# 习题课　矩阵的秩与线性映射初步

## §1　作业六　1, 2题

1. 设 $U, V, W$ 是 $\mathbb{R}^n$ 的子空间.

证明: (i) $(U+V)\cap W \supseteq U\cap W + V\cap W$

(ii) $(U\cap V)+W \subseteq (U+W)\cap(V+W)$

并举例分别取等号、严格包含

证: (i) 设 $\vec{x} \in U\cap W + V\cap W$

则 $\exists\, \vec{y} \in U\cap W$, $\vec{z} \in V\cap W$ 使得

$$\vec{x} = \vec{y} + \vec{z}$$

$\because \vec{y}\in U$, $\vec{z}\in V$ $\therefore \vec{x} \in U+V$

$\because \vec{y}, \vec{z} \in W$ $\therefore \vec{x} \in W$

由此 $\vec{x} \in (U+V)\cap W$　(i) 成立

(ii) 设 $\vec{x} \in (U\cap V)+W$

$\exists\, \vec{y} \in (U\cap V)$, $\vec{z} \in W$ 使得

$$\vec{x} = \vec{y} + \vec{z}$$

~~由此 可知~~ $\because \vec{y} \in U$, $\vec{z} \in W$

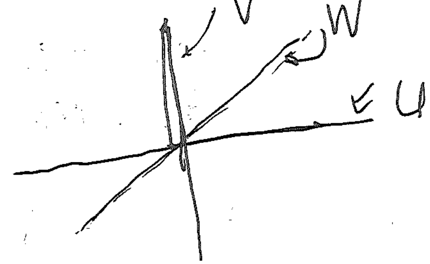
$\therefore \vec{x} \in U+W$

同理 $\vec{x} \in V+W$. 于是 $\vec{x} \in (U+W)\cap(V+W)$

(ii) 成立

取 $U=V=W$ 时　等号成立

设 $U=\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle$, $V=\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle$, $W=\langle\begin{pmatrix}1\\1\end{pmatrix}\rangle$ 时 严格包含成立




2. 设 $S_0=\{\vec{v}_1,\dots,\vec{v}_t\} \subset \mathbb{R}^n$. 且 $V=\langle S_0\rangle$.

令 $d = \dim V$

证明 (i) $d \le t$　(ii) 当 $d=t$ 时, $\vec{v}_1,\dots,\vec{v}_t$ 线性无关

证: 设 $B$ 是 $S_0$ 中一个极大线性无关组

$B$ 是 $V$ 的一组基

由命题1.8. $\therefore d \le t$

(i) $\because d=|B| \le |S_0| = t$

(ii) 设 $d=t$. 则 $|B| = |S_0|$

$\because S_0$ 有限且 $B \subset S_0$ $\therefore B=S_0 \Rightarrow \vec{v}_1,\dots,\vec{v}_t$ 线性无关

# §2. 矩阵的秩

## §2.1 矩阵秩的定义:

设 $A \in \mathbb{R}^{m\times n}$

$$V_r(A) := \langle \vec{A}_1, \cdots, \vec{A}_m \rangle \subset \mathbb{R}^{1\times n}$$

$$V_c(A) := \langle \vec{A}^{(1)}, \cdots, \vec{A}^{(n)} \rangle \subset \mathbb{R}^{m}$$

定理: $\dim V_r(A) = \dim V_c(A)$

定义: $\operatorname{rank}(A) \overset{\text{def}}{=} \dim V_r(A)$

① "容易引理": 初等行(列)变换不改变 $V_r(A)$ $[V_c(A)]$

② "难引理": 初等行变换不改变列空间的维数.

③ $\boxed{\operatorname{rank}(A) \leq \min(m, n)}$

## §2.2 秩的计算和作业六中第3题

求 $\operatorname{rank}(A)$ 的方法

对 $A$ 实施行或列变换.

设 $\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\alpha}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{\alpha}_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}$

求 $\cancel{\boxed{A = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3)}}$ $\dim \underbrace{\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle}_{V_1}$

令 $A = \langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle$. 则 $V_1 = V_c(A)$

$\Rightarrow \dim V_1 = \operatorname{rank}(A)$

方法一: 行变换

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_4 - \frac{1}{3} r_2]{r_3 - \frac{2}{3} r_2}$$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_r} \quad \Rightarrow \operatorname{rank}(A_r) = 2 \Rightarrow \operatorname{rank}(A) = 2 \Rightarrow \dim V_1 = 2.$$

方法2 列变换

$$A \xrightarrow{c_2+c_2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{c_3-c_2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow \operatorname{rank}(A)=2 \Rightarrow \dim V_1 = 2$

由"容易"引理： $V_1$ 有基 $\underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{\lambda}_1}, \underbrace{\begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}}_{\vec{\lambda}_2}$

类似 $\vec{\beta}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{\beta}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 7 \end{pmatrix}, V_2 = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$

$$B = \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 3 \\ 1 & 7 \end{pmatrix} \Rightarrow \operatorname{rank}(B)=2 \Rightarrow \dim V_2 = 2$$

因 $V_2$ 有基： $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 7 \end{pmatrix}$

求 $V_1+V_2$ 的基

设 $C = (\vec{\lambda}_1, \vec{\lambda}_2, \vec{\beta}_1, \vec{\beta}_2)$

则 $V_c(C) = V_1+V_2$

$$C = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & -1 & -1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 7 \end{pmatrix} \xrightarrow[c_4-c_1]{c_3-2c_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & -5 & -3 \\ 1 & 2 & -2 & 2 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[c_4+c_2]{c_3+\frac{5}{3}c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & \frac{4}{3} & 4 \\ 0 & 1 & \frac{8}{3} & 8 \end{pmatrix} \xrightarrow{c_4-3c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{8}{3} & 0 \end{pmatrix}$$

$V_1+V_2$ 基 $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{4}{3} \\ \frac{8}{3} \end{pmatrix}$ 可换为 $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow \dim(V_1+V_2) = 3$

求 $V_1 \cap V_2$ 的基

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1+V_2)$$
$$= 2 + 2 - 3 = 1.$$

求以 C 为矩阵的齐次线性方程组的解

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & -1 & -1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 7 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & -5 & -3 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[\text{互换}]{r_2,\ r_4} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 2 & -2 & 2 \\ 0 & 3 & -5 & -3 \end{pmatrix} \xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -4 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_3 + 7x_4 = 0 \\ x_3 + 3x_4 = 0 \end{cases}$$

令 $x_4 = 1$

$$\begin{cases} x_1 = 5 \\ x_2 = -4 \\ x_3 = -3 \\ x_4 = 1 \end{cases}$$

$\Rightarrow 5\vec{\alpha}_1 - 4\vec{\alpha}_2 - 3\vec{\beta}_1 + \vec{\beta}_2 = 0$

$\Rightarrow 3\vec{\beta}_1 - \vec{\beta}_2 \in V_1 \cap V_2$

$\because \dim(V_1 \cap V_2) = 1$

$\therefore V_1 \cap V_2$ 的基

$$3\vec{\beta}_1 - \vec{\beta}_2 = 3\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \\ 7 \end{pmatrix} = \cancel{\begin{pmatrix} 7 \\ -4 \\ 3 \\ 10 \end{pmatrix}} \begin{pmatrix} 5 \\ -2 \\ -3 \\ -4 \end{pmatrix}$$

第3题 2  $A = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3)$, $B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$

$\dim(V_1) = \dim(V_2) = 3$

$V_1$ 基底 $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$，$V_2$ 基底 $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$

$C = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3)$

$\operatorname{rank}(C) = 4$.   $C = \mathbb{R}^4 = \langle \vec{e}^{(1)}, \vec{e}^{(2)}, \vec{e}^{(3)}, \vec{e}^{(4)} \rangle$

$\dim(V_1 \cap V_2) = 3 + 3 - 4 = 2$.

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & -2 & 3 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 2 & 3 & -2 & -6 & 4 & -5 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -3 & 1 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & -8 & 2 & -9 \end{pmatrix}$$

$$\xrightarrow[\text{互换}]{r_2\ r_3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & -4 & -8 & 2 & -9 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & -3 & -8 & 4 & -8 \end{pmatrix}$$

$$\xrightarrow{r_4+3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0 \\ -x_2 + x_3 + 2x_5 + x_6 = 0 \\ x_4 + x_5 + x_6 = 0 \end{cases}$$

$$\vec{V_1} = \begin{pmatrix} x \\ x \\ x \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{V_2} = \begin{pmatrix} x \\ x \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$-\vec{\beta_1} + \vec{\beta_2}$, $-\vec{\beta_1} + \vec{\beta_3}$ 是 $V_1 \cap V_2$ 的基

例：设 $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$

令 $C = (\vec{A}^{(1)}, \cdots \vec{A}^{(n)}, \vec{B}^{(1)}, \cdots, \vec{B}^{(k)})$

证明 $\mathrm{rank}(C) \leqslant \mathrm{rank}(A) + \mathrm{rank}(B)$

证：$V_c(C) = V_c(A) + V_c(B)$

$$\begin{aligned} \mathrm{rank}(C) &= \dim V_c(C) \\ &= \dim(V_c(A) + V_c(B)) \\ &= \dim V_c(A) + \dim V_c(B) - \dim(V_c(A) \cap V_c(B)) \\ &\geqslant \dim V_c(A) + \dim V_c(B) \\ &= \mathrm{rank}(A) + \mathrm{rank}(B). \end{aligned}$$

例：~~斜斯特罗金 p61~~. 练习4

设 $A = \begin{pmatrix} \alpha_1 & \alpha_2, & \cdots & \alpha_n \\ \beta_1, & \beta_2, & \cdots & \beta_n \end{pmatrix}$

$B = \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & \alpha_n \\ \beta_1, & \beta_2, & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}$

求 $A$ 和 $B$ 等秩的条件.

解：考虑方程组

$$\begin{cases} \alpha_1 x + \beta_1 y = \gamma_1 \\ \vdots \\ \alpha_n x + \beta_n y = \gamma_n \end{cases} \quad (*)$$

系数矩阵 $\widetilde{A} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \vdots & \vdots \\ \alpha_n & \beta_n \end{pmatrix}$　增广矩阵 $\widetilde{B} = \left(\begin{array}{cc|c} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots \\ \alpha_n & \beta_n & \gamma_n \end{array}\right)$

(*) 有解 $\Longleftrightarrow$ $\operatorname{rank}(A) = \operatorname{rank}(B)$

注意到 $\operatorname{rank}(A) = \operatorname{rank}(\widetilde{A})$，$\operatorname{rank}(B) = \operatorname{rank}(\widetilde{B})$

所以 $\operatorname{rank}(A) = \operatorname{rank}(B)$

$\Longleftrightarrow$ (*) 有解

$\Longleftrightarrow$ 对应的 n 条直线有公共交点。

## §3 线性映射.

1. 线性映射的验证.
2. 线性映射保持线性组合和线性相关性（命题3.1）
3. ~~线性空间~~ 子空间在线性映射下的像和逆像仍是子空间

例：设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性映射，$V \subset \mathbb{R}^m$ 是子空间，$V \subset \operatorname{im}(\varphi)$

~~满足~~ ~~证明~~ $V \subset \mathbb{R}^m$

证明：$\dim(\varphi^{-1}(V)) \geq \dim V$

证：设 $\vec{v}_1 \ldots \vec{v}_d$ 是 $V$ 的一组基

因为 $V \subset \operatorname{im}(\varphi)$，所以

$\exists\ \vec{u}_1, \ldots \vec{u}_d \in \varphi^{-1}(V)$ 使得

$\varphi(\vec{u}_1) = \vec{v}_1, \ldots, \varphi(\vec{u}_d) = \vec{v}_d$

$\because$ $\vec{v}_1, \ldots \vec{v}_d$ 线性无关

$\therefore$ $\vec{u}_1, \ldots \vec{u}_d$ 线性无关

由此可知 $\dim \varphi^{-1}(V) \geq d$ [基扩充定理]!

例：$\varphi: \mathbb{R}^6 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ x_3 \\ x_5 \end{pmatrix}$$

验证：$\varphi$ 是线性映射，求 $\ker(\varphi)$ 和 $\operatorname{im}(\varphi)$ 的基

解：设 $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix}$，$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_6 \end{pmatrix}$，$\alpha, \beta \in \mathbb{R}$

$$\varphi(\alpha\vec{x} + \beta\vec{y}) = \varphi\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_6 + \beta y_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha x_3 + \beta y_3 \\ \alpha x_5 + \beta y_5 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 0 \\ x_3 \\ x_5 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ y_3 \\ y_5 \end{pmatrix}$$

$$= \alpha\varphi(\vec{x}) + \beta\varphi(\vec{y})$$

$$A_\varphi = (\varphi(\vec{e}^{(1)}), \dots, \varphi(\vec{e}^{(n)}))$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\operatorname{rank}(A_\varphi) = 2$

$\Rightarrow \dim(\operatorname{im}(\varphi)) = 2$ 和 $\dim\ker(\varphi) = 4$

$\ker(\varphi)$ ~~= L(e~~ 基 $\vec{e}^{(1)}, \vec{e}^{(2)}, \vec{e}^{(4)}, \vec{e}^{(6)}$

$\operatorname{im}(\varphi) =$ ⓔ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\vec{\varepsilon}^{(2)}} \quad \vec{\varepsilon}^{(3)}$

例：设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$，

$\forall\, \vec{x} \in \mathbb{R}^n \quad \exists\, C_{\vec{x}} \in \mathbb{R}$ 使得

$$\varphi(\vec{x}) = C_{\vec{x}}\,\vec{x}$$

证明 存在 $C \in \mathbb{R}$，与 $\vec{x}$ 无关

满足 $\varphi(\vec{x}) = C\vec{x}$.

证：~~回顾 对偶定理（映射参数）~~

~~证明中的一个结论~~

证：设 $\varphi(\vec{e}^{(j)}) = c_j \vec{e}^{(j)}$, $j=1,2,\ldots n$.

其中 $c_j \in \mathbb{R}$

令 $i, j \in \{1, \ldots, n\}$ $i \neq j$

$$\varphi(\vec{e}^{(i)} + \vec{e}^{(j)}) = c_{ij}(\vec{e}^{(i)} + \vec{e}^{(j)})$$

$$\varphi(\vec{e}^{(i)} + \vec{e}^{(j)}) = \varphi(\vec{e}^{(i)}) + \varphi(\vec{e}^{(j)})$$

$$= c_i \vec{e}^{(i)} + c_j \vec{e}^{(j)}$$

$$\Rightarrow (c_{ij} - c_i)\vec{e}^{(i)} + (c_{ij} - c_j)\vec{e}^{(j)} = \vec{0}$$

$$\Rightarrow c_i = c_{ij}, \quad c_j = c_{ij} \quad [\vec{e}^{(i)}, \vec{e}^{(j)} \text{ 线性无关}]$$

$$\Rightarrow c_i = c_j$$

即，$c_1 = c_2 = \cdots = c_n$ 记为 $c$

则 $\varphi(\vec{e}^{(j)}) = c\vec{e}^{(j)}$. $j=1,2,\ldots n$

~~故~~ $\vec{x} \in \mathbb{R}^n$, $\exists x_1 \ldots x_n \in \mathbb{R}$

$$\vec{x} = x_1 \vec{e}^{(1)} + \cdots + x_n \vec{e}^{(n)}$$

$$\varphi(\vec{x}) = x_1 \varphi(\vec{e}^{(1)}) + \cdots + x_n \varphi(\vec{e}^{(n)})$$

$$= x_1 c \vec{e}^{(1)} + \cdots + x_n c \vec{e}^{(n)}$$

$$= c(x_1 \vec{e}^{(1)} + \cdots + x_n \vec{e}^{(n)})$$

$$= c\vec{x}.$$

$\boxed{/\!/}$

## 思考题

证明：$\mathbb{R}^n$ 不可能是有限个真子空间的并

证：假设 $V_1,\dots,V_s$ 是 $\mathbb{R}^n$ 的真子空间

使得

$$V_1\cup\cdots\cup V_{s-1}\cup V_s=\mathbb{R}^n$$

且 $s$ 极小. 则 $s\neq 1$. 且

$(V_1\cup\cdots\cup V_{s-1})$ 和 $V_s$ 彼此

不互相包含. ~~者~~ 否则 $s$ 不是最

小) 于是

$$\exists\, \vec{v}\in (V_1\cup\cdots\cup V_{s-1})\setminus V_s$$

$$\vec{w}\in V_s\setminus(V_1\cup\cdots\cup V_{s-1})$$

$\forall \alpha\in\mathbb{R}$, 令 $\vec{u_\alpha}=\vec{v}+\alpha\vec{w}$

断言 (i) $\vec{u_\alpha}\notin V_s$

(ii) 若 $\vec{u_\alpha},\vec{u_\beta}\in V_j$, 其中 $j$ 是 $\{1,2,\dots,s-1\}$ 中某个数

则 $\vec{u_\alpha}=\vec{u_\beta}$

断言的证明

(i) 假设 $\vec{u_\alpha}\in V_s$, 则

$\vec{v}=\vec{u_\alpha}-\alpha\vec{w}\in V_s$. 与 $\vec{v}$ 的取法矛盾

(ii) $\cancel{\boxed{\beta u_\alpha-\alpha u_\beta\in V_j}}$

$\because \vec{u_\alpha},\vec{u_\beta}\in V_j$ $\therefore \beta\vec{u_\alpha}-\alpha\vec{u_\beta}\in V_j$

于是 $(\beta-\alpha)\vec{w}\in V_j$

$\because \vec{w}\notin V_j$ $\therefore \beta-\alpha=0$ 断言成立

由断言和 $\mathbb{R}$ 有无穷多个元素可知

无穷多个 $\vec{u_\alpha}$ 要在 $V_1\cup V_2\cup\cdots\cup V_{s-1}$

中且每个 $V_j$ 至多包含一个.

这是不可能的. $\blacksquare$