第四次作业.

1. $\mathbb{R}^3$中. 设 $\vec{u}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{u}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{u}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$. $\vec{v}_1=\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\vec{v}_2=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\vec{v}_3=\begin{pmatrix}0\\1\\2\end{pmatrix}$.

求由基 $(\vec{u}_1,\vec{u}_2,\vec{u}_3)$ 到基 $(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ 的过渡矩阵. 并求 $\vec{\xi}=(1,2,3)^t$ 在 $(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ 下坐标.

解: $(\vec{v}_1,\vec{v}_2,\vec{v}_3)=(\vec{u}_1,\vec{u}_2,\vec{u}_3)\begin{pmatrix}1&1&0\\2&1&1\\2&1&2\end{pmatrix}$. 设 $A=\begin{pmatrix}1&1&0\\2&1&1\\2&1&2\end{pmatrix}$. 则 A 为转换矩阵.

设 $\vec{\xi}$ 在 $\vec{u}_1,\vec{u}_2,\vec{u}_3$ 下的坐标为 $\vec{\alpha}$, 则 $\vec{\xi}=(\vec{u}_1,\vec{u}_2,\vec{u}_3)\vec{\alpha}=(\vec{u}_1,\vec{u}_2,\vec{u}_3)\begin{pmatrix}1\\2\\3\end{pmatrix}$.

$\vec{v}_1,\vec{v}_2,\vec{v}_3$ $\quad\quad$ $\vec{\beta}$. $\quad$ $\vec{\xi}=(\vec{v}_1,\vec{v}_2,\vec{v}_3)\vec{\beta}=(\vec{u}_1,\vec{u}_2,\vec{u}_3)A\vec{\beta}$

$$\therefore \vec{\beta}=A^{-1}\vec{\alpha}=\begin{pmatrix}-1&2&-1\\+2&-2&1\\0&-1&1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

2. 设 V, W 是域 F 上的有限维线性空间. 证明 $\dim(V\times W)=\dim V+\dim W$.

Pf: 设 $\dim V=n$. $\dim W=m$. 且 $\{\vec{v}_1,\cdots,\vec{v}_n\}$ 为 V 的一组基.

$\{\vec{w}_1,\cdots,\vec{w}_m\}$ 为 W 的一组基.

下面证明 $\begin{pmatrix}\vec{v}_1\\\vec{0}\end{pmatrix},\cdots,\begin{pmatrix}\vec{v}_n\\\vec{0}\end{pmatrix},\begin{pmatrix}\vec{0}\\\vec{w}_1\end{pmatrix},\cdots\begin{pmatrix}\vec{0}\\\vec{w}_m\end{pmatrix}$ 为 $V\times W$ 的一组基.

线性无关: 设 $\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_m\in F$, s.t.

$$\alpha_1\begin{pmatrix}\vec{v}_1\\\vec{0}\end{pmatrix}+\cdots+\alpha_n\begin{pmatrix}\vec{v}_n\\\vec{0}\end{pmatrix}+\beta_1\begin{pmatrix}\vec{0}\\\vec{w}_1\end{pmatrix}+\cdots+\beta_n\begin{pmatrix}\vec{0}\\\vec{w}_m\end{pmatrix}=\begin{pmatrix}\vec{0}_V\\\vec{0}_W\end{pmatrix}$$

则 $\begin{pmatrix}\sum_{i=1}^n\alpha_i\vec{v}_i\\\sum_{i=1}^m\beta_i\vec{w}_i\end{pmatrix}=\begin{pmatrix}\vec{0}\\\vec{0}\end{pmatrix}$ 　又 $\{\vec{v}_1,\cdots,\vec{v}_n\}$, $\{\vec{w}_1,\cdots,\vec{w}_m\}$ 均线性无关

$\therefore \alpha_1=\cdots=\alpha_n=\beta_1=\cdots=\beta_m=0$. $\therefore \begin{pmatrix}\vec{v}_1\\\vec{0}\end{pmatrix},\cdots,\begin{pmatrix}\vec{v}_n\\\vec{0}\end{pmatrix},\begin{pmatrix}\vec{0}\\\vec{w}_1\end{pmatrix},\cdots,\begin{pmatrix}\vec{0}\\\vec{w}_m\end{pmatrix}$ 线性无关.

$\forall \begin{pmatrix}\vec{v}\\\vec{w}\end{pmatrix}\in V\times W$, 则 $\vec{v}\in V$, $\vec{w}\in W$. $\therefore \exists!\,\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_m\in F$

s.t. $\vec{v}=\sum_{i=1}^n\alpha_i\vec{v}_i$, $\vec{w}=\sum_{i=1}^m\beta_i\vec{w}_i$ $\therefore \begin{pmatrix}\vec{v}\\\vec{w}\end{pmatrix}=\sum_{i=1}^n\alpha_i\begin{pmatrix}\vec{v}_i\\\vec{0}\end{pmatrix}+\sum_{i=1}^m\beta_i\begin{pmatrix}\vec{0}\\\vec{w}_i\end{pmatrix}$

综上, $\begin{pmatrix}\vec{v}_1\\\vec{0}\end{pmatrix},\cdots,\begin{pmatrix}\vec{v}_n\\\vec{0}\end{pmatrix},\begin{pmatrix}\vec{0}\\\vec{w}_1\end{pmatrix},\cdots,\begin{pmatrix}\vec{0}\\\vec{w}_m\end{pmatrix}$ 是 $V\times W$ 的一组基

$\therefore \dim(V\times W)=n+m=\dim V+\dim W$ $\quad\blacksquare$.

3. 求迹为0的n阶实方阵构成空间的维数和一组基底.

解：迹为0的实方阵 $V=\{A\in M_n(\mathbb{R})\mid tr(A)=\sum_{i=1}^{n}a_{ii}=0\}$.

从对偶空间 U^0 零化的角度.

$\forall A\in M_n(\mathbb{R})$ 有 n^2 个待定元素. 若 $A\in V$. 则满足一个约束条件,

即 $trA=a_{11}+\cdots+a_{nn}=0$. $\therefore$ 解空间维数为 n^2-1.

$a_{11}=-(a_{22}+\cdots+a_{nn})$

$S=\{E_{11}-E_{kk}\mid k=2,3,\cdots,n\}\cup\{E_{ij}\mid 1\le i,j\le n \text{ 且 } i\ne j\}$.

E_{ij} 表示第i行j列元为1其余元为0的n阶实方阵.

若 $\exists\ a_{ij},a_{kk}\in\mathbb{R}$, $k=2,\cdots,n$, $1\le i,j\le n$ 且 $i\ne j$,

s.t. $\sum_{i,j}a_{ij}E_{ij}+\sum_{k=2}^{n}a_{kk}(E_{11}-E_{kk})=O_{n\times n}$ 即 $\begin{pmatrix}\sum_{k=2}^{n}a_{kk} & & (a_{ij})\\ & -a_{22} & \\ & & \ddots \\ (\tilde{a}_{ij}) & & -a_{nn}\end{pmatrix}=O_{n\times n}$

$\therefore a_{kk}=0$, $a_{ij}=0$, $\forall k\in\{2,\cdots,n\}$, $1\le i,j\le n$. $\therefore S$ 线性无关.

又 $|S|=n^2-1$. $\therefore$ S 是 V 的一组基.

(方2). $tr: M_n(\mathbb{R})\twoheadrightarrow\mathbb{R}$. 满射.线性 $\ker tr=V$

$A\longmapsto tr(A)$. $\therefore \dim V=\dim M_n(\mathbb{R})-\dim(im(tr))=n^2-1$.

4. 由所有单变元t的次数 $\le n$ 的满足 $f(1)=0$ 的多项式组成的空间维数是多少? 找出一组基底.

解：方1. 设 $V=\{f(t)\in\mathbb{R}[t]\mid f(1)=0 \text{ 且 } \deg f(t)\le n\}\subseteq P_{n+1}$ 是一个线性空间.

怎么由所找基生成. $=\{(t-1)g(t)\mid \forall g(t)\in\mathbb{R}[x] \text{ 且 } \deg g(t)\le n-1, \text{即 } g\in P_n\}$.

又 $P_n=\langle 1,t,t^2,\cdots,t^{n-1}\rangle_{\mathbb{R}}$ 为 P_n 的一组基.

$V=\langle t-1,(t-1)t,(t-1)t^2,\cdots,(t-1)t^{n-1}\rangle_{\mathbb{R}}$ 为 V 的一组基

显然线性无关.

$\dim V=n$.

方: $f=a_0+a_1t+\cdots+a_nt^n$.
$f(1)=a_0+a_1+\cdots+a_n=0$
$\therefore a_0=-a_1-\cdots-a_n$.
$f(t)=a_n(t^n-1)+\cdots+a_1(t-1)$.

方2. 令 $x=t-1$ 则 $t=x+1$
$f(t)=a_0+a_1(x+1)+\cdots+a_n(x+1)^n=b_0+b_1x+\cdots+b_nx^n$ $\triangleq g(x)$
$\therefore 0=f(1)=g(0)=b_0$
$\therefore f(t)=g(x)=b_1x+\cdots+b_nx^n$
$\{g(x)\mid g(0)=0\}$ 维数为n.
基 $\{x,x^2,\cdots,x^n\}$ $=\{1-t,(1-t)^2,\cdots,(1-t)^n\}$

方2. $V\subseteq P_n$. 且满足一个约束条件. $\forall f(t)=f_nt^n+\cdots+f_1t+f_0\in V$

有 $f_n+\cdots+f_1+f_0=0$ $\therefore \dim V=\dim(P_{n+1})-1=n+1-1=n$.

由 Taylor 展开, $f(t)=f(1)+f'(1)\cdot(t-1)+\cdots+\frac{1}{n!}f^{(n)}(1)(t-1)^n$ 且 $f(1)=0$.

$\forall f\in V$, $f=\alpha_1(t-1)+\cdots+\alpha_n(t-1)^n$, $\alpha_i\in F$.

又 $t-1,\cdots,(t-1)^n$ 线性无关. $\therefore\{t-1,(t-1)^2,\cdots,(t-1)^n\}$ 为 V 的一组基. ▢

5. 找出 P_n 的基 $(1, t, t^2, \cdots, t^{n-1})$ 到 $(1, t-\alpha, (t-\alpha)^2, \cdots, (t-\alpha)^{n-1})$ 的转换矩阵.

解: 对 $1 \leq k \leq n-1$, 由于 $(t-\alpha)^k = \sum_{i=0}^{k} \binom{k}{i} (-\alpha)^{k-i} t^i$, 故

$$(1, t-\alpha, (t-\alpha)^2, \cdots, (t-\alpha)^{n-1}) = (1, t, t^2, \cdots, t^{n-1}) \begin{pmatrix} 1 & -\alpha & (-\alpha)^2 & (-\alpha)^3 & \cdots & (-\alpha)^{n-1} \\ 0 & 1 & \binom{2}{1}(-\alpha) & \binom{3}{1}(-\alpha)^2 & \cdots & \binom{n-1}{1}(-\alpha)^{n-2} \\ 0 & 0 & 1 & \binom{3}{2}(-\alpha) & \cdots & \binom{n-1}{2}(-\alpha)^{n-3} \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

6. 设 $V_1, \cdots, V_k$ 为 n 维向量空间 V 的子空间

证明: 若 $\sum_{i=1}^{k} \dim V_i > n(k-1)$, 则 $V_1 \cap \cdots \cap V_k \neq \{\vec{0}\}$.

Pf: (方1). 先用数学归纳法证明 或直接证 $\dim(\bigcap_{i=1}^{k} V_i) \geq \sum_{i=1}^{k} \dim V_i - n(k-1)$.

$$\dim(\bigcap_{i=1}^{k} V_i) = \sum_{i=1}^{k} \dim V_i - \dim(V_1+V_2) - \dim(V_1 \cap V_2 + V_3) - \cdots - \dim(\bigcap_{i=1}^{k-1} V_i + V_k).$$

$k=2$ 时, 由维数公式, $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1+V_2)$.

设 $k-1$ 时, $\dim(\bigcap_{i=1}^{k-1} V_i) = \sum_{i=1}^{k-1} V_i - \dim(V_1+V_2) - \dim(V_1 \cap V_2 + V_3) - \cdots - \dim(\bigcap_{i=1}^{k-2} V_i + V_{k-1})$

则 $\dim(\bigcap_{i=1}^{k} V_i) = \dim((\bigcap_{i=1}^{k-1} V_i) \cap V_k) = \dim V_k + \dim(\bigcap_{i=1}^{k-1} V_i) - \dim(\bigcap_{i=1}^{k-1} V_i + V_k)$.

由归纳假设 $= \dim V_k + \sum_{i=1}^{k-1} \dim(V_i) - \dim(V_1+V_2) - \cdots - \dim(V_k + \bigcap_{i=1}^{k-2} V_i) - \dim(\bigcap_{i=1}^{k-1} V_i + V_k)$

$\therefore \dim(\bigcap_{i=1}^{k} V_i) = \sum_{i=1}^{k} \dim V_i - \dim(V_1+V_2) - \dim(V_1 \cap V_2 + V_3) - \cdots - \dim(\bigcap_{i=1}^{k-1} V_i + V_k)$

又 $\sum_{i=1}^{k} \dim V_i > n(k-1)$, 且 $V_1+V_2, V_1 \cap V_2 + V_3, \cdots, \bigcap_{i=1}^{k-1} V_i + V_k \subseteq V$ 子空间.

$\therefore \dim(\bigcap_{i=1}^{k} V_i) > n(k-1) - (k-1) \cdot n = 0$. $\therefore \bigcap_{i=1}^{k} V_i \neq \{\vec{0}\}$

(方2). $\varphi: V_1 \times \cdots \times V_k \longrightarrow V^{k-1}$

$(\vec{v}_1, \cdots, \vec{v}_k) \longmapsto (\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \cdots, \vec{v}_{k-1} - \vec{v}_k)$

显然, φ 是 $V_1 \times \cdots \times V_k$ 到 V^{k-1} 的线性映射. (自己验证)

且若 $(\vec{v}_1, \cdots, \vec{v}_k) \in V_1 \times \cdots \times V_k$, s.t. $\varphi(\vec{v}_1, \cdots, \vec{v}_k) = \vec{0}$, 则 $\vec{v}_1 = \vec{v}_2 = \cdots = \vec{v}_k \in V_1 \cap \cdots \cap V_k$.

$\therefore \ker\varphi \subseteq (V_1 \cap \cdots \cap V_k)^k$. $\therefore \dim(\ker\varphi) \leq k \cdot \dim(\bigcap_{i=1}^{k} V_i)$.

又 $\because \dim(\ker\varphi) = \dim(V_1 \times \cdots \times V_k) - \dim(\mathrm{im}\,\varphi)$ 且 $\mathrm{im}\,\varphi \subseteq V^{k-1}$

$$\therefore k \cdot \dim(\bigcap_{i=1}^{k} V_i) \geq \dim(\ker\varphi) = \dim(V_1 \times \cdots \times V_k) - \dim(\mathrm{im}\,\varphi)$$
$$= \sum_{i=1}^{k} \dim V_i - \dim(\mathrm{im}\,\varphi)$$
$$> n(k-1) - \dim(\mathrm{im}\,\varphi) \geq n(k-1) - n(k-1) = 0.$$

$\therefore \dim(\bigcap_{i=1}^{k} V_i) > 0$ $\therefore \bigcap_{i=1}^{k} V_i \neq \{\vec{0}\}$ ▯

幻方与半幻方

Def. 设矩阵 $A\in M_n(\mathbb{Q})$. 若 A 的每一列, 每一行元素之和均为某个固定的数 $\sigma(A)$,
则称 A 为半幻方 (semi-magic square). 称 $\sigma(A)$ 为 A 的值.
若半幻方 A 两条对角线上元素的和也为 $\sigma(A)$, 则称 A 为幻方.
记所有半幻方矩阵构成的集合为 $SMag_n(\mathbb{Q})$. 所有幻方构成集合为 $Mag_n(\mathbb{Q})$.

i.e. $SMag_n(\mathbb{Q})=\{A\in M_n(\mathbb{Q})\mid A=(a_{ij}),\ s.t.\ \sum_{k=1}^n a_{ik}=\sum_{k=1}^n a_{kj}=\sigma(A)\in\mathbb{Q},\ \forall i,j\in\{1,\cdots,n\}\}$

$Mag_n(\mathbb{Q})=\{A=(a_{ij})\in M_n(\mathbb{Q})\mid \sum_{k=1}^n a_{ik}=\sum_{k=1}^n a_{kj}=\sum_{k=1}^n a_{kk}=\sum_{k=1}^n a_{k,n-k+1}\ (=Tr(A))\ \forall i,j\}$.

易证 $SMag_n(\mathbb{Q})$ 和 $Mag_n(\mathbb{Q})$ 均为 $M_n(\mathbb{Q})$ 中的子空间 (自己验证).

eg1. 若 $n=2$. 则 $SMag_2(\mathbb{Q})=\langle E,\begin{pmatrix}0&1\\1&0\end{pmatrix}\rangle$ $\quad Mag_2(\mathbb{Q})=\langle\begin{pmatrix}1&1\\1&1\end{pmatrix}\rangle$.

若 $n=3$. 中国最常见的九宫图 (河图. 洛书. 纵横图).

$$\begin{pmatrix}4&9&2\\3&5&7\\8&1&6\end{pmatrix}.$$

黄蓉: 九宫之义. 法以灵龟. 二四为肩. 六八为足
戴九履一. 左三右七. 五居中央.

八卦 宋. 杨辉称之

定义如下四个线性子空间 (自己验证).

$SMag_n^0(\mathbb{Q}):=\{A\in SMag_n(\mathbb{Q})\mid tr(A)=0\}$.

$SMag_n^*(\mathbb{Q}):=\{A\in SMag_n(\mathbb{Q})\mid \sigma(A)=0\}$.

$Mag_n^0(\mathbb{Q}):=\{A\in Mag_n(\mathbb{Q})\mid tr(A)=0\}$.

$Mag_n^*(\mathbb{Q}):=\{A\in Mag_n(\mathbb{Q})\mid \sigma(A)=0\}$.

再给定一个矩阵

$$S=\begin{pmatrix}1&\cdots&1\\\vdots&\ddots&\vdots\\1&\cdots&1\end{pmatrix}=(1)_{n\times n}.$$

显然 S 是一个幻方.

lemma1. (1). $SMag_n(\mathbb{Q})=SMag_n^0(\mathbb{Q})\oplus\langle S\rangle=SMag_n^*(\mathbb{Q})\oplus\langle S\rangle$. 从对偶空间的角度来描述: 解空间元素为矩阵元素.

(2). $Mag_n(\mathbb{Q})=Mag_n^0(\mathbb{Q})\oplus\langle S\rangle=Mag_n^*(\mathbb{Q})\oplus\langle S\rangle$.

Pf: (1). 显然 $SMag_n^0(\mathbb{Q})\cap\langle S\rangle=SMag_n^*(\mathbb{Q})\cap\langle S\rangle=\{0\}$.

对 $\forall\ A\in SMag_n(\mathbb{Q})$. 令 $A_0=A-\frac{tr(A)}{n}S$

$\because$ tr 是线性映射 $\quad\therefore tr(A_0)=tr(A)-\frac{tr(A)}{n}\underbrace{tr(S)}_{n}=0$.

$\therefore A_0\in SMag_n^0(\mathbb{Q})$ $\quad$ 即 $A=A_0+\frac{tr(A)}{n}S$, 其中 $A_0\in SMag_n^0(\mathbb{Q})$

$\frac{tr(A)}{n}S\in\langle S\rangle$ $\quad\therefore SMag_n(\mathbb{Q})=SMag_n^0(\mathbb{Q})\oplus\langle S\rangle$.

同理. 令 $A_* = A - \frac{\sigma(A)}{n} S$　　下验证 $\sigma: SMag_n(\mathbb{Q}) \longrightarrow \mathbb{Q}$ 线性映射.

$\forall\ A, B \in SMag_n(\mathbb{Q})$, $\alpha, \beta \in \mathbb{Q}$. $\sigma(\alpha A + \beta B) = \sigma\big((\alpha a_{ij} + \beta b_{ij})_{n\times n}\big) = \sum_{k=1}^{n} (\alpha a_{ik} + \beta b_{ik})$

$= \alpha \sum_{k=1}^{n} a_{ik} + \beta \sum_{k=1}^{n} b_{ik} = \alpha \cdot \sigma(A) + \beta \cdot \sigma(B) \Rightarrow \sigma$ 为线性映射.

$\therefore\ \sigma(A_*) = \sigma(A) - \frac{\sigma(A)}{n} \underbrace{\sigma(S)}^{n} = 0$　　$\therefore A_* \in SMag_n^*(\mathbb{Q})$.

$\therefore\ A = A_* + \frac{\sigma(A)}{n} S \in SMag_n^*(\mathbb{Q}) \oplus \langle S \rangle$　　$\therefore SMag_n(\mathbb{Q}) = SMag_n^*(\mathbb{Q}) \oplus \langle S \rangle$.

(2). 证明同 $\underline{V}$.

lemma 2. $SMag_n(\mathbb{Q}) = Mag_n(\mathbb{Q}) \oplus \langle E_n \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle$, $n \geq 3$ 时成立.

Pf: 令 $W = \langle E_n \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle$　显然 $n \geq 2$ 时, $\langle E_n \rangle \cap \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle = \{0\}$

设 $A \in Mag_n(\mathbb{Q}) \cap W$. 令 $A = \lambda E + \mu \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = \begin{pmatrix} \lambda & & \mu \\ & \ddots\ \iddots & \\ \mu & & \lambda \end{pmatrix}$, λ, μ 不全为 0.

又 $\because A$ 幻方　$\therefore$ 当 $\begin{cases} n \text{ 为偶数} & \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}, \quad \lambda + \mu = n\lambda = n\mu. \\ n \text{ 为奇数} & \begin{pmatrix} \lambda & & \mu \\ & \lambda+\mu & \\ \mu & & \lambda \end{pmatrix}, \quad \lambda + \mu = n\lambda + \mu = n\mu + \lambda \end{cases} \Rightarrow \begin{cases} n = 2 \\ n = 1. \end{cases}$

$(n-1)\lambda = \mu$　$\lambda, \mu \neq 0 \Rightarrow n = 1$

$(n-1)\mu = \lambda$

$\Rightarrow n = 1$

与 $n \geq 3$ 矛盾.　$\therefore\ Mag_n(\mathbb{Q}) \cap W = \{0\}$.

设 $A \in Mag_n(\mathbb{Q}) \oplus \langle E \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle$ 即 $\overset{(a_{ij})}{A} = \overset{(m_{ij})}{M} + \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} + \begin{pmatrix} & & \mu \\ & \iddots & \\ \mu & & \end{pmatrix}$

则 $\forall\ i, j$, $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} m_{ik} + \lambda + \mu$, $\sum_{k=1}^{n} a_{kj} = \sum_{k=1}^{n} m_{kj} + \lambda + \mu$.

$\because\ M \in Mag_n(\mathbb{Q})$, $\therefore$ 各行各列相等. $\therefore\ A \in SMag_n(\mathbb{Q})$.

$\therefore\ Mag_n(\mathbb{Q}) \oplus \langle E_n \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle \subseteq SMag_n(\mathbb{Q})$

$\therefore\ \dim SMag_n(\mathbb{Q}) \geq \dim Mag_n(\mathbb{Q}) + 2$

($Mag_n(\mathbb{Q})$ 与 $SMag_n(\mathbb{Q})$ 均为方程解空间

rank ≤ 方程个数).

而 半幻方若为幻方需加 2 个约束方程　　$\operatorname{rank}(M) \leq \operatorname{rank}(SM) + 2$.

$\therefore\ \dim SMag_n(\mathbb{Q}) - 2 \leq \dim Mag_n(\mathbb{Q})$.

$\therefore\ \dim SMag_n(\mathbb{Q}) = \dim Mag_n(\mathbb{Q}) + 2 = \dim \left(Mag_n(\mathbb{Q}) \oplus \langle E \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle \right)$.

$\therefore\ SMag_n(\mathbb{Q}) = Mag_n(\mathbb{Q}) \oplus \langle E \rangle \oplus \left\langle \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\rangle$.　$\blacksquare$.

Thm 1. $\dim(SMag_n(\mathbb{Q})) = n^2-2n+2$. $\dim(Mag_n(\mathbb{Q})) = n^2-2n$.

Pf: 由 lemma 1 可知, $SMag_n(\mathbb{Q}) = SMag_n^*(\mathbb{Q}) \oplus \langle S \rangle$.

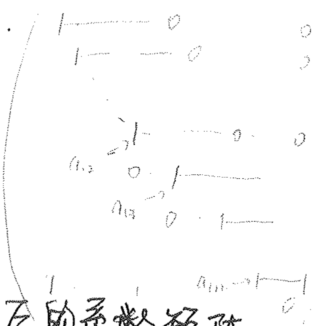
(方1) 对 $\forall A = (a_{ij}) \in SMag_n^*(\mathbb{Q})$, 则 $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 0$, $\forall i,j \in \{1,\cdots,n\}$.

2n个方程

每个齐次线性方程对应一个线性函数 $f \in U^*$, 其解空间为 U.

$$\begin{cases} a_{11}+a_{12}+\cdots+a_{1n}=0 & (1) \\ a_{21}+a_{22}+\cdots+a_{2n}=0 & (2) \\ \vdots & \\ a_{n1}+a_{n2}+\cdots+a_{nn}=0 & (n) \\ a_{11}+a_{21}+\cdots+a_{n1}=0 & (*) \\ a_{12}+a_{22}+\cdots+a_{n2}=0 & 2' \\ \vdots & \\ a_{1n}+a_{2n}+\cdots+a_{nn}=0 & n' \end{cases}$$

即 $\tilde{A} \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{n2} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0$.



$\tilde{A}$ 为将(*)移到最后一行后的系数矩阵.

此时, $\tilde{A}$ 前 $2n-1$ 行为阶梯形, 线性无关. $\therefore \operatorname{rank}(\tilde{A}) \geq 2n-1$.

又 $a_{11} \overset{\text{由}(1)}{=\!=} -(a_{12}+\cdots+a_{1n}) = (a_{22}+a_{32}+\cdots+a_{n2}) + (a_{23}+\cdots+a_{n3}) + \cdots + (a_{2n}+\cdots+a_{nn}) = \sum_{i,j\geq 2} a_{ij}$

即前n行之和与后n行之和相同 $a_{11} \overset{\text{由}(*)}{=\!=} -(a_{21}+\cdots+a_{n1}) = (a_{22}+a_{23}+\cdots+a_{2n}) + (a_{32}+\cdots+a_{3n}) + \cdots + (a_{n2}+\cdots+a_{nn}) = \sum_{i,j\geq 2} a_{ij}$

$\therefore$ $\tilde{A}$ 的 $2n$ 个行线性相关, $\therefore \operatorname{rank}(\tilde{A}) = 2n-1$ $\therefore \dim(SMag_n^*(\mathbb{Q})) = n^2-(2n-1)$.

$\therefore \dim(SMag_n(\mathbb{Q})) = n^2-(2n-1)+1 = n^2-2n+2$. $\dim(Mag_n(\mathbb{Q})) = n^2-2n$.

(方2). 对 $\forall \tilde{A} \in SMag_n^*(\mathbb{Q})$, $\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

自由变元.

← 由这 $(n-1)^2$ 个元素来决定其余元素.

~~下证 $T=\{E_{ij}, i,j \in \{2,3,\cdots,n\}\}$ 为 $SMag_n^*(\mathbb{Q})$ 的一组基.~~

$$\begin{cases} a_{12} = -(a_{22}+\cdots+a_{n2}) \\ \vdots \\ a_{1n} = -(a_{2n}+\cdots+a_{nn}) \\ a_{21} = -(a_{22}+\cdots+a_{2n}) \\ \vdots \\ a_{n1} = -(a_{n2}+\cdots+a_{nn}) \\ a_{11} = +\sum_{i,j\geq 2} a_{ij} \end{cases}$$

这 $(n-1)^2$ 个待定元可任意取值,

由左边约束条件得到的方阵 $\tilde{A} \in SMag_n^*(\mathbb{Q})$.

$\therefore \dim(SMag_n^*(\mathbb{Q})) = (n-1)^2 = n^2-2n+1$.

投影.

设 $V=W_1\oplus W_2\oplus\cdots\oplus W_m$ 是 m 个子空间的直和分解. 即 $\forall\vec{x}\in V$,

$\exists!\ \vec{x}_1,\cdots,\vec{x}_m$, 且 $\vec{x}_i\in W_i$, $i=1,\cdots,m$. s.t. $\vec{x}=\vec{x}_1+\cdots+\vec{x}_m$

定义(投影): 称映射 $P_i: V\longrightarrow W_i$ $[\vec{x}\longmapsto\vec{x}_i]$ 为 V 到子空间 W_i 的投影.

Prop. 投影 P_i 为线性映射.

Pf: $\forall\alpha,\beta\in F$, $\vec{x},\vec{y}\in V=W_1\oplus\cdots\oplus W_m$. 则 $\exists!\ \vec{x}_i,\vec{y}_i\in W_i$, $i=1,\cdots,m$.

s.t. $\vec{x}=\vec{x}_1+\cdots+\vec{x}_m$. $\exists!\ \vec{y}_i\in W_i$, $i=1,\cdots,m$, s.t. $\vec{y}=\vec{y}_1+\cdots+\vec{y}_m$

$\because\ \alpha\vec{x}+\beta\vec{y}=(\alpha\vec{x}_1+\beta\vec{y}_1)+\cdots+(\alpha\vec{x}_m+\beta\vec{y}_m)$ 分解唯一. 且 $\alpha\vec{x}_i+\beta\vec{y}_i\in W_i$. $i=1,\cdots,n$.

$\therefore\ P_i(\alpha\vec{x}+\beta\vec{y})=\alpha\vec{x}_i+\beta\vec{y}_i=\alpha P_i(\vec{x})+\beta P_i(\vec{y})$. $\therefore P_i$ 线性映射.

Thm. 若 P_i 为投影, 则 $\sum_{i=1}^m P_i=id$, $P_iP_j=\delta_{ij}P_j$. 反过来, 若 $P_i: V\longrightarrow V$, $i=1,\cdots,m$

为有限个线性映射, 且满足 $\sum_{i=1}^m P_i=id$, $P_iP_j=\delta_{ij}P_j$ 则 $V=W_1\oplus\cdots\oplus W_m$, 其中 $W_i=im(P_i)$

Pf: $\forall\vec{x}\in V$, $(\sum_{i=1}^m P_i)(\vec{x})=P_1(\vec{x})+\cdots+P_m(\vec{x})=\vec{x}_1+\cdots+\vec{x}_m=\vec{x}=id(\vec{x})$. $\therefore\sum_{i=1}^m P_i=id$

若 $i\neq j$, $P_iP_j(\vec{x})=P_i(\vec{x}_j)=P_i(\vec{0}+\underset{\substack{\uparrow\\ i}}{\vec{0}}+\underset{\substack{\uparrow\\ j}}{\vec{x}_j}+\cdots+\vec{0})=\vec{0}=O(\vec{x})$.

若 $i=j$, $P_iP_i(\vec{x})=P_i(\vec{x}_j)=\vec{x}_j=P_j(\vec{x})$ $\Rightarrow$ $P_i^2=P_i$, $P_i^s=P_i$

反过来. 对 $\forall\vec{x}\in V$, $(\sum_{i=1}^m P_i)(\vec{x})=P_1(\vec{x})+\cdots+P_m(\vec{x})=\vec{x}$

且 $P_i(\vec{x})\in im(P_i)\subseteq V$. $\therefore\ \vec{x}=P_1(\vec{x})+\cdots+P_m(\vec{x})\in im(P_1)+\cdots+im(P_m)\subseteq V$.

$\therefore$ 由 $\vec{x}$ 的任意性, $V\subseteq im(P_1)+\cdots+im(P_m)$. $\therefore V=im(P_1)+\cdots+im(P_m)$.

$\forall i\in\{1,\cdots,m\}$.

$\forall\vec{x}\in im(P_i)\cap(\sum_{j\neq i}im(P_j))$, 则 $P_i(\vec{x})=\vec{x}$. 且 $\exists\ \vec{\alpha}_j\in V$, $j\neq i$

s.t. $\vec{x}=\sum_{j\neq i}P_j(\vec{\alpha}_j)$ $\therefore\ \vec{x}=P_i(\sum_{j\neq i}P_j(\vec{\alpha}_j))=\sum_{j\neq i}P_iP_j(\vec{\alpha}_j)=\vec{0}$

$\therefore\ im(P_i)\cap\sum_{j\neq i}im(P_j)=\{\vec{0}\}$. 对 $\forall\ i\in\{1,\cdots,m\}$

$\therefore\ V=im(P_1)\oplus\cdots\oplus im(P_m)$. $\square$