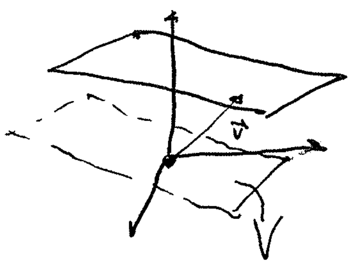
更正：例 设 $A, B \in M_n(\mathbb{R})$ 且 $AB=BA$

证明：$\operatorname{rank}(A+B)+\operatorname{rank}(AB) \leq \operatorname{rank}(A)+\operatorname{rank}(B)$

注2 线性方程组的几何意义

回忆：设 $V \subseteq \mathbb{R}^n$ 是子空间，$\vec{v} \in \mathbb{R}^n$

称 $\vec{v}+V$ 是线性流形



定理2.5. 设 $A \in \mathbb{R}^{m\times n}$, $\vec{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} \in \mathbb{R}^m$

则方程组 $A\vec{x}=\vec{b}$ 的解或者是空集 或者是线性流形

下面验证定理2.5的逆命题

即任一线性流形都是某个线性方程组的解

设线性流形

$M=\vec{v}+V$, 其中 $\vec{v}=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, ①

$V \subseteq \mathbb{R}^n$ 是 $\mathbb{R}^n$ 中 d 维子空间.

由期中考试第8题(ii)

$V = U_1 \cap \cdots \cap U_{n-d}$,

其中 $U_1, \cdots U_{n-d}$ 是 $n-1$ 维子空间

由该题第一问：

U_i 是方程 $a_{i1}x_1+\cdots+a_{in}x_n=0$ 的解空间

$(i=1,2,\cdots n-d)$

令 $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \cdots \\ a_{n-d,1} & \cdots & a_{n-d,n} \end{pmatrix}$ 则

V 是 $A\vec{x}=\vec{0}_{n-d}$ 的解空间

设 $\vec{b}=A\vec{v}$. 则

$\vec{v}$ 是 $A\vec{x}=\vec{b}$ 的一个解

由第二次习题(4)

$A\vec{x}=\vec{b}$ 的在 $\mathbb{R}^n$ 中的所有解

是 $\vec{v}+V$. □

(高斯公式)

注：$a_1x_1+\cdots+a_nx_n=b$ 在 $\mathbb{R}^n$ 中的解的集合称为 $\mathbb{R}^n$ 中超平面.（$a_1,\cdots,a_n$ 不全为零）

$n=1$，点，$n=2$，直线，$n=3$ 平面

于是 $A\vec{x}=\vec{b}$ 是求若干个超平面的交

线性流形（或空集）和若干个超平面的交是一回事儿.

第3章 行列式 (determinant)

回忆：设 $A=\begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}$.

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

当 $\det(A)\neq 0$ 时 方程组

$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ 的唯一解是

$$x_1=\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\det(A)},\quad x_2=\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det(A)}$$

关于3阶行列式的定义见 p.18

设 $A=(a_{ij})\in M_n(\mathbb{R})$

$$\det(A):=\sum_{\sigma\in S_n}\varepsilon_\sigma\, a_{\sigma(1),1}\cdots a_{\sigma(n),n}$$

§0 $\mathbb{R}^n$ 上的多重线性函数

定义：设 $f:\underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{m}\longrightarrow\mathbb{R}$

$(\vec{x}_1,\cdots,\vec{x}_m)\rightarrow f(\vec{x}_1,\cdots,\vec{x}_m)$

称为 m 重线性函数，如果

$\forall i\in\{1,\cdots,m\}$，$\vec{y}\in\mathbb{R}^n$，$\alpha,\beta\in\mathbb{R}$

$$f(\vec{x}_1,\cdots,\vec{x}_{i-1},\alpha\vec{x}_i+\beta\vec{y},\vec{x}_{i+1},\cdots,\vec{x}_m)$$
$$=\alpha f(\vec{x}_1,\cdots,\vec{x}_{i-1},\vec{x}_i,\vec{x}_{i+1},\cdots,\vec{x}_m)$$
$$+\beta f(\vec{x}_1,\cdots,\vec{x}_{i-1},\vec{y},\vec{x}_{i+1},\cdots,\vec{x}_m)$$

那固定 $\vec{x}_1,\cdots,\vec{x}_m$ 任意 $m-1$ 个向量，

f 是关于那个没固定的向量的线性函数

例：设 $f: \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ 2重线性

(双线性)。展开 $f(\vec{u}+\vec{v}, \vec{x}+\vec{y})$

$$f(\vec{u}+\vec{v}, \vec{x}+\vec{y}) = f(\vec{u}, \vec{x}+\vec{y}) + f(\vec{v}, \vec{x}+\vec{y})$$

$$= f(\vec{u},\vec{x}) + f(\vec{u},\vec{y}) + f(\vec{v},\vec{x}) + f(\vec{v},\vec{y})$$

设 $\alpha, \beta \in \mathbb{R}$

$$f(\alpha\vec{x}, \beta\vec{y}) = \alpha f(\vec{x}, \beta\vec{y}) = \alpha\beta f(\vec{x},\vec{y})$$

$\boxed{\text{定理 0.1}}$ 设 $f: \underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{m} \longrightarrow \mathbb{R}$ m重线性

令 $\vec{x_j} = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix}$，$j = 1, 2, \ldots m$

则 $$f(\vec{x_1}, \ldots \vec{x_m}) = \sum_{i_1=1}^{m} \cdots \sum_{i_m=1}^{m} a_{i_1,\ldots,i_m} x_{i_1,1} \cdots x_{i_m,m}$$

其中 $$a_{i_1,\ldots i_m} = f(\vec{e}^{(i_1)}, \ldots, \vec{e}^{(i_m)})$$

[注：$(a_{i_1,\ldots i_m})_{i_1\ldots i_m \in \{1,\ldots m\}}$ 在2维上称为一个 (m,n)型张量]

证：由 $\vec{x_j} = \sum_{i_j=1}^{m} x_{i_j} \vec{e}^{(i)}$，$j=1,\ldots,m$ ③

$$f(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_m})$$

$$= f\left(\sum_{i_1=1}^{n} x_{i_1,1} \vec{e}^{(i_1)}, \vec{x_2}, \ldots, \vec{x_m}\right)$$

$$= \sum_{i_1=1}^{n} x_{i_1,1} f(\vec{e}^{(i_1)}, \vec{x_2}, \ldots, \vec{x_m})$$

$$= \sum_{i_1=1}^{m} x_{i_1,1} f\left(\vec{e}^{(i_1)}, \sum_{i_2=1}^{m} x_{i_2,2} \vec{e}^{(i_2)}, \vec{x_3}, \ldots, \vec{x_m}\right)$$

$$= \sum_{i_1=1}^{m} x_{i_1,1} \left[\sum_{i_2=1}^{m} x_{i_2,2} f(\vec{e}^{(i_1)}, \vec{e}^{(i_2)}, \vec{x_3}, \ldots, \vec{x_m})\right]$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} f(\vec{e}^{(i_1)}, \vec{e}^{(i_2)}, \vec{x_3}, \ldots \vec{x_m}) x_{i_1,1} x_{i_2,2}$$

$$\cdots$$

$$= \sum_{i_1=1}^{m} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} f(\vec{e}^{(i_1)}, \ldots, \vec{e}^{(i_m)}) x_{i_1,1} \cdots x_{i_m,m}$$ ∎

定义：设 $f: \mathbb{R}^m\times\cdots\times\mathbb{R}^m \longrightarrow \mathbb{R}$
$(\vec{x_1}, \ldots, \vec{x_m}) \longmapsto f(\vec{x_1}, \ldots \vec{x_m})$

是全斜对称的，如果 $\forall\, i,j \in \{1,\ldots,m\}$

有 $$f(\vec{x_1}, \ldots \vec{x_i}, \ldots \vec{x_j}, \ldots x_m) = -f(\vec{x_1}, \ldots \vec{x_j} \ldots \vec{x_i} \ldots \vec{x_m})$$

例：设 $\vec{x_1}=\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix}$，$\vec{x_2}=\begin{pmatrix}x_{12}\\x_{22}\end{pmatrix}$

定义：$\det:\ \mathbb{R}^2\times\mathbb{R}^2\longrightarrow\mathbb{R}$

$$(\vec{x_1},\vec{x_2})\longmapsto\begin{vmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{vmatrix}$$

验证 det 双线性 全斜对称

证：设 $\alpha,\beta\in\mathbb{R}$，$\vec{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}$

$$\det(\alpha\vec{x_1}+\beta\vec{y},\vec{x_2})=\left|\alpha\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix}+\beta\begin{pmatrix}y_1\\y_2\end{pmatrix},\ \begin{pmatrix}x_{12}\\x_{22}\end{pmatrix}\right|$$

$$=\begin{vmatrix}\alpha x_{11}+\beta y_1 & x_{12}\\ \alpha x_{21}+\beta y_2 & x_{22}\end{vmatrix}=(\alpha x_{11}+\beta y_1)x_{22}-x_{12}(\alpha x_{21}+\beta y_2)$$

$$=\alpha(x_{11}x_{22}-x_{12}x_{21})+\beta(y_1x_{22}-y_2x_{12})$$

$$=\alpha\begin{vmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{vmatrix}+\beta\begin{vmatrix}y_1&x_{12}\\y_2&x_{22}\end{vmatrix}$$

$$=\alpha\det(\vec{x_1},\vec{x_2})+\beta\det(\vec{y},\vec{x_2})$$

$$\det(\vec{x_2},\vec{x_1})=\begin{vmatrix}x_{12}&x_{11}\\x_{22}&x_{21}\end{vmatrix}$$

$$=x_{12}x_{21}-x_{11}x_{22}=-\begin{vmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{vmatrix}$$

引理 0.1 设 $f:\ \underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{m}\longrightarrow\mathbb{R}$ ~~斜~~全斜对称。$i,j\in\{1,\dots,m\}$ 且 $i\neq j$

~~如果~~ 则 $f(\vec{x_1},\dots,\underset{i}{\vec{v}},\dots,\underset{j}{\vec{v}},\dots\vec{x_m})=0$

证：因为 f 斜对称 所以

$$f(\vec{x_1},\dots\underset{i}{\vec{v}},\dots\underset{j}{\vec{v}}\dots\vec{x_m})=-f(\vec{x_1},\dots\underset{i}{\vec{v}}\dots\underset{j}{\vec{v}},\dots)$$

$$\Rightarrow\quad 2f(\vec{x_1},\dots,\vec{v},\dots,\vec{v},\dots\vec{x_m})=0$$

$\because 2\neq0\ \therefore f(\vec{x_1},\dots\vec{v},\dots\vec{v},\dots\vec{x_m})=0$ □

$\boxed{\text{引理 0.2}}$ 利用定理 0.1 中符号，~~斜~~ 并设 $m=n$。~~则~~ 如果 f n重线性全斜对称，

则 $$f(\vec{x_1},\dots\vec{x_m})=\lambda\sum_{\sigma\in S_n}\varepsilon_\sigma x_{\sigma(1),1}\cdots x_{\sigma(n),n}$$

其中 $\lambda=a_{1,2,\dots,n}=f(\vec{e}^{(1)},\vec{e}^{(2)},\dots,\vec{e}^{(n)})$

证：由定理①

$$f(\vec{x_1},\cdots,\vec{x_n})=\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}a_{i_1,\cdots i_n}x_{i_1,1}\cdots x_{i_n,n}$$

∵ f 斜对称 ∴ 当 $i_1,\cdots i_n$ 中有两个相同时 则 $a_{i_1,\cdots i_n}=0$.

当 $i_1,\cdots i_n$ 两两不同时

$\sigma=\begin{pmatrix}1 & 2 & \cdots & n\\ i_1 & i_2 & \cdots & i_n\end{pmatrix}$ 是一个置换

满足 $\sigma(k)=i_k$, $k=1,2\cdots n$

于是 $f(\vec{x_1},\cdots\vec{x_n})$

$$=\sum_{\sigma\in S_n}a_{\sigma(1),\cdots\sigma(n)}x_{\sigma(1),1},\cdots x_{\sigma(n),n}$$

断言：$a_{\sigma(1)\cdots\sigma(n)}=\varepsilon_\sigma a_{12\cdots n}$

断言的证明

设 $\sigma=\tau_1\cdots\tau_k$，其中 $\tau_1,\cdots,\tau_k$ 是对换 ③

若 $k=0$. 则

$a_{\sigma(1),\cdots\sigma(n)}=a_{1,2,\cdots,n}$ 断言成立

设 $k=1$. 则 $\sigma=(i,j)$, $i,j\in\{1,\cdots n\}$, $i\ne j$

$$a_{\sigma(1),\cdots\sigma(n)}=f(\vec{e}^{(1)},\cdots,\underset{i}{\vec{e}^{(j)}},\cdots\underset{j}{\vec{e}^{(i)}},\cdots\vec{e}^{(n)})$$

$$=-f(\vec{e}^{(1)},\cdots\underset{i}{\vec{e}^{(i)}},\cdots\underset{j}{\vec{e}^{(j)}},\cdots\vec{e}^{(n)})$$

$$=-a_{1,2,\cdots,n}=\varepsilon_\sigma a_{1,2,\cdots,n}.$$

设 $k-1$ 时断言成立 且 $\pi=\tau_2\cdots\tau_k$

则 $a_\tau=\varepsilon_\tau a_{12\cdots n}$

$$a_{\sigma(1),\cdots,\sigma(n)}=f(\vec{e}^{[\sigma(1)]},\cdots,\vec{e}^{[\sigma(n)]})$$

$$=f(\vec{e}^{[\tau_1(\pi(1))]},\cdots,\vec{e}^{[\tau_1\pi(n)]})$$

$$\overset{k=1}{=}-f(\vec{e}^{[\pi(1)]},\cdots,\vec{e}^{[\pi(n)]})$$

$$=-\varepsilon_\pi a_{1,2,\cdots,n}=\varepsilon_\sigma a_{1,2,\cdots,n}.$$

断言成立. ▨

~~行列~~ $\boxed{\text{行列式定义}}$ det 是 $\mathbb{R}^n$ 上 n 重线性斜对称函数. 满足 $\det(\vec{e}^{(1)}, \dots, \vec{e}^{(n)}) = 1$

设 $A \in M_n(\mathbb{R})$, A 的行列式是 $\det(\vec{A}^{(1)}, \dots, \vec{A}^{(n)})$ 记为 $\det(A)$ 或 $|A|$.

注 1. $\det(E) = 1$,

注 2. (*) $\det(A) = \sum_{\sigma \in S} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$

其中 $A = (a_{ij})_{n\times n}$.

例: 利用 (*) 展开

$$\left| \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} \right|$$

$$S_3 = \{ e, (12), (23), (13), \underset{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}{\underset{\shortparallel}{(12)(13)}}, \underset{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}{\underset{\shortparallel}{(12)(23)}} \}$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$

例: 设 A 中两列相同. 证明 $\det(A) = 0$

证: ~~法1. 由~~ 法1. 设 B 是由 A ~~交换 A 中两列:~~

$\det(A)$ 是关于列的斜对称函数, 由引理 0.1, $\det(A) = 0$

另证: 不依赖性质 $2 \neq 0$

由 (*)

$$\det(A) = \sum_{\sigma \in S} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

不妨设 $\vec{A}^{(1)} = \vec{A}^{(2)}$. $\tau = (12)$

则令 $P_\sigma = \varepsilon_\sigma a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$

$$P_{\sigma\tau} = \varepsilon_{\sigma\tau} a_{\sigma\tau(1),1} a_{\sigma\tau(2),2} \cdots a_{\sigma\tau(n),n}$$
$$= -\varepsilon_\sigma a_{\sigma(2),1} a_{\sigma(1),2} a_{\sigma(3),3} \cdots a_{\sigma(n),n}$$

$$= -\varepsilon_\sigma\, a_{\sigma(1),1}\, a_{\sigma(2),2}\, a_{\sigma(3),3} \cdots a_{\sigma(n),n}$$

$$(\because a_{\sigma(2),1} = a_{\sigma(2),2},\quad a_{\sigma(1),1} = a_{\sigma(1),2})$$

$$= -P_\sigma.$$

因此在 (*) 中这两项对消. 而

$\forall\ \cancel{\sigma}\ \lambda, \mu \in S_n \quad \lambda\tau = \mu\tau \iff \lambda = \mu$

于是 (*) 中由 Σ 可以写成 $\frac{n!}{2}$ 个两两对消的项. 于是 $\det(A) = 0$. ▨

§1. 行列式的 $\boxed{\text{初等性质}}$

D_0 (正规性) $\det(E) = 1$

D_1 (列等) 若 $A \in M_n(\mathbb{R})$ 中两列相同 则 $\det(A) = 0$

D_2 (斜对称) 设 B 是通过交换 A 的两列得到的矩阵, 则

$$\det(B) = -\det(A)$$

D_3 (多重线性)

~~D_{31} 若 A 的中有一列是 $\vec{0}_n$, 则 $\det(A)=0$~~

~~固定其它(非零列), A 是关于 $\vec{0}_n$ 所在列的线性~~

$\forall\ \vec{x}, \vec{y} \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}$

$$\det(\vec{A}^{(1)}, \ldots, \vec{A}^{(i-1)}, \alpha\vec{x} + \beta\vec{y}, \vec{A}^{(i+1)}, \ldots, \vec{A}^{(n)})$$
$$= \alpha \det(\vec{A}^{(1)}, \ldots \vec{A}^{(i-1)}, \vec{x}, \vec{A}^{(i+1)}, \ldots, \vec{A}^{(n)})$$
$$+ \beta \det(\vec{A}^{(1)}, \ldots \vec{A}^{(i-1)}, \vec{y}, \vec{A}^{(i+1)}, \ldots, \vec{A}^{(n)})$$

D_{31}. $\det(\vec{A}^{(1)}, \ldots \vec{A}^{(i-1)}, \alpha\vec{A}^{(i)}, \vec{A}^{(i+1)}, \ldots \vec{A}^{(n)}) = \alpha \det(A)$

[取 $\vec{x} = \vec{A}^{(i)}$, $\vec{y} = \vec{0}_n$]

D_{32} $\det(\alpha A) = \alpha^n \det(A)$.

[$\det(\alpha A) = \det(\alpha\vec{A}^{(1)}, \ldots, \alpha\vec{A}^{(n)}) = \alpha^n \det(A)$ [$\because D_{31}$

D_{33} 若 $\vec{A}^{(i)} = \vec{0}_n$, 则 (对某个 $i \in \{1, \ldots, n\}$)

则 $\det(A) = 0$

[取 ~~$\alpha = 0$, $\vec{x}$ 任意~~ $\alpha = \beta = 0$]

D_{34} 设 $\vec{V} = \langle \vec{A}^{(1)}, \dots, \vec{A}^{(j-1)}, \vec{A}^{(j+1)}, \dots, \vec{A}^{(n)} \rangle$

则 $\det(A) = \det(\vec{A}^{(1)}, \dots, \vec{A}^{(j-1)}, \vec{A}^{(j)}+\vec{v}, \vec{A}^{(j+1)}, \dots, \vec{A}^{(n)})$

[验证：不妨设 $j=1$, $\vec{v} = \alpha_2 \vec{A}^{(2)} + \dots + \alpha_n \vec{A}^{(n)}, \alpha_i \in \mathbb{R}$]

$$\det(\vec{A}^{(1)}+\vec{v}, \vec{A}^{(2)}, \dots, \vec{A}^{(n)}) =$$

$$\det(\vec{A}^{(1)}, \vec{A}^{(2)}, \dots, \vec{A}^{(n)}) + \det(\vec{v}, \vec{A}^{(2)}, \dots, \vec{A}^{(n)})$$

$$= \det(A) + \det\left(\sum_{k=2}^{n} \alpha_k \vec{A}^{(k)}, \vec{A}^{(2)}, \dots, \vec{A}^{(n)}\right)$$

$$= \det(A) + \sum_{k=2}^{n} \det(\vec{A}^{(k)}, \vec{A}^{(2)}, \dots, \vec{A}^{(n)})$$

$$= \det(A) \qquad \text{[列等]}$$

定理1.1 设 $A \in M_n(\mathbb{R})$. 则 $\det(A) = \det(A^t)$

证：设 $A = (a_{ij})_{n\times n}$ 则 $A^t = (a'_{ij})$,

其中 $a'_{ij} = a_{ji}$, $i, j \in \{1, \dots, n\}$.

$$|A^t| = \sum_{\sigma \in S_n} \varepsilon_\sigma a'_{\sigma(1),1} \cdots a'_{\sigma(n),n}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma \underbrace{a_{\sigma^{-1}(\sigma(1)),\sigma(1)}}_{1,\sigma(1)} \cdots a_{\sigma^{-1}(\sigma(n)),\sigma(n)}$$

换序后 $\sigma(1), \dots, \sigma(n)$ 即为 $1, 2, \dots, n$

$$= \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma^{-1}(1),1} \cdots a_{\sigma^{-1}(n),n}$$

$$= \sum_{\tau \in S_n} \varepsilon_\sigma a_{\tau(1),1} \cdots a_{\tau(n),n}$$

(其中 $\tau = \sigma^{-1}$)

$$= \sum_{\tau \in S_n} \varepsilon_\tau a_{\tau(1),1} \cdots a_{\tau(n),n}.$$

$(\because \varepsilon_\sigma = \varepsilon_{\sigma^{-1}})$

$$= |A| \qquad \blacksquare$$

注：D_1, D_2, D_3 中的所有性质对行变也适用

例：~~证明~~ 设 $A\in M(\mathbb{R}^n)$，$\vec{A_1}=\vec{A_2}$，则

$$\det(A)=0$$

证：$A^t=\left((\vec{A_1})^t,(\vec{A_2})^t,(\vec{A_3})^t,\cdots(\vec{A_n})^t\right)$

∵ ~~det~~ $\vec{A_1}=\vec{A_2}$ ∴ $(\vec{A_1})^t=(\vec{A_2})^t$

∴ $\det(A^t)=0 \Rightarrow \det(A)=0$

注：对 A 作一次第1类初等行(列)变换 det(A) 变号

……第2类…… det(A)不变

D4. 设

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & 0 & & \ddots & \vdots \\ & & & & a_{nn} \end{pmatrix}$$

（上三角方阵）则 $\det(A)=a_{11}a_{22}\cdots a_{nn}$

验证：$\det(A)=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma(1),1}\cdots a_{\sigma(n),n}$

如果 ~~σ不是~~ σ 不是恒同映射

则 $\exists\, k\in\{1,\cdots n\}$ 使得 $\sigma(k)\neq k$

若 $k<\sigma(k)$，则 ~~$a_{k,\sigma(k)}=0$~~ $a_{\sigma(k),k}=0$

若 $k>\sigma(k)$ 则 $\exists\, l\in\{1,\cdots n\}$

使得 $l<\sigma(l)$ 则 ~~$a_{l,\sigma(l)}$~~ $a_{\sigma(l),l}=0$

由此可知，当 σ 不是恒同映射时

$$a_{\sigma(1),1}\cdots a_{\sigma(n),n}=0$$

$\Rightarrow \det(A)=a_{11}\cdots a_{nn}$ ☐

例：设 $D=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$，求D的值

$$D=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}=1\cdot(-2)^3=-8$$

例：设 $A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$. 计算 $\det(A)$

$$\det(A)=-\det(E_3)=-1$$

例 设 $A=\begin{pmatrix}a&b&b&\cdots&b\\b&a&b&\cdots&b\\&&\cdots&&\\b&b&b&\cdots&a\end{pmatrix}$

展开 $\det(A)$.

$$\det(A)=\begin{vmatrix}a+(n-1)b&b&b&\cdots&b\\a+(n-1)b&a&b&\cdots&b\\&&\cdots&&\\a+(n-1)b&b&b&\cdots&a\end{vmatrix}$$

$$=(a+(n-1)b)\begin{vmatrix}1&b&b&\cdots&b\\1&a&b&\cdots&b\\1&b&a&\cdots&b\\&&\cdots&&\\1&b&b&\cdots&a\end{vmatrix}$$

$$=(a+(n-1)b)\begin{vmatrix}1&b&b&\cdots&b\\0&a-b&0&\cdots&0\\0&0&a-b&0\cdots&0\\&&\cdots&&\\0&0&0&\cdots 0&a-b\end{vmatrix}$$

$$=(a+(n-1)b)(a-b)^{n-1}$$

定理 1.2 设 $A\in M_n(\mathbb{R})$ 则

$$\det(A)=0 \iff \operatorname{rank}(A)<n$$

证："$\Leftarrow$" $\cancel{\dim(A}$ $\dim V_c(A)<n$

不妨设 $\vec{A}^{(1)}+\sum_{i=2}^{n}\alpha_i\vec{A}^{(i)}=0 \quad \alpha_i\in\mathbb{R}$

$$\Rightarrow \det(A)=\det\left(\vec{A}^{(1)}+\sum_{i=2}^{n}\alpha_i\vec{A}^{(i)},\ \vec{A}^{(2)},\dots,\vec{A}^{(n)}\right)$$

$$=\det\left(\vec{0}_n,\ \vec{A}^{(2)},\dots,\vec{A}^{(n)}\right)=0$$

"$\Rightarrow$" 只要证 $\operatorname{rank}(A)=n \Rightarrow \det(A)\neq 0$

由第1，2类初等行（列）变换 由 A 得到 $B=\begin{pmatrix}\alpha_1&*&\cdots&&*\\&\alpha_2&&&\\&&\ddots\ \alpha_r&&\\&&&\ddots&\\0&&&&\alpha_n\end{pmatrix}$

因为 $\operatorname{rank}(A)=n$. 所以 $\operatorname{rank}(B)=n$,

于是 $\alpha_1,\dots,\alpha_n$ 都是不是零 $\Rightarrow \det(B)\neq 0$

而 $\det(A)=\pm\det(B)$. $\Rightarrow \det(A)\neq 0$ □

注 由定理1.2可知，上例中
A 满秩 $\Longleftrightarrow$ $a\neq b$ 且 $a+(n-1)b\neq 0$.

§2. 行列式的进一步性质

§2.1 按一行(列)展开

设 $A=(a_{ij})_{n\times n}$ 从 A 中删去第 i 行和第 j 列后得到 ~~的行列式，记为~~ $n-1$ 阶矩阵的行列式，记为 M_{ij}，称为 A 关于 (i,j) 的余子式. $(-1)^{i+j}M_{ij}$ 称为 A 的关于 (i,j) 的代数余子式，记为 A_{ij}.

例：设 $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$

$$M_{11}=\begin{vmatrix}5&6\\8&9\end{vmatrix}\qquad A_{11}=(-1)^{1+1}M_{11}=M_{11}$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&2\\7&8\end{vmatrix}=-\begin{vmatrix}1&2\\7&8\end{vmatrix}$$

定理2.1 设 $A\in M_n(\mathbb{R})$ 则

(i) $\det(A)=a_{i1}A_{i1}+\cdots+a_{in}A_{in}$
$=a_{1j}A_{1j}+\cdots+a_{nj}A_{nj}$

~~(ii) $a_{i1}A_{k1}+\cdots+a_{in}A_{kn}=0$ $(i\neq k)$~~
~~$=a_{1j}A_{1l}+\cdots+a_{nj}A_{nl}=0$ $(j\neq l)$~~

证：断言1. 设 $A=\begin{pmatrix}a_{11}&\cdots&a_{1,n-1}&0\\ \vdots&&&\\ a_{n-1,1}&\cdots&a_{n-1,n-1}&0\\ a_{n,1}&\cdots&a_{n,n-1}&a_{nn}\end{pmatrix}$

则 $\det(A)=a_{nn}A_{nn}$

断言1的证明.

$$\det(A)=\sum_{\sigma\in S_n}\varepsilon_\sigma a_{\sigma(1),1}\cdots a_{\sigma(n-1),n-1}a_{\sigma(n),n}$$

$$=\sum_{\substack{\sigma\in S_n\\ \sigma(n)=n}}\varepsilon_\sigma a_{\sigma(1),1}\cdots a_{\sigma(n-1),n-1}a_{n,n}$$

$$=a_{n,n}\sum_{\tau\in S_{n-1}}\varepsilon_\tau a_{\tau(1),1}\cdots a_{\tau(n-1),(n-1)}$$

[当 $\sigma(n)=n$ 时 $\varepsilon_\sigma=\varepsilon_\tau$
其中 $\tau=\sigma|_{\{1,2,\ldots,n-1\}}$

$= a_{nn} M_{nn} = a_{nn} A_{nn}$. 断言1成立

断言2 设 $A = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & & \vdots & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ & & \cdots & \cdots & \cdots & & \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$

则 $\det(A) = a_{ij} A_{ij}$

断言2的证明：把 a_{ij} 通过行列对调换到 n行n列的位置得到矩阵B

$$\det(B) = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} & 0 \\ \vdots & & & & & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} & 0 \\ \vdots & & & & & & \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} & a_{ij} \end{vmatrix}$$

$= a_{ij} M_{ij}$ ~~$= a_{ij}$~~ 断言1,

其中 M_{ij} 是 A 关于 (i,j) 的余子式

~~$\det(B) = (-1)^{(n-i)+(n-j)} \det(A)\ a_{ij}$~~

~~$= = (-1)^{(n-i)+(n-j)}$~~

~~$\det(A) = \det(A) = (-1)^{n-i+n-j} a_{ij} M_{ij}$~~

$=$

$\det(A) = (-1)^{n-i+n-j} \det(B)$

$= a_{ij} (-1)^{i+j} M_{ij} = a_{ij} A_{ij}$

断言2成立

考虑一般情形

$$\det(A) = \det\left(\vec{A}^{(1)}, \cdots \vec{A}^{(j-1)} \begin{pmatrix} a_{1j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{A}^{(j+1)}, \cdots \vec{A}^{(n)}\right)$$

$$+ \cdots + \det\left(\vec{A}^{(1)}, \cdots \vec{A}^{(j-1)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{nj} \end{pmatrix}, \vec{A}^{(j+1)} \cdots \vec{A}^{(n)}\right)$$

$= a_{1j} A_{1j} + \cdots + a_{nj} A_{nj}$.

类似 $\det(A) = a_{1i} A_{1i} + \cdots + a_{ni} A_{ni}$

∎

~~(1) 成立~~

~~(2) 设 $k \neq i$~~

$a_{1i} A_{1}$

例：按第一行展开

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

例

$$D = \begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix} = -2\begin{vmatrix} 5 & 3 & -1 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & -1 & 4 \\ 0 & 2 & 3 & 5 \end{vmatrix}$$

$$= -2\times 5\begin{vmatrix} -2 & 3 & 1 \\ -4 & -1 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 20\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= 20\begin{vmatrix} 1 & 3 & 1 \\ 0 & -7 & 2 \\ 0 & 6 & 6 \end{vmatrix} = 20\begin{vmatrix} -7 & 2 \\ 6 & 6 \end{vmatrix} = -1080$$

例 $\boxed{\text{Vandemonde 行列式}}$ $n \geqslant 2$

$$A_n = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ & & \cdots & & \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}, \quad V_n = \det(A_n)$$

解：~~A_n的生[illegible]~~

$$V_2 = \begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} = \alpha_2 - \alpha_1$$

$$V_3 = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha_1 & 0 \\ 1 & \alpha_2 & \alpha_2^2 - \alpha_1\alpha_2 \\ 1 & \alpha_3 & \alpha_3^2 - \alpha_1\alpha_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1\alpha_2 \\ 1 & \alpha_3 - \alpha_1 & \alpha_3^2 - \alpha_1\alpha_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1\alpha_2 \\ \alpha_3 - \alpha_1 & \alpha_3^2 - \alpha_1\alpha_3 \end{vmatrix} = \begin{vmatrix} \alpha_2 - \alpha_1 & \alpha_2(\alpha_2 - \alpha_1) \\ \alpha_3 - \alpha_1 & \alpha_3(\alpha_3 - \alpha_1) \end{vmatrix}$$

$$= (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)\begin{vmatrix} 1 & \alpha_2 \\ 1 & \alpha_3 \end{vmatrix}$$

$$= (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2).$$

注. 设 $a_1, \cdots a_k \in \mathbb{R}$. $a_1 \cdots a_k = \prod_{i=1}^{k} a_k$

结论 $V_n = \prod_{1 \le i < j \le n} (x_j - x_i)$

$n = 2, 3$ ✓

$$V_n = \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & & \alpha_2^{n-2} & \alpha_2^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \alpha_n & & \alpha_n^{n-2} & \alpha_n^{n-1} \end{vmatrix}$$

$=$ 第j列 − 第j-1列 $\times \alpha_1$

$(j = \cancel{n-1, n-2, \cdots 1}$ $n, n-1, \cdots, 2)$

$$\begin{vmatrix} 1 & 0 & & 0 & 0 \\ 1 & \alpha_2 - \alpha_1 & & \alpha_2^{n-2} - \alpha_1\alpha_2^{n-3} & \alpha_2^{n-1} - \alpha_1\alpha_2^{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n - \alpha_1 & & \alpha_n^{n-2} - \alpha_1\alpha_n^{n-3} & \alpha_n^{n-1} - \alpha_1\alpha_n^{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_2 - \alpha_1 & \cdots & \alpha_2^{n-3}(\alpha_2 - \alpha_1) & \alpha_2^{n-2}(\alpha_2 - \alpha_1) \\ \alpha_3 - \alpha_1 & \cdots & \alpha_3^{n-3}(\alpha_3 - \alpha_1) & \alpha_3^{n-2}(\alpha_3 - \alpha_1) \\ \vdots & & \vdots & \vdots \\ \alpha_n - \alpha_1 & & \alpha_n^{n-3}(\alpha_n - \alpha_1) & \alpha_n^{n-2}(\alpha_n - \alpha_1) \end{vmatrix}$$

$$= (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) \cdots (\alpha_n - \alpha_1) \begin{vmatrix} 1 & \alpha_2 & \cdots & \alpha_2^{n-2} \\ 1 & \alpha_3 & & \alpha_3^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_n & & \alpha_3^{n-2} \end{vmatrix}$$

由 $= \prod_{k=2}^{n} (\alpha_k - \alpha_1) \prod_{2 \le i < j \le n} (\alpha_j - \alpha_i)$

$= \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i) = V_n$ 归纳完毕 ✓

注: 设多项式

$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$

满足 $f(\alpha_i) = \beta_i$, $i = 1, 2, \cdots, n$

求 $a_0, a_1, \cdots a_{n-2}, a_{n-1}$

$f(\alpha_i) = a_0 + a_1\alpha_i + \cdots + a_{n-2}\alpha_i^{n-2} + a_{n-1}\alpha_i^{n-1}$ $i = 1, 2, \cdots n$

$$A_n \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

, A_n是插值问题对应线性方程组的多项式.

例: $\Delta_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ & & \ddots & \ddots & \ddots & 1 \\ & & & & 1 & 2 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$ $\quad \Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 6-2 = 4$

猜: $\Delta_n = n+1$

$\Delta_n = 2\Delta_{n-1} - \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & 2 & 1 \\ 0 & \cdots & & \cdots & 1 & 2 \end{vmatrix}$

$= 2\Delta_{n-1} - \Delta_{n-2}$

$\cancel{n=2=3, \quad \Delta_2 = 3.}$

$\cancel{\Delta_n = 2(n-1)+1}$

$\because \Delta_n = 2n - (n-1)$

$= n+1$ ✓