回忆：定理 3.2

设 $\vec{v}_1, \cdots, \vec{v}_n \in \mathbb{R}^m$ 则 $\exists!$ 线性映射

$$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

使得 $\varphi(\vec{e}^{(j)}) = \vec{v}_j,\ j=1,\cdots,n$

注：由证明可知 $\vec{x} \in \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$\varphi(\vec{x}) = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

φ 的矩阵是 $\cancel{A}(\vec{v}_1, \cdots \vec{v}_n)$

定义：设 A 是 $m\times n$ 阶实矩阵

~~由~~ 则 $\vec{A}^{(1)}, \cdots, \vec{A}^{(n)} \in \mathbb{R}^m$

线性映射 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$

满足 $\varphi(\vec{e}^{(j)}) = \vec{A}^{(j)},\ j=1,\cdots,n$

称为由 A 确定的线性映射

记为 φ_A

φ_A 的矩阵

$$(\varphi(\vec{e}^{(1)}), \cdots, \varphi(\vec{e}^{(n)})) = (\vec{A}^{(1)}, \cdots \vec{A}^{(n)})$$

$$= A$$

$$[\ \boxed{\varphi_{(}}\ A(\varphi_A) = A\ ]$$

设 φ 是 $\mathbb{R}^n \to \mathbb{R}^m$ 的线性映射

$$A_\varphi = (\varphi(\vec{e}^{(1)}), \cdots, \varphi(\vec{e}^{(n)}))$$

A_φ 对~~应~~的线性映射 ψ 满足

$$\psi(\vec{e}^{(j)}) = A_\varphi^{(j)} = \varphi(\vec{e}^{(j)}),\ j=1,\cdots,n.$$

由定理 3.2. $\psi = \varphi$.

$$[\ \varphi_{(A_\varphi)} = \varphi\ ]$$

定理 3.3. 设 $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ 是从 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的线性映射的集合，$\mathbb{R}^{m\times n}$ 是 $m\times n$ 阶实数矩阵的集合，则映射

$$\Phi: \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}^{m\times n}$$
$$\varphi \longmapsto A_\varphi$$

是双射.

证: $\Psi: \mathbb{R}^{m\times n} \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$$A \longmapsto \varphi_A$$

$$\Psi\circ\Phi(\varphi) = \Psi(A_\varphi) = \varphi_{(A_\varphi)} = \varphi$$

$$\Phi\circ\Psi(A) = \Phi(\varphi_A) = A_{(\varphi_A)} = A$$

于是 $\Psi\circ\Phi = id_{\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)}$

$\Phi\circ\Psi = id_{\mathbb{R}^{m\times n}}$ ▨

例: 利用定理 3.1. 证明定理 3.4

考虑: $\varphi_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\ker(\varphi_A) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \,\middle|\, x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)} = \vec{0}_m \right\}$$
$$= V_A$$

$$\mathrm{im}(\varphi_A) = \left\{ x_1\vec{A}^{(1)} + \cdots + x_n\vec{A}^{(n)} \,\middle|\, x_1, \ldots x_n \in \mathbb{R} \right\}$$
$$= V_c(A)$$

由定理 3.1

$$\dim[\mathrm{im}(\varphi_A)] + \dim[\ker(\varphi_A)] = n$$

$$\dim V_c(A) = n$$

可得 $\dim V_A + \dim V_c(A) = n$

即 $\dim V_A + \mathrm{rank}(A) = n$. ▨

证明定理 3.4 的思路

回忆: 从 $V_c(A)$ 的一组基 $\vec{A}^{(1)}, \ldots \vec{A}^{(r)}$ 出发

构造 V_A 的一组基 $\vec{V}_{r+1}, \ldots \vec{V}_n$

(从 $\mathrm{im}(\varphi_A)$ 的~~基~~基底得到 $\ker(\varphi_A)$ 的基底)

证明定理 3.1 的思路

从 $\ker(\varphi)$ 的基底出发, $\vec{v}_1, \ldots \vec{v}_d$

扩充为 $\mathbb{R}^n$ 的一组基, $\vec{v}_1, \ldots \vec{v}_d, \vec{v}_{d+1}, \ldots \vec{v}_n$

证明 $\varphi(\vec{v}_{d+1}), \ldots \varphi(\vec{v}_n)$ 是 $\mathrm{im}(\varphi)$ 的基底

(从 $V_{(A_\varphi)}$ 的基底出发, 得出 $V_c(A)$ 的一组基)

给定 线性映射 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

计算 $\ker(\varphi)$ 的一组基. $\operatorname{im}(\varphi)$ 的一组基

① 求 φ 的矩阵 A

② 求 V_A 的一组基（解齐次线性方程组）

得出 $\ker(\varphi)$ 的基

③ 求 $V_c(A)$ 的一组基（列高斯消去）

得出 $\operatorname{im}(\varphi)$ 的基

例：设 $\varphi: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 + x_3 + x_4 + x_5 \\ x_1 - x_2 - x_3 - x_4 - x_5 \\ 4x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \end{pmatrix}$$

求 $\ker(\varphi)$ 和 $\operatorname{im}(\varphi)$ 的基底

$$A_\varphi = \left(\varphi(\vec{e}^{(1)}), \varphi(\vec{e}^{(2)}), \varphi(\vec{e}^{(3)}), \varphi(\vec{e}^{(4)}), \varphi(\vec{e}^{(5)}) \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 4 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$A \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rank}(A) = 2$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 + x_3 + x_4 + x_5 = 0 \end{cases} \qquad \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$\dim(\operatorname{im}(\varphi)) = 2 \Rightarrow \vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ 是 $V_c(A)$ 的基

于是 $\ker(\varphi)$ 的一组基是 $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$\operatorname{im}(\varphi)$ — — — — $\vec{w}_1, \vec{w}_2$.

命题3.4（线性单射和满射的矩阵刻化）

设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 是线性映射，A 是 φ 的矩阵

(i) φ 是单射 $\Longleftrightarrow$ A 列满秩

(ii) φ 是满射 $\Longleftrightarrow$ A 行满秩

证：(i) φ 是单射 $\Longleftrightarrow \ker(\varphi) = \{\vec{0}_n\}$（命题3.3）

$\Longleftrightarrow V_A = \{\vec{0}_n\}$

$\Longleftrightarrow \operatorname{rank}(A) = n$ ← 列数

（定理2.4）

(ii) φ 是满射 $\Longleftrightarrow \operatorname{im}(\varphi) = \mathbb{R}^m$

$\Longleftrightarrow$ ~~dim~~ $V_c(A) = \mathbb{R}^m$

$\Longleftrightarrow \dim V_c(A) = m$ ~~（定义）~~

$\Longleftrightarrow \operatorname{rank}(A) = m$ ← 行数

~~例：设 $A \in \mathbb{R}^{m\times n}$. 如果对于任意~~

~~$\forall \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$~~

注 行初等变换 对应求 $\ker(\varphi_A)$

列初等变换 ⋯⋯ $\operatorname{im}(\varphi_A)$.

注：设 $\varphi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$，$A$ 是 φ 的矩阵：$\operatorname{rank}(\varphi) := \operatorname{rank}(A)$

§4. 矩阵运算

§4.1. 矩阵的加法和数乘.

设 $A = (a_{ij})_{m\times n}$，$B = (b_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$

φ_A，φ_B 是两个从 $\mathbb{R}^n \to \mathbb{R}^m$ 的线性映射

命题4.1. 设 $\varphi, \psi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

则 $\varphi + \psi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \longmapsto \varphi(\vec{x}) + \psi(\vec{x})$

也是线性映射

证：设 $\alpha, \beta \in \mathbb{R}$，$\vec{x}, \vec{y} \in \mathbb{R}^n$

$(\varphi+\psi)(\alpha\vec{x} + \beta\vec{y}) = \varphi(\alpha\vec{x}+\beta\vec{y}) + \psi(\alpha\vec{x}+\beta\vec{y})$

$= \alpha\varphi(\vec{x}) + \beta\varphi(\vec{y}) + \alpha\psi(\vec{x}) + \beta\psi(\vec{y})$

$= \alpha(\varphi(\vec{x}) + \psi(\vec{x})) + \beta(\varphi(\vec{y}) + \psi(\vec{y}))$

$= \alpha(\varphi+\psi)(\vec{x}) + \beta(\varphi+\psi)(\vec{y})$ ▨

~~$\varphi + \psi$ 的矩阵是~~

~~φ~~

$\varphi_A+\varphi_B$ 的矩阵是

$$\left((\varphi_A+\varphi_B)(\vec{e}^{(1)}),\ \dots,\ (\varphi_A+\varphi_B)(\vec{e}^{(n)})\right)$$

$$=\left(\varphi_A(\vec{e}^{(1)})+\varphi_B(\vec{e}^{(1)}),\ \dots,\ \varphi_A(\vec{e}^{(n)})+\varphi_B(\vec{e}^{(n)})\right)$$

$$=\left(\vec{A}^{(1)}+\vec{B}^{(1)},\ \dots,\ \vec{A}^{(n)}+\vec{B}^{(n)}\right)$$

$$=(a_{ij}+b_{ij})_{m\times n}$$

定义：设 $A, B\in\mathbb{R}^{m\times n}$ 定义

$$A+B=(a_{ij}+b_{ij})_{m\times n}$$

注：$A+B=\left(\vec{A}^{(1)}+\vec{B}^{(1)},\ \dots,\ \vec{A}^{(n)}+\vec{B}^{(n)}\right)$

$$=\begin{pmatrix}\vec{A}_1+\vec{B}_1\\ \vdots\\ \vec{A}_m+\vec{B}_m\end{pmatrix}$$

命题 4.2 设 $\varphi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$, $\lambda\in\mathbb{R}$

则 $\lambda\varphi:\ \mathbb{R}^n\longrightarrow\mathbb{R}^m$, $\vec{x}\longmapsto\lambda\varphi(\vec{x})$ 是线性映射

证：设 $\alpha,\beta\in\mathbb{R}$, $\vec{x},\vec{y}\in\mathbb{R}^n$

$$(\lambda\varphi)(\alpha\vec{x}+\beta\vec{y})=\lambda\varphi(\alpha\vec{x}+\beta\vec{y})$$

$$=\alpha\lambda\varphi(\vec{x})+\beta\lambda\varphi(\vec{y})$$

$$=\lambda\alpha\varphi(\vec{x})+\lambda\beta\varphi(\vec{y})$$

$$=\alpha(\lambda\varphi)(\vec{x})+\beta(\lambda\varphi)(\vec{y})$$

$\square$.

$\lambda\varphi_A$ 的矩阵是

$$\left((\lambda\varphi_A)(\vec{e}^{(1)}),\ \dots\ (\lambda\varphi_A)(\vec{e}^{(n)})\right)$$

$$=\left(\lambda\varphi_A(\vec{e}^{(1)}),\ \dots,\ \lambda\varphi_A(\vec{e}^{(n)})\right)$$

$$=\left(\lambda\vec{A}^{(1)},\ \dots\ \lambda\vec{A}^{(n)}\right)=(\lambda a_{ij})_{m\times n}$$

定义：设 $A\in\mathbb{R}^{m\times n}$, $\lambda\in\mathbb{R}$

$$\lambda A=(\lambda a_{ij})_{m\times n}$$

注 $\lambda A=\left(\lambda\vec{A}^{(1)},\ \dots,\ \lambda\vec{A}^{(n)}\right)=\begin{pmatrix}\lambda\vec{A}_1\\ \vdots\\ \lambda\vec{A}_m\end{pmatrix}$

例 设 $A=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$, $B=\begin{pmatrix}-1&2\\-3&1\\5&0\end{pmatrix}$

求 $2A-B$

例: $2A-B = 2\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix} + (-1)\begin{pmatrix}-1 & 2\\ -3 & 1\\ 5 & 0\end{pmatrix}$

$$= \begin{pmatrix}2 & 4\\ 6 & 8\\ 10 & 12\end{pmatrix} + \begin{pmatrix}1 & -2\\ 3 & -1\\ -5 & 0\end{pmatrix} = \begin{pmatrix}3 & 2\\ 9 & 7\\ 5 & 12\end{pmatrix}$$

例: 设 $A, B \in \mathbb{R}^{m\times n}$. 证明

$$\mathrm{rank}(A+B) \leqslant \mathrm{rank}(A) + \mathrm{rank}(B)$$

证: $A+B = (\vec{A}^{(1)}+\vec{B}^{(1)}, \cdots, \vec{A}^{(n)}+\vec{B}^{(n)})$

$$V_c(A+B) = \langle \vec{A}^{(1)}+\vec{B}^{(1)}, \cdots, \vec{A}^{(n)}+\vec{B}^{(n)} \rangle$$

$$\subseteq \langle \vec{A}^{(1)}, \cdots \vec{A}^{(n)} \rangle + \langle \vec{B}^{(1)}, \cdots \vec{B}^{(n)} \rangle$$

$$= V_c(A) + V_c(B)$$

$$\mathrm{rank}(A+B) = \dim V_c(A+B)$$

$$\leqslant \dim[V_c(A) + V_c(B)]$$

$$= \dim V_c(A) + \dim V_c(B) - \dim(V_c(A) \cap V_c(B))$$

$$\leqslant \dim V_c(A) + \dim V_c(B) = \mathrm{rank}(A) + \mathrm{rank}(B)$$

∎

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~~§4.2 矩阵的乘法~~

~~命题4.3 设 φ~~

注: 由矩阵加法和数乘的定义

可知. $\varphi_A + \varphi_B = \varphi_{A+B}$

$\alpha\varphi_A = \varphi_{\alpha A}$

换言之 设 $\varphi, \psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

φ, ψ 的矩阵是 A, B 则

~~$\forall \alpha, \beta$~~ $\varphi+\psi$ 的矩阵是 $A+B$

$\alpha\varphi$ 的矩阵是 αA, 其

中 $\alpha \in \mathbb{R}$

以下规律显然成立

$\forall A, B, C \in \mathbb{R}^{m\times n}$

$A+B = B+A$, $(A+B)+C = A+(B+C)$

$A + O_{m\times n} = A$

$\forall \alpha, \beta \in \mathbb{R}$

$\alpha(\beta A) = (\alpha\beta)A$

$\alpha(A+B) = \alpha A + \alpha B$, $(\alpha+\beta)A = \alpha A + \beta A$

注：$\mathbb{R}^{m\times n}$ 可以看成 ~~$\mathbb{R}^{mn}$ 维坐标空间~~ mn 维坐标空间.

§4.2 矩阵~~的~~乘法的定义

~~命题~~ 4.3. 设 $\varphi \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^m)$

$\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$.

则 $\varphi\circ\psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

证：

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\psi"] \arrow[dr, "\varphi\circ\psi"'] & \mathbb{R}^s \arrow[d, "\varphi"] \\ & \mathbb{R}^m \end{tikzcd}$$

设 $\vec{x}, \vec{y} \in \mathbb{R}^n$

$\alpha, \beta \in \mathbb{R}$

$\varphi\circ\psi(\alpha\vec{x} + \beta\vec{y})$

$= \varphi(\psi(\alpha\vec{x}+\beta\vec{y}))$

$= \varphi(\alpha\psi(\vec{x}) + \beta\psi(\vec{y}))$

$= \alpha\varphi(\psi(\vec{x})) + \beta\varphi(\psi(\vec{y}))$

$= \alpha\,\varphi\circ\psi(\vec{x}) + \beta\,\varphi\circ\psi(\vec{y})$ $\boxtimes$

设 φ 的矩阵为 A，则 $A \in \mathbb{R}^{m\times s}$

ψ 的矩阵为 B，则 $B \in \mathbb{R}^{s\times n}$

$A = (a_{ik})_{\substack{i=1\cdots m\\k=1\cdots s}}$, $B = (b_{kj})_{\substack{k=1\cdots s\\j=1\cdots n}}$

问 $\varphi\circ\psi$ 的矩阵是什么？

~~§4.2 矩阵的乘法~~:

设 $C \in \mathbb{R}^{m\times n}$ 是 $\varphi \circ \psi$ 的矩阵

$$C = (\varphi\circ\psi(\vec{e}^{(1)}), \cdots, \varphi\circ\psi(\vec{e}^{(n)}))$$

$$= (\varphi(\psi(\vec{e}^{(1)})), \cdots, \varphi(\psi(\vec{e}^{(n)})))$$

(复合的定义)

$$= (\varphi(\vec{B}^{(1)}), \cdots \varphi(\vec{B}^{(n)}))$$

(B 的定义)

设 $\vec{B}^{(1)} = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{s1} \end{pmatrix}, \cdots, \vec{B}^{(n)} = \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{sn} \end{pmatrix}$

$$\varphi(\vec{B}^{(j)}) = b_{1j}\vec{A}^{(1)} + \cdots + b_{sj}\vec{A}^{(s)}$$

$j = 1, \cdots, n$

~~$C = (b_{1}$~~ $\quad C = (C^{(1)}, \cdots C^{(n)})$

其中 $\quad C^{(j)} = b_{1j}\vec{A}^{(1)} + \cdots + b_{sj}\vec{A}^{(s)}$

$j = 1, \cdots, n$

于是

$$c_{ij} = b_{1j}a_{i1} + \cdots + b_{sj}a_{is}$$

$$= a_{i1}b_{1j} + \cdots + a_{is}b_{sj}$$

$$= \sum_{k=1}^{s} a_{ik}b_{kj}$$

~~$C \neq \psi C$~~ 定义：设 $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times n}$

AB (A, B 的积) 是 $m\times n$ 阶的矩阵 $C = (c_{ij})_{m\times n}$.

其中 $\quad c_{ij} = a_{i1}b_{1j} + \cdots + a_{is}b_{sj}$

$i = 1, \cdots, m, \quad j = 1, \cdots n$

"$c_{ij} = (a_{i1}, \cdots, a_{is}) \begin{pmatrix} b_{1j} \\ \vdots \\ b_{js} \end{pmatrix} = a_{i1}b_{1j} + \cdots + a_{is}b_{js}$"

ϕ

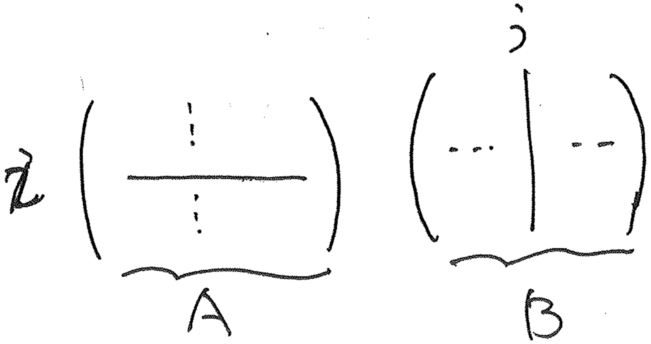
回忆：设 $A \in \mathbb{R}^{m\times s}$，$B \in \mathbb{R}^{s\times n}$

$$C = AB \in \mathbb{R}^{m\times n}$$

设 $A = (a_{ik})_{\substack{i=1,\dots,m \\ k=1,\dots,s}}$　$B = (b_{kj})_{\substack{k=1,\dots,s \\ j=1,\dots,n}}$

$$C = (c_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

则 $c_{ij} = \sum_{k=1}^{s} a_{ik} b_{kj}$



$$c_{ij} = \vec{A}_i \vec{B}^{(j)} = (a_{i1}, a_{i2}, \cdots a_{is}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{is} b_{sj}$$

例 $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$　$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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计算 AB

① AB 的行数和列数

$A \in \mathbb{R}^{2\times 3}$，$B \in \mathbb{R}^{3\times 3}$，$AB \in \mathbb{R}^{2\times 3}$

② 计算

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 3 \\ 4 & 9 & 6 \end{pmatrix}$$

注 BA 没有定义

例 如果 A, B 都是 n 阶方阵

则 AB 和 BA 也都是 n 阶方阵

设 $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$$AB=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$BA=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

注1 $AB \neq BA$ 交换律不成立

注2, $A\neq O_{2\times2}$, $B\neq O_{2\times2}$ 但

$$BA=O_{2\times2}$$

注3. 消去律不成立

即 $AB=AC \not\Rightarrow B=C$

$AB=CB \not\Rightarrow A=C$

例 符号化简

设 $A=(a_{ij})_{m\times n}$. $\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$

$$A\vec{x}=\begin{pmatrix}a_{11}, \cdots\cdots a_{1n}\\ \cdots\\ a_{m1}, \cdots a_{mn}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$$=\begin{pmatrix}a_{11}x_1+\cdots+a_{1n}x_n\\ \cdots\\ a_{m1}x_1+\cdots+a_{mn}x_n\end{pmatrix}$$

于是以 A 为系数的齐次线性方程可写为

$$A\vec{x}=\vec{O}_m$$

设 $\vec{b}=\begin{pmatrix}b_1\\ \vdots\\ b_m\end{pmatrix}$ 则

$$(L)\begin{cases}a_{11}x_1+\cdots+a_{1n}x_n=b_1\\ \vdots\\ a_{m1}x_1+\cdots+a_{mn}x_n=b_m\end{cases}$$ 可写为 $A\vec{x}=\vec{b}$

~~§4.3 矢阵~~

设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 是线性映射.

A 是 φ 的矩阵. $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\varphi(\vec{x}) = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} = A\vec{x}$$

§4.3 矩阵乘法的运算规律

~~1. 结合律~~: 由矩阵乘法的定义可知

如果 $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times n}$

则 $\varphi_A: \mathbb{R}^s \longrightarrow \mathbb{R}^m$, $\varphi_B: \mathbb{R}^n \longrightarrow \mathbb{R}^s$

线性映射. 且 $\varphi_A \circ \varphi_B$ 的矩阵是

AB

即 $\varphi_A \circ \varphi_B = \varphi_{AB}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_B"] \arrow[dr] & \mathbb{R}^s \arrow[d, "\varphi_A"] \\ & \mathbb{R}^m \end{tikzcd}$$

$\varphi_{AB} = \varphi_A \circ \varphi_B$

结合律: 设 $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times k}$, $C \in \mathbb{R}^{k\times n}$

则 $(AB)C = A(BC)$

验证:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_C"] \arrow[dr, "\varphi_{BC} = \varphi_B \circ \varphi_C"'] \arrow[drr, dashed] & \mathbb{R}^k \arrow[d, "\varphi_B"] \arrow[dr, dashed, "\varphi_A \circ \varphi_B = \varphi_{AB}"] & \\ & \mathbb{R}^s \arrow[r, "\varphi_A"] & \mathbb{R}^m \end{tikzcd}$$

由映射的结合律:

$$\varphi_A \circ (\varphi_B \circ \varphi_C) = (\varphi_A \circ \varphi_B) \circ \varphi_C$$

$$\varphi_A \circ (\varphi_{BC}) = (\varphi_{AB}) \circ \varphi_C$$

$$\varphi_{A(BC)} = \varphi_{(AB)C}$$

由定理3.2 $A(BC) = (AB)C$

例: 设 $A \in \mathbb{R}^{m\times n}$, E_m 是 m 阶单位方阵. E_n 是 n 阶单位方阵

则 $E_m A = A E_n = A$

证：设 $id_{E_m}: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ 是恒同映射，则 id_{E_m} 的矩阵是 E_m

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_A"] \arrow[dr, "id_{E_m}\circ\varphi_A"'] & \mathbb{R}^m \arrow[d, "id_{E_m}"] \\ & \mathbb{R}^m \end{tikzcd}$$

$$\varphi_A = id_{E_m}\circ \varphi_A = \varphi_{E_m A} \Longrightarrow A = E_m A$$

类似可证 $AE_n = A$

矩阵乘法 设 $B = (b_{ij})_{m\times n} = AE_n$

$$b_{ij} = \vec{A}_i \vec{E}_n^j = \vec{A}_i \cdot \vec{e}^{(j)}$$

$$= (a_{i1}, a_{i2}, \cdots a_{in}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} j = a_{ij} \quad \boxed{}$$

分配律 设 $A, B \in \mathbb{R}^{m\times n}$

设 $C \in \mathbb{R}^{k\times m}$

$$C(A+B) = CA + CB$$

设 $\alpha \in \mathbb{R}$

$$C(\alpha A) = \alpha(CA)$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_A+\varphi_B"] \arrow[dr, "\varphi_C\circ(\varphi_A+\varphi_B)"'] & \mathbb{R}^m \arrow[d, "\varphi_C"] \\ & \mathbb{R}^k \end{tikzcd}$$

$$\varphi_C \circ (\varphi_A + \varphi_B) = \varphi_C \circ (\varphi_{A+B}) = \varphi_{C(A+B)}$$

$\forall \vec{x} \in \mathbb{R}^n$

$$\varphi_C \circ (\varphi_A + \varphi_B)(\vec{x}) = \varphi_C((\varphi_A+\varphi_B)(\vec{x}))$$
$$= \varphi_C(\varphi_A(\vec{x}) + \varphi_B(\vec{x}))$$
$$= \varphi_C(\varphi_A(\vec{x})) + \varphi_C(\varphi_B(\vec{x}))$$
$$= \varphi_C\circ\varphi_A(\vec{x}) + \varphi_C\circ\varphi_B(\vec{x})$$

于是 $\varphi_C \circ (\varphi_A + \varphi_B) = \varphi_C\circ\varphi_A + \varphi_C\circ\varphi_B$

$$\Longrightarrow C(A+B) = CA + CB$$

$$\varphi_C \circ (\alpha\varphi_A)(\vec{x}) = \varphi_C(\alpha\varphi_A(\vec{x}))$$
$$= \alpha\varphi_C(\varphi_A(\vec{x})) = \alpha(\varphi_C\circ\varphi_A)(\vec{x})$$

于是

③ $\varphi_C \circ \varphi_{\alpha A} = \varphi_{\alpha C} \circ \varphi_A$

$\Rightarrow C(\alpha A) = \alpha(CA)$

同理. 设 $C \in \mathbb{R}^{n\times k}$

$$(A+B)C = AC+BC$$

$$A(\alpha C) = \alpha AC$$

转置： 设 $A \in \mathbb{R}^{m\times s}$, $B \in \mathbb{R}^{s\times n}$

则 $(AB)^t = B^t A^t$

证： 设 $A = (a_{ik})_{\substack{i=1,\cdots m\\ k=1\cdots s}}$, $B=(b_{kj})_{\substack{k=1\cdots s\\ j=1\cdots n}}$

$A^t = (a'_{ki})_{\substack{k=1\cdots s\\ i=1,\cdots m}}$ $a'_{ki}=a_{ik}$

$B^t = (b'_{jk})_{\substack{j=1,\cdots n\\ k=1\cdots s}}$ $b'_{jk}=b_{kj}$

$C = AB = (c_{ij})_{\substack{i=1,\cdots m\\ j=1,\cdots n}}$

$C^t = (c'_{ji})_{\substack{j=1,\cdots n\\ i=1\cdots m}}$. $c_{ij} = c'_{ji}$

设 $D = B^t A^t = (d_{ji})_{\substack{j=1\cdots m\\ i=1\cdots m}}$

$$d_{ji} = \sum_{k=1}^{s} b'_{jk} a'_{ki} = \sum_{k=1}^{s} b_{kj} a_{ik}$$

$$= \sum_{k=1}^{s} a_{ik} b_{kj} = c_{ij} = c'_{ji}$$

其中 $j=1,\cdots,m$. $i=1,\cdots,m$. 于是

$D = C^t \Rightarrow B^t A^t = (AB)^t$

~~注：消去律不成立，A, B ∈ A∈$\mathbb{R}^{m\times s}$~~, $B, C \in \mathbb{R}^{s\times n}$, $A \neq O_{m\times s}$

$\cancel{AB = AC}$ ~~(左消去律不成立~~

$\Rightarrow \cancel{B = C}$

~~同理 右消去律也不成立.~~

例: Hadamard's product

设 $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{m\times n}$

$A\odot B=(a_{ij}b_{ij})_{m\times n}$

(Nickname Chodren's product)

设 $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

计算 AA 和 $A\odot A$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\odot\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$$

例: n阶方阵

$$\begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$$ 称为对角阵

也写为 $\mathrm{diag}(\lambda_1,\dots\lambda_n)$

特别地 $\begin{pmatrix} \lambda & & O \\ & \ddots & \\ O & & \lambda \end{pmatrix}_{n\times n}$ 记为

$\mathrm{diag}_n(\lambda)$ 或 λE_n 称为数乘矩阵

设 $A\in\mathbb{R}^{m\times n}$

证明 ① $\mathrm{diag}(\lambda_1,\dots\lambda_m)A=\begin{pmatrix} \lambda_1\vec{A}_1 \\ \vdots \\ \lambda_m\vec{A}_m \end{pmatrix}$

② $A\,\mathrm{diag}(\lambda_1,\dots\lambda_n)=(\lambda_1\vec{A}^{(1)},\dots\lambda_n\vec{A}^{(n)})$

证: ① 设 $A=(a_{ij})_{m\times n}$.

$\mathrm{diag}(\lambda_1,\dots\lambda_m)A=(b_{ij})_{m\times n}$

$B=\mathrm{diag}(\lambda_1,\dots\lambda_m)$

$$b_{ij}=\vec{B}_i\vec{A}^{(j)}=(\underbrace{0\cdots0\,\lambda_i\,0\cdots0}_{m})\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$=\lambda_i a_{ij}$

$\vec{B}_i=(\lambda_i a_{i1},\dots\lambda_i a_{in})$

$=\lambda_i(a_{i1},\dots,a_{in})=\lambda_i\vec{A}_i$

$i=1,\dots,n$

② 可以类似的证明

但我们利用转置来证.

首先介绍转置的一些基本符号和术语

$$(\alpha_1, \cdots \alpha_n)^t = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^t = (\beta_1, \cdots \beta_m)$$

$$A = \begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}_{m\times n}, \quad A^t = \left((\vec{A}_1)^t, \cdots (\vec{A}_m)^t\right)_{n\times m}$$

$$A = (\vec{A}^{(1)}, \cdots \vec{A}^{(n)}), \quad A^t = \begin{pmatrix} (\vec{A}^{(1)})^t \\ \vdots \\ (\vec{A}^{(n)})^t \end{pmatrix}$$

例如:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}^t = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \vdots & \vdots \\ \alpha_n & \beta_n \end{pmatrix}$$

由此可知: $(A^t)^t = A$

~~[illegible]~~

注 $\begin{pmatrix} \lambda_1 & 0 & 0 \cdots 0 & 0 \\ 0 & \lambda_2 & 0 \cdots 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 \cdots 0 & \lambda_n \end{pmatrix}^t$ 等于其本身

设 $B = A \operatorname{diag}(\lambda_1, \cdots \lambda_n)$

$$B^t = [A \operatorname{diag}(\lambda_1, \cdots \lambda_n)]^t$$

$$= \operatorname{diag}(\lambda_1, \cdots \lambda_n)^t \cdot A^t$$

$$= \operatorname{diag}(\lambda_1, \cdots \lambda_n) \cdot A^t$$

$$= \begin{pmatrix} \lambda_1 \vec{A}^t_1 \\ \vdots \\ \lambda_n \vec{A}^t_n \end{pmatrix} = \begin{pmatrix} \lambda_1 (\vec{A}^{(1)})^t \\ \vdots \\ \lambda_n (\vec{A}^{(n)})^t \end{pmatrix}$$

$$B = (B^t)^t = \begin{pmatrix} \lambda_1 ((\vec{A}^{(1)})^t)^t \\ \vdots \\ \lambda_n ((\vec{A}^{(n)})^t)^t \end{pmatrix} = (\lambda_1 \vec{A}^{(1)}, \cdots \lambda_n \vec{A}^{(n)})$$

例: 设 $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. $D=\begin{pmatrix}1&&O\\&2&\\O&&3\end{pmatrix}$

求 DA 和 AD

解 $DA=\begin{pmatrix}1&2&3\\8&10&12\\21&24&27\end{pmatrix}$ $AD=\begin{pmatrix}1&4&9\\4&10&18\\7&16&27\end{pmatrix}$

$DA\neq AD$

例 设 $D=\lambda E_n$, $A\in\mathbb{R}^{n\times n}$

证明 $DA=AD$

$DA=(\lambda E_n)A=\lambda(E_nA)=\lambda A$

$AD=(A(\lambda E_n))=\lambda(AE_n)=\lambda A$

$\Rightarrow DA=AD$