回忆： 设 $f: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \longrightarrow \mathbb{R}$

n重线性、斜对称，则

$$f(\vec{x}_1, \dots, \vec{x}_n) = \lambda \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1} \cdots x_{\sigma(n),n}.$$

其中 $\vec{x}_k = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{nk} \end{pmatrix}$, $k=1,2,\dots n$

$$\lambda = f(\vec{e}^{(1)}, \dots, \vec{e}^{(n)}).$$

定义：det：是 $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n}$ 到 $\mathbb{R}^n$ 的 n重线性，斜对称函数，满足

$$\det(\vec{e}^{(1)}, \dots, \vec{e}^{(n)}) = 1.$$

即

$$\det(\vec{x}_1, \dots \vec{x}_n) = \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1),1} \cdots x_{\sigma(n),n}$$

其中 $\vec{x}_k$ 同上

例：$n=2$ 时 $S_2 = \{\underset{\text{恒同}}{\underset{\downarrow}{e}}, \underset{\sigma}{\underset{\downarrow}{(12)}}\}$ ①

$$\det(\vec{x}_1, \vec{x}_2) = \varepsilon_e x_{e(1),1} x_{e(2),2} + \varepsilon_\sigma x_{\sigma(1),1} x_{\sigma(2),2}$$

$$= x_{11}x_{12} - x_{21}x_{12} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

定义：设 $A = (a_{ij})_{n\times n} \in M_n(\mathbb{R})$.

A 的行列式定义为

$$\sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

即 $\det(\vec{A}^{(1)}, \dots \vec{A}^{(n)})$. 记为 $|A|$ 或 $\det(A)$.

由此可知：$\det(A)$ 是关于 A 的列的多重线性函数

例：设 $A=(a_{ij})_{3\times3}$. 求 $|A|$ 的展开式

$$S_3=\{\, e,\ \underset{\sigma_1}{\overset{\shortparallel}{(12)}},\ \underset{\sigma_2}{\overset{\shortparallel}{(13)}},\ \underset{\sigma_3}{\overset{\shortparallel}{(23)}},\ \underset{\sigma_4}{\overset{\shortparallel}{(12)(13)}},\ \underset{\sigma_5}{\overset{\shortparallel}{(12)(23)}} \,\}$$

$$(12)(13)=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\quad (12)(23)=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

$$|A|=\varepsilon_e\, a_{e(1),1}\, a_{e(2),2}\, a_{e(3),3}$$

$$+\ \varepsilon_{\sigma_1} a_{\sigma_1(1),1}\, a_{\sigma_1(2),2}\, a_{\sigma_1(3),3}$$

$$+\ \cdots\ +\ \varepsilon_{\sigma_5} a_{\sigma_5(1),1}\, a_{\sigma_5(2),3}\, a_{\sigma_5(3),3}$$

$$= a_{11}a_{22}a_{33}\ [illegible]\ a_{21}a_{12}a_{33}-a_{31}a_{22}a_{13}$$

$$-\ a_{11}a_{32}a_{23}+a_{31}a_{21}a_{23}+a_{21}a_{32}a_{13}$$

$$\begin{array}{ccc|ccc}
 & & - & - & - & \\
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & + \\
 & & + & + & &
\end{array}$$

注：行列式的组合描述

在每行每列中取且仅取一个元素构成积，配上符号，把所有可能的积加起来.

注：若 $a_{ij}\in\mathbb{Z}$, $i,j=1,\cdots n$, 则 $|A|\in\mathbb{Z}$

§1 行列式的基本性质.

D1. 正规性：$\det(E_n)=1$.

D2 多重线性：$\forall\ \vec{x},\vec{y}\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{R}$

$$\det(\vec{A}^{(1)},\cdots\vec{A}^{(j-1)},\ \alpha\vec{x}+\beta\vec{y},\ \vec{A}^{(j+1)},\cdots,\vec{A}^{(n)})$$

$$=\alpha\det(\vec{A}^{(1)},\cdots,\vec{A}^{(j-1)},\vec{x},\vec{A}^{(j+1)},\cdots\vec{A}^{(n)})$$

$$+\ \beta\det(\vec{A}^{(1)},\cdots,\vec{A}^{(j-1)},\vec{y},\vec{A}^{(j+1)},\cdots,\vec{A}^{(n)}).$$

特别地：$\textcircled{D_{21}}$ $\det(\vec{A}^{(1)},\cdots\vec{A}^{(j-1)},\ \alpha\vec{A}^{(j)},\vec{A}^{(j+1)},\cdots\vec{A}^{(n)})$

$$=\alpha\det(A).$$

$\textcircled{D_{22}}$ $\det(\alpha A)=\alpha^n\det(A)$

$\textcircled{D_{23}}$ $\det(\vec{A}^{(1)},\cdots,\vec{A}^{(j-1)},\vec{0},\vec{A}^{(j+1)},\cdots,\vec{A}^{(n)})=\vec{0}$

（在 D_{21} 中设 $\alpha=0$）

D_3 斜对称性: $i \neq j$

$$\det(\vec{A}^{(1)}, \cdots, \underset{i}{\vec{A}^{(i)}}, \cdots \underset{j}{\vec{A}^{(j)}} \cdots \vec{A}^{(n)})$$

$$= -\det(\vec{A}^{(1)}, \cdots, \underset{i}{\vec{A}^{(j)}} \cdots, \underset{j}{\vec{A}^{(i)}}, \cdots \vec{A}^{(n)})$$

D_4 列等性. 如果 A 中两列相同

则 $\det(A) = 0$

<u>证法1</u> 由引理 0.1 直接可得

但要用 $2 \neq 0$.

<u>证法2</u> 由斜对称性. 不妨设 $\vec{A}^{(1)} = \vec{A}^{(2)}$

设 $\tau = (12)$, $\sigma \in S_n$. 考虑

$$(*) \quad |A| = \sum_{\pi \in S_n} \varepsilon_\pi a_{\pi(1),1} \cdots a_{\pi(n),n}$$

中两项. $P_\sigma = \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$

$$P_{\sigma\tau} = \varepsilon_{\sigma\tau} a_{\sigma\tau(1),1} \cdots a_{\sigma\tau(n),n}$$

$$= -\varepsilon_\sigma a_{\sigma(2),1} a_{\sigma(1),2}, \cdots a_{\sigma(n),n}$$

$$(\because \vec{A}^{(1)} = \vec{A}^{(2)}) \quad = -\varepsilon_\sigma \sigma_{\sigma(1),1}, \sigma_{\sigma(2),2} \cdots \sigma_{\sigma(n),n}$$

$$\Rightarrow P_\sigma + P_{\sigma\tau} = 0$$

$$|A| = P_\sigma + P_{\sigma\tau} + \sum_{\substack{\pi \in S_n \\ \pi \neq \sigma \\ \pi \neq \sigma\tau}} \varepsilon_\pi a_{\pi(1),1} \cdots a_{\pi(n),n}$$

注意到 $\forall \lambda, \mu \in S_n \quad \lambda\tau = \mu\tau \Leftrightarrow \lambda = \mu$ ($\because \tau$ 可逆)

于是当 $\pi \neq \sigma$, $\pi \neq \sigma\tau$ 时

$\pi\tau \neq \sigma$, $\pi\tau \neq \sigma\tau$

由此可知 (*) 右侧可以写成 $\frac{n!}{2}$ 对

彼此反号的项之和 $\Rightarrow |A| = 0$.

引理 1.1 设 $A \in M_n(\mathbb{R})$. 如果 $\operatorname{rank}(A) < n$

则 $\det(A) = 0$.

证: 不妨设 $\vec{A}^{(1)}$ 是 $\vec{A}^{(2)}, \cdots, \vec{A}^{(n)}$ 的线性组合; 即 $\exists \alpha_2, \cdots \alpha_n \in \mathbb{R}$ 使得

$$\vec{A}^{(1)} = \alpha_2 \vec{A}^{(2)} + \cdots + \alpha_n \vec{A}^{(n)}$$

则 $\det(A) = \overset{\det}{\vee}(\alpha_2 \vec{A}^{(2)} + \cdots + \alpha_n \vec{A}^{(n)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)})$

$= \alpha_2 \det(\vec{A}^{(2)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)}) + \cdots + \alpha_n \det(\vec{A}^{(n)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)})$

(多重线性)

D_3 斜对称性: $i \neq j$

$$\det(\vec{A}^{(1)}, \cdots, \underset{i}{\vec{A}^{(i)}}, \cdots \underset{j}{\vec{A}^{(j)}} \cdots \vec{A}^{(n)})$$

$$= -\det(\vec{A}^{(1)}, \cdots, \underset{i}{\vec{A}^{(j)}} \cdots \underset{j}{\vec{A}^{(i)}}, \cdots \vec{A}^{(n)})$$

D_4 列等性. 如果 A 中两列相同

则 $\det(A) = 0$

证法1 由引理0.1直接可得

但要用 $2 \neq 0$.

证法2 由斜对称性. 不妨设 $\vec{A}^{(1)} = \vec{A}^{(2)}$

设 $\tau = (12)$, $\sigma \in S_n$. 考虑

(*) $|A| = \sum\limits_{\pi \in S_n} \varepsilon_\pi a_{\pi(1),1} \cdots a_{\pi(n),n}$

中两项. $P_\sigma = \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$

$$P_{\sigma\tau} = \varepsilon_{\sigma\tau} a_{\sigma\tau(1),1} \cdots a_{\sigma\tau(n),n}$$

$$= -\varepsilon_\sigma a_{\sigma(2),1} a_{\sigma(1),2}, \cdots a_{\sigma(n),n}$$

$(\because \vec{A}^{(1)} = \vec{A}^{(2)})$ $= -\varepsilon_\sigma \sigma_{\sigma(1),1}, \sigma_{\sigma(2),2} \cdots a_{\sigma(n),n}$

$\Rightarrow P_\sigma + P_{\sigma\tau} = 0$

$$|A| = P_\sigma + P_{\sigma\tau} + \sum_{\substack{\pi \in S_n \\ \pi \neq \sigma \\ \pi \neq \sigma\tau}} \varepsilon_\pi a_{\pi(1),1} \cdots a_{\pi(n),n}$$

注意到 $\forall \lambda, \mu \in S_n$ $\lambda\tau = \mu\tau \Leftrightarrow \lambda = \mu$
$(\because \tau$ 可逆$)$

于是当 $\pi \neq \sigma$, $\pi \neq \sigma\tau$ 时
$\pi\tau \neq \sigma$, $\pi\tau \neq \sigma\tau$

由此可知 (*) 右侧可以写成 $\frac{n!}{2}$ 对
彼此反号的项之和 $\Rightarrow |A| = 0$.

引理1.1 设 $A \in M_n(\mathbb{R})$. 如果 $\mathrm{rank}(A) < n$

则 $\det(A) = 0$.

证: 不妨设 $\vec{A}^{(1)}$ 是 $\vec{A}^{(2)}, \cdots, \vec{A}^{(n)}$ 的线性组合. 即 $\exists \alpha_2, \cdots \alpha_n \in \mathbb{R}$ 使得

$$\vec{A}^{(1)} = \alpha_2 \vec{A}^{(2)} + \cdots + \alpha_n \vec{A}^{(n)}$$

则 $\det(A) \overset{\det}{=} (\alpha_2 \vec{A}^{(2)} + \cdots + \alpha_n \vec{A}^{(n)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)})$

$= \alpha_2 \det(\vec{A}^{(2)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)}) + \cdots + \alpha_n \det(\vec{A}^{(n)}, \vec{A}^{(2)}, \cdots \vec{A}^{(n)})$

(多重线性)

$= 0$ （列等）.

D_5 列变换　设 $\vec v \in \langle \vec A^{(1)}, \dots \vec A^{(j+1)}, \vec A^{(j+1)}, \dots \vec A^{(n)} \rangle$

则 $\det(A) = \det(\vec A^{(1)}, \dots \vec A^{(j-1)}, \vec A^{(j)}+\vec v, \vec A^{(j+1)}, \dots \vec A^{(n)})$

证：设 $\vec v = \alpha_1 \vec A^{(1)} + \cdots + \alpha_{j-1} \vec A^{(j-1)} + \alpha_{j+1} \vec A^{(j+1)} + \cdots + \alpha_n \vec A^{(n)}$

其中 $\alpha_1, \dots \alpha_{j-1}, \alpha_{j+1}, \dots \alpha_n \in \mathbb{R}$

$$\det(\vec A^{(1)}, \dots \vec A^{(j-1)}, \vec A^{(j)}+\vec v, \vec A^{(j+1)}, \dots \vec A^{(n)})$$
$$= \alpha_1 \det(\underline{\vec A^{(1)}}, \dots \vec A^{(j-1)}, \underline{\vec A^{(1)}}, \vec A^{(j+1)}, \dots \vec A^{(n)}) + \cdots$$
$$+ \alpha_{j-1} \det(\vec A^{(1)}, \underline{\vec A^{(j-1)}}, \underline{\vec A^{(j-1)}}, \vec A^{(j+1)}, \dots \vec A^{(n)})$$
$$+ \det(\vec A^{(1)}, \dots \vec A^{(j-1)}, \vec A^{(j)}, \vec A^{(j+1)}, \dots \vec A^{(n)})$$
$$+ \alpha_{j+1} \det(\vec A^{(1)} \dots \vec A^{(j-1)}, \underline{\vec A^{(j+1)}}, \underline{\vec A^{(j+1)}} \dots \vec A^{(n)}) + \cdots$$
$$+ \alpha_n \det(\vec A^{(1)}, \dots, \vec A^{(j-1)}, \underline{\vec A^{(n)}}, \vec A^{(j+1)}, \dots \underline{\vec A^{(n)}})$$
$$= \det(\vec A^{(1)}, \dots \vec A^{(j-1)}, \vec A^{(j)}, \vec A^{(j+1)}, \dots \vec A^{(n)})$$
$$= \det(A).$$

~~命~~题1.1　设 $A \in M_n(\mathbb{R})$，则 $\det(A) = \det(A^t)$ ④

证：设 $A = (a_{ij})_{n\times n}$　则 $A^t = (a'_{ij})_{n\times n}$

则 $a'_{ij} = a_{ji}$,　$i, j \in \{1, 2, \dots, n\}$.

$$\det(A^t) = \sum_{\sigma \in S_n} \varepsilon_\sigma a'_{\sigma(1),1} \cdots a'_{\sigma(n),n}$$
$$= \sum_{\sigma \in S_n} \varepsilon_\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$
$$= \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma^{-1}(\sigma(1)),\sigma(1)} \cdots a_{\sigma^{-1}(\sigma(n)),\sigma(n)}$$
$$= \sum_{\sigma^{-1} \in S_n} \varepsilon_{\sigma^{-1}} a_{\sigma^{-1}(1),1} \cdots a_{\sigma^{-1}(n),n}$$
$$= \det(A).$$ □

注：由此可以得出，行列式关于列的性质，也适用于行

例： 设 $A \in M_n(\mathbb{R})$ 中有两行相等

则 $|A| = 0$

证：因为 A 中有两行相等，

所以 A^t 中有两列相等.

由列多性，$|A^t| = 0 \Rightarrow |A| = 0$ [命题列]

注： 对 A 作一次一类初等行(列)

变换得 B.　$\det(A) = -\det(B)$

一次二类得 B　$\det(A) = \det(B)$

一次三类　$\det(B) = \lambda \det(A)$

D6 设 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$

则 $|A| = a_{11} a_{22} \cdots a_{nn}$.

证：设 $\varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$

是 $|A|$ 的展开式中的一项

若 $\sigma \neq e$

$\exists\, k \in \{1, \cdots, n\}$ 使得 $\sigma(k) \neq k$

若 $\sigma(k) > k$ 则该项为零

若 $\sigma(k) < k$. 则 $\exists\, l \in \{1, \cdots, n\} \setminus k$ 使得

$\sigma(l) > l$. 于是 该项也为零.

由此可知：$|A| = a_{11} \cdots a_{nn}$. □

展开行列式的方法：通过第二、三类初等变换把 A 化为上(下)三角形

例：设 $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$

$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -8.$

例：设 $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$|A| = -\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$

定理1.1 设 $A\in M_n(\mathbb{R})$. 则

$$\det(A)=0 \iff \operatorname{rank}(A)<n$$

证: "$\Leftarrow$" 引理1

"$\Rightarrow$" 通过第一二类初等行(列)变换

$$A \longrightarrow B = \begin{pmatrix} \square & & * \\ & \ddots & \\ 0 & & \square \\ & & \end{pmatrix} \}r$$

其中 $\operatorname{rank}(A)=r$

∵ $\det(A)=0$ ∴ $\det(B)=0$

$\Rightarrow r<n$ (性质D_6) $\Rightarrow \operatorname{rank}(A)<n$ ☒

例: 设 $A=\begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ & & \ddots & \\ b & b & \cdots & a \end{pmatrix}$

问 a, b 取何值时, A 可逆

解 把 A 中第2, 3…n列加至第1列 得

$$\det(A)=\begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & \cdots & b \\ & & \ddots & \\ a+(n-1)b & b & \cdots & a \end{vmatrix} = [a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & & b \\ \vdots & \vdots & & \vdots \\ 1 & b & & a \end{vmatrix}$$

$$=(a+(n-1)b)\begin{vmatrix} 1 & b-0 & \cdots & b \\ 0 & a-b & 0 \cdots & 0 \\ & & \ddots & O \\ & O & & a-b \end{vmatrix}$$

$$=(a+(n-1)b)(a-b)^{n-1}.$$

当 $a\neq b$ 且 $a+(n-1)b\neq 0$ 时, A可逆

§2 行列式的进一步性质

§2.1 按行或列展开

设 $A\in M_n(\mathbb{R})$, $i, j\in\{1,\cdots,n\}$

记 M_{ij} 是 A 中去掉第i行, 第j列 得到的 $(n-1)$阶方阵的行列式, 称之为 A 关于 (i,j) 的余子式. 而 $(-1)^{i+j}$ A_{ij} 称为 A 关于 (i,j) 的代数余子式.

例：设 $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$ $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

~~$= - | 1$~~ $= - \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$.

定理2.1 设 $A \in M_n(\mathbb{R})$. 则 $\forall i, j \in \{1, \dots, n\}$

$$\det(A) = \sum_{k=1}^{n} a_{ik} A_{ik} = \sum_{k=1}^{n} a_{kj} A_{\cancel{jk}\,kj}$$

按第i行展开　　按第j列展开

证：断言1 设

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n-1} & 0 \\ & \cdots & & \\ a_{n-1,1} & \cdots & a_{n-1,n-1}, & 0 \\ a_{n,1} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

则 $\det(A) = a_{nn} A_{nn}$

断言1的证明

$$|A| = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n-1),n-1} a_{\sigma(n),n}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(n)=n}} \varepsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n-1),n-1} a_{n,n}$$

$$= a_{n,n} \sum_{\tau \in S_{n-1}} \varepsilon_\tau a_{\tau(1),1} \cdots a_{\tau(n-1),n-1}$$

$= a_{n,n} A_{n,n}$. ✓

断言2：设 $A = \begin{pmatrix} a_{11} & \cdots & a_{1j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ & & \cdots & & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1}, & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1}, & \cdots & a_{i+1,n} \\ & & \cdots & & & & \\ a_{n,1} & \cdots & a_{nj-1} & 0 & a_{n,j+1}, & \cdots & a_{n,n} \end{pmatrix}$

则 $|A| = a_{ij} A_{ij}$

断言2的证明：通过第一类初等行列变换

$$A \longrightarrow B = \begin{pmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} & 0 \\ & & \cdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} & 0 \\ a_{i1} & \cdots & a_{ij-1} & a_{ij+1} & \cdots & a_{i,n} & a_{ij} \end{pmatrix}$$

$$\det(B) \overset{a_{ij}}{\overset{\vee}{=}} (-1)^{n-i+n-j} M_{ij} \overset{a_{ij}}{\overset{\vee}{=}} (-1)^{i+j} M_{ij} = a_{ij} A_{ij}$$

✓

回忆

$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & 0 & a_{1,j+1}, & \cdots, & a_{1n} \\ & - & - & - & - & - & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1}, & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & \boxed{a_{ij}} & a_{i,j+1}, & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ & & - & - & - & & \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1}, & \cdots & a_{n,n} \end{vmatrix}$$

$$= (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots a_{1n} & 0 \\ & & - & - & \cdot & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots a_{i-1,n} & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots a_{i+1,n} & 0 \\ & & - & - & - & \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots a_{n,n} & 0 \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots a_{i,n} & \boxed{a_{ij}} \end{vmatrix}$$

$$= (-1)^{i+j} a_{ij} M_{ij} = a_{ij} A_{ij}$$

例：

$$\begin{vmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{vmatrix} = (-1)^2 \begin{vmatrix} a_{11} & a_{13} & a_{14} & 0 \\ a_{21} & a_{23} & a_{24} & 0 \\ a_{31} & a_{33} & a_{34} & a_{32} \\ a_{41} & a_{43} & a_{44} & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} a_{11} & a_{13} & a_{14} & 0 \\ a_{21} & a_{23} & a_{24} & 0 \\ a_{41} & a_{42} & a_{44} & 0 \\ a_{31} & a_{33} & a_{34} & a_{32} \end{vmatrix} = -a_{32} M_{32}$$

$$= + a_{32} A_{32}$$

考虑一般情形

$$\det(A)=\det\left(\vec{A}^{(1)},\cdots,\vec{A}^{(j-1)},\begin{pmatrix}a_{1j}\\0\\\vdots\\0\end{pmatrix},\vec{A}^{(j+1)},\cdots\vec{A}^{(n)}\right)$$

$$+\cdots+\det\left(\vec{A}^{(1)},\cdots\vec{A}^{(j-1)}\begin{pmatrix}0\\\vdots\\0\\a_{nj}\end{pmatrix}\vec{A}^{(j+1)},\cdots\vec{A}^{(n)}\right)$$

$$=a_{1j}A_{1j}+\cdots+a_{nj}A_{nj}=\sum_{k=1}^{n}a_{kj}\cdot A_{kj}$$

类似可证 $\det(A)=\sum_{k=1}^{n}a_{ik}A_{ik}$. ▨

例：按第一行展开

$$\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}$$

$$+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}.$$

例： $D=\begin{vmatrix}5&3&-1&2&0\\1&7&2&5&2\\0&-2&3&1&0\\0&-4&-1&4&0\\0&2&3&5&0\end{vmatrix}$

$$=-2\begin{vmatrix}5&3&-1&2\\0&-2&3&1\\0&-4&-1&4\\0&2&3&5\end{vmatrix}=-10\begin{vmatrix}-2&3&1\\-4&-1&4\\2&3&5\end{vmatrix}$$

$$=+20\begin{vmatrix}1&3&1\\2&-1&4\\-1&3&5\end{vmatrix}=20\begin{vmatrix}1&3&1\\0&-7&2\\0&6&6\end{vmatrix}=20\begin{vmatrix}-7&2\\6&6\end{vmatrix}$$

$$=-1080$$

例： Vandermonde 行列式

设 $A_n=\begin{pmatrix}1&\alpha_1,&\alpha_1^2,\cdots&\alpha_1^{n-1}\\1&\alpha_2&\alpha_2^2,\cdots&\alpha_2^{n-1}\\&&\cdots&\\1&\alpha_n&\alpha_n^2\cdots&\alpha_n^{n-1}\end{pmatrix}$

$V_n(\alpha_1\cdots\alpha_n)$ ~~V_n~~ $=\det(A_n)$.

解 $n=2$ $\begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} = \alpha_2 - \alpha_1$

$n=3$ $\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 0 & \alpha_2-\alpha_1 & \alpha_2^2-\alpha_1^2 \\ 0 & \alpha_3-\alpha_1 & \alpha_3^2-\alpha_1^2 \end{vmatrix}$

$= (\alpha_2-\alpha_1)(\alpha_3-\alpha_1) \begin{vmatrix} 1 & \alpha_1 & \alpha_2^2 \\ 0 & 1 & \alpha_2+\alpha_1 \\ 0 & 1 & \alpha_3+\alpha_1 \end{vmatrix}$

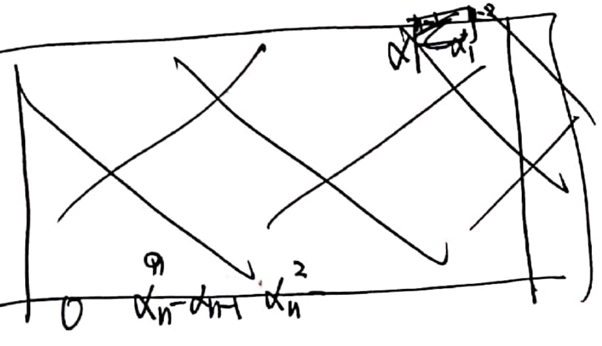
$= (\alpha_2-\alpha_1)(\alpha_3-\alpha_1) \begin{vmatrix} 1 & \alpha_2+\alpha_1 \\ 1 & \alpha_3+\alpha_1 \end{vmatrix}$

$= (\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)$

证: $\overset{V_n(\alpha_1 \cdots \alpha_n)}{V_n} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$

$n=2, 3$ 成立. 设 $\boxed{V_{n-1}}$ $n-1$ 时 成立. 当 n 时

$V_n =$



$V_n = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_2-\alpha_1 & \cdots & \alpha_2^{n-2}-\alpha_1\alpha_2^{n-3} & \alpha_2^{n-1}-\alpha_1\alpha_2^{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n-\alpha_1 & \cdots & \alpha_n^{n-2}-\alpha_1\alpha_n^{n-3} & \alpha_n^{n-1}-\alpha_1\alpha_n^{n-2} \end{vmatrix}$

$= (\alpha_2-\alpha_1) \cdots (\alpha_n-\alpha_1) \begin{vmatrix} 1 & 0 & \cdots & \alpha_2^{n-3} & \alpha_2^{n-2} \\ 1 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & & \\ 1 & 1 & & \alpha_n^{n-3} & \alpha_n^{n-2} \end{vmatrix}$

$= (\alpha_2-\alpha_1) \cdots (\alpha_n-\alpha_1) \, V_{n-1}(\alpha_2, \cdots \alpha_n)$

$= (\alpha_2-\alpha_1) \cdots (\alpha_n-\alpha_1) \prod_{2 \le i < j \le n} (\alpha_j - \alpha_i)$

$= \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$ ∎

例: 设 $A = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 \cdots & 0 \\ 0 & 1 & 2 & 1 \cdots & 0 \\ & & \ddots & \ddots & \ddots \\ & O & & 1 & 2 \end{pmatrix}$

求 $D_n = |A|$.

n=1. $D_1=2$

n=2 $D_2=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3$

n=3 $D_3=\begin{vmatrix}2&1&0\\1&2&1\\0&1&2\end{vmatrix}=2\begin{vmatrix}2&1\\1&2\end{vmatrix}-\begin{vmatrix}1&0\\1&2\end{vmatrix}$

$=2\times3-2=4$ …

猜 $D_n=n+1$ $n=1,2,3$ ✓

设 ~~D~~ n-1时 公式成立

$$D_n=2D_{n-1}+\begin{vmatrix}1&0&0&0&\cdots&0\\1&2&1&0&\cdots&0\\0&1&2&1&\cdots&0\\&&\ddots&\ddots&\ddots&1\\&O&&\ddots&&1\end{vmatrix}$$

$$D_n=2D_{n-1}-D_{n-2}=2n-(n-1)$$

$=n+1$ ▨

§2.2 分块矩阵的行列式

定理2.2 设 $A\in M_m(\mathbb{R})$, $B\in M_n(\mathbb{R})$

$C\in\mathbb{R}^{m\times n}$. 设

$$D=\begin{pmatrix}A&C\\O&B\end{pmatrix}_{(m+n)\times(m+n)}$$

则 $\det(D)=\det(A)\det(B)$

证: 法1 (矩阵法). 对m归纳

m=1 $|D|=\begin{vmatrix}a&c_1\cdots c_n\\0&B\end{vmatrix}$ ✓ [定理2.1]

$=a\det(B)$

设 m-1时 定理成立. 当m时

$$|D|=\begin{vmatrix}a_{11}&\cdots&a_{1m}&C\\a_{21}&\cdots&a_{2m}&\\\vdots&&\vdots&\\a_{m1}&\cdots&a_{mm}&\\&O&&B\end{vmatrix}$$

$= a_{11}D_{11} + a_{21}D_{21} + \cdots + a_{m1}D_{m1}$

（其中 D_{ij} 是 D 的代数余子式）

$= a_{11}A_{11}|B| + \cdots + a_{m1}A_{m1}|B|$

$= (a_{11}A_{11} + \cdots + a_{m1}A_{m1})|B| = |A||B|$ ▨

证法2. 映射法：

$$f: \underbrace{R^m \times \cdots \times R^m}_{m} \longrightarrow \mathbb{R}$$

$$\vec{A}^{(1)}, \cdots \vec{A}^{(m)} \mapsto \det(D) = \det\begin{pmatrix} A & C \\ O & B \end{pmatrix}$$

则 f，m重线性 全斜对称

$\Rightarrow \quad f(\vec{A}^{(1)}, \cdots \vec{A}^{(m)}) = \lambda \det(A)$

其中 $\lambda \in \mathbb{R}$ 待定

取 $A = E_m$. 则

$f(\vec{e}^{(1)}, \cdots \vec{e}^{(m)}) = \det\begin{pmatrix} E_m & C \\ O & B \end{pmatrix}$

$= \det(B)$ ~~det(A)~~ $= \lambda$

（按 1, 2, ⋯ m 行逐次展开）

$\Rightarrow \quad \det(D) = \det(B)\det(A)$ ▨

[归纳例] 推论2.1. 设 $A \in M_m(\mathbb{R})$, $B \in M_n(\mathbb{R})$

(i) 设 $C \in \mathbb{R}^{n\times m}$ 则

$$\begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A||B|$$

(ii) 设 $C \in \mathbb{R}^{m\times n}$. 则

$$\begin{vmatrix} C & A \\ B & O \end{vmatrix} = (-1)^{mn}|A||B|$$

证：(i) 利用命题1.1

(ii) $\begin{vmatrix} \boxed{C} & \boxed{A} \\ \boxed{B} & O \end{vmatrix} = (-1)^{mn}\begin{vmatrix} A & C \\ O & B \end{vmatrix}$

$= (-1)^{mn}\det(A)\det(B)$ ▨

例 计算

$$\begin{vmatrix} 0 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix}$$

$= -12.$

§2.3 乘法定理

定理2.3 设 $A, B \in M_n(\mathbb{R})$，则

$$\det(AB) = \det(A)\det(B)$$

证：法1（矩阵法）

若 A 或 B 不满秩．则 AB也不满秩

（ $\operatorname{rank}(AB) \le \min(\operatorname{rank}(A), \operatorname{rank}(B))$

则 $\det(AB)=0$　　$\det(A)\det(B)=0$

（定理1.1）．

于是定理成立．

考虑 A，B 都满秩情形．

由第一章定理5.2．A和B 可逆

由讲义2-5 p16 例 A和B都是若干初等矩阵乘积

断言：设 C 是 n阶初等方阵

则 $\det(AC)=\det(A)\det(C)$
$\det(CA)=\det(A)\det(C)$

断言的证明：

情形1： $C=F_{ij}$

$\det(AF_{ij}) = -\det(A)$，　$\det(F_{ij})=-1$

$\Rightarrow \det(AF_{ij}) = \det(A)\det(F_{ij})$

情形2． 设 $C=F_{ij}(\lambda)$，$i\neq j$

$\det(AF_{ij}) = \det(A)$．　［D_5 列变换］

$\det(F_{ij})=1$．

$\Rightarrow \det(AF_{ij}) = \det(A)\det(F_{ij})$

情形2．　$C=F_j(\lambda)$．　$\lambda\neq 0$

$\det(AF_{ij}) = \lambda\det(F_{ij})$．

$\det(F_{ij}) = \lambda$．　$\Rightarrow$　$\det(AF_{ij})=\det(A)\det$

断言成立

设 $B = C_1\cdots C_k$，其中 $C_1\cdots C_k$ 是初等矩阵

$\det(AB) = \det(\underbrace{AC_1\cdots C_{k-1}}C_k)$

$\det(AC_1\cdots C_{k-1})\det(C_k)$

$= \cancel{\det(A)\det}$

$= \det(AC_1\cdots C_{k-2})\det(C_{k-1})\det(C_k)$

$= \det(AC_1\cdots C_{k-2})\det(C_{k-1}C_k)$

$\cdots$

$= \det(A)\det(C_1\cdots C_{k-2}C_{k-1}C_k)$

$= \det(A)\det(C)$．

证法2 （映射法）

$$f:\ \underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{n}\longrightarrow\mathbb{R}^n$$

$$\overrightarrow{B}^{(1)},\cdots,\overrightarrow{B}^{(n)}\mapsto \det(AB)$$

$$f(\overrightarrow{B}^{(1)},\cdots\overrightarrow{B}^{(n)})=\det(A\overrightarrow{B}^{(1)},\cdots A\overrightarrow{B}^{(n)})$$

设 $\overrightarrow{B}^{(j)}=\alpha\overrightarrow{x}+\beta\overrightarrow{y}$, $\alpha,\beta\in\mathbb{R}$, $\overrightarrow{x},\overrightarrow{y}\in\mathbb{R}^n$

则 $f(\overrightarrow{B}^{(1)},\cdots\overrightarrow{B}^{(j-1)},\alpha\overrightarrow{x}+\beta\overrightarrow{y},\overrightarrow{B}^{(j+1)},\cdots,\overrightarrow{B}^{(n)})$

$=\det(A\overrightarrow{B}^{(1)},\cdots A\overrightarrow{B}^{(j-1)},A(\alpha\overrightarrow{x}+\beta\overrightarrow{y}),A\overrightarrow{B}^{(j+1)},\cdots A\overrightarrow{B}^{(n)})$

$=\det(A\overrightarrow{B}^{(1)},\cdots A\overrightarrow{B}^{(j-1)}\ \alpha A\overrightarrow{x}+\beta A\overrightarrow{y},A\overrightarrow{B}^{(j+1)},\cdots A\overrightarrow{B}^{(n)})$

$=\alpha\det(A\overrightarrow{B}^{(1)},\cdots A\overrightarrow{B}^{(j-1)},A\overrightarrow{x},A\overrightarrow{B}^{(j+1)},\cdots,A\overrightarrow{B}^{(n)})$

$+\beta\det(A\overrightarrow{B}^{(1)},\cdots A\overrightarrow{B}^{(j-1)},A\overrightarrow{y},A\overrightarrow{B}^{(j+1)},\cdots A\overrightarrow{B}^{(n)})$

$=\alpha f(\overrightarrow{B}^{(1)},\cdots\overrightarrow{B}^{(j-1)},\overrightarrow{x},\overrightarrow{B}^{(j+1)},\cdots\overrightarrow{B}^{(n)})$

$+\beta f(\overrightarrow{B}^{(1)},\cdots\overrightarrow{B}^{(j-1)},\overrightarrow{y},\overrightarrow{B}^{(j+1)},\cdots\overrightarrow{B}^{(n)})$

$\Rightarrow f$ n重线性

$$f(\cdots\underset{i}{\overrightarrow{B}^{(j)}},\cdots\underset{j}{\overrightarrow{B}^{(i)}})$$

$$=\det(\cdots\underset{i}{A\overrightarrow{B}^{(j)}},\cdots\underset{j}{A\overrightarrow{B}^{(i)}}\cdots)$$

$$=-\det(\cdots\underset{i}{A\overrightarrow{B}^{(i)}}\cdots\underset{j}{A\overrightarrow{B}^{(j)}}\cdots)=-f(\cdots\underset{i}{\overrightarrow{B}^{(i)}}\cdots\overrightarrow{B}^{(j)}\cdots)$$

$\Rightarrow f$ 交错的

$\Rightarrow f(\overrightarrow{B}^{(1)},\cdots\overrightarrow{B}^{(n)})=\lambda\det(B)$. $\lambda\in\mathbb{R}$ 与B无关

取 $B=E_n$ $f(\overrightarrow{e}^{(1)},\cdots\overrightarrow{e}^{(n)})=\lambda=\det(A)$

$\Rightarrow$ $\det(AB)=\det(A)\det(B)$ ☑

例：设 $A=\begin{pmatrix}1&\cos\theta_1&\cos2\theta_1\\1&\cos\theta_2&\cos2\theta_2\\1&\cos\theta_3&\cos2\theta_3\end{pmatrix}$

求 $|A|$

$$|A|=\begin{vmatrix}1&\cos\theta_1&2\cos^2\theta_1-1\\1&\cos\theta_2&2\cos^2\theta_2-1\\1&\cos\theta_3&2\cos^2\theta_3-1\end{vmatrix}$$

$$=\left|\begin{pmatrix}1&\cos\theta_1&\cos^2\theta_1\\1&\cos\theta_2&\cos^2\theta_2\\1&\cos\theta_3&\cos^2\theta_3\end{pmatrix}\begin{pmatrix}1&0&-1\\0&1&0\\0&0&2\end{pmatrix}\right|$$

$$= \begin{vmatrix} 1 & \cos\theta_1 & \cos^2\theta_1 \\ 1 & \cos\theta_2 & \cos^2\theta_2 \\ 1 & \cos\theta_3 & \cos^2\theta_3 \end{vmatrix} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$=2(\cos\theta_2-\cos\theta_1)(\cos\theta_3-\cos\theta_2)(\cos\theta_3-\cos\theta_1)$$

例：设 $A = \left((\alpha_i+\beta_j)^{n-1} \right)_{n\times n}$

求 $|A|$.

$$a_{ij} = (\alpha_i+\beta_j)^{n-1} = \alpha_i^{n-1} + \binom{n-1}{1}\alpha_i^{n-2}\beta_j + \cdots + \binom{n-1}{n-2}\alpha_i\beta_j^{n-2} + \beta_j^{n-1}$$

$$= \underbrace{\left(\alpha_i^{n-1}, \binom{n-1}{1}\alpha_i^{n-2}, \cdots, \binom{n-1}{n-2}\alpha_i, 1\right)}_{n} \begin{pmatrix} 1 \\ \beta_j \\ \vdots \\ \beta_j^{n-1} \end{pmatrix} \Big\} n$$

$$A = \underbrace{\begin{pmatrix} \alpha_1^{n-1}, & \binom{n-1}{1}\alpha_1^{n-2}, & \cdots, & \binom{n-1}{n-2}\alpha_1, & 1 \\ \alpha_2^{n-1}, & \binom{n-1}{1}\alpha_2^{n-2} & \cdots & \binom{n-1}{n-2}\alpha_2, & 1 \\ - & - & - & - & \\ \alpha_n^{n-1}, & \binom{n-1}{1}\alpha_n^{n-1} & \cdots & \binom{n-1}{n-2}\alpha_n, & 1 \end{pmatrix}}_{B} \times \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & & \beta_n \\ \vdots & \vdots & & \vdots \\ \beta_1^{n-1} & \beta_2^{n-1} & & \beta_n^{n-1} \end{pmatrix}}_{C}$$

$$|A| = \boxed{n-1}$$

$$|A| = |B||C| = \binom{n-1}{1}\cdots\binom{n-1}{n-2} \begin{vmatrix} \alpha_1^{n-1}, & \alpha_1^{n-2}, & \cdots & 1 \\ & \cdots & \cdots & \\ \alpha_n^{n-1} & \alpha_n^{n-2} & \cdots & 1 \end{vmatrix} V_n(\beta_1\cdots\beta_n)$$

$$= (-1)^{\frac{n(n-1)}{2}} \binom{n-1}{1}\cdots\binom{n-1}{n-2} V_n(\alpha_1\cdots\alpha_n)V_n(\beta_1\cdots\beta_n).$$

注：$|\alpha A| = \alpha^n|A|$，对任意 $A\in M_n(\mathbb{R})$

特别地 $|-A| = (-1)^n|A|$

例：设 $A\in M_{2n+1}(\mathbb{R})$，斜对称

证明 A 不可逆

证：$|-A^t| = (-1)^{2n+1}|A^t| = |A|$ $(\because A=-A^t)$

$\Rightarrow (-1)|A^t| = |A| \Rightarrow$ ~~2~~$|A| = -|A|$

$\Rightarrow 2|A|=0 \Rightarrow |A|=0$

$(\because 2\neq 0)$.

注：设 $A, B \in M_n(\mathbb{R})$

$\det(AB) = \det(BA)$

尽管 $AB \neq BA$.

定义：设 $A = (a_{ij})_{n\times n}$. A 的迹(trace)

为 $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

自己验证：$tr(AB) = tr(BA)$

注：一般而言 $rank(AB) \neq rank(BA)$.

§2.4 伴随矩阵

记号：设 $i, j \in \{1, 2, \cdots n\}$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

称之为 Kronecker 符号

引理2.1 设 $A = (a_{ij})_{n\times n}$. 则 $\forall$ ~~a_{ij}~~ $i, j \in \{1, \cdots, n\}$

i) $\sum\limits_{k=1}^{n} a_{ik} A_{jk} = \delta_{ij} |A|$

(ii) $\sum\limits_{k=1}^{n} a_{ki} A_{kj} = \delta_{ij} |A|$

<u>证</u>：i) 设 $\vec{b} = (b_1, \cdots, b_n)$

$$B = \begin{pmatrix} \vec{A_1} \\ \vdots \\ \vec{A_{j-1}} \\ \vec{b} \\ \vec{A_{j+1}} \\ \vdots \\ \vec{A_n} \end{pmatrix}$$

由定理2.1.

$\det(B) = b_1 A_{j1} + \cdots + b_n A_{jn}$

<u>情形1</u> 设 $i \neq j$. $\vec{b} = \vec{A_i}$

则 $\det(B) = 0$. 即

$a_{i1} A_{j1} + \cdots + a_{in} A_{jn} = 0$

$\Rightarrow \sum\limits_{k=1}^{n} a_{ik} A_{jk} = \delta_{ij} |A|$.