第七次作业.

1. $A=\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n\end{pmatrix}$ $B=\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n\end{pmatrix}$

试用平面上n条直线所成的集合的几何性质给出 $rank(A)=rank(B)$ 的条件

解：考虑 $A^t=\begin{pmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \vdots & \vdots \\ \alpha_n & \beta_n\end{pmatrix}$, $B^t=\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots \\ \alpha_n & \beta_n & \gamma_n\end{pmatrix}$

$\uparrow$ 二元齐次线性方程组的系数矩阵　　$\uparrow$ 二元非齐次线性方程组的增广矩阵.

非齐次线性方程组相容 $\Longleftrightarrow rank(A^t)=rank(B^t) \Longleftrightarrow rank(A)=rank(B)$

$\Updownarrow$

$\exists (x_0,y_0)\in\mathbb{R}^2$ s.t. $\alpha_i x_0+\beta_i y_0=\gamma_i$ $(i=1,2,\cdots,n)$ 均成立.

$\Updownarrow$

平面上n条直线 $\alpha_i x+\beta_i y=\gamma_i$ $(i=1,2,\cdots,n)$ 有交点.

2. 下列哪些是线性映射？

(1) $[x_1,\cdots,x_n] \xrightarrow{\varphi} [x_n,\cdots,x_1]$

$\forall\ \vec{x}=\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}, \vec{y}=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{R}^n$.

$$\varphi(\alpha\vec{x}+\beta\vec{y})=\varphi\left(\begin{pmatrix}\alpha x_1+\beta y_1\\ \vdots \\ \alpha x_n+\beta y_n\end{pmatrix}\right)=\begin{pmatrix}\alpha x_n+\beta y_n\\ \vdots\\ \alpha x_1+\beta y_1\end{pmatrix}$$

$$\alpha\varphi(\vec{x})+\beta\varphi(\vec{y})=\alpha\begin{pmatrix}x_n\\ \vdots\\ x_1\end{pmatrix}+\beta\begin{pmatrix}y_n\\ \vdots\\ y_1\end{pmatrix}=\begin{pmatrix}\alpha x_n+\beta y_n\\ \vdots\\ \alpha x_1+\beta y_1\end{pmatrix}$$

$\Rightarrow \varphi(\alpha\vec{x}+\beta\vec{y})=\alpha\varphi(\vec{x})+\beta\varphi(\vec{y})$　∴ φ 是线性映射.

(2) $[x_1, \cdots, x_n] \xrightarrow{\varphi} [x_1, x_2^2, \cdots, x_n^n]$

$\forall \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}$.

$$\varphi(\alpha\vec{x}+\beta\vec{y}) = \varphi\begin{pmatrix} \alpha x_1+\beta y_1 \\ \vdots \\ \alpha x_n+\beta y_n \end{pmatrix} = \begin{pmatrix} \alpha x_1+\beta y_1 \\ (\alpha x_2+\beta y_2)^2 \\ \vdots \\ (\alpha x_n+\beta y_n)^n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha^2 x_2^2 \\ \vdots \\ \alpha^n x_n^n \end{pmatrix} + \begin{pmatrix} 0 \\ 2\alpha\beta x_2 y_2 \\ \vdots \\ \sum_{k=1}^{n-1}\binom{n}{k}(\alpha x_n)^k(\beta y_n)^{n-k} \end{pmatrix} + \begin{pmatrix} \beta y_1 \\ \beta^2 y_2^2 \\ \vdots \\ \beta^n y_n^n \end{pmatrix}$$

$$\alpha\varphi(\vec{x})+\beta\varphi(\vec{y}) = \alpha\begin{pmatrix} x_1 \\ x_2^2 \\ \vdots \\ x_n^n \end{pmatrix} + \beta\begin{pmatrix} y_1 \\ y_2^2 \\ \vdots \\ y_n^n \end{pmatrix} = \begin{pmatrix} \alpha x_1+\beta y_1 \\ \alpha x_2^2+\beta y_2^2 \\ \vdots \\ \alpha x_n^n+\beta y_n^n \end{pmatrix}$$

$\Rightarrow \varphi$ 不是线性映射.

(3) $[x_1, \cdots, x_n] \xrightarrow{\varphi} [x_1, x_1+x_2, \cdots, x_1+x_2+\cdots+x_n]$

$\forall \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}$

$$\varphi(\alpha\vec{x}+\beta\vec{y}) = \varphi\left(\begin{pmatrix} \alpha x_1+\beta y_1 \\ \alpha x_2+\beta y_2 \\ \vdots \\ \alpha x_n+\beta y_n \end{pmatrix}\right) = \begin{pmatrix} \alpha x_1+\beta y_1 \\ \alpha(x_1+x_2)+\beta(y_1+y_2) \\ \\ \alpha(x_1+\cdots+x_n)+\beta(y_1+\cdots+y_n) \end{pmatrix}$$

$$\alpha\varphi(\vec{x})+\beta\varphi(\vec{y}) = \alpha\begin{pmatrix} x_1 \\ x_1+x_2 \\ \vdots \\ x_1+\cdots+x_n \end{pmatrix} + \beta\begin{pmatrix} y_1 \\ y_1+y_2 \\ \vdots \\ y_1+\cdots+y_n \end{pmatrix} = \begin{pmatrix} \alpha x_1+\beta y_1 \\ \alpha(x_1+x_2)+\beta(y_1+y_2) \\ \vdots \\ \alpha(x_1+\cdots+x_n)+\beta(y_1+\cdots+y_n) \end{pmatrix}$$

3. 求(1)解空间的一组基和(2)的全部解.

(1)
$$A = \begin{pmatrix} 1 & 3 & 5 & -4 & 0 \\ 1 & 3 & 2 & -2 & 1 \\ 1 & -2 & 3 & -1 & -1 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 4 & -3 & -1 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & -4 & 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 4 & -3 & -1 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 18 & -12 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 4 & -3 & -1 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & -3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank}(A)=3$, $\therefore$ 解空间维数为 $5-3=2$.

(1) 解得：$\begin{cases} x_1 = -2x_3 + x_4 \\ x_2 = -x_3 + x_4 \\ x_5 = 3x_3 - 2x_4 \end{cases}$ 解空间一组基底为 $\underset{\vec{v_1}}{\underset{\parallel}{\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 3 \end{pmatrix}}}$, $\underset{\vec{v_2}}{\underset{\parallel}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}}}$

(2)

$$B = (A \mid b) = \left(\begin{array}{ccccc|c} 1 & 3 & 5 & -4 & 0 & 2 \\ 1 & 3 & 2 & -2 & 1 & 1 \\ 1 & -2 & 3 & -1 & -1 & 3 \\ 1 & 2 & 1 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\text{行变换}} \left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\mathrm{rank}(A) = \mathrm{rank}(B)$，$\therefore$ 非齐次线性方程组 (2) 相容.

(2) 解得：$\begin{cases} x_1 = -2x_3 + x_4 + 2 \\ x_2 = -x_3 + x_4 \\ x_5 = 3x_2 - 2x_4 - 1 \end{cases}$. 令 $x_3 = x_4 = 0$，得到 $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ 是 (2) 一个特解.

则 (2) 的解为 $\vec{v} + \langle \vec{v_1}, \vec{v_2} \rangle$

4. 设 $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 是线性映射且双射，ϕ^{-1} 是 ϕ 逆映射，证：

(1) ϕ^{-1} 是线性映射； (2) $n = m$.

证：(1) $\phi^{-1}: \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

$\forall \vec{y_1}, \vec{y_2} \in \mathbb{R}^m$，$\forall \alpha, \beta \in \mathbb{R}$.

$\exists!\ \vec{x_1}, \vec{x_2} \in \mathbb{R}^n$ s.t. $\phi(\vec{x_1}) = \vec{y_1}$，$\phi(\vec{x_2}) = \vec{y_2}$.

~~∵ φ是双射~~ $\therefore \vec{x_1} = \phi^{-1}(\vec{y_1})$，$\vec{x_2} = \phi^{-1}(\vec{y_2})$

$\because \phi(\alpha\vec{x_1} + \beta\vec{x_2}) = \alpha\phi(\vec{x_1}) + \beta\phi(\vec{x_2}) = \alpha\vec{y_1} + \beta\vec{y_2}$ $\cancel{\therefore \alpha\vec{x_1} + \beta\vec{x_2} = \phi^{-1}(\alpha\vec{y_1} + \beta\vec{y_2})}$

$\therefore \phi^{-1}(\alpha\vec{y_1} + \beta\vec{y_2}) = \alpha\vec{x_1} + \beta\vec{x_2} = \alpha\phi^{-1}(\vec{y_1}) + \beta\phi^{-1}(\vec{y_2})$

$\Rightarrow \phi^{-1}$ 是线性映射.

(2) 由对偶定理可知：$\dim(\ker(\phi)) + \dim(\mathrm{im}(\phi)) = n$.

ϕ 是单射 $\Rightarrow \ker(\phi) = \{\vec{0}\} \Rightarrow \dim(\ker(\phi)) = 0$.

ϕ 是满射 $\Rightarrow \mathrm{im}(\phi) = \mathbb{R}^m \Rightarrow \dim(\mathrm{im}(\phi)) = m$

因此 $m = n$.

注： $\dim(\ker(\phi))+\dim(\mathrm{im}(\phi))=n$. $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

① ϕ是单射，则 $n \le m$.

ϕ是单射 $\Leftrightarrow \ker(\phi)=\{\vec{0}\} \Leftrightarrow \dim(\ker(\phi))=0 \Leftrightarrow \dim(\mathrm{im}(\phi))=n$.

$\because \mathrm{im}(\phi) \subset \mathbb{R}^m$. $\therefore n=\dim(\mathrm{im}(\phi)) \le m$.

② ϕ是满射，则 $n \ge m$.

ϕ是满射 $\Leftrightarrow \mathrm{im}(\phi)=\mathbb{R}^m \Leftrightarrow \dim(\mathrm{im}(\phi))=m$.

$\therefore \dim(\ker(\phi))+m=n \Longrightarrow n \ge m$.

综上： $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

ϕ单： $n \le m$ ， ϕ满： $n \ge m$ ， ϕ双： $n=m$

5. 设 $\phi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ 线性映射，且 $\forall \vec{x} \in \mathbb{R}^n, \phi(\vec{x}) \in \langle \vec{x} \rangle$. 证明：

$\exists \lambda \in \mathbb{R}$ s.t. $\forall \vec{x} \in \mathbb{R}^n$, $\phi(\vec{x})=\lambda \vec{x}$.

分析： $\forall \vec{x} \in \mathbb{R}^n$, $\phi(\vec{x})=\lambda_{\vec{x}} \vec{x}$ ， $\lambda_{\vec{x}} \in \mathbb{R}$ 与 $\vec{x}$ 有关.

要证的是对所有 $\vec{x} \in \mathbb{R}^n$, 可以取一样的 λ .

证：设 $\vec{v}_1, \cdots, \vec{v}_n$ 是 $\mathbb{R}^n$ 的一组基，且 $\phi(\vec{v}_i)=\lambda_i \vec{v}_i$ ， $\lambda_i \in \mathbb{R}$, $i=1, \cdots, n$.

下证 $\lambda_1=\cdots=\lambda_n$.

取 $\vec{x}=\vec{v}_1+\cdots+\vec{v}_n$ ，则 $\exists \lambda \in \mathbb{R}$ s.t. $\phi(\vec{x})=\lambda \vec{x}$.

则 $\phi(\vec{v}_1+\cdots+\vec{v}_n)=\phi(\vec{v}_1)+\cdots+\phi(\vec{v}_n)=\lambda_1\vec{v}_1+\cdots+\lambda_n\vec{v}_n=\lambda(\vec{v}_1+\cdots+\vec{v}_n)$

$\Longrightarrow (\lambda_1-\lambda)\vec{v}_1+\cdots+(\lambda_n-\lambda)\vec{v}_n=0$

$\because \vec{v}_1, \cdots, \vec{v}_n$ 线性无关 $\therefore \lambda_1=\lambda_2=\cdots=\lambda_n=\lambda$.

$\forall \vec{y} \in \mathbb{R}^n$. $\exists! \alpha_1, \cdots, \alpha_n \in \mathbb{R}$ s.t. $\vec{y}=\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n$.

$$\Longrightarrow \phi(\vec{y})=\phi(\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n)=\alpha_1\phi(\vec{v}_1)+\cdots+\alpha_n\phi(\vec{v}_n)$$
$$=\lambda(\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n)$$
$$=\lambda\vec{y}.$$

$\mathbb{R}^n$ 的任意子空间 V 都可以看成是某个齐次线性方程组的解空间.

证: 先证明 V 是某个线性映射的核空间.

设 $\dim V = k \leq n$ 且 $\vec{v}_1, \cdots, \vec{v}_k \in \mathbb{R}^n$ 是 V 的一组基

由基扩充定理, $\exists\, \vec{v}_{k+1}, \cdots, \vec{v}_n \in \mathbb{R}^n$ s.t. $\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n$ 是 $\mathbb{R}^n$ 的一组基

由第二章命题5.9可知, $\exists!$ 线性映射 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.

$$\varphi(\vec{v}_i) = \begin{cases} \vec{0}, & 1 \leq i \leq k \\ \vec{v}_i, & k+1 \leq i \leq n. \end{cases}$$

下面证明 $V = \ker\varphi$.

首先, 由于 $\vec{v}_1, \cdots, \vec{v}_k \in \ker\varphi$. 所以 $V \subset \ker\varphi$. $\Rightarrow \dim(\ker\varphi) \geq k$

又 $\because \vec{v}_{k+1}, \cdots, \vec{v}_n \in \operatorname{im}\varphi \quad \therefore \dim(\operatorname{im}\varphi) \geq n-k$

由对偶定理可知

$$\dim(\ker\varphi) + \dim(\operatorname{im}\varphi) = n.$$

$\Rightarrow \dim(\ker\varphi) \leq k$.

$\therefore \dim(\ker\varphi) = k$. 又 $\because V \subset \ker\varphi$ 且 $\dim V = \dim(\ker\varphi) = k$

$\therefore V = \ker\varphi$.

设 A 是 φ 在标准基下的矩阵.

则 $\forall \vec{x} \in \mathbb{R}^n$, $\varphi(\vec{x}) = A\vec{x}$.

因此 $\ker(\varphi) = \{\vec{x} \in \mathbb{R}^n \mid \cancel{\varphi(\vec{x})}\ A\vec{x} = \vec{0}\}$

是以 A 为系数矩阵的齐次方程组的解空间.

注：① 证明过程中φ的构造不唯一，只要满足下列要求即可．

$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\varphi(\vec{v_i}) = \begin{cases} \vec{0}, & 1 \le i \le k \\ \vec{w_i}, & k+1 \le i \le n \end{cases}$ ．其中 $\vec{w}_{k+1}, \cdots, \vec{w_n}$ 线性无关．

$\Downarrow$

这就要求 $m \geq n-k$

② ．$V = \ker\varphi$ 第二种证法：

由 $\vec{v_1}, \cdots, \vec{v_k} \in \ker\varphi$，可得 $V \subset \ker\varphi$

$\because \operatorname{im}\varphi = \langle \varphi(\vec{v_1}), \cdots, \varphi(\vec{v_k}), \varphi(\vec{v}_{k+1}), \cdots, \varphi(\vec{v_n}) \rangle$

$= \langle \vec{v}_{k+1}, \cdots, \vec{v_n} \rangle$ ．

且 $\vec{v}_{k+1}, \cdots, \vec{v_n}$ 线性无关 $\therefore \vec{v}_{k+1}, \cdots, \vec{v_n}$ 是 $\operatorname{im}\varphi$ 的一组基且

$\dim(\operatorname{im}\varphi) = n-k$ ．

由对偶定理可知

$\dim(\ker\varphi) + \dim(\operatorname{im}\varphi) = n$ ．

$\Rightarrow \dim(\ker\varphi) = k$

又 $\because V \subset \ker\varphi$ 且 $\dim V = k$ ．$\therefore V = \ker\varphi$ ．