回忆 定理 2.4

设 A 是 $m\times n$ 阶实矩阵. 则

$$\dim V_A + \operatorname{rank}(A) = n$$

例 求 (L)

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_4 = 0 \end{cases}$$ 的解空间

解: $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$

$\operatorname{rank}(A) = 2 \Rightarrow \dim V_A = 4-2 = 2$

由 A 可得 $\begin{cases} x_3 = x_1 + x_2 \\ x_4 = -x_1 + x_2 \end{cases}$

取 $x_1 = 1,\ x_2 = 0$ 得解 $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

取 $x_1 = 0,\ x_2 = 1$ 得解 $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$V_A = \{ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \mid \alpha_1, \alpha_2 \in \mathbb{R} \}.$$

记号 设 $\vec{v} \in \mathbb{R}^n$, $S \subseteq \mathbb{R}^n$, $S \neq \varnothing$

把 $\{\vec{v}\} + S$ 简记为 $\vec{v} + S$. 称为

S 关于 $\vec{v}$ 的平移

定理 2.5 设 (L) 是方程组

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

设 $A = \begin{pmatrix} a_{11}, \cdots a_{1n} \\ \cdots \\ a_{m1}, \cdots a_{mn} \end{pmatrix}$, 且 $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ 是

(L) 的一个解. 则 (L) 的所有解是

$$\vec{v} + V_A$$

证: 设 $\vec{w} \in V_A$. 则 $\vec{v} + \vec{w}$ 是 (L) 的解

(见第2次作业)

反之. 设 $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ 是 (L) 的解.

则 $\vec{u} - \vec{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{pmatrix} \in V_A$ (第2次作业)

$$\vec{u} = \vec{v} + \vec{u} - \vec{v} \in \vec{v} + V_A \qquad \boxed{}$$

例 求 (L) $\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_1 - x_2 + x_4 = 2 \end{cases}$

的所有解:

解: 由上例可知, $\mathrm{rank}(A) = 2$.

于是 $\mathrm{rank}\left(A \,\middle|\, \begin{matrix}1\\2\end{matrix}\right) = 2$. (L) 有解

$\vec{v} = \begin{pmatrix}0\\0\\-1\\2\end{pmatrix}$ 是 (L) 的一个解. 于是 (L) 的

所有解是 $\vec{v} + V_A = \begin{pmatrix}0\\0\\-1\\2\end{pmatrix} + \left\{\alpha_1\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix} \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R}\right\}$

关于矩阵秩的若干等式和不等式

1. 设 A 是 $m\times n$ 阶矩阵. 则

$\mathrm{rank}(A) \leq \min(m, n)$

证: $V_r(A) \subset \mathbb{R}^{1\times n} \Rightarrow \mathrm{rank}(A) \leq n$

$V_c(A) \subset \mathbb{R}^m \Rightarrow \mathrm{rank}(A) \leq m$

☒

②. 定义: ~~如果~~ $\mathrm{rank}(A) = m$, 则称 A 行满秩

… $\mathrm{rank}(A) = n$, 则称 A 列满秩

如果 $\mathrm{rank}(A) = \min(m, n)$, 则称 A 满秩

定义: 设 $A = \begin{pmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}, & \cdots, & a_{2n} \\ & \cdots & & \\ a_{m1}, & a_{m2}, & \cdots & a_{mn} \end{pmatrix}$

则矩阵 $\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}_{n\times m}$ 称为 A 的转置

记为 A^t

注. 设 $A^t = (a'_{ji})_{\substack{j=1,\dots,n \\ i=1,\dots,m}}$ 则

$a'_{ji} = a_{ij}$

2. $\mathrm{rank}(A) = \mathrm{rank}(A^t)$

证: $\dim V_r(A^t) = \dim V_c(A)$

例：(L) $\begin{cases} x+y=1 \\ 2x-y=2 \\ 5x+2y=5 \end{cases}$

$$B=\left(\underbrace{\begin{matrix}1&1\\2&-1\\5&2\end{matrix}}_{A}\ \middle|\ \begin{matrix}1\\2\\5\end{matrix}\right)\to\left(\begin{matrix}1&1\\0&-3\\0&-3\end{matrix}\ \middle|\ \begin{matrix}1\\0\\0\end{matrix}\right)$$

$$\to\left(\begin{matrix}1&1\\0&-3\\0&0\end{matrix}\ \middle|\ \begin{matrix}1\\0\\0\end{matrix}\right)\quad \text{rank}(A)=\text{rank}(B)=2$$

(L) 有解.

$(\tilde{L})$ $\begin{cases} x+y=1 \\ 2x-y=2.01 \\ 5x+2.01y=5 \end{cases}$

$$\tilde{B}=\left(\begin{matrix}1&1\\2&-1\\5&2.01\end{matrix}\ \middle|\ \begin{matrix}1\\2.01\\5\end{matrix}\right)\to\left(\begin{matrix}1&1\\0&-3\\0&-2.99\end{matrix}\ \middle|\ \begin{matrix}1\\0.01\\0\end{matrix}\right)$$

$$\to\left(\begin{matrix}1&1\\0&-3\\0&0\end{matrix}\ \middle|\ \begin{matrix}1\\0.01\\ \underbrace{-0.01\times\frac{2.99}{3}}_{\neq 0}\end{matrix}\right)\quad \begin{matrix}\text{rank}(A)=2\\ \text{rank}(\tilde{B})=3\end{matrix}$$

$(\tilde{L})$ 无解

§3 线性映射初步

§3.1 定义与基本性质

定义：设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 如果 $\forall\, \alpha\in\mathbb{R}$，$\vec{x},\vec{y}\in\mathbb{R}^n$

(i) $\varphi(\vec{x}+\vec{y})=\varphi(\vec{x})+\varphi(\vec{y})$

(ii) $\varphi(\alpha\vec{x})=\alpha\varphi(\vec{x})$.

则称 φ 是线性映射.

记号 $\vec{O}_n=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}\Big\}n$. $\vec{O}_m=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}\Big\}m$

注：由 (ii) 中取 $\alpha=0$ 可知

$$\varphi(\vec{O}_n)=\vec{O}_m$$

命题 3.1 设 $\varphi: \mathbb{R}^n\longrightarrow\mathbb{R}^m$ 是线性的

$\alpha_1,\dots,\alpha_k\in\mathbb{R}$，$\vec{v}_1,\dots\vec{v}_k\in\mathbb{R}^n$

则 (i) $\varphi\left(\sum_{i=1}^k \alpha_i\vec{v}_i\right)=\sum_{i=1}^k\alpha_i\varphi(\vec{v}_i)$

(ii) 如果 $\vec{v}_1,\dots,\vec{v}_k$ 线性相关，则 $\varphi(\vec{v}_1),\dots,\varphi(\vec{v}_k)$ 也线性相关

(iii) 如果 $\varphi(\vec{v}_1),\cdots,\varphi(\vec{v}_k)$ 线性无关，则 $\vec{v}_1,\cdots,\vec{v}_k$ 也线性无关

证：(i) 对 k 归纳，$k=1$，由定义中(ii) 可知 $\varphi(\alpha_1\vec{v}_1)=\alpha\varphi(\vec{v}_1)$.

设 $k-1$ 时结论成立，当 k 时

$\varphi(\alpha_1\vec{v}_1+\cdots+\alpha_{k-1}\vec{v}_{k-1}+\alpha_k\vec{v}_k)$

$=\varphi(\alpha_1\vec{v}_1+\cdots+\alpha_{k-1}\vec{v}_{k-1})+\varphi(\alpha_k\vec{v}_k)$ [定义中(i)]

$=\alpha_1\varphi(\vec{v}_1)+\cdots+\alpha_{k-1}\varphi(\vec{v}_{k-1})+\alpha_k\varphi(\vec{v}_k)$ [归纳法]

(ii) 设 $\beta_1\vec{v}_1+\cdots+\beta_k\vec{v}_k=\vec{O}_n$，其中 $\beta_1\cdots\beta_k\in\mathbb{R}$ 不全为零. 由(i)可知

$\vec{O}_m=\varphi(\beta_1\vec{v}_1+\cdots+\beta_k\vec{v}_k)=\beta_1\varphi(\vec{v}_1)+\cdots+\beta_k\varphi(\vec{v}_k)$

(iii) 是(ii)的逆否命题 ☒

例：零映射：$\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$
$\vec{x}\mapsto\vec{O}_m$

$\varphi(\vec{x}+\vec{y})=\vec{O}_m=\vec{O}_m+\vec{O}_m=\varphi(\vec{x})+\varphi(\vec{y})$

$\varphi(\alpha\vec{x})=\vec{O}_m=\alpha\vec{O}_m=\alpha\varphi(\vec{x})$.

恒同映射. $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$
$\vec{x}\longmapsto\vec{x}$

$\varphi(\vec{x}+\vec{y})=\vec{x}+\vec{y}=\varphi(\vec{x})+\varphi(\vec{y})$

$\varphi(\alpha\vec{x})=\alpha\vec{x}=\alpha\varphi(\vec{x})$

嵌入：设 $n\leq m$.

$$\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^m$$
$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\longmapsto\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\\0\\\vdots\\0\end{pmatrix}\left.\begin{matrix}\\\\\end{matrix}\right\}m-n$$

$$\varphi\left(\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}+\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\right)=\varphi\left(\begin{pmatrix}x_1+y_1\\\vdots\\x_n+y_n\end{pmatrix}\right)=\begin{pmatrix}x_1+y_1\\\vdots\\x_n+y_n\\0\\\vdots\\0\end{pmatrix}$$

$$=\begin{pmatrix}x_1\\\vdots\\x_n\\0\\\vdots\\0\end{pmatrix}+\begin{pmatrix}y_1\\\vdots\\y_n\\0\\\vdots\\0\end{pmatrix}=\varphi\left(\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\right)+\varphi\left(\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\right).$$

$$\varphi\left(\alpha\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\right)=\varphi\left(\begin{pmatrix}\alpha x_1\\\vdots\\\alpha x_n\end{pmatrix}\right)=\begin{pmatrix}\alpha x_1\\\vdots\\\alpha x_n\\0\\\vdots\\0\end{pmatrix}=\alpha\begin{pmatrix}x_1\\\vdots\\x_n\\0\\\vdots\\0\end{pmatrix}$$

投影: $n \geq m$

$$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{pmatrix} \right) = \varphi\left(\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_m+y_m \\ \vdots \\ x_n+y_n \end{pmatrix} \right) = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_m+y_m \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{pmatrix} \right) + \varphi\left(\begin{pmatrix} y_1 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{pmatrix} \right)$$

$$\varphi\left(\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{pmatrix} \right) = \varphi\left(\begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_m \\ \vdots \\ \alpha x_n \end{pmatrix} \right) = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_m \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$= \alpha \varphi \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_n \end{pmatrix}$$

线性函数. 设 $\alpha_1, \ldots \alpha_n \in \mathbb{R}$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \alpha_1 x_1 + \cdots + \alpha x_n$$

设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$f(\vec{x}+\vec{y}) = f\left(\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \right) = \alpha_1(x_1+y_1) + \cdots + \alpha_n(x_n+y_n)$$

$$= (\alpha_1 x_1 + \cdots + \alpha_n x_n) + (\alpha_1 y_1 + \cdots + \alpha_n y_n)$$

$$= f(\vec{x}) + f(\vec{y})$$

设 $\lambda \in \mathbb{R}$

$$f(\lambda \vec{x}) = f\left(\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} \right) = \alpha_1 \lambda x_1 + \cdots + \alpha_n \lambda x_n$$

$$= \lambda(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \lambda f(\vec{x}).$$

命题 3.2. 设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$$

则 φ 是线性映射 $\Leftrightarrow$ $f_1, \ldots f_m$ 是线性函数

证: "$\Rightarrow$" 设 $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\varphi(\vec{x}+\vec{y}) = \begin{pmatrix} f_1(\vec{x}+\vec{y}) \\ \vdots \\ f_m(\vec{x}+\vec{y}) \end{pmatrix} \Rightarrow f_i(\vec{x}+\vec{y}) = f_i(\vec{x}) + f_i(\vec{y}), \quad i = 1, \cdots m.$$

$$\|$$

$$\varphi(\vec{x}) + \varphi(\vec{y}) = \begin{pmatrix} f_1(\vec{x}) + f_1(\vec{y}) \\ \vdots \\ f_m(\vec{x}) + f_m(\vec{y}) \end{pmatrix}$$

$$\varphi(\alpha\vec{x}) = \begin{pmatrix} f_1(\alpha\vec{x}) \\ \vdots \\ f_m(\alpha\vec{x}) \end{pmatrix} \Rightarrow f_i(\alpha\vec{x}) = \alpha f_i(\vec{x})$$

$$\|$$

$$\alpha\varphi(\vec{x}) = \begin{pmatrix} \alpha f_1(\vec{x}) \\ \vdots \\ \alpha f_m(\vec{x}) \end{pmatrix}$$

$$\Leftarrow \quad \varphi(\vec{x}+\vec{y}) = \begin{pmatrix} f_1(\vec{x}+\vec{y}) \\ \vdots \\ f_m(\vec{x}+\vec{y}) \end{pmatrix} = \begin{pmatrix} f_1(\vec{x}) + f_1(\vec{y}) \\ \vdots \\ f_m(\vec{x}) + f_m(\vec{y}) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} + \begin{pmatrix} f_1(\vec{y}) \\ \vdots \\ f_m(\vec{y}) \end{pmatrix} = \varphi(\vec{x}) + \varphi(\vec{y})$$

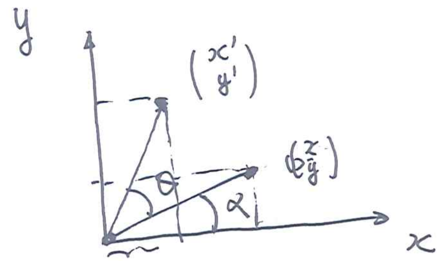
$$\varphi(\alpha\vec{x}) = \begin{pmatrix} f_1(\alpha\vec{x}) \\ \vdots \\ f_m(\alpha\vec{x}) \end{pmatrix} = \begin{pmatrix} \alpha f_1(\vec{x}) \\ \vdots \\ \alpha f_m(\vec{x}) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} = \alpha\varphi(\vec{x}). \quad \square$$

例: 旋转 $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

是线性映射



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$r = \sqrt{x^2+y^2}$$

$$x' = r\cos(\theta+\alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$
$$= x\cos\theta - y\sin\alpha$$
$$y' = r\sin(\theta+\alpha) = x\sin\theta + y\cos\theta.$$

例: 平移 设 $\vec{v} \in \mathbb{R}^n \setminus \{\vec{0}\}$

$$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto \vec{x} + \vec{v}$$

不是线性映射 因为 $\varphi(\vec{0}_n) = \vec{v} \neq \vec{0}_n$

例: 二次函数 $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$

$$f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$$
$$f(x) + f(y) = x^2 + y^2.$$

$\because 2 \neq 0 \quad \therefore f$ 不是线性的.

命题3.3 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射

U 是 $\mathbb{R}^n$ ~~线性映射~~ 子空间，V 是 $\mathbb{R}^m$

的子空间 则

(i) $\varphi(U)$ 是 $\mathbb{R}^m$ 的子空间，$\dim U \geq \dim \varphi(U)$

(ii) $\varphi^{-1}(V)$ 是 $\mathbb{R}^n$ 的子空间

证：(i) 设 $\vec{v_1}, \vec{v_2} \in \varphi(U)$，$\alpha \in \mathbb{R}$

$\exists\ \vec{u_1}, \vec{u_2} \in U$ 使得

$\varphi(\vec{u_1}) = \vec{v_1}$，$\varphi(\vec{u_2}) = \vec{v_2}$

$\Rightarrow \vec{v_1} + \vec{v_2} = \varphi(\vec{u_1}) + \varphi(\vec{u_2}) = \varphi(\vec{u_1} + \vec{u_2})$

$\Rightarrow \vec{v_1} + \vec{v_2} \in \varphi(U)$

$\alpha \vec{v_1} = \alpha\varphi(\vec{u_1}) = \varphi(\alpha \vec{u_1})$

$\Rightarrow \alpha \vec{v_1} \in \varphi(U)$

由此可知 $\varphi(U)$ 是子空间

~~(ii)~~

设 $\vec{v_1}, \dots \vec{v_d}$ 是 $\varphi(U)$ 的一组基

~~则~~ 且 $\varphi(\vec{u_i}) = \vec{v_i}$，$i = 1, 2, \dots d$，$\vec{u_i} \in U$

由命题3.1 (iii)

$\vec{u_1}, \dots \vec{u_d} \in U$ 线性无关

则 $\dim U \geq d$.

(ii) 期中考试用

§3.2 核与像

定义：设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射

$\mathrm{im}(\varphi)$ 即 $\varphi(\mathbb{R}^n)$ 称为 φ 的像空间

（注. $\mathrm{im}(\varphi)$ 是子空间）

$\varphi^{-1}(\{\vec{0}_m\})$ 称为 φ 的核空间，记为

$\ker(\varphi)$

验证：$\ker(\varphi)$ 是 $\mathbb{R}^n$ 的子空间

设 $\vec{v_1}, \vec{v_2} \in \ker(\varphi)$

$\varphi(\vec{v_1} + \vec{v_2}) = \varphi(\vec{v_1}) + \varphi(\vec{v_2}) = \vec{0}_m + \vec{0}_m = \vec{0}_m$

$\varphi(\alpha \vec{v_1}) = \alpha\varphi(\vec{v_1}) = \vec{0}_m$

$\Rightarrow \vec{v_1} + \vec{v_2}, \alpha \vec{v_1} \in \ker(\varphi)$.

例：零映射 φ：$\ker(\varphi)=\mathbb{R}^n$；$\mathrm{im}(\varphi)=\{\vec{0}_m\}$

恒同映射：$\ker(\varphi)=\{\vec{0}_n\}$，$\mathrm{im}(\varphi)=\mathbb{R}^m$

嵌入：$\ker(\varphi)=\{\vec{0}_n\}$，$\mathrm{im}(\varphi)=\left\{\begin{pmatrix} a_1\\ \vdots\\ a_n\\ 0\\ \vdots\\ 0\end{pmatrix} \middle| a_1\cdots a_n\in\mathbb{R}\right\}$

投射：$\ker(\varphi)=\left\{\begin{pmatrix} 0\\ \vdots\\ 0\\ a_{m+1}\\ \vdots\\ a_n\end{pmatrix}^n \middle| a_{m+1},\dots,a_n\in\mathbb{R}\right\}$

$\mathrm{im}(\varphi)=\mathbb{R}^m$

命题3.3 设 $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ 是线性映射

则 φ 是单射 $\Longleftrightarrow \ker(\varphi)=\{\vec{0}_n\}$

证：$\Rightarrow$ $\because$ φ 单射且 $\varphi(\vec{0}_n)=\vec{0}_m$

$\therefore \ker(\varphi)=\{\vec{0}_n\}$

$\Leftarrow$. 设 $\vec{u},\vec{v}\in\mathbb{R}^n$ 且 $\varphi(\vec{u})=\varphi(\vec{v})$

则 $\varphi(\vec{u})-\varphi(\vec{v})=\vec{0}_m$

$\Rightarrow \varphi(\vec{u}-\vec{v})=\vec{0}_m$

$\Rightarrow \vec{u}-\vec{v}\in\ker(\varphi)=\{\vec{0}_n\}$

$\Rightarrow \vec{u}-\vec{v}=\vec{0}_n$

$\Rightarrow \vec{u}=\vec{v}$ ▨

例：设 $\varphi:\mathbb{R}^2\to\mathbb{R}^n$ 是旋转.

证明 φ 是单射

证：$\varphi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}0\\0\end{pmatrix}$

则 $\begin{cases} x\cos\theta-y\sin\theta=0 & \times\cos\theta \\ & + \\ x\sin\theta-y\cos\theta=0 & \times\sin\theta\end{cases}$

$\Rightarrow x(\cos^2\theta+\sin^2\theta)=0$

$\Rightarrow x=0.$ $\Rightarrow y=0$

$\Rightarrow \ker(\varphi)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} \Rightarrow \varphi$ 单射.

~~定理3.1 设 $\varphi:\mathbb{R}^m\to\mathbb{R}^m$ 线性~~

~~则 $\cancel{\ker}$ $\dim(\ker(\varphi))+\dim(\mathrm{im}(\varphi))=n$~~

~~证：设 $\vec{v}_1,\dots,\vec{v}_d$ 是 $\ker(\varphi)$ 的一组基~~

~~$\vec{v}_1,\dots,\vec{v}_d,\vec{v}_{d+1},\dots,\vec{v}_n$ 是 $\mathbb{R}^n$ 的一组基~~

~~(注 若 $\ker(\varphi)=\{\vec{0}_n\}$ 则取 $d=0$)~~

~~断言：$\varphi(\vec{v}_{d+1}),\dots,\varphi(\vec{v}_n)$ 线性无关~~

回忆：$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 是线性映射

如果 $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$，$\alpha \in \mathbb{R}$

$\varphi(\vec{x}+\vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$，$\varphi(\alpha\vec{x}) = \alpha\varphi(\vec{x})$

注1. φ 线性 $\Rightarrow \varphi(\vec{0}_n) = \vec{0}_m$

证法1. 令 $\alpha = 0$，

$\varphi(0\vec{x}) = 0 \cdot \varphi(\vec{x}) \Rightarrow \varphi(\vec{0}_n) = \vec{0}_m$

证法2. $\varphi(\vec{0}_n + \vec{0}_n) = \varphi(\vec{0}_n) + \varphi(\vec{0}_n)$

$\parallel$

$\varphi(\vec{0}_n)$

$\Rightarrow \varphi(\vec{0}_n) = \vec{0}_m$

注2 线性映射把"直线"映成"直线"，"平面"映成"平面"，且零向量映成零向量

注3 命题3.3 (i) 补充：

设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性，$\mathcal{U}$ 是 $\mathbb{R}^n$ 的子空间，则

$$\dim \varphi(\mathcal{U}) \leq \dim \mathcal{U}.$$

证：设 $\vec{v}_1, \dots, \vec{v}_d$ 是 $\varphi(\mathcal{U})$ 的一组基

因为 $\vec{v}_1, \dots, \vec{v}_d \in \varphi(\mathcal{U})$，所以存在

$\vec{u}_1, \dots, \vec{u}_d \in \mathcal{U}$，使得

$\varphi(\vec{u}_1) = \vec{v}_1, \quad \dots \quad \varphi(\vec{u}_d) = \vec{v}_d$

由命题3.1 (iii)

$\vec{u}_1, \dots, \vec{u}_d$ 线性无关.

则由基扩充定理. $\mathcal{U}$ 有基底

$\vec{u}_1, \dots, \vec{u}_d, \vec{u}_{d+1}, \dots \vec{u}_k$，$k \geq d$

$\Rightarrow \dim \mathcal{U} = k \geq d = \dim \varphi(\mathcal{U})$. $\square$

定理3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性映射

则 $\dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi)) = n$

证：设 $\vec{v}_1, \dots, \vec{v}_d$ 是 $\ker(\varphi)$ 的一组基

由基扩充定理. $\mathbb{R}^n$ 有基底

$\vec{v}_1, \dots, \vec{v}_d, \vec{v}_{d+1}, \dots, \vec{v}_n$

(注：若 $\ker(\varphi) = \{\vec{0}_n\}$. 则取 $d = 0$)

断言1：$\varphi(\vec{v}_{d+1}), \dots, \varphi(\vec{v}_n)$ 线性无关.

设 $\alpha_{d+1}, \cdots \alpha_n \in \mathbb{R}$ 使得

$$\alpha_{d+1}\varphi(\vec{v}_{d+1}) + \cdots + \alpha_n \varphi(\vec{v}_n) = \vec{0}_m$$

则 $\varphi(\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_n \vec{v}_n) = \vec{0}_m$

即 $\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_n\vec{v}_n \in \ker(\varphi)$

$\Rightarrow \exists\ \alpha_1, \cdots \alpha_d \in \mathbb{R}$ 使得

$$\alpha_{d+1}\vec{v}_{d+1} + \cdots + \alpha_n\vec{v}_n = \alpha_1\vec{v}_1 + \cdots + \alpha_d\vec{v}_d$$

$$\Rightarrow \alpha_1\vec{v}_1 + \cdots + \alpha_d\vec{v}_d + (-\alpha_{d+1})\vec{v}_{d+1} + \cdots + (-\alpha_n)\vec{v}_n = \vec{0}_n$$

$$\Rightarrow \alpha_{d+1} = \cdots = \alpha_n = 0$$

于是断言1成立

断言2 $\operatorname{im}(\varphi) = \langle \varphi(\vec{v}_{d+1}), \cdots, \varphi(\vec{v}_n) \rangle$

设 $\vec{w} \in \operatorname{im}(\varphi)$. 则 $\exists \vec{v} \in \mathbb{R}^n$ 使得

$$\varphi(\vec{v}) = \vec{w}$$

$\because \vec{v}_1, \cdots \vec{v}_n$ 是 $\mathbb{R}^n$ 的一组基

$\therefore \exists\ \beta_1 \cdots \beta_n$ 使得

$$\vec{v} = \beta_1\vec{v}_1 + \cdots + \beta_d\vec{v}_d + \beta_{d+1}\vec{v}_{d+1} + \cdots + \beta_n\vec{v}_n$$

$$\vec{w} = \varphi(\vec{v}) = \varphi(\beta_1\vec{v}_1 + \cdots + \beta_d\vec{v}_d) + \varphi(\beta_{d+1}\vec{v}_{d+1} + \cdots + \beta_n\vec{v}_n)$$

$$= \beta_{d+1}\varphi(\vec{v}_{d+1}) + \cdots + \beta_n\varphi(\vec{v}_n)$$

$$\in \langle \varphi(\vec{v}_{d+1}), \cdots, \varphi(\vec{v}_n) \rangle$$

由上述两个断言可知.

$\varphi(\vec{v}_{d+1}), \cdots, \varphi(\vec{v}_n)$ 是 $\operatorname{im}(\varphi)$ 的一组基

从而 $\dim(\operatorname{im}(\varphi)) = n-d$. ▨

例: 证明旋转 φ 是满射

$\because \ker(\varphi)$ 是单射

$\therefore \dim(\ker(\varphi)) = 0$

$$0 + \dim(\operatorname{im}(\varphi)) = 2$$

再由 $\operatorname{im}(\varphi) \subset \mathbb{R}^2$. 所以 $\operatorname{im}(\varphi) = \mathbb{R}^2$.

推论3.1 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性

则 (i) φ 是单射 $\Leftrightarrow \dim \mathrm{im}(\varphi) = n$

(ii) 当 $m=n$ 时

φ 是单射 $\Leftrightarrow$ φ 是满射

证: (i) φ 是单射 $\overset{\text{命题3.3}}{\Longleftrightarrow} \ker(\varphi) = \{\vec{0}_n\}$

$\Leftrightarrow \dim(\ker(\varphi)) = 0$

$\underset{\text{定理3.1}}{\Leftrightarrow} \dim \mathrm{im}(\varphi) = n$

(ii) ~~由(i)~~ 此时 $\mathrm{im}(\varphi) \subset \mathbb{R}^n$

由(i) φ 是单射 $\Leftrightarrow \mathrm{im}(\varphi) = \mathbb{R}^n$ $\square$

例: 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ 是单射或满射

则 (i) φ 是双射

(ii) φ^{-1} 也是线性的

证: 由推论3.1.(ii)

φ 是单射或满射 则 φ 是双射.

设 $\vec{x}, \vec{y} \in \mathbb{R}^n$. 则存在 $\vec{u}, \vec{v} \in \mathbb{R}^n$

使得 $\vec{x} = \varphi(\vec{u}), \vec{y} = \varphi(\vec{v}) \Rightarrow \varphi^{-1}(\vec{x}) = \vec{u}$, $\varphi^{-1}(\vec{y}) = \vec{v}$

则 $\varphi(\vec{u}+\vec{v}) = \varphi(\vec{u}) + \varphi(\vec{v}) = \vec{x}+\vec{y}$

$\varphi^{-1}(\vec{x}+\vec{y}) = \cancel{\vec{u}+\vec{v}}\ \varphi^{-1}\circ\varphi(\vec{u}+\vec{v}) = \vec{u}+\vec{v}$

$= \varphi^{-1}(\vec{x}) + \varphi^{-1}(\vec{y})$

类似地 可证 $\forall \alpha \in \mathbb{R}$

$\varphi^{-1}(\alpha\vec{x}) = \alpha\, \varphi^{-1}(\vec{x})$ $\square$

§3.3. 线性映射的矩阵表示

设 $\vec{e}^{(1)}, \dots, \vec{e}^{(n)}$ 是 $\mathbb{R}^n$ 的标准基

$\vec{\varepsilon}^{(1)}, \dots, \vec{\varepsilon}^{(m)}$ 是 $\mathbb{R}^m$ 的标准基

定义: 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射

$$A = (\varphi(\vec{e}^{(1)}), \cdots, \varphi(\vec{e}^{(n)}))$$ 称为

φ 在 $\vec{e}^{(1)}, \cdots, \vec{e}^{(n)}$; $\vec{\varepsilon}^{(1)}, \cdots \vec{\varepsilon}^{(m)}$ 下的矩阵

简称为 φ 的矩阵，记为 A_φ.

注：A 是 $m \times n$ 阶的矩阵.

设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\varphi(\vec{x}) = \varphi(x_1\vec{e}^{(1)} + \cdots + x_n\vec{e}^{(n)})$$
$$= x_1\varphi(\vec{e}^{(1)}) + \cdots + x_n\varphi(\vec{e}^{(n)})$$
$$= x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)}$$

令 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

则 $\varphi(\vec{x}) = \cancel{x_1\vec{A}}\ x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

例：零映射 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$

$$A_\varphi = (\underbrace{\vec{0}_m, \cdots \vec{0}_m}_{n}) = O_{m\times n}$$

恒同映射 $\varphi: \mathbb{R}^n \to \mathbb{R}^n$

$$A_\varphi = (\vec{e}^{(1)}, \cdots, \vec{e}^{(n)}) = E_n = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}$$

嵌入. $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ $n \leq m$

$$A_\varphi = (\vec{\varepsilon}^{(1)}, \cdots, \vec{\varepsilon}^{(n)}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

投影 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ $n \geq m$

$$A_\varphi = (\vec{e}^{(1)} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 & 0 & & 0 \end{pmatrix} = (E_n, O_{m\times(n-m)})$$

旋转 $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

$$A_\varphi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

例：利用定理2.4证明定理3.1

证：$\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性

记 A_φ 为 A. 设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

则 $\varphi(\vec{x}) = x_1 \vec{A}^{(1)} + \cdots + x_n \vec{A}^{(n)}$

于是 $\ker(\varphi) = V_A \quad \operatorname{im}(\varphi) = V_c(A)$

由定理2.4

$$\begin{array}{ccccc} \dim V_A & + & \operatorname{rank}(A) & = n \\ \| & & \| & \\ \dim \ker(\varphi) & & \dim V_c(A) & \\ & & \| & \\ & & \dim \operatorname{im}(\varphi) & \end{array}$$

于是 $\dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi)) = n$

定理3.2 设 $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$. 则存在

唯一的线性映射 φ，使得

$$\varphi(\vec{e}^{(j)}) = \vec{v}_j, \quad j = 1, \dots, n$$

证 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n$$

设 $\vec{v}_j = \begin{pmatrix} v_{1j} \\ \vdots \\ v_{mj} \end{pmatrix}$.

$$\varphi(\vec{x}) = \begin{pmatrix} v_{11} x_1 + \cdots + v_{1n} x_n \\ \cdots \\ v_{m1} x_1 + \cdots + v_{mn} x_n \end{pmatrix}$$

由命题3.3. φ 是线性的

设 ψ 是线性映射 满足 $\psi(\vec{e}^{(j)}) = \vec{v}_j, \ j = 1 \dots n$

$$\forall \ \psi(\vec{x}) = \psi(x_1 \vec{e}^{(1)} + \cdots + x_n \vec{e}^{(n)})$$

$$= x_1 \psi(\vec{e}^{(1)}) + \cdots + x_n \psi(\vec{e}^{(n)})$$

$$= x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n$$

$$= \varphi(\vec{x}) \qquad \blacksquare$$