第八次作业

1. $\varphi: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1+x_2 \\ x_1-x_3 \\ x_2+x_3 \\ x_1+2x_2+x_3 \end{pmatrix}$$

证：(1) φ 是线性映射.

$\forall\ \vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\vec{y}=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$.

验证 $\varphi(\alpha\vec{x}+\beta\vec{y}) = \alpha\varphi(\vec{x})+\beta\varphi(\vec{y})$ 即可.（具体过程同学们可自行验证）

(2) 由定义可知：

$$A_\varphi = (\varphi(\vec{e}_1), \varphi(\vec{e}_2), \varphi(\vec{e}_3)) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

(3). 利用初等行变换得.

$$A_\varphi \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\because \operatorname{rank}(A_\varphi)=2$ $\therefore \dim(\operatorname{Im}(\varphi))=2$.

由对偶定理可得. $\dim(\ker(\varphi)) = 3-2=1$

(4). $\ker(\varphi)$ 对应的齐次线性方程组是.

$$\begin{cases} x_1+x_2=0 \\ x_2+x_3=0 \end{cases} \Longleftrightarrow \begin{cases} x_1=-x_2 \\ x_3=-x_2 \end{cases}$$

于是. $\ker(\varphi)$ 一组基是 $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\because \operatorname{Im}(\varphi)=V_c(A_\varphi)$ 且 $\dim(\operatorname{Im}(\varphi))=2$.

$\therefore$ A_φ 中任意两个线性无关的列向量是 $\operatorname{Im}(\varphi)$ 的一组基.

例 $\operatorname{Im}(\varphi)$ 的一组基是 $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ 或 $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ 或 $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

[注 此题中任取 A_φ 的两个列向量都是 $\operatorname{Im}(\varphi)$ 的一组基]

1

注：

(1) 复习：

命题6.6. 设 $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 线性映射. 它在标准基下的矩阵是 $A \in \mathbb{R}^{m\times n}$.

(i) $\mathrm{im}(\varphi) = V_c(A)$. 从而 $\dim(\mathrm{im}(\varphi)) = \mathrm{rank}(A)$.

特别地，φ 满射 $\Longleftrightarrow$ A 行满秩.

(ii) $\ker(\varphi)$ 是 A 对应的齐次线性方程组的解空间. 从而 $\dim(\ker(\varphi)) = n - \mathrm{rank}(A)$.

特别地，φ 单射 $\Longleftrightarrow$ A 列满秩

(iii) φ 双射 $\Longleftrightarrow$ $m=n$. 且 A 满秩.

(2) 求矩阵的秩可以做行变换或列变换

求解空间只能做行变换.

$\Rightarrow$ 第(3)问做行变换后第(4)问可直接用第(3)问所变换后的矩阵.

(3). 初等行变换不改变列秩，但会改变~~行~~列空间.

$\because \mathrm{im}(\varphi) = V_c(A_\varphi)$. $\therefore \mathrm{im}(\varphi)$ 的基要选 A_φ 的列向量，而不是所变换后的矩阵的列向量.

但可从变换后的矩阵判断 A_φ 的哪些列向量线性无关.

$$A_\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \triangleq B.$$

$A_\varphi = (\vec{A}_\varphi^{(1)}, \vec{A}_\varphi^{(2)}, \vec{A}_\varphi^{(3)})$

$B = (\vec{B}^{(1)}, \vec{B}^{(2)}, \vec{B}^{(3)})$.

$\vec{B}^{(1)}, \vec{B}^{(2)}$ 线性无关 $\Longrightarrow$ $\vec{A}_\varphi^{(1)}, \vec{A}_\varphi^{(2)}$ 线性无关.

这是因为 $(\vec{A}_\varphi^{(1)}, \vec{A}_\varphi^{(2)}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = (\vec{B}^{(1)}, \vec{B}^{(2)})$

因为 $\vec{B}^{(1)}, \vec{B}^{(2)}, \vec{B}^{(3)}$ 任取两个线性无关 $\Longrightarrow$ $\vec{A}_\varphi^{(1)}, \vec{A}_\varphi^{(2)}, \vec{A}_\varphi^{(3)}$ 任取两个线性无关.

$\therefore$ 任取两个都是 $\mathrm{im}(\varphi)$ 的一组基.

2.

(1) $\begin{pmatrix}-1 & 1\\ -2 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}2 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}0 & -2\\ 2 & -3\\ 2 & 5\end{pmatrix}$

(2) $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}=\begin{pmatrix}\cos\alpha\cos\beta-\sin\alpha\sin\beta & -\cos\alpha\sin\beta-\sin\alpha\cos\beta\\ \sin\alpha\cos\beta+\cos\alpha\sin\beta & -\sin\beta\sin\alpha+\cos\alpha\cos\beta\end{pmatrix}$

$=\begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix}$

(3) $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n=\begin{pmatrix}\cos n\alpha & -\sin n\alpha\\ \sin n\alpha & \cos n\alpha\end{pmatrix}$ 用归纳法证.

$n=2$时，(2)中令$\alpha=\beta$即可.

假设$n-1$成立.

$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^{n-1}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$

$\overset{\text{归纳假设}}{=}\begin{pmatrix}\cos(n-1)\alpha & -\sin(n-1)\alpha\\ \sin(n-1)\alpha & \cos(n-1)\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$

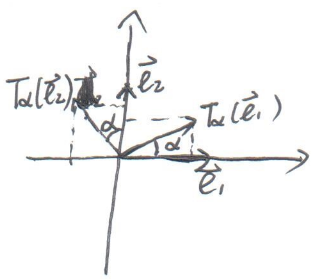
$=\begin{pmatrix}\cos n\alpha & -\sin n\alpha\\ \sin n\alpha & \cos n\alpha\end{pmatrix}$

注：设 $T_\alpha:\mathbb{R}^2\to\mathbb{R}^2$. 是旋转，即.

$A=\left(T_\alpha(\vec{e_1}),T_\alpha(\vec{e_2})\right)=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ i.e. $T_\alpha(\vec{e_1})=\cos\alpha\,\vec{e_1}+\sin\alpha\,\vec{e_2}$

$T_\alpha(\vec{e_2})=-\sin\alpha\,\vec{e_1}+\cos\alpha\,\vec{e_2}$.

$\vec{e_1},\vec{e_2}$ 逆时针旋转α角.



$T_\alpha^k:\mathbb{R}^2\to\mathbb{R}^2$. $k\geq 1$.

$\vec{x}\to A^k\vec{x}$.

$\vec{e_1},\vec{e_2}$ 逆时针旋转 $k\alpha$角.

3. (1) 令 $A=\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}(y_1\ y_2\ \cdots\ y_n)$. 求矩阵A的秩.

解：令 $A_1=\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}$. $A_2=(y_1,y_2,\cdots,y_n)$. $A=A_1A_2$.

若 $A_1=0$ 或 $A_2=0$. 则 $A=0$, $\Rightarrow rank(A)=0$.

法一：若 $A_1\neq 0$, 且 $A_2\neq 0$. 则 $rank(A_1)=rank(A_2)=1$.

$\because rank(A_1)+rank(A_2)-1\leq rank(A)\leq \min(rank(A_1), rank(A_2))$

i.e. $1\leq rank(A)\leq 1 \quad \Rightarrow rank(A)=1$.

法二：若 $A_1\neq 0$, $A_2\neq 0$.

$$A=\begin{pmatrix}x_1y_1 & x_1y_2 & \cdots & x_1y_n\\x_2y_1 & x_2y_2 & \cdots & x_2y_n\\\vdots & \vdots & & \vdots\\x_my_1 & x_my_2 & \cdots & x_my_n\end{pmatrix}\xrightarrow[\substack{r_i-\frac{x_i}{x_1}r_1\\ i=2,\cdots,m}]{\text{不妨设 }x_1\neq 0.}\begin{pmatrix}x_1y_1 & x_1y_2 & \cdots & x_1y_n\\0 & 0 & \cdots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & \cdots & 0\end{pmatrix}$$

$\because x_1\neq 0$. $y_1,\cdots,y_n$ 不全为0 $\therefore$ 第一行不是零向量 $\Rightarrow rank(A)=1$.

综上：$rank(A)=\begin{cases}0, & A_1=0 \text{ 或 } A_2=0.\\1, & A_1\neq 0, A_2\neq 0.\end{cases}$

(2). $A\in\mathbb{R}^{m\times n}$, $rank(A)=1$. 则A可表示成 $\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}(y_1\ y_2\ \cdots\ y_n)$. x_i, 不全为0
y_i 不全为0.

方法一：用行向量看.

设 $A=\begin{pmatrix}\vec{A_1}\\\vec{A_2}\\\vdots\\\vec{A_m}\end{pmatrix}$. $\because rank(A)=1$ $\therefore$ A任意非0行向量都是 $V_r(A)$ 的一组基.
不妨设 $\vec{A_1}$ 是一组基 i.e. $V_r(A)=\langle\vec{A_1}\rangle$.
则 $\exists!\ k_2,\cdots,k_m\in\mathbb{R}$ s.t. $\vec{A_i}=k_i\vec{A_1}$. $i=2,\cdots,m$.

$$\Rightarrow A=\begin{pmatrix}\vec{A_1}\\k_2\vec{A_1}\\\vdots\\k_m\vec{A_1}\end{pmatrix}=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\k_2a_{11} & \cdots & k_2a_{1n}\\\vdots & & \vdots\\k_ma_{11} & \cdots & k_ma_{1n}\end{pmatrix}=\begin{pmatrix}1\\k_2\\\vdots\\k_m\end{pmatrix}(a_{11}\ a_{12}\ \cdots\ a_{1n})$$

方法二：用列向量看.

$A=(\vec{A}^{(1)},\cdots,\vec{A}^{(n)})$. 不妨设 $V_c(A)=\langle\vec{A}^{(1)}\rangle$. 则 $\exists!\ l_2,\cdots,l_n$ s.t.

$\vec{A}^{(j)}=l_j\vec{A}^{(1)},\ j=2,\cdots,n.$

$$A=\begin{pmatrix} a_{11} & l_2a_{11} & \cdots & l_na_{11} \\ a_{21} & l_2a_{21} & \cdots & l_na_{21} \\ \vdots & \vdots & & \vdots \\ a_{m1} & l_2a_{m1} & \cdots & l_na_{m1} \end{pmatrix}=\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}(1\ l_2\ \cdots\ l_n)$$

注：由(1)(2)可知. $\operatorname{rank}(A)=1 \Longleftrightarrow A=\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}(y_1,\cdots,y_n)$ x_i不全为0, y_j不全为0.

(3) $A\in\mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=r$. A可以写成r个秩为1的矩阵的和

但不能写成$r-1$个秩为1的矩阵的和

方法一：用行向量看.

设 $A=\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_r \\ \vec{A}_{r+1} \\ \vdots \\ \vec{A}_m \end{pmatrix}$ 不妨设 $\vec{A}_1,\cdots,\vec{A}_r$是$V_r(A)$的一组基 且

$\vec{A}_i=\alpha_{i1}\vec{A}_1+\cdots+\alpha_{ir}\vec{A}_r$, $i=r+1,\cdots,m$. $\alpha_{ij}\in\mathbb{R}$.

$$\Rightarrow A=\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_r \\ \alpha_{r+1,1}\vec{A}_1+\cdots+\alpha_{r+1,r}\vec{A}_r \\ \vdots \\ \alpha_{m,1}\vec{A}_1+\cdots+\alpha_{m,r}\vec{A}_r \end{pmatrix}=\underbrace{\begin{pmatrix} \vec{A}_1 \\ \vec{0} \\ \vdots \\ \vec{0} \\ \alpha_{r+1,1}\vec{A}_1 \\ \vdots \\ \alpha_{m,1}\vec{A}_1 \end{pmatrix}}_{B_1}+\underbrace{\begin{pmatrix} \vec{0} \\ \vec{A}_2 \\ \vdots \\ \vec{0} \\ \alpha_{r+1,2}\vec{A}_2 \\ \vdots \\ \alpha_{m,2}\vec{A}_2 \end{pmatrix}}_{B_2}+\cdots+\underbrace{\begin{pmatrix} \vec{0} \\ \vec{0} \\ \vdots \\ \vec{A}_r \\ \alpha_{r+1,r}\vec{A}_r \\ \vdots \\ \alpha_{m,r}\vec{A}_r \end{pmatrix}}_{B_r}$$

i.e. $A=B_1+B_2+\cdots+B_r$. 且 $\operatorname{rank}(B_i)=1$. $i=1,2,\cdots,r$.

[注：$\because\ \vec{A}_1,\cdots,\vec{A}_r$是基 $\therefore\ \vec{A}_1,\cdots,\vec{A}_r$都是非$\vec{0}$向量]

方法二：用列向量看

类似设 $\vec{A}^{(1)},\cdots,\vec{A}^{(r)}$是$V_c(A)$一组基. $\vec{A}^{(j)}=\beta_{j,1}\vec{A}^{(1)}+\cdots+\beta_{j,r}\vec{A}^{(r)}$, $j=r+1,\cdots,n$ $\beta_{j,i}\in\mathbb{R}$.

$A=(\vec{A}^{(1)},\cdots,\vec{A}^{(r)},\beta_{r+1,1}\vec{A}^{(1)}+\cdots+\beta_{r+1,r}\vec{A}^{(r)},\cdots,\beta_{n,1}\vec{A}^{(1)}+\cdots+\beta_{n,r}\vec{A}^{(r)})$

$=(\vec{A}^{(1)},\vec{0},\cdots,\vec{0},\beta_{r+1,1}\vec{A}^{(1)},\cdots,\beta_{n,1}\vec{A}^{(1)})+\cdots+(\vec{0},\vec{0},\cdots,\vec{A}^{(r)},\beta_{r+1,r}\vec{A}^{(r)},\cdots,\beta_{n,r}\vec{A}^{(r)})$

如果 $A = B_1 + \cdots + B_{r-1}$，$\mathrm{rank}(B_i) = 1$.

则 $\mathrm{rank}(A) \le \mathrm{rank}(B_1) + \cdots + \mathrm{rank}(B_{r-1}) = r-1$.

与 $\mathrm{rank}(A) = r$ 矛盾.

注：例6.13. $A, B \in \mathbb{R}^{m\times n}$，$\mathrm{rank}(A+B) \le \mathrm{rank}(A) + \mathrm{rank}(B)$.

4. $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ 线性映射. 证：

(1) $\ker(\varphi) \subset \ker(\varphi^2) \subset \cdots$

(2) $\mathrm{im}(\varphi) \supset \mathrm{im}(\varphi^2) \supset \cdots$

(3) $\exists k \in \mathbb{Z}^+$ s.t. $\ker(\varphi^k) = \ker(\varphi^{k+1}) = \cdots$

$\mathrm{im}(\varphi^k) = \mathrm{im}(\varphi^{k+1}) = \cdots$

证：(1). 证. $\ker(\varphi^i) \subset \ker(\varphi^{i+1})$ $\forall i \ge 1$.

$\forall \vec{x} \in \ker(\varphi^i)$，有 $\varphi^i(\vec{x}) = 0$，则 $\varphi^{i+1}(\vec{x}) = \varphi(\varphi^i(\vec{x})) = \varphi(\vec{0}) = \vec{0}$.

$\Rightarrow \vec{x} \in \ker(\varphi^{i+1})$ i.e. $\ker(\varphi^i) \subset \ker(\varphi^{i+1})$.

(2). 证 $\mathrm{im}(\varphi^i) \supset \mathrm{im}(\varphi^{i+1})$. $\forall i \ge 1$.

$\forall \vec{y} \in \mathrm{im}(\varphi^{i+1})$. $\exists \vec{x} \in \mathbb{R}^n$ s.t. $\varphi^{i+1}(\vec{x}) = \vec{y}$. i.e. $\varphi^i(\varphi(\vec{x})) = \vec{y}$,

其中 $\varphi(\vec{x}) \in \mathbb{R}^n$. $\Rightarrow \vec{y} \in \mathrm{im}(\varphi^i)$ i.e. $\mathrm{im}(\varphi^{i+1}) \subset \mathrm{im}(\varphi^i)$

(3). 分析：$\ker(\varphi^i)$ 是 $\mathbb{R}^n$ 的子空间. $\dim(\ker(\varphi^i)) \le n$.

$\mathrm{im}(\varphi^i) \supset \{0\}$. $\therefore \dim(\mathrm{im}(\varphi^i)) \ge 0$.

注：(2) 方法二：证 $\mathrm{im}(\varphi^i) \supset \mathrm{im}(\varphi^{i+1})$ $\forall i \ge 1$.

$\mathrm{im}(\varphi^{i+1}) = \varphi^i(\mathrm{im}\,\varphi) \subset \varphi^i(\mathbb{R}^n) = \mathrm{im}(\varphi^i)$

(反证法). 若不存在 $k\in\mathbb{Z}^+$ s.t. $\ker(\varphi^k)=\ker(\varphi^{k+1})=\cdots$.

则可以找到一个严格包含的序列:

$$\ker(\varphi^{i_0})\subsetneq\ker(\varphi^{i_1})\subsetneq\cdots\subsetneq\ker(\varphi^{i_n})\subsetneq\ker(\varphi^{i_{n+1}})\subsetneq\cdots$$

其中. $\dim(\ker(\varphi^{i_s}))\geq s$. $s=0,1,2,\cdots$

则 $\dim(\ker(\varphi^{i_{n+1}}))\geq n+1$, 与 $\ker(\varphi^{i_{n+1}})\subset\mathbb{R}^n$ 矛盾.

$\therefore \exists k\in\mathbb{Z}^+$, s.t. $\ker(\varphi^k)=\ker(\varphi^{k+1})=\cdots$

由对偶定理可知. $\dim(\ker(\varphi^i))+\dim(\operatorname{im}(\varphi^i))=n$. $\forall i\geq 1$.

$\Longrightarrow \dim(\operatorname{im}(\varphi^k))=\dim(\operatorname{im}(\varphi^{k+1}))=\cdots=n-\dim(\ker(\varphi^k))$.

又 $\because$ $\operatorname{im}(\varphi^k)\supset\operatorname{im}(\varphi^{k+1})\supset\cdots$

$\therefore$ $\operatorname{im}(\varphi^k)=\operatorname{im}(\varphi^{k+1})=\cdots$

注: (1) 若 $\exists k\in\mathbb{Z}^+$, s.t. $\ker(\varphi^k)=\ker(\varphi^{k+1})$, 则

$$\ker(\varphi^k)=\ker(\varphi^{k+1})=\ker(\varphi^{k+2})=\cdots$$
$$\operatorname{im}(\varphi^k)=\operatorname{im}(\varphi^{k+1})=\operatorname{im}(\varphi^{k+2})=\cdots$$

(2) 若 $\exists k\in\mathbb{Z}^+$ s.t. $\operatorname{im}(\varphi^k)=\operatorname{im}(\varphi^{k+1})$

则

$\operatorname{im}(\varphi^k)=\operatorname{im}(\varphi^{k+1})=\cdots$

$\ker(\varphi^k)=\ker(\varphi^{k+1})=\cdots$

证: $\operatorname{im}(\varphi^k)=\varphi^k(\mathbb{R}^n)$

$\operatorname{im}(\varphi^{k+1})=\varphi^{k+1}(\mathbb{R}^n)$

$\varphi^k(\mathbb{R}^n)=\varphi^{k+1}(\mathbb{R}^n)\Rightarrow\varphi^{k+1}(\mathbb{R}^n)=\varphi^{k+2}(\mathbb{R}^n)$

i.e. $\operatorname{im}(\varphi^{k+1})=\operatorname{im}(\varphi^{k+2})$

证: (1) 先证 $\ker(\varphi^{k+1})=\ker(\varphi^{k+2})$.

已知 $\ker(\varphi^{k+1})\subset\ker(\varphi^{k+2})$. 下证 $\ker(\varphi^{k+2})\subset\ker(\varphi^{k+1})$

$\forall\vec{x}\in\ker(\varphi^{k+2})$, $\varphi^{k+2}(\vec{x})=\varphi^{k+1}(\varphi(\vec{x}))=\vec{0}\Longrightarrow\varphi(\vec{x})\in\ker(\varphi^{k+1})=\ker(\varphi^k)$

$\Longrightarrow\varphi^k(\varphi(\vec{x}))=\varphi^{k+1}(\vec{x})=\vec{0}\Longrightarrow\vec{x}\in\ker(\varphi^{k+1})$

$\therefore\ker(\varphi^{k+1})=\ker(\varphi^{k+2})$

类似可证 $\ker(\varphi^{k+2})=\ker(\varphi^{k+3})=\cdots$

因此. $\ker(\varphi^i)$ 的关系如下:

$$\ker(\varphi)\subsetneq\ker(\varphi^2)\subsetneq\cdots\subsetneq\ker(\varphi^k)=\ker(\varphi^{k+1})=\cdots$$

[一旦出现 $\ker(\varphi^k)=\ker(\varphi^{k+1})$, 则从 $\ker(\varphi^k)$ 开始稳定 i.e. $\ker(\varphi^k)=\ker(\varphi^{k+1})=\cdots$]

因此 $\dim(\ker(\varphi)) < \dim(\ker(\varphi^2)) < \cdots < \dim(\ker(\varphi^k)) = \dim(\ker(\varphi^{k+1})) = \cdots$

由对偶定理可知

$$\dim(\mathrm{im}(\varphi)) > \dim(\mathrm{im}(\varphi^2)) > \cdots > \dim(\mathrm{im}(\varphi^k)) = \dim(\mathrm{im}(\varphi^{k+1})) = \cdots$$

又 $\because$ $\mathrm{im}(\varphi) \supset \mathrm{im}(\varphi^2) \supset \cdots$

由维数的性质可知

$$\mathrm{im}(\varphi) \supsetneqq \mathrm{im}(\varphi^2) \supsetneqq \cdots \supsetneqq \mathrm{im}(\varphi^k) = \mathrm{im}(\varphi^{k+1}) = \cdots$$

例：

$$\varphi: \begin{array}{ccccccccc}
\mathbb{R}^5 & \xrightarrow{\varphi} & \mathbb{R}^5 & \xrightarrow{\varphi} & \mathbb{R}^5 & \xrightarrow{\varphi} & \mathbb{R}^5 & \xrightarrow{\varphi} & \mathbb{R}^5 \\
\vec{e}_1 & \longmapsto & \vec{e}_1 & \longmapsto & \vec{e}_1 & \longmapsto & \vec{e}_1 & \longmapsto & \vec{e}_1 \\
\vec{e}_2 & \longmapsto & \vec{e}_2 & \longmapsto & \vec{e}_2 & \longmapsto & \vec{e}_2 & \longmapsto & \vec{e}_2 \\
\vec{e}_3 & \longmapsto & \vec{0} & \longmapsto & \vec{0} & \longmapsto & \vec{0} & \longmapsto & \vec{0} \\
\vec{e}_4 & \longmapsto & \vec{e}_3 & \longmapsto & \vec{0} & \longmapsto & \vec{0} & \longmapsto & \vec{0} \\
\vec{e}_5 & \longmapsto & \vec{e}_4 & \longmapsto & \vec{e}_3 & \longmapsto & \vec{0} & \longmapsto & \vec{0} \\
 & & \vec{e}_5 & \longmapsto & \vec{e}_4 & \longmapsto & \vec{e}_3 & \longmapsto & \vec{0} \\
 & & & & \vec{e}_5 & \longmapsto & \vec{e}_4 & \longmapsto & \vec{e}_3 \\
 & & & & & & \vec{e}_5 & \longmapsto & \vec{e}_4
\end{array}$$

$\ker\varphi = \langle \vec{e}_3 \rangle$，$\ker\varphi^2 = \langle \vec{e}_3, \vec{e}_4 \rangle$，$\ker\varphi^3 = \langle \vec{e}_3, \vec{e}_4, \vec{e}_5 \rangle$，$\ker\varphi^4 = \langle \vec{e}_3, \vec{e}_4, \vec{e}_5 \rangle \cdots$

$\mathrm{im}\varphi = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \rangle$，$\mathrm{im}\varphi^2 = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$，$\mathrm{im}\varphi^3 = \langle \vec{e}_1, \vec{e}_2 \rangle$，$\mathrm{im}\varphi^4 = \langle \vec{e}_1, \vec{e}_2 \rangle \cdots$

$\Rightarrow$ $\ker\varphi \subsetneqq \ker\varphi^2 \subsetneqq \ker\varphi^3 = \ker\varphi^4 = \cdots$

$\mathrm{im}\varphi \supsetneqq \mathrm{im}\varphi^2 \supsetneqq \mathrm{im}\varphi^3 = \mathrm{im}\varphi^4 = \cdots$