $\mathscr{B}$ 的不变子空间. ② 四

3. 设 V 是 F 上的线性空间. $\mathscr{A}\in\mathcal{L}(V)$. 证: $\mathscr{A}$ 可逆 $\Leftrightarrow \mu_{\mathscr{A}}(0)\neq 0$.

Pf: "$\Rightarrow$" 设 $\mu_{\mathscr{A}}(0)=0$. 则设 $\mu_{\mathscr{A}} = t^m + a_{m-1}t^{m-1}+\cdots+a_1 t$

有 $\mathscr{A}^m + a_{m-1}\mathscr{A}^{m-1}+\cdots+a_1\mathscr{A}=0 \Rightarrow \mathscr{A}(\mathscr{A}^{m-1}+a_{m-1}\mathscr{A}^{m-2}+\cdots+a_1\mathcal{E})=0$.

$\because \mathscr{A}$ 可逆 $\therefore \mathscr{A}^{m-1}+a_{m-1}\mathscr{A}^{m-2}+\cdots+a_1\mathcal{E}=0$. 与 $\mu_{\mathscr{A}}$ 的极小性矛盾.

"$\Leftarrow$" 设 $\mu_{\mathscr{A}}(t)= t^m + a_{m-1}t^{m-1}+\cdots+a_1 t+a_0$ 且 $a_0 \overset{=\mu_{\mathscr{A}}(0)}{\neq 0}$.

则 $\mu_{\mathscr{A}}(\mathscr{A}) = \mathscr{A}^m+\cdots+a_1\mathscr{A}+a_0\mathcal{E}=0 \Rightarrow \mathscr{A}(\mathscr{A}^{m-1}+\cdots+a_1\mathcal{E})=(-a_0)\cdot\mathcal{E}$.

$\Rightarrow \mathscr{A}\Big(\underbrace{\frac{1}{-a_0}\mathscr{A}^{m-1}+\cdots+\left(\frac{a_1}{-a_0}\right)\mathcal{E}}_{\to \mathscr{A}^{-1}}\Big)=\mathcal{E} \Rightarrow \mathscr{A}$ 可逆

4. $\mathscr{A}\in\mathcal{L}(V)$. 若 $\exists\, p\in\mathbb{N}$, s.t. $\operatorname{im}\mathscr{A}^p = \operatorname{im}\mathscr{A}^{p+1}$

证明: $V=\ker\mathscr{A}^p\oplus\operatorname{im}\mathscr{A}^p$ 是两个 ~~A~~ 一不变子空间的直和.

Pf: 首先证明 $\ker\mathscr{A}^p$, $\operatorname{im}\mathscr{A}^p$ 是 $\mathscr{A}$-子空间. (对 $\forall p\in\mathbb{N}$).

对 $\forall\vec{x}\in\ker\mathscr{A}^p$. 有 $\mathscr{A}^p\vec{x}=\vec{0}$. $\mathscr{A}^p(\mathscr{A}\vec{x})=\mathscr{A}^{p+1}\vec{x}=\mathscr{A}(\mathscr{A}^p\vec{x})=\vec{0}$.

$\therefore \mathscr{A}\vec{x}\in\ker\mathscr{A}^p$ $\therefore \ker\mathscr{A}^p$ 是 $\mathscr{A}$-子空间.

$\forall\vec{x}\in\operatorname{im}\mathscr{A}^p$, $\exists\,\vec{y}\in V$, s.t. $\vec{x}=\mathscr{A}^p\vec{y}$

$\therefore \mathscr{A}\vec{x}=\mathscr{A}(\mathscr{A}^p\vec{y})=\mathscr{A}^{p+1}(\vec{y})\in\operatorname{im}\mathscr{A}^{p+1}=\operatorname{im}\mathscr{A}^p$ $\therefore \operatorname{im}\mathscr{A}^p$ 是 $\mathscr{A}$-子空间.

再证明 $V=\ker\mathscr{A}^p\oplus\operatorname{im}\mathscr{A}^p$. ~~即证 $\ker\mathscr{A}^p\cap\operatorname{im}\mathscr{A}^p=\{\vec{0}\}$~~.

断言. $p\in\mathbb{N}$. 若 $\operatorname{im}\mathscr{A}^p=\operatorname{im}\mathscr{A}^{p+1}$, 则 $\forall q\in\mathbb{N}$, $\operatorname{im}\mathscr{A}^p=\operatorname{im}\mathscr{A}^{p+q}$.

数学归纳法. $q=1$ 时, 显然成立.

假设 $q-1$ 时成立. 即 $\operatorname{im}\mathscr{A}^p=\operatorname{im}\mathscr{A}^{p+q-1}$.

则 $\operatorname{im}\mathscr{A}^{p+q}=\operatorname{im}\mathscr{A}\circ\mathscr{A}^{p+q-1}=\mathscr{A}(\operatorname{im}\mathscr{A}^{p+q-1})=\mathscr{A}(\operatorname{im}\mathscr{A}^p)$

$=\operatorname{im}(\mathscr{A}\circ\mathscr{A}^p)=\operatorname{im}\mathscr{A}^{p+1}=\operatorname{im}\mathscr{A}^p$. 得证.

特别地. 令 $q=p$. 则 $\operatorname{im}\mathscr{A}^p=\operatorname{im}\mathscr{A}^{2p}=\operatorname{im}((\mathscr{A}^p)^2)$. $\mathscr{A}^p\in\mathcal{L}(V)$.

$\Rightarrow \operatorname{rank}(\mathscr{A}^p) = \operatorname{rank}((\mathscr{A}^p)^2)$. 由核像分解 $V = \operatorname{im}\mathscr{A}^p \oplus \ker \mathscr{A}^p$ 四.

第十二次作业.

1. 设 V 为 F 上有限维线性空间. $\mathscr{A} \in \mathcal{L}(V)$. 若 $\mathscr{A}$ 可逆 则 $\mathscr{A}$ 的不变子空间也是 $\mathscr{A}^{-1}$ 的不变子空间. $\mathscr{A}$ 的特征向量也是 $\mathscr{A}^{-1}$ 的.

Pf: 设 $U \subseteq V$ 为 $\mathscr{A}$ 的任一不变子空间. 则 $\mathscr{A}|_U \in \mathcal{L}(U)$.

$\because \mathscr{A}$ 可逆 $\therefore \ker \mathscr{A} = \{\vec{0}\} \Rightarrow \ker \mathscr{A}|_U = \{\vec{0}\} \Rightarrow \dim(\operatorname{im}\mathscr{A}|_U) = \dim U$.

又 $\because \operatorname{im}\mathscr{A}|_U \subseteq U$. ($U$ 是 $\mathscr{A}$-子空间). $\therefore \operatorname{im}\mathscr{A}|_U = U$. $\therefore \mathscr{A}|_U$ 满

$\therefore \mathscr{A}|_U$ 可逆. 则对 $\forall \vec{u} \in U$, $\exists \vec{v} \in U$. s.t. $\overset{\mathscr{A}|_U(\vec{v})}{\mathscr{A}(\vec{v})} = \vec{u} \Rightarrow \vec{v} = \mathscr{A}^{-1}(\vec{u}) \in U$.

$\therefore U$ 是 $\mathscr{A}^{-1}$-子空间.

(设 $\vec{x}$ 是 $\mathscr{A}$ 的特征向量. 则 $\exists \lambda \in F\setminus\{0\}$. $\mathscr{A}\vec{x} = \lambda\vec{x}$, $\vec{x} = \lambda\mathscr{A}^{-1}\vec{x}$, 即 $\mathscr{A}^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$).

特别地, $\vec{v}$ 为 $\mathscr{A}$ 的特征向量 $\Longleftrightarrow$ $\langle\vec{v}\rangle$ 为 $\mathscr{A}$-子空间

$\Updownarrow$

$\vec{v}$ 为 $\mathscr{A}^{-1}$ 的特征向量. $\Longleftrightarrow$ $\langle\vec{v}\rangle$ 为 $\mathscr{A}^{-1}$-子空间

2. $\mathscr{A} \in \mathcal{L}(V)$. 且 $\mathscr{A}^2 = \mathcal{E}$. 证 $\forall \vec{v} \in V$. $\vec{v} - \mathscr{A}\vec{v} = \vec{0}$ 或是以 -1 为特征值的特征向量, 若 $\operatorname{char}(K) \neq 2$. 则 $V = V^1 \oplus V^{-1}$.

Pf: $\forall \vec{v} \in V$. 若 $\vec{v} - \mathscr{A}\vec{v} = \vec{0}$, 则 $\mathscr{A}\vec{v} = 1\cdot\vec{v}$ 即 $\vec{v} \in V^1$.

若 $\vec{v} - \mathscr{A}\vec{v} \neq \vec{0}$, 则 $\mathscr{A}(\vec{v} - \mathscr{A}\vec{v}) = \mathscr{A}\vec{v} - \mathscr{A}^2\vec{v} = \mathscr{A}\vec{v} - \vec{v} = (-1)(\vec{v} - \mathscr{A}\vec{v})$.

则 $\vec{v} - \mathscr{A}\vec{v} \in V^{-1}$.

$\forall \vec{x} \in V^1 \cap V^{-1}$. 则 $\mathscr{A}\vec{x} = \vec{x} = -\vec{x} \Rightarrow 2\vec{x} = \vec{0} \xRightarrow{\operatorname{char}(K)\neq 2} \vec{x} = \vec{0}$

$\therefore V^1 \cap V^{-1} = \{\vec{0}\}$. $\forall \vec{x} \in V$, 若 $\vec{x} = \mathscr{A}\vec{x}$. 则 $\vec{x} \in V^1 \subseteq V^1 \oplus V^{-1}$.

若 $\vec{x} \neq \mathscr{A}\vec{x}$, 则 $\vec{x} = \dfrac{\vec{x} - \mathscr{A}\vec{x}}{2} + \dfrac{\mathscr{A}\vec{x} + \vec{x}}{2} \in V^{-1} + V^1$

$\therefore V \subseteq V^1 \oplus V^{-1}$ $\therefore V = V^1 \oplus V^{-1}$.

注: $\lambda_1 \neq \lambda_2 \in \operatorname{spec}_F(\mathscr{A})$. 则 $V^{\lambda_1} \cap V^{\lambda_2} = \{\vec{0}\}$. 四

方2. (核核分解). $f(t) = (t-1)(t+1)$

$V = \underset{V^{-1}}{\ker(\mathscr{A} + \mathcal{E})} \oplus \underset{V^1}{\ker(\mathscr{A} - \mathcal{E})}$

3. 求特征多项式、特征值、特征向量.

(1). $A=\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$. $\chi_A = |\lambda E - A| = \begin{vmatrix} \lambda+2 & -2 \\ 2 & \lambda-3 \end{vmatrix} = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)$.

$\chi_A=0 \Rightarrow \lambda_1=2,\ \lambda_2=-1.$ $(\lambda_1 E - A)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

设 $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2$.

$(\lambda_2 E - A)\vec{x} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(2). $A=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. $\chi_A = |\lambda E - A| = \begin{vmatrix} \lambda-\cos\theta & \sin\theta \\ -\sin\theta & \lambda-\cos\theta \end{vmatrix} = \lambda^2 - 2\lambda\cos\theta + 1$.

则 $\chi_A = 0 \Rightarrow \lambda_{1,2} = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$

设 $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2$. $(\lambda_1 E - A)\vec{x} = \begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$(\lambda_2 E - A)\vec{x} = \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

$\theta = k\pi$. $\lambda_i E - A = O_{2\times 2}$. $\vec{x}$ 为 V 中任意向量.

(5). $A=\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$. $\chi_A = |\lambda E - A| = \begin{vmatrix} \lambda-4 & 5 & -2 \\ -5 & \lambda+7 & -3 \\ -6 & 9 & \lambda-4 \end{vmatrix} = \lambda^2(\lambda-1)$.

$\chi_A = 0 \Rightarrow \lambda_1=\lambda_2=0,\ \lambda_3=1.$ 设 $\vec{x} = (x_1, x_2, x_3)^t \in \mathbb{C}^3$

$(\lambda_1 E - A)\vec{x} = \begin{pmatrix} -4 & 5 & -2 \\ -5 & 7 & -3 \\ -6 & 9 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$.

$(\lambda_3 E - A)\vec{x} = \begin{pmatrix} -3 & 5 & -2 \\ -5 & 8 & -3 \\ -6 & 9 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

eg1. $A=\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & & & \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$ 令 $\xi \overset{n\text{次单位根}}{=} e^{\frac{2\pi}{n}i}$，$\varepsilon_k=\xi^k$.

则 $\vec{v}_k=\begin{pmatrix} 1 \\ \varepsilon_k \\ \vdots \\ \varepsilon_k^{n-1} \end{pmatrix}$ $(k=0,1,\cdots,n-1)$. 为 A 的线性无关的特征向量.

求 $\vec{v}_k$ 对应的特征值并求 $|A|$.

解: (注: 欧拉公式 $e^{\pi i}+1=0$. (上帝创造的公式).)

$\xi=e^{\frac{2\pi}{n}i}=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$ (n次单位根之一. $\xi^n=1$).

且 $1,\xi,\xi^2,\cdots,\xi^{n-1}$ 构成全部的 n 次单位根. 两两不同.

首先证明 $\vec{v}_0,\cdots,\vec{v}_{n-1}$ 线性无关.

令 $V=(\vec{v}_0,\vec{v}_1,\cdots,\vec{v}_{n-1})=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \varepsilon_0 & \varepsilon_1 & & \varepsilon_{n-1} \\ \vdots & \vdots & & \vdots \\ \varepsilon_0^{n-1} & \varepsilon_1^{n-1} & \cdots & \varepsilon_{n-1}^{n-1} \end{pmatrix} \Rightarrow |V|=\prod_{1\le i<j\le n}(\varepsilon_j-\varepsilon_i)\neq 0$.

(Vandermonde 行列式)

再证 v_k $(k=0,\cdots,n-1)$ 是 A 的特征向量.

$$A\vec{v}_k=\begin{pmatrix} a_0+a_1\varepsilon_k+\cdots+a_{n-1}\varepsilon_k^{n-1} \\ a_{n-1}+a_0\varepsilon_k+\cdots+a_{n-2}\varepsilon_k^{n-1} \\ \vdots \\ a_1+a_2\varepsilon_k+\cdots+a_0\varepsilon_k^{n-1} \end{pmatrix} \overset{\boxed{\varepsilon_k^n=1}}{=} \begin{pmatrix} a_0+a_1\varepsilon_k+\cdots+a_{n-1}\varepsilon_k^{n-1} \\ a_0\varepsilon_k+a_1\varepsilon_k^2+\cdots+a_{n-1}\underline{\varepsilon_k^n} \\ a_0\varepsilon_k^2+a_1\varepsilon_k^3+\cdots+a_{n-2}\varepsilon_k^n+a_{n-1}\varepsilon_k^{n+1} \\ \vdots \\ a_0\varepsilon_k^{n-1}+a_1\varepsilon_k^n+\cdots+a_{n-1}\varepsilon_k^{2n-2} \end{pmatrix}$$

$$=(a_0+a_1\varepsilon_k+\cdots+a_{n-1}\varepsilon_k^{n-1})\begin{pmatrix} 1 \\ \varepsilon_k \\ \vdots \\ \varepsilon_k^{n-1} \end{pmatrix}=\left(\sum_{j=0}^{n-1}a_j\varepsilon_k^j\right)\cdot\vec{v}_k$$

则 $\vec{v}_k$ 为 A 的特征向量，对应特征值为 $\sum_{j=0}^{n-1}a_j\varepsilon_k^j \triangleq \lambda_k$.

最后求 $|A|$: 若 A 可对角化. $A=P^{-1}\Lambda P$ $|A|=|\Lambda|=\lambda_1\cdots\lambda_n$

考虑 $AV=A(\vec{v}_0,\vec{v}_1,\cdots,\vec{v}_{n-1})=(A\vec{v}_0,A\vec{v}_1,\cdots,A\vec{v}_{n-1})=(\lambda_0\vec{v}_0,\lambda_1\vec{v}_1,\cdots,\lambda_{n-1}\vec{v}_{n-1})$

则 $|AV|=|A|\cdot|V|=|\lambda_0\vec{v}_0,\lambda_1\vec{v}_1,\cdots,\lambda_{n-1}\vec{v}_{n-1}|=(\lambda_0\lambda_1\cdots\lambda_{n-1})|\vec{v}_0,\vec{v}_1,\cdots,\vec{v}_{n-1}|$

$=(\lambda_0\lambda_1\cdots\lambda_{n-1})\cdot|V|$

χ_A在F中有n个根时，$|A| = \prod_{\lambda\in spec_F(A)} \lambda$

$\because |V| \neq 0 \quad \therefore |A| = \lambda_0\lambda_1\cdots\lambda_{n-1} = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j \varepsilon_k^j\right)$ ☐.

注：$AV = (\lambda_0\vec{v}_0, \cdots, \lambda_{n-1}\vec{v}_{n-1}) = \cancel{\begin{pmatrix}\lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \\ & & & \lambda_{n-1}\end{pmatrix}} \underbrace{(\vec{v}_0, \vec{v}_1, \cdots, \vec{v}_{n-1})}_{V} \begin{pmatrix}\lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1}\end{pmatrix}$.

则 A可对角化为 $V^{-1}AV = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{n-1})$

任意矩阵 若可对角化 则相似于一个对角矩阵.（合同是作用于对称矩阵）

eg2. 设 $A \in M_n(F)$ 可对角化 i.e. $A \sim_s \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ 对 $\forall f(t) \in F[t]$.

证明 $f(A) \sim_s \begin{pmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}$.

Pf: $\exists P \in GL_n(F)$，s.t. $A = P^{-1}\Lambda P$，其中 $\Lambda = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$.

对 $\forall f(t) \in F[t]$. 设 $f(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_0$.

则 $a_i A^i = a_i(P^{-1}\Lambda P)^i = a_i(P^{-1}\Lambda^i P) = a_i P^{-1}\begin{pmatrix}\lambda_1^i & & & \\ & \lambda_2^i & & \\ & & \ddots & \\ & & & \lambda_n^i\end{pmatrix}P$.

$$= P^{-1}\begin{pmatrix}a_i\lambda_1^i & & & \\ & a_i\lambda_2^i & & \\ & & \ddots & \\ & & & a_i\lambda_n^i\end{pmatrix}P = P^{-1}(a_i A^i)P.$$

于是 $f(A) = \sum_{i=0}^m a_i A^i = \sum_{i=0}^m P^{-1}\begin{pmatrix}a_i\lambda_1^i & & \\ & \ddots & \\ & & a_i\lambda_n^i\end{pmatrix}P$

$$= P^{-1}\left(\sum_{i=0}^m \begin{pmatrix}a_i\lambda_1^i & & \\ & \ddots & \\ & & a_i\lambda_n^i\end{pmatrix}\right)P = P^{-1}\begin{pmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}P$$

$\therefore f(A) \sim_s \begin{pmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix} = (f(\Lambda))$. ☐.

注：$A \sim_s \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ 则 $\mu_A = \mathrm{Lcm}(\mu_{\lambda_1}, \cdots, \mu_{\lambda_n})$. $\chi_A = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$

$= \mathrm{lcm}(t-\lambda_1, \cdots, t-\lambda_n)$

$\Rightarrow \mu_A \mid \chi_A$.