第三次作业.

1. (2). 验证 $f: P_n \longrightarrow P_n$ $= \{ u \in \mathbb{R}[t] \mid \deg(u) < n \}$. $f(u(t)) = t u'(t) - u(t)$ 是线性映射并且求出 $\ker f$.

(3). 非退化矩阵 $C \in M_n(\mathbb{R})$ 确定的映射 $f_C: X \longrightarrow C^{-1}XC$. 验证 f_C 在 $M_n(\mathbb{R})$ 上是线性的且 $f_C(XY) = f_C(X) f_C(Y)$

Pf: (2). 先验证 f 是线性的. $\forall\ c_1, c_2 \in \mathbb{R}$, $u_1(t), u_2(t) \in P_n$.

$$f(c_1u_1(t)+c_2u_2(t)) = t\cdot(c_1u_1(t)+c_2u_2(t))' - (c_1u_1(t)+c_2u_2(t))$$
$$= t(c_1u_1'(t)+c_2u_2'(t)) - c_1u_1(t) - c_2u_2(t)$$
$$= c_1(tu_1'(t)-u_1(t)) + c_2(tu_2'(t)-u_2(t)) = c_1 f(u_1(t)) + c_2 f(u_2(t)).$$

若 $u(t) \in P_n$, s.t. $f(u(t)) = 0$. 即 $tu'(t) - u(t) = 0$.

法2: 若 $u \neq 0$. $\frac{u'}{u} = \frac{1}{t} = \frac{t'}{t}$ $\Rightarrow \frac{u'}{u} - \frac{t'}{t} = \frac{(u/t)'}{(u/t)} = 0 \Rightarrow (u/t)' = 0$, $u/t = c \in \mathbb{R}$

设 $u(t) = u_0 + u_1 t + \cdots + u_{n-1}t^{n-1}$, $u_i \in \mathbb{R}$. $i = 1, \cdots, n-1$.

则 $u_0 + u_1 t + \cdots + u_{n-1}t^{n-1} = t[u_1 + 2u_2 t + \cdots + (n-1)u_{n-1}t^{n-2}] = u_1t + 2u_2t^2 + \cdots + (n-1)u_{n-1}t^{n-1}$

$\Rightarrow u_0 = 0$, $u_1 = u_1$, $u_i = iu_i$, $i \in \{2, 3, \cdots, n-1\}$. $\Rightarrow u_2 = u_3 = \cdots = u_{n-1} = 0$.

$\therefore \ker f = \{u(t) \in P_n \mid f(u(t)) = tu'(t) - u(t) = 0\} = \{u(t) = u_1 t \mid u_1 \in \mathbb{R}\} = \langle t \rangle_{\mathbb{R}}$.

(3). f_C 是线性的: $\forall\ X_1, X_2 \in M_n(\mathbb{R})$, $a, b \in \mathbb{R}$.

$$f_C(aX_1 + bX_2) = C^{-1}(aX_1+bX_2)C = C^{-1}(aX_1)C + C^{-1}(bX_2)C$$

$M_n(\mathbb{R})$ 是一个环 满足分配律.

$$= aC^{-1}X_1C + bC^{-1}X_2C = af_C(X_1) + bf_C(X_2)$$

$X, Y \in M_n(\mathbb{R})$. $f_C(XY) = C^{-1}(XY)C = C^{-1}XCC^{-1}YC = (C^{-1}XC)(C^{-1}YC) = f_C(X)f_C(Y)$.

2. $U = \left\langle \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix} \right\rangle$, $V = \left\langle \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-3\end{pmatrix} \right\rangle$ 求 $U+V$, $U \cap V$ 的一组基.

解: ∵ $U, V \subseteq \mathbb{R}^3$ 子空间. 设 $u_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, $u_3 = \begin{pmatrix}1\\3\\3\end{pmatrix}$. $v_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\3\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\-3\end{pmatrix}$

$U + V = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle$.

设 $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 3 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -4 \end{pmatrix}$

$\therefore (1,2,1), (0,1,2), (0,0,1)$ 线性无关

$\mathrm{rank}(A) = 3$.

$\therefore W_1$ 的一组基为 $(1,2,1), (0,1,2), (0,0,1)$.

2). $\forall\ \vec{w} \in U \cap V$. $\exists\ x_1, x_2, x_3 \in \mathbb{R}$, $y_1, y_2, y_3 \in \mathbb{R}$. s.t. $\vec{w} = x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3$ 即 $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 - y_1\vec{v}_1 - y_2\vec{v}_2 - y_3\vec{v}_3 = \vec{0}$.

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & -2 & -1 \\ 2 & 1 & 3 & -2 & -3 & -1 \\ 1 & -1 & 3 & -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \Rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \vec{0}.$$

$$\Rightarrow \begin{cases} x_1 = \cancel{-2x_2+2y_2+2y_3}\ 2x_3+2y_3 \\ x_2 = x_3+y_2+y_3 \\ y_1 = -y_2+2y_3. \end{cases} \Rightarrow \vec{u} = y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3$$

$$= (-y_2+2y_3)\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 = y_2(\vec{v}_1+\vec{v}_2) + y_3(2\vec{v}_1+\vec{v}_3)$$

$$= y_2(1,1,-1)^t + y_3(3,5,1)^t$$

$\therefore \{(1,1,-1)^t, (3,5,1)^t\}$ 是 $U\cap V$ 的一组基. $U\cap V$ 是2维向量空间.

3. 设 $V=F[x]$, $U=\{f(x^2) \mid f\in V\}$. 分别求 U 和 V/U 的一组基.

解: 设 $\forall f\in V$, $f=a_0+a_1x+\cdots+a_nx^n$, $a_0,\cdots,a_n\in F$.

则 $f(x^2)=a_0+a_1x^2+\cdots+a_nx^{2n}$ 又 $\{1,x^2,\cdots,x^{2n},\cdots\}$ 线性无关

$\therefore \{1,x^2,x^4,\cdots,x^{2n},\cdots\}$ 是 U 的一组基.

$V/U=\{f(x)+U \mid \forall f(x)\in V\}$ $\forall f=a_0+a_1x+\cdots+a_nx^n\in V$.

不妨设 $n=2k+1$, $k\in\mathbb{N}$. 则可设 $g=a_0+a_2x^2+\cdots+a_{2k}x^{2k}\in U$

$h=a_1x+a_3x^3+\cdots+a_{2k+1}x^{2k+1}$, 则 $f=g+h$, $f+U=(g+h)+U=h+U$.

$\therefore \forall \bar{h}\in V/U$ 可由 $\{x+U, x^3+U,\cdots,x^{2k+1}+U,\cdots\}$ 线性生成, 且线性无关.

$\therefore \{x+U, x^3+U,\cdots,x^{2k+1}+U,\cdots\}$ 为 V/U 的一组基. 基中一定不含零元.

4. 设 V,W 是两个域 F 上的线性空间, $\phi: V\longrightarrow W$ 线性映射. 令 $\bar{\phi}: V/\ker\phi \longrightarrow W$
$\vec{v}+\ker\phi \longmapsto \phi(\vec{v})$.

证明 (i). $\bar{\phi}$ 是线性映射且为单射.

(ii). 记 π 是 V 到 $V/\ker\phi$ 自然的商映射, 则 $\phi=\bar{\phi}\circ\pi$.

即任意线性映射可分解成一个单射和一个满射的复合.

Pf: (i). 良定义. $\vec{v}_1+\ker\phi=\vec{v}_2+\ker\phi$, 则 $\vec{v}_1-\vec{v}_2\in\ker\phi$. 即 $\phi(\vec{v}_1-\vec{v}_2)=\vec{0}$

即 $\phi(\vec{v}_1)=\phi(\vec{v}_2)$. $\Rightarrow \bar{\phi}(\vec{v}_1+\ker\phi)=\phi(\vec{v}_1)=\phi(\vec{v}_2)=\bar{\phi}(\vec{v}_2+\ker\phi)$

线性. $\forall\ \alpha,\beta\in F$. $\vec{v}_1+\ker\phi, \vec{v}_2+\ker\phi\in V/\ker\phi$.

$$\bar{\phi}(\alpha(\vec{v}_1+\ker\phi)+\beta(\vec{v}_2+\ker\phi)) = \bar{\phi}((\alpha\vec{v}_1+\beta\vec{v}_2)+\ker\phi) = \phi(\alpha\vec{v}_1+\beta\vec{v}_2)$$

$$= \alpha\phi(\vec{v}_1)+\beta\phi(\vec{v}_2) = \alpha\bar{\phi}(\vec{v}_1+\ker\phi)+\beta\bar{\phi}(\vec{v}_2+\ker\phi).$$

单射: $\ker\bar{\phi} = \{\vec{v}+\ker\phi \mid \vec{v}\in V$ 且 $\bar{\phi}(\vec{v}+\ker\phi)=\phi(\vec{v})=\vec{0}\}$

$= \{\vec{v}+\ker\phi \mid \vec{v}\in\ker\phi\} = \{\vec{0}+\ker\phi\}$

$\Leftrightarrow$ $\bar{\phi}$ 是单射.

(ii). $\pi: V \longrightarrow V/\ker\phi$ 线性

$\vec{v} \longmapsto \vec{v}+\ker\phi$.

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "\phi"'] & V/\ker\phi \arrow[d, "\bar{\phi}"] \\ & W \end{tikzcd}$$

$\forall\, \vec{v}\in V$, $\bar{\phi}\circ\pi(\vec{v}) = \bar{\phi}(\pi(\vec{v})) = \bar{\phi}(\vec{v}+\ker\phi) = \phi(\vec{v})$.

由 $\vec{v}$ 的任意性, 知 $\phi = \bar{\phi}\circ\pi$. $\bar{\phi}$ 是单射 且 π 为满射.

($V/\ker\phi \cong \mathrm{im}\,\phi \subseteq W$. $V/\ker\phi \overset{\bar{\phi}}{\hookrightarrow} W$.).

5. 证明. $\mathbb{Z}_2\times\mathbb{Z}_2$ 可写成 3 个真子空间的并.

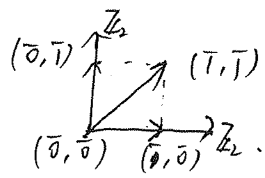
Pf: $\mathbb{Z}_2$ 作为交换群是域 $\mathbb{Z}_2$ 上的线性空间.

则 $\mathbb{Z}_2\times\mathbb{Z}_2$ 是域 $\mathbb{Z}_2$ 上的线性空间. $\mathbb{Z}_2\times\mathbb{Z}_2 = \{(\bar{0},\bar{0}),(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{1})\}$.

其子空间: $V_0 = \{(\bar{0},\bar{0})\}$. $V_1 = \{(\bar{0},\bar{0}),(\bar{0},\bar{1})\}$, $V_2 = \{(\bar{0},\bar{0}),(\bar{1},\bar{0})\}$

$V_3 = \{(\bar{0},\bar{0}),(\bar{1},\bar{1})\}$, $V_4 = \mathbb{Z}_2\times\mathbb{Z}_2$.

则 $\mathbb{Z}_2\times\mathbb{Z}_2 = V_1\cup V_2\cup V_3$.



(这里 $\mathbb{Z}_2$ 是有限域. 其线性空间可写成互不包含的子空间的并).

维数.

· 定义 (基底) 称 $S\subseteq V$ 是 V 的一组基底，若 S 是 V 中的一个极大线性无关集.

(维数). $\dim_F V=\begin{cases}+\infty & 若\ |S|=\infty\\ n & 若\ |S|=n\\ 0 & 若\ V=S=\{\vec{0}\}\end{cases}$.

定理 1. (基扩充定理). 设 V 是有限维线性空间，$W\subseteq V$ 子空间. 设 $\vec{w}_1,\cdots,\vec{w}_d$ 是 W 的一组基，则 $\exists\ \vec{v}_1,\cdots,\vec{v}_{n-d}$, s.t. $\vec{w}_1,\cdots,\vec{w}_d,\vec{v}_1,\cdots,\vec{v}_{n-d}$ 是 V 的一组基 ($\dim V=n$).

2. (维数公式). $V_1,V_2\subseteq V$ 子空间 且 $\dim V<+\infty$. 则

$$\dim V_1+\dim V_2=\dim(V_1+V_2)+\dim(V_1\cap V_2).$$

(商空间维数公式). $W\subseteq V$ 子空间. 则 $\dim V/W=\dim V-\dim W$.

(n个维数的公式) V,W 有限维向量空间. $V\cong W \Leftrightarrow \dim V=\dim W$

(注: 判断 $U=V$. $U\subseteq V$ 且 $\dim U=\dim V$.
求一组基. $\{\vec{e}_1,\cdots,\vec{e}_n\}$ 线性无关 且 $\dim U=n$).

$\dim(V\times W)=\dim V+\dim W$.

$U\subseteq V$ 子空间 $\Rightarrow \dim U\le\dim V$.

$U\neq V \Leftrightarrow \dim U<\dim V$.

$\varphi\in \mathrm{Hom}(V,W)$, $U\subset V$ 子空间，则 $\dim U\ge \dim\varphi(U)$

(直和的等价条件) $\dim(U_1+\cdots+U_k)=\dim U_1+\cdots+\dim U_k$. $U_i\subseteq V$ 子空间.

eg1. 设 V 是域 F 上的线性空间，$V_1,V_2\subseteq V$ 子空间 且 $\dim V_1=s$, $\dim V_2=t$.

设 $V_1=\langle\vec{v}_1,\cdots,\vec{v}_s\rangle_F$, $V_2=\langle\vec{w}_1,\cdots,\vec{w}_t\rangle_F$ 为基.

令 $U=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_s\\ y_1\\ \vdots\\ y_t\end{pmatrix}\in F^{s+t}\ \middle|\ x_1\vec{v}_1+\cdots+x_s\vec{v}_s=y_1\vec{w}_1+\cdots+y_t\vec{w}_t\right\}$. 则 $\dim U=\dim(V_1\cap V_2)$

Pf: (方1). 构造线性映射 $\varphi:\ U\longrightarrow V_1\cap V_2$.

$$(x_1,\cdots,x_s,y_1,\cdots,y_t)^t\longmapsto \sum_{i=1}^s x_i\vec{v}_i\ \left(=\sum_{j=1}^t y_j\vec{w}_j\right).$$

良定义: $\forall\ \vec{u}=(x_1,\cdots,x_s,y_1,\cdots,y_t)^t\in U$. $\varphi(\vec{u})=\sum_{i=1}^s x_i\vec{v}_i=\sum_{j=1}^t y_jw_j\in V_1\cap V_2$

若 $\vec{u}=\vec{u}'$, $\vec{u}'=(x_1',\cdots,x_s',y_1',\cdots,y_t')^t\in U$.

则 $\varphi(\vec{u}')=\sum_{i=1}^s x_i'\vec{v}_i=\sum_{i=1}^s x_i\vec{v}_i=\varphi(\vec{u})$. 即 $\{\vec{v}_1,\cdots,\vec{v}_s\}$ 是一组基. 线性表示唯一. 良定义显然

线性映射: $\forall\ \alpha,\beta\in F$. $\vec{u}=(x_1,\cdots,x_s,y_1,\cdots,y_t)^t$, $\vec{u}'=(x_1',\cdots,x_s',y_1',\cdots,y_t')^t\in U$.

$$\varphi(\alpha\vec{u}+\beta\vec{u}')=\sum_{i=1}^s(\alpha x_i+\beta x_i')\vec{v}_i=\sum_{i=1}^s\alpha x_i\vec{v}_i+\sum_{i=1}^s\beta x_i'\vec{v}_i=\alpha\sum_{i=1}^s x_i\vec{v}_i+\beta\sum_{i=1}^s x_i'\vec{v}_i$$

$$\therefore\ \varphi(\alpha\vec{u}+\beta\vec{u}')=\alpha\varphi(\vec{u})+\beta\varphi(\vec{u}')$$

单射： $\ker\varphi = \{\vec u = (x_1,\cdots,x_s,y_1,\cdots,y_t)^t \in U \mid \varphi(\vec u)=\sum_{i=1}^s x_i v_i = \sum_{j=1}^t y_j \vec w_j = \vec 0\}$.

$\xlongequal[\text{均线性无关}]{\{\vec v_i\},\{\vec w_j\}} \{\vec u\in U \mid x_1=\cdots=x_s=y_1=\cdots=y_t=0\} = \{\vec 0\}$.

$\Leftrightarrow$ φ 为单射.

满射： $\forall f\in V_1\cap V_2$. $\exists x_1,\cdots,x_s,y_1,\cdots,y_t \in F$ s.t. $f=\sum_{i=1}^s x_i\vec v_i = \sum_{j=1}^t y_j\vec w_j$

$\Rightarrow$ $\vec u = (x_1,\cdots,x_s,y_1,\cdots,y_t)^t \in U$ 即 $\varphi(\vec u)=f$. 则 φ 满射.

综上, f 为线性同构. $\therefore U \cong V_1\cap V_2$. $\Leftrightarrow$ $\dim U = \dim V_1\cap V_2$.

(方2). $\sum_{i=1}^s x_i\vec v_i + \sum_{j=1}^t (-y_j)\vec w_j = \vec 0 \Rightarrow \underbrace{(\vec v_1,\cdots,\vec v_s,-\vec w_1,\cdots,-\vec w_t)}_{A}\begin{pmatrix} x_1\\ \vdots\\ x_s\\ y_1\\ \vdots\\ y_t\end{pmatrix} = \vec 0$.

即 U 为以 A 为系数矩阵的齐次线性方程组的解空间.

$\therefore \dim U = s+t-\operatorname{rank}A = s+t-\dim(\langle \vec v_1,\cdots,\vec v_s,\vec w_1,\cdots,\vec w_t\rangle)$
$= s+t-\dim(V_1+V_2) = \dim(V_1\cap V_2)$.

基变换与坐标变换.

设 V 为域 F 上的 n 维线性空间, $\{\vec e_1,\cdots,\vec e_n\}$ 为 V 的一组基, 则 $\forall \vec v\in V$, $\exists!\ \alpha_1,\cdots,\alpha_n\in F$, s.t. $\vec v = \alpha_1\vec e_1+\cdots+\alpha_n\vec e_n = (\vec e_1,\cdots,\vec e_n)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

称 $\vec\alpha = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$ 为 $\vec v$ 在基 $\{\vec e_1,\cdots,\vec e_n\}$ 下的坐标. 若 $\{\vec e_1',\cdots,\vec e_n'\}$ 为 V 的另一组基, 则存在可逆矩阵 A. s.t. $(\vec e_1',\cdots,\vec e_n') = (\vec e_1,\cdots,\vec e_n)A$ $\vec e_i' = (\vec e_1,\cdots,\vec e_n)\vec A^{(i)}$, $i=1,\cdots,n$.

习惯线性表示用矩阵来表示.

称 A 为 $\{\vec e_1,\cdots,\vec e_n\}$ 到 $\{\vec e_1',\cdots,\vec e_n'\}$ 的转换矩阵. $\vec v = (\vec e_1,\cdots,\vec e_n)\vec\alpha = (\vec e_1',\cdots,\vec e_n')\vec\alpha' = (\vec e_1,\cdots,\vec e_n)A\vec\alpha'$, $\vec\alpha = A\vec\alpha'$ ($\vec e_i$ 无关)

此时, $\vec v$ 在 $\{\vec e_1',\cdots,\vec e_n'\}$ 下的坐标为 $\vec\alpha' = A^{-1}\vec\alpha = A^{-1}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$. $\vec\alpha = A\vec\alpha'$

Thm. 设 V 是 F 上的有限维线性空间. $\vec v_1,\cdots,\vec v_n,\vec w_1,\cdots,\vec w_m\in V$ 且 $\vec w_1,\cdots,\vec w_m$ 线性无关. 若 $\exists A\in F^{m\times n}$, s.t. $(\vec v_1,\cdots,\vec v_n) = (\vec w_1,\cdots,\vec w_m)A$

则 $\operatorname{rank}A = \dim_F\langle \vec v_1,\cdots,\vec v_n\rangle$.

Pf: (方1) 设 $\dim(\langle\vec v_1,\cdots,\vec v_n\rangle) = k$. 不妨设 $\vec v_1,\cdots,\vec v_k$ 线性无关.

(im角度) 则 $\forall j\in\{1,\cdots,n\}$, $\vec v_j = \sum_{s=1}^k \alpha_{sj}\vec v_s$ $\exists\ \alpha_{1j},\cdots,\alpha_{kj}\in F$. s.t.

$\therefore \underbrace{\vec v_j}_{(\vec w_1,\cdots,\vec w_m)\vec A^{(j)}} = \sum_{s=1}^k \alpha_{sj}\vec v_s = \sum_{s=1}^k \alpha_{sj}(\vec w_1,\cdots,\vec w_m)\vec A^{(s)} = (\vec w_1,\cdots,\vec w_m)\sum_{s=1}^k\alpha_{sj}\vec A^{(s)}$

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由 $\vec{w}_1,\cdots,\vec{w}_m$ 线性无关知，$\vec{A}^{(j)}=\sum_{s=1}^{k}\alpha_{sj}\vec{A}^{(s)}$ 对 $\forall j\in\{1,\cdots,n\}$ 成立.

$\therefore$ ~~dim~~ $\mathrm{rank}(A)\le k$. 假设 $\mathrm{rank}A<k$. 则存在不全为 0 的 $\beta_1,\cdots,\beta_k$.

s.t. $\beta_1\vec{A}^{(1)}+\cdots+\beta_k\vec{A}^{(k)}=\vec{0}$ 则 $(\vec{w}_1,\cdots,\vec{w}_m)(\beta_1\vec{A}^{(1)}+\cdots+\beta_k\vec{A}^{(k)})=\vec{0}$

即 $\beta_1(\vec{w}_1,\cdots,\vec{w}_m)\vec{A}^{(1)}+\cdots+\beta_k(\vec{w}_1,\cdots,\vec{w}_m)\vec{A}^{k}=\beta_1\vec{v}_1+\cdots+\beta_k\vec{v}_k=\vec{0}$. ($\rightarrow\leftarrow$). $\square$

(方2). 设 $V_1=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in F^n \;\middle|\; (\vec{v}_1,\cdots,\vec{v}_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\vec{0}\right\}$

(ker角度) $V_2=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in F^n \;\middle|\; (\vec{w}_1,\cdots,\vec{w}_m)A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\vec{0}\right\}$. ~~则 $V_1=V_2$~~.

$\because (\vec{v}_1,\cdots,\vec{v}_n)=(\vec{w}_1,\cdots,\vec{w}_m)A$ $\therefore V_1=V_2$.

下证 $V_A=V_2$. 显然 $V_A\subseteq V_2$. $\forall\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in V_2$.

有 $(\vec{w}_1,\cdots,\vec{w}_m)A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=(\vec{w}_1,\cdots,\vec{w}_m)\left(A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\right)=\vec{0}$. 由 $(\vec{w}_1,\cdots,\vec{w}_m)$ 线性无关,

知 $A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\vec{0}$ $\therefore \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in V_A$ $\therefore V_2\subseteq V_A$ $\therefore V_2=V_A$

$\therefore V_1=V_2=V_A$. $\therefore \dim V_1=\dim V_A$ $\therefore n-\dim(\langle\vec{v}_1,\cdots,\vec{v}_n\rangle)=n-\mathrm{rank}A$.

$\therefore \dim\langle\vec{v}_1,\cdots,\vec{v}_n\rangle=\mathrm{rank}A$. $\square$

Cor. $\vec{v}_1,\cdots,\vec{v}_n,\vec{w}_1,\cdots,\vec{w}_m\in V$, $(\vec{v}_1,\cdots,\vec{v}_n)=(\vec{w}_1,\cdots,\vec{w}_m)A$. 若 $\vec{w}_1,\cdots,\vec{w}_m$ 为 U 的一组基且 A 可逆

则 $n=m$ 且 $\vec{v}_1,\cdots,\vec{v}_m$ 也为 U 的一组基，反过来也成立.

eg 2. V 为有限维向量空间. $U_1,U_2,U_3\le V$ 为子空间. 证明

$\dim(U_1+U_2+U_3)\le\dim U_1+\dim U_2+\dim U_3-\dim U_1\cap U_2-\dim U_1\cap U_3-\dim U_2\cap U_3+\dim U_1\cap U_2\cap U_3$

Pf: 由维数公式: $\dim((U_1+U_2)+U_3)=\dim(U_1+U_2)+\dim U_3-\dim((U_1+U_2)\cap U_3)$

$=\dim U_1+\dim U_2-\dim U_1\cap U_2+\dim U_3-\dim((U_1+U_2)\cap U_3)$.

$\because U_1\cap U_3,\ U_2\cap U_3\subseteq(U_1+U_2)\cap U_3$ 子空间.

$\therefore \dim(U_1+U_2)\cap U_3)\ge\dim(U_1\cap U_3+U_2\cap U_3)=\dim(U_1\cap U_3)+\dim(U_2\cap U_3)-\dim(U_1\cap U_2\cap U_3)$.

$\therefore \dim(U_1+U_2+U_3)\le\dim U_1+\dim U_2+\dim U_3-\dim U_1\cap U_2-\dim U_1\cap U_3-\dim U_2\cap U_3+\dim(U_1\cap U_2\cap U_3)$

当 $U_1\cap U_3+U_2\cap U_3\ne(U_1+U_2)\cap U_3$ 时，上式不等号成立.

取 $U_1=\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle$, $U_2=\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle$, $U_3=\langle\begin{pmatrix}1\\1\end{pmatrix}\rangle$.

左 $=\dim(U_1+U_2+U_3)=\dim(\mathbb{R}^2)=2$. $<$ 右 $=3-0+0=3$. $\square$

掌握维数公式.