回忆：设 $\vec{e_1},\cdots,\vec{e_n}$，$\vec{e_1'},\cdots,\vec{e_n'}$ 是 V 的两组基

则 $\exists!\ A\in GL_n(F)$ 使得

$$(\vec{e_1'},\cdots,\vec{e_n'})=(\vec{e_1},\cdots,\vec{e_n})A$$

↳ 转换矩阵

设 $\vec{v}=\alpha_1\vec{e_1}+\cdots+\alpha_n\vec{e_n}=\alpha_1'\vec{e_1'}+\cdots+\alpha_n'\vec{e_n'}$

$$=(\vec{e_1},\cdots,\vec{e_n})\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}=(\vec{e_1'},\cdots,\vec{e_n'})\begin{pmatrix}\alpha_1'\\ \vdots\\ \alpha_n'\end{pmatrix}$$

则 $\begin{pmatrix}\alpha_1'\\ \vdots\\ \alpha_n'\end{pmatrix}=A^{-1}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

定理8.2 设 $\vec{e_1},\cdots,\vec{e_n}$ 是 V 的一组基

$A\in M_n(F)$

$$(\vec{v_1},\cdots,\vec{v_n})=(\vec{e_1},\cdots,\vec{e_n})A$$

则 $\vec{v_1},\cdots,\vec{v_n}$ 是 V 的基 $\Leftrightarrow$ A 可逆

证："$\Rightarrow$" 定理8.1的结论

"$\Leftarrow$" 设 $\beta_1,\cdots,\beta_n\in F$

使得 $(\vec{v_1},\cdots,\vec{v_n})\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=\vec{0}$

则 $\vec{0}=(\vec{v_1},\cdots,\vec{v_n})\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=(\vec{e_1},\cdots,\vec{e_n})A\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}$

因为 $\vec{e_1},\cdots,\vec{e_n}$ 线性无关

所以 $A\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=A^{-1}\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

$\Rightarrow\vec{v_1},\cdots,\vec{v_n}$ 线性无关

由定理6.1．$\vec{v_1},\cdots,\vec{v_n}$ 是 V 的一组基

例：在 $F[x]^{(3)}$ 中，设

$p_1=x(x-1)$，$p_2=x(x-2)$，$p_3=x(x-3)+1$

是不是一组基．如果是求 $1,x,x^2$ 到 p_1,p_2,p_3 的转换矩阵

解：$p_1=x^2-x=(1,x,x^2)\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}$

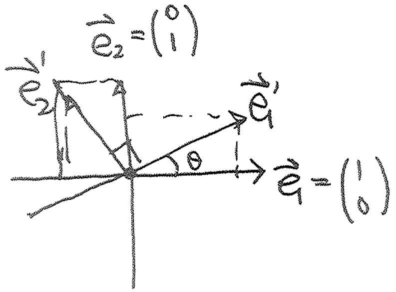
$p_2=x^2-2x=(1,x,x^2)\begin{pmatrix}0\\ -2\\ 1\end{pmatrix}$

$p_3=x^2-3x+1=(1,x,x^2)\begin{pmatrix}1\\ -3\\ 1\end{pmatrix}$

$$(p_1, p_2, p_3) = (1, x, x^2)\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ -1 & -2 & -3 \\ 1 & 1 & 1 \end{pmatrix}}_{A}$$

$\because \ \mathrm{rank}(A) = 3 \quad \therefore \ p_1, p_2, p_3$ 是 $F[x]^{(3)}$ 的基

例：$\mathbb{R}^2$ 中两组基如图，求它们之间的转换矩阵，$\vec{e_1}, \vec{e_2}, \vec{e_1}', \vec{e_2}'$ 长度都是 1



$$\vec{e_1}' = \cos\theta\,\vec{e_1} + \sin\theta\,\vec{e_2}$$
$$\vec{e_2}' = -\sin\theta\,\vec{e_1} + \cos\theta\,\vec{e_2}$$

$$(\vec{e_1}', \vec{e_2}') = (\vec{e_1}, \vec{e_2})\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A}$$

$$(\vec{e_1}, \vec{e_2}) = \cancel{A^{-1}}\,(\vec{e_1}', \vec{e_2}')\,A^{-1}$$
$$= (\vec{e_1}', \vec{e_2}')\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

自己验证：对上述矩阵 A

$$A^{-1} = A^t$$

例：设 $V = F^2$. 求 $\begin{pmatrix}2\\3\end{pmatrix}$

在 $\vec{e_1}' = 2\vec{e_1} + 4\vec{e_2}$，$\vec{e_2}' = \vec{e_1} - \vec{e_2}$ 下的坐标

其中 $\vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$

$$(\vec{e_1}', \vec{e_2}') = (\vec{e_1}, \vec{e_2})\underbrace{\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}}_{A}$$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right)$$

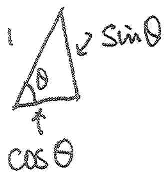
$$\to \left(\begin{array}{cc|cc} 2 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & -3 & -2 & 1 \end{array}\right)$$

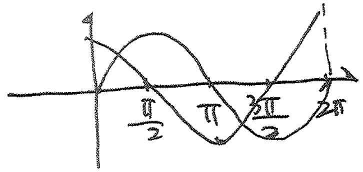
$$\to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & +\frac{2}{3} & \frac{-1}{3} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$\begin{pmatrix}2\\3\end{pmatrix}$ 在 $\vec{e_1}', \vec{e_2}'$ 下坐标 $A^{-1}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}\frac{5}{6}\\ \frac{1}{3}\end{pmatrix}$

§9 对偶空间





§9.1 对偶基

定义：设 V 是 n 维线性空间，基域是 F

$V^* := \mathrm{Hom}(V, F)$ 称为 V 的对偶空间

0^* 代表零映射

注：设 $\vec{e}_1, \cdots, \vec{e}_n$ 是 V 的一组基

$f \in V^*$，$f(\vec{e}_i) = \alpha_i$，$i = 1, 2, \cdots n$

设 $\vec{x} = (\vec{e}_1, \cdots, \vec{e}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$f(\vec{x}) = \cancel{f(\vec{e}_1}(f(\vec{e}_1), \cdots, f(\vec{e}_n))\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= (\alpha_1, \cdots \alpha_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= \alpha_1 x_1 + \cdots + \alpha_n x_n$

即当 V 中坐标选定时

V^* 中元素可以看成一次齐次多项式

（也称线性齐次多项式）

例：设 $V = \mathbb{R}$，$V^* = \{\alpha x \mid \alpha \in \mathbb{R}\}$

↳ 标准基

$V = \mathbb{R}^2$，$V^* = \{\alpha_1 x_1 + \alpha_2 x_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}$

设 V 的一组基是 $\vec{e}_1, \cdots, \vec{e}_n$，定义

$\cancel{e_i^* \in \mathrm{Hom}}$ $e_i^* \in V^*$ 满足

$e_i^*(\vec{e}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j. \end{cases}$ $i, j \in \{1, 2, \cdots n\}$

由定理 6.2

$e_1^*, \cdots, e_n^*$ 存在且唯一

验证：$e_1^*, \cdots, e_n^*$ 是 V^* $\cancel{e_1, \cdots, e_n}$ 的一组基

设 $\alpha_1, \cdots \alpha_n \in F$

$\cancel{f =} f = \alpha_1 e_1^* + \cdots + \alpha_n e_n^* = 0^*$

则 $f(\vec{e}_i) = 0$，$\forall i = 1, 2, \cdots n$.

另一方面

$$f(\vec{e_i}) = \alpha_1 e_1^*(\vec{e_i}) + \cdots + \alpha_n e_n^*(\vec{e_i})$$
$$= \sum_{j=1}^n \alpha_j e_j^*(\vec{e_i}) = \sum_{j=1}^n \alpha_j \delta_{ji} = \alpha_i$$

$\Rightarrow \alpha_i = 0, \quad i=1,2,\cdots n$

$\Rightarrow e_1^*, \cdots, e_n^*$ 线性无关

设 $g \in V^*$, $\quad g(\vec{e_i}) = \beta_i, \quad i=1,2\cdots n$

设 $h = \beta_1 e_1^* + \cdots + \beta_n e_n^*$

$$h(\vec{e_i}) = \left[\sum_{j=1}^n \beta_j e_j^*\right](\vec{e_i}) = \sum_{j=1}^n \beta_j e_j^*(\vec{e_i})$$
$$= \sum_{j=1}^n \beta_j \delta_{ji} = \beta_i. \quad i=1\cdots,n$$

由定理6.2. $\quad g = h$.

于是 $\quad e_1^*, \cdots, e_n^*$ 是 V^* 的一组基

定义：设 $\vec{e_1}, \cdots, \vec{e_n}$ 是 V 的一组基

则 $e_1^*, \cdots, e_n^*$ 称为 $\vec{e_1}, \cdots, \vec{e_n}$ 在 V^* 中的对偶基

定理9.1. 如果 $\dim V = n$, 则 $\dim V^* = n$

例：设 $V = F^n$, $\vec{e_1} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \cdots, \vec{e_n} = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

$$f_1 : V \to F, \quad \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} \mapsto x_1 \quad , \cdots, \quad f_n : V \to F, \quad \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} \mapsto x_n$$

验证：$f_1, \cdots, f_n$ 是 $\vec{e_1}, \cdots, \vec{e_n}$ 的对偶基

证：$f_i(\vec{e_j})$ 的值是 $\vec{e_j}$ 的第 i 个坐标

于是 $f_i(\vec{e_j}) = \delta_{ij}$, $\quad i, j \in \{1, \cdots, n\}$.

例：设 $V = F[x]^{(n)}$. $\quad \forall i \in \{0, 1, \cdots, n-1\}$

$$[x^i] : V \longrightarrow F, \quad \sum_{i=0}^{n-1} \alpha_i x^i \mapsto \alpha_i$$

自己验证：$1, x, \cdots x^{n-1}$ 的对偶基

是 $[x^0], [x], \cdots [x^{n-1}]$.

注: $e_i^*(\alpha_1\vec{e}_1+\cdots+\alpha_n\vec{e}_n)$

$$= e_i^*\left(\sum_{j=1}^n \alpha_j \vec{e}_j\right) = \sum_{j=1}^n \alpha_j e_i^*(\vec{e}_j)$$

$$= \sum_{j=1}^n \alpha_j \delta_{ij} = \alpha_i$$

即 e_i^* 是取 $\vec{v}=\alpha_1\vec{e}_1+\cdots+\alpha_n\vec{e}_n$ 在 $\vec{e}_1,\cdots,\vec{e}_n$ 下的第 i 个坐标

例: 设 $\vec{e}_1,\cdots,\vec{e}_n$; $\vec{\varepsilon}_1,\cdots,\vec{\varepsilon}_n$ 是 V 的两组基

$e_1^*,\cdots,e_n^*$; $\varepsilon_1^*,\cdots,\varepsilon_n^*$ 是它们的对偶基

设 ~~$(\vec{e}_1,\cdots,\vec{e}_n$~~

$$(\vec{\varepsilon}_1,\cdots,\vec{\varepsilon}_n) = (\vec{e}_1,\cdots,\vec{e}_n)A, \quad A\in GL_n(F)$$

求 $B\in GL_n(F)$ 使得

$$(\varepsilon_1^*,\cdots,\varepsilon_n^*) = (e_1^*,\cdots,e_n^*)B$$

解: $\forall\, i,j\in\{1,\cdots,n\}$

$$\delta_{ij} = \varepsilon_i^*(\vec{\varepsilon}_j) = \left[(e_1^*,\cdots,e_n^*)\vec{B}^{(i)}\right](\vec{\varepsilon}_j)$$

$$= \left[(e_1^*,\cdots,e_n^*)\vec{B}^{(i)}\right]\left((\vec{e}_1,\cdots,\vec{e}_n)\vec{A}^{(j)}\right)$$

设 $B=(b_{ki})_{n\times n}$, $A=(a_{lj})_{n\times n}$

$$(e_1^*,\cdots,e_n^*)\vec{B}^{(i)} = (e_1^*,\cdots,e_n^*)\begin{pmatrix} b_{1i} \\ \vdots \\ b_{ni}\end{pmatrix}$$

$$= b_{1i}e_1^*+\cdots+b_{ni}e_n^* = \sum_{k=1}^n b_{ki}e_k^*$$

$$(\vec{e}_1,\cdots,\vec{e}_n)\vec{A}^{(j)} = (\vec{e}_1,\cdots,\vec{e}_n)\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj}\end{pmatrix}$$

$$= a_{1j}\vec{e}_1+\cdots+a_{nj}\vec{e}_n = \sum_{l=1}^n a_{lj}\vec{e}_l$$

$$\delta_{ij} = \left(\sum_{k=1}^n b_{ki}e_k^*\right)\left(\sum_{l=1}^n a_{lj}\vec{e}_l\right)$$

$$= \sum_{k=1}^n b_{ki}e_k^*\left(\sum_{l=1}^n a_{lj}\vec{e}_l\right)$$

$$= \sum_{k=1}^n b_{ki}a_{kj}, \quad i,j\in\{1,2,\cdots n\}$$

$$\Rightarrow \quad B = (A^t)^{-1}$$

☒

§9.2 线性相关性的对偶描述

引理 9.1 设 $\vec{v} \in V$，则以下断言等价

(i) $\vec{v} = \vec{0}$

(ii) $\forall f \in V^*$，$f(\vec{v}) = 0$

(iii) 设 $f_1, \cdots f_n$ 是 V^* 的一组基

$$f_1(\vec{v})=0, \ldots, f_n(\vec{v})=0$$

证：(i) $\Rightarrow$ (ii) ✓　　(ii) $\Rightarrow$ (iii) ✓

(iii) $\Rightarrow$ (i)．假设 $\vec{v}$ 不是 $\vec{0}$

则 $\{\vec{v}\}$ 是线性无关集

$\exists$ V 的基底

$\vec{v}=\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$

设 $e_1^*, e_2^*, \ldots, \vec{e}_n^*$ 是对偶基

特别有 $e_1^*(\vec{e_1}) = 1 \neq 0$

设 $\alpha_1, \ldots \alpha_n \in F$

$$e_1^* = \alpha_1 f_1 + \cdots + \alpha_n f_n$$

$$e_1^*(\vec{e_1}) = \alpha_1 f_1(\vec{e_1}) + \cdots + \alpha_n f_n(\vec{e_n}) = 0 \neq 1 \rightarrow\leftarrow$$

推论 9.1. 设 $\vec{u}, \vec{v} \in V$．则下列断言等价

(i) $\vec{u} = \vec{v}$

(ii) $\forall f \in V^*$，$f(\vec{u}) = f(\vec{v})$

(iii) 设 $f_1, \ldots f_n$ 是 V^* 的一组基

$$f_i(\vec{u}) = f_i(\vec{v}), \quad i=1,2,\ldots,n$$

证：对 $\vec{u}-\vec{v}$ 用引理 9.1 即可

引理 9.2 设 $f_1, \ldots, f_m \in V^*$，$\vec{v_1}, \ldots, \vec{v_k} \in V$

设 $A = (f_i(\vec{v_j}))_{m \times k}$．再设

$\vec{v} = \alpha_1 \vec{v_1} + \cdots + \alpha_k \vec{v_k}$，其中 $\alpha_1, \ldots \alpha_k \in F$

则 $$\begin{pmatrix} f_1(\vec{v}) \\ \vdots \\ f_m(\vec{v}) \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}$$

证：$$f_i(\vec{v}) = f_i\left((\vec{v_1}, \ldots \vec{v_k}) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}\right) = (f_i(\vec{v_1}), \ldots f_i(\vec{v_k})) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}$$

$$= \vec{A_i} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}, \quad i=1, \ldots m$$

$$\Rightarrow \begin{pmatrix} f_1(\vec{v}) \\ \vdots \\ f_m(\vec{v}) \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \quad \boxtimes$$

引理9.3 设 $\vec{v}_1,\dots,\vec{v}_k\in V$. 则下列断言~~成立~~等价

(i) $\vec{v}_1,\dots,\vec{v}_k$ 线性相关

(ii) $\forall f_1,\dots,f_k\in V^*$, 矩阵

$A=(f_i(\vec{v}_j))_{k\times k}$ 不满秩

(iii) 设 $g_1,\dots,g_n$ 是 V^* 的一组基

$B=(g_i(\vec{v}_j))_{n\times k}$ 的秩小于 k

证: (i) $\Rightarrow$ (ii)

设 $\alpha_1,\dots,\alpha_k\in F$, 不全为零 使得

$\vec{0}=\alpha_1\vec{v}_1+\dots+\alpha_k\vec{v}_k$

由引理9.2

$$\begin{bmatrix} f_1(\vec{0}) \\ \vdots \\ f_k(\vec{0}) \end{bmatrix} = A\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_k\end{pmatrix}=\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

$\Rightarrow \operatorname{rank}(A)<k$

$\left(\because \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_k\end{pmatrix}\neq\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}\right)$

(ii) $\Rightarrow$ (iii) $\because$ B 只有k列, 由(ii)得

B中任意k阶子式都是零. 于是 $\operatorname{rank}(B)<k$

(iii) $\Rightarrow$ (i) $\because \operatorname{rank}(B)<k$

$\therefore\ \exists\, \beta_1,\dots,\beta_k\in F$, 不全为零 使得

$$B\begin{pmatrix}\beta_1\\ \vdots \\ \beta_k\end{pmatrix}=\underbrace{\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}}_{n}$$

令 $\vec{u}=\beta_1\vec{v}_1+\dots+\beta_k\vec{v}_k$? . 由引理9.2

$$\begin{pmatrix} g_1(\vec{u}) \\ \vdots \\ g_n(\vec{u}) \end{pmatrix} = B\begin{pmatrix}\beta_1\\ \vdots \\ \beta_k\end{pmatrix}=\underbrace{\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}}_{n}$$

$\Rightarrow g_1(\vec{u})=\dots=g_n(\vec{u})=0 \Rightarrow \vec{u}=0$

引理9.1

$\Rightarrow \vec{v}_1,\dots,\vec{v}_k$ 线性相关.

定理9.2 设 $\vec{v}_1,\dots,\vec{v}_k\in V$，且 $f_1,\dots,f_n\in V^*$ 是 V^* 的一组基。令

$$A=(f_i(\vec{v}_j))_{n\times k}$$

则 $\dim\langle\vec{v}_1,\dots,\vec{v}_k\rangle=\mathrm{rank}(A)$.

证： 设 $r=\mathrm{rank}(A)$. 不妨设 $\vec{A}^{(1)},\dots,\vec{A}^{(r)}$ 线性无关. 令 $B=(\vec{A}^{(1)},\dots,\vec{A}^{(r)})$

则 $B=(f_i(\vec{v}_l))_{n\times r}$ 且 $\mathrm{rank}(B)=r$

则 于是 $\vec{v}_1,\dots,\vec{v}_r$ 线性无关（引理9.3 (iii)和(i)等价）

$\forall m\in\{r+1,\dots,k\}$. 设 $B_m=(\vec{A}^{(1)},\dots,\vec{A}^{(r)},\vec{A}^{(m)})$

则 $\cancel{\mathrm{rank}}(B_m$

$$B_m=\left(\begin{array}{c|c} B & \begin{array}{c} f_1(\vec{v}_m)\\ \vdots\\ f_n(\vec{v}_m)\end{array}\end{array}\right)$$ 且 $\mathrm{rank}(B)=r$

$\Rightarrow$ $\vec{v}_1,\dots,\vec{v}_r,\vec{v}_m$ 线性相关（引理9.3 (iii)和(i)等价）

$\Rightarrow$ $\vec{v}_m\in\langle\vec{v}_1,\dots,\vec{v}_r\rangle\Rightarrow\dim\langle\vec{v}_1,\dots,\vec{v}_r\rangle=r$ □

例 设 $V=F[x]^{(m)}$，$\alpha_1,\dots,\alpha_m\in F$ 两两不同

~~$\forall\alpha\in F$~~ $\varphi_i: V\longrightarrow F$，$p(x)\longmapsto p(\alpha_i)$，$i=1,2,\dots,m$

证明 (i) $\varphi_1,\dots,\varphi_m$ 是 V^* 的一组基

(ii) 设 $p_1,p_2,\dots,p_m\in V$

则 $p_1,p_2,\dots,p_m$ 是 V 的基 $\Longleftrightarrow$

$$\det(p_i(\alpha_j))_{m\times m}\neq 0$$

证： 由讲义3. p3 例子. $\varphi_1,\dots,\varphi_m\in V^*$

由定理9.1 $\dim V^*=m$. 于是只要证

$\varphi_1,\dots,\varphi_m$ 线性无关即可

假设. $\exists\,\beta_1,\dots,\beta_m\in F$，不全为零

使得 $\beta_1\varphi_1+\dots+\beta_m\varphi_m=0$

不妨设 $\beta_m\neq 0$

令 $p=(x-\alpha_1)\cdots(x-\alpha_{m-1})\in V$

$0=\beta_1p(\alpha_1)+\dots+\beta_{m-1}p(\alpha_{m-1})+\beta_mp(\alpha_m)$

$\Rightarrow \beta_m p(\alpha_m) = 0$

$\because \alpha_m \neq \alpha_i, \quad i=1,2,\cdots m-1$

$\therefore p(\alpha_m) \neq 0 \quad \therefore \beta_m = 0$ ⟶⟵ ▨

(ii) 由(i)和定理9.2直接可得

§9.3 自然同构

设 V, W 是两个 F 上的线性空间

$\varphi \in \mathrm{Hom}(V, W)$

称 ~~如果~~ φ 是"自然的"，如果 φ 的定义与基底无关.

例：设 (i) $O_{V,W}$ $\quad V \to W$, $\vec{v} \mapsto \vec{0}_W$

(ii) $\varepsilon_V : V \to V$, $\vec{v} \mapsto \vec{v}$

(iii) 设 V 是 W 的子空间

$\pi: W \to W/V$, $\vec{w} \mapsto \vec{w} + V$

(iv) 设 $V = F[x]$, $\alpha \in F$

$\varphi_\alpha : V \to F$, $p(x) \mapsto p(\alpha)$.

(v) 设 $V = \mathbb{R}[x]$

$' \quad V \to V$, $p \mapsto p'$

如果 φ 是自然的线性同构，则称 V, W 自然同构.

例：设 U, V 是 W 的子空间

且 $W = U \oplus V$

则 V 与 W/U 自然同构

证：~~$\forall \vec{w} \in W, \exists! \vec{u} \in U, \vec{v} \in V$ 使得 $\vec{w} = \vec{u} + \vec{v}$~~

$$\varphi: V \longrightarrow W/U$$
$$\vec{w} \longmapsto \vec{v}+U$$

良定义: $\vec{v} \underset{U}{\sim} \vec{v}' \Rightarrow \vec{v}-\vec{v}' \in U \quad (\vec{v}' \in V)$

$\therefore \quad \vec{v}-\vec{v}' \in V \cap U = \{\vec{0}\}$

$\Rightarrow \vec{v}=\vec{v}'$.

φ是线性的可直接验证:

如果 $\varphi(\vec{v})=\vec{0}+U$, 则 $\vec{v}+U=\vec{0}+U$

$\Rightarrow \vec{v} \in U \Rightarrow \vec{v}=\vec{0}$ (直和等价条件)

于是 φ 是单射

$\forall\ \vec{w} \in W \quad \exists\ \vec{u} \in U,\ \vec{v} \in V$

使得 $\vec{w} = \vec{u}+\vec{v}$

$\varphi(\vec{v}) = \vec{v}+U = (\vec{v}+U)+(\vec{0}+U)$

$= (\vec{v}+U)+(\vec{u}+U) = (\vec{v}+\vec{u})+U = \vec{w}+U$

$\Rightarrow \varphi$ 满 $\quad \square$

设 $\dim V = n < \infty$

定理9.3 V 与 V^{**} 自然同构

证: $\vec{v} \in V$, $\quad \varepsilon_{\vec{v}}: V^* \longrightarrow F$
$$f \longmapsto f(\vec{v})$$

验证: $\varepsilon_{\vec{v}} \in V^{**}$

$\forall\ \alpha, \beta \in F,\ f, g \in V^*$

$\varepsilon_{\vec{v}}(\alpha f+\beta g) = (\alpha f+\beta g)(\vec{v})$

$= \alpha f(\vec{v}) + \beta g(\vec{v}) = \alpha \varepsilon_{\vec{v}}(f) + \beta \varepsilon_{\vec{v}}(g)$

验证完毕

$$\varphi: V \longrightarrow V^{**}$$
$$\vec{v} \longmapsto \varepsilon_{\vec{v}}$$

验证 φ 是线性同构

$\forall \alpha, \beta \in F, \quad \vec{u}, \vec{v} \in V$

$\varphi(\alpha\vec{u}+\beta\vec{v}) = \varepsilon_{\alpha\vec{u}+\beta\vec{v}}$

$\forall f \in V^* \quad \varepsilon_{\alpha\vec{u}+\beta\vec{v}}(f) = f(\alpha\vec{u}+\beta\vec{v})$

$= \alpha f(\vec{u}) + \beta f(\vec{v}) = \alpha \varepsilon_{\vec{u}}(f) + \beta \varepsilon_{\vec{v}}(f)$

$= \alpha[\varphi(\vec{u})](f) + \beta[\varphi(\vec{v})](f)$

由 f 的任意性 $\varphi(\alpha\vec{u}+\beta\vec{v}) = \alpha\varphi(\vec{u}) + \beta\varphi(\vec{v})$

φ 线性.

设 $\varphi(\vec{v}) = 0$ 则 $\forall f \in V^*$, $[\varphi(\vec{v})](f) = \xi_{\vec{v}}(f) = 0$

即 $\forall f \in V^*$ $f(\vec{v}) = 0$

$\Rightarrow \vec{v} = \vec{0}$ (引理 9.1)

$\because$ $\dim V^* = n$ (定理 9.1) $\therefore$ $\dim V^{**} = n$

$\because$ $\dim V^{**} = \dim V < \infty$ $\therefore$ φ 单 $\Rightarrow$ φ 满射

证: $\dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi)) = \dim V$

$\because$ $\dim(\ker(\varphi)) = 0$

$\therefore$ $\dim(\operatorname{im}(\varphi)) = \dim V$

$\because$ $\operatorname{im}(\varphi) \subset V^{**}$ 且 $\dim V^{**} = \dim V$

$\Rightarrow$ $\operatorname{im}(\varphi) = V^{**}$, φ 满

自己证明: 设 $\dim V = \dim W$, $\varphi \in \operatorname{Hom}(V, W)$

则 φ 是单射 $\Leftrightarrow$ φ 是满射.

§9.4. 子空间的对偶

设 $U \subset V$ 是子空间. 定义

$$U^\circ = \{ f \in V^* \mid \forall \vec{u} \in U, f(\vec{u}) = 0 \}$$

~~例: 设 子空间~~

验证 U° 是 V^* 的子空间

设 $f, g \in U^\circ$, $\alpha, \beta \in F$

$\forall \vec{u} \in U$ $(\alpha f + \beta g)(\vec{u})$

$= \alpha f(\vec{u}) + \beta g(\vec{u})$

$= \alpha \cdot 0 + \beta \cdot 0 = 0.$

$\Rightarrow \alpha f + \beta g \in U^\circ$ 验证完

称 U° 是 U 的零化空间

例：设 $V=\mathbb{R}^2$，$U=\langle\begin{pmatrix}2\\3\end{pmatrix}\rangle$

求 U^0 的一组基.

解：在标准基下 设 U^0 中的元素为 $\alpha_1 x_1+\alpha_2 x_2$

$$2\alpha_1+3\alpha_2=0$$

$\Rightarrow$ 得向 $\begin{pmatrix}-3\\2\end{pmatrix}$

$\Rightarrow U^0=\langle -3x_1+2x_2\rangle$.

即 U^0 的维数为1. 且 U^0 的元素是 ~~以~~ 使得 $\langle\begin{pmatrix}2\\3\end{pmatrix}\rangle$ $\begin{pmatrix}2\\3\end{pmatrix}$ 为解的所有齐次线性方程

而 $-3x_1+2x_3=0$ 的解空间.恰恰是 U.

例：设 $V=\mathbb{R}^3$，$U=\langle\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\rangle$.

求 U^0 的一组基

解 设 $\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$ 是 U^0 中的元素

则 $\begin{cases}\alpha_1+\alpha_3=0\\ \alpha_1+2\alpha_2+\alpha_3=0\end{cases}$ $\Rightarrow$ $\begin{matrix}\alpha_2=0.\\ \alpha_1=-\alpha_3.\end{matrix}$

$U^0=\{\beta x_1-\beta x_3 \mid \beta\in\mathbb{R}\}=\langle x_1-x_3\rangle$.

$x_1-x_3=0$ 的解空间恰是 U.

定理 9.4 设 $U\subset V$ 是子空间

则 $\dim U+\dim U^0=\dim V$

证：设 $\vec{e}_1,\dots,\vec{e}_d$ 是 U 的一组基

由基扩充性质.

$$\vec{e}_1,\dots,\vec{e}_d,\vec{e}_{d+1},\dots,\vec{e}_n$$

是 V 的基. 设其对偶基为

$$e_1^*,\dots,e_d^*,e_{d+1}^*,\dots e_n^*$$

则 $e_{d+1}^*,\dots,e_n^*\in U^0$.

于是需要证 $U^0=\langle e_{d+1}^*,\dots,e_n^*\rangle$ 即可

设 $f\in U^0$

$$f=\alpha_1e_1^*+\dots+\alpha_de_d^*+\alpha_{d+1}e_{d+1}^*+\dots+\alpha_ne_n^*$$

$\forall i\in\{1,\dots d\}$ $f(\vec{e}_i)=0$

$\Rightarrow \alpha_1e_1^*(\vec{e}_i)+\dots+\alpha_de_d^*(\vec{e}_i)+\alpha_{d+1}e_{d+1}^*(\vec{e}_i)+\dots+\alpha_ne_n^*(\vec{e}_i)$

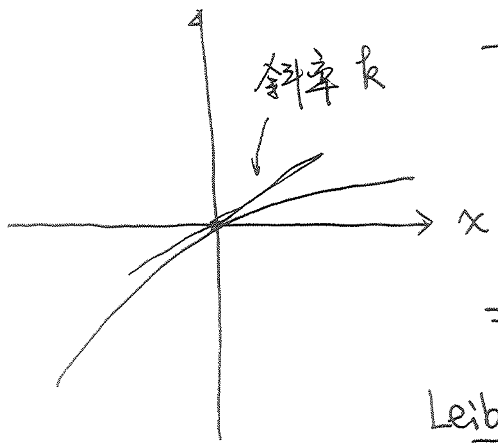
$=0$

$\Rightarrow$ ~~$\alpha_i(\vec{e}_i$~~ $\alpha_i=0$ $\square$

例：设 $y(x): \mathbb{R} \to \mathbb{R}$ 且在原点可导

$y'(0) = k$

$y(x)$ 在原点的切线



是 $\langle \begin{pmatrix} 1 \\ k \end{pmatrix} \rangle =: T$

$T^0 = \langle kx - y \rangle$

$= \{ \alpha(kx-y) \mid \alpha \in \mathbb{R} \}$

$\underset{\text{Leibniz}}{=} \langle k(dx) - (dy) \rangle$

切线方程 $y = kx$ 即 $dy = k\,dx$

注：对于二元函数 $z = z(x,y)$

在原点偏导数：$\frac{\partial z}{\partial x}(\vec{0}) = \lim_{x \to 0} \frac{z(x,0) - z(0,0)}{x}$

$\frac{\partial z}{\partial y}(\vec{0}) = \lim_{y \to 0} \frac{z(0,y) - z(0,0)}{y}$

过原点的切平面 $\langle \begin{pmatrix} 1 \\ 0 \\ \frac{\partial z}{\partial x}(\vec{0}) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial z}{\partial y}(\vec{0}) \end{pmatrix} \rangle = T$

$T^0 = \langle \alpha \left(\frac{\partial z}{\partial x}(\vec{0})(dx) + \frac{\partial z}{\partial x}(\vec{0})(dy) - dz \right) \mid \alpha \in \mathbb{R} \rangle$

切平面方程

$z = \frac{\partial z}{\partial x}(\vec{0})\, x + \frac{\partial y}{\partial y}(\vec{0})\, y$，微分 $dz = \frac{\partial z}{\partial x}(\vec{0})dx + \frac{\partial z}{\partial y}(\vec{0})dy$

例：设 U 是 F^n 中 d 维子空间.

则 U 是 F^n 中 $n-d$ 个超平面的交

~~其中 $n = \dim V$~~.

证：由定理 9.4. U^0 维数为 $n-d$

设 $f_1 = a_{11}x_1 + \cdots + a_{1n}x_n$　　$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n-d,1} & \cdots & a_{n-d,n} \end{pmatrix}$

$\vdots$

$f_{n-d} = a_{n-d,1}x_1 + \cdots + a_{n-d,n}x_n$

是 U^0 的一组基

$\because \forall i \in \{1, \ldots, n-d\}$　$f_i|_U \equiv 0$

于是 $U \subset V_A$

$\because f_1, \ldots, f_{n-d}$ 线性无关 $\therefore \operatorname{rank}(A) = n-d$

$\Rightarrow \dim V_A = d \Rightarrow U = V_A$ ▨

注：设 $U_1, U_2 \subset V$ 子空间

U_1，是方程组 $A_1\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ 的解空间

U_2 是 ⋯ $A_2\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ ⋯

$U_1\cap U_2$ 是方程组 $\begin{pmatrix}A_1\\ A_2\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

的解空间.