赵若寰 54.

p83. 12.

$$(A|E_5)=\left(\begin{array}{ccccc|ccccc} 5&4&3&2&1&1&0&0&0&0\\ 4&8&6&4&2&0&1&0&0&0\\ 3&6&9&6&3&0&0&1&0&0\\ 2&4&6&8&4&0&0&0&1&0\\ 1&2&3&4&5&0&0&0&0&1 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|ccccc} 1&2&3&4&5&0&0&0&0&1\\ 0&6&12&18&24&-1&0&0&0&5\\ 0&0&6&12&18&0&-1&0&0&4\\ 0&0&0&6&12&0&0&-1&0&3\\ 0&0&0&0&6&0&0&0&-1&2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|ccccc} 1&2&3&4&5&0&0&0&0&1\\ 0&1&2&3&4&-\frac{1}{6}&0&0&0&\frac{5}{6}\\ 0&0&1&2&3&0&-\frac{1}{6}&0&0&\frac{2}{3}\\ 0&0&0&1&2&0&0&-\frac{1}{6}&0&\frac{1}{2}\\ 0&0&0&0&1&0&0&0&-\frac{1}{6}&\frac{1}{3} \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|ccccc} 1&1&1&1&1&\frac{1}{6}&0&0&0&\frac{1}{6}\\ 0&1&1&1&1&-\frac{1}{6}&\frac{1}{6}&0&0&\frac{1}{6}\\ 0&0&1&1&1&0&-\frac{1}{6}&\frac{1}{6}&0&\frac{1}{6}\\ 0&0&0&1&1&0&0&-\frac{1}{6}&\frac{1}{6}&\frac{1}{6}\\ 0&0&0&0&1&0&0&0&-\frac{1}{6}&\frac{1}{3} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|ccccc} 1&0&0&0&0&\frac{1}{3}&-\frac{1}{6}&0&0&0\\ 0&1&0&0&0&-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}&0&0\\ 0&0&1&0&0&0&-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}&0\\ 0&0&0&1&0&0&0&-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}\\ 0&0&0&0&1&0&0&0&-\frac{1}{6}&\frac{1}{3} \end{array}\right) \quad \therefore A^{-1}=\begin{pmatrix} \frac{1}{3}&-\frac{1}{6}&0&0&0\\ -\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}&0&0\\ 0&-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}&0\\ 0&0&-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}\\ 0&0&0&-\frac{1}{6}&\frac{1}{3} \end{pmatrix}$$

$$(F|E_4)=\left(\begin{array}{cccc|cccc} 2&3&2&1&1&0&0&0\\ 3&6&4&2&0&1&0&0\\ 4&8&6&3&0&0&1&0\\ 2&4&3&2&0&0&0&1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 2&3&2&1&1&0&0&0\\ 0&1&1&1&-1&0&0&1\\ 0&2&2&1&-2&0&1&0\\ 0&3&2&1&-3&2&0&0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1&0&0&0&2&-1&0&0\\ 0&1&0&0&-1&2&-1&0\\ 0&2&2&1&-2&0&1&0\\ 0&0&0&1&0&0&-1&2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1&0&0&0&2&-1&0&0\\ 0&1&0&0&-1&2&-1&0\\ 0&1&1&0&-1&0&1&-1\\ 0&0&0&1&0&0&-1&2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1&0&0&0&2&-1&0&0\\ 0&1&0&0&-1&2&-1&0\\ 0&0&1&0&0&-2&2&-1\\ 0&0&0&1&0&0&-1&2 \end{array}\right) \quad \therefore F^{-1}=\begin{pmatrix} 2&-1&0&0\\ -1&2&-1&0\\ 0&-2&2&-1\\ 0&0&-1&2 \end{pmatrix}$$

14. $A=\begin{pmatrix} a & b\\ c & d \end{pmatrix}$

$$A^2=\begin{pmatrix} a & b\\ c & d \end{pmatrix}\begin{pmatrix} a & b\\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc, & ab+bd\\ ac+dc, & bc+d^2 \end{pmatrix}$$

$$(a+d)A-(ad-bc)E=\begin{pmatrix} a^2+ad, & ab+bd\\ ac+cd, & ad+d^2 \end{pmatrix}-\begin{pmatrix} ad-bc & 0\\ 0 & ad-bc \end{pmatrix}=\begin{pmatrix} a^2+bc, & ab+bd\\ ac+cd, & bc+d^2 \end{pmatrix}$$

$\therefore A^2 = (a+d)A - (ad-bc)E$

15. $A^2-(a+d)A+(ad-bc)E=0$ $\quad \cancel{ad-bc\neq 0}$ $\quad ad-bc\neq 0$

$$\therefore E=\frac{1}{ad-bc}\left[(a+d)A-A^2\right]$$

$$=\frac{1}{ad-bc}A\left[(a+d)E-A\right]$$

$\because \cancel{ad-bc\neq 0}$

$\therefore$ A存在逆矩阵

$$\therefore A^{-1}=\frac{1}{ad-bc}\left[(a+d)E-A\right]=\frac{1}{ad-bc}\left[\begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix}-\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

$$=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

16. 证明: 当 $m=1$ 时: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=0$. $\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2=0$ 成立

当 $m=2$ 时: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^m=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2=0$ 成立

当 $m>2$ 时: 若 $ad-bc\neq 0$. 则 A 存在可逆矩阵 即 A 满秩

则 A^m 也满秩

但 $A^m=0$. $\operatorname{rank}(A^m)=0$. $\rightarrow\times\leftarrow$

$\therefore ad-bc=0$.

$\therefore A^2=(a+d)A$

$\because A^m=0$

$\therefore A^m=A^{m-2}A^2=A^{m-2}(a+d)A=(a+d)A^{m-1}=(a+d)A^{m-3}\cdot A^2=(a+d)^2A^{m-2}$

$$=(a+d)^2A^{m-4}\cdot A^2=(a+d)^3A^{m-3}\ \cdots\ =(a+d)^{m-2}A^2$$

$\therefore (a+d)^{m-2}A^2=0$

① 若 $a+d=0$ $\quad A^2=\begin{pmatrix} a^2+bc, & ab+bd \\ ac+cd, & bc+d^2 \end{pmatrix}=\begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore ad=-a^2=-d^2$

$\therefore A^2=0$

② 若 $a+d\neq 0$ $\quad A^2=0$

综上: $A^2=0$.

期中考题：

5. (ii) 已知 $um+vn=d$ (*). $g=\gcd(m,n)$. 设 $m=gk$，$\overbrace{\quad}^{n=gl}$

带余除法 $u=ql+a$，$a\in\{0,1,\cdots,l-1\}$.

把上述 u 代入 (*) 可得

$$d=(ql+a)m+vn$$
$$=am+ql\cdot gk+vn \qquad (m=gk,\ n=gl)$$
$$=am+(qk+v)n$$

令 $b=qk+v$ 即可.

6. (ii) 方法一：由 (i) 知 $\dim\langle\vec{v}_1,\cdots,\vec{v}_k\rangle\le k-1$. 只需证 $\dim\langle\vec{v}_1,\cdots,\vec{v}_k\rangle\ne k-1$

(反证法). 若 $\dim\langle\vec{v}_1,\cdots,\vec{v}_k\rangle=k-1$. 不妨设 $\vec{v}_1,\cdots,\vec{v}_{k-1}$ 线性无关

因为 $\vec{v}_1,\cdots,\vec{v}_{k-1},\vec{v}_k$ 线性相关，故 $\vec{v}_k$ 是 $\vec{v}_1,\cdots,\vec{v}_{k-1}$ 的线性组合且组合系数唯一

$\because \alpha_k\ne0$ (若 $\alpha_k=0$，则由 $\vec{v}_1,\cdots,\vec{v}_{k-1}$ 线无 $\Rightarrow \alpha_1=\cdots=\alpha_{k-1}=0$ 矛盾)

$\therefore \vec{v}_k=-\frac{1}{\alpha_k}(\alpha_1\vec{v}_1+\cdots+\alpha_{k-1}\vec{v}_{k-1})$.

若 $\beta_k=0$，则 $\beta_1=\cdots=\beta_{k-1}=0$，与 $\operatorname{rank}\begin{pmatrix}\alpha_1&\cdots&\alpha_k\\\beta_1&\cdots&\beta_k\end{pmatrix}=2$ 矛盾 $\therefore \beta_k\ne0$.

则 $\vec{v}_k=-\frac{1}{\beta_k}(\beta_1\vec{v}_1+\cdots+\beta_{k-1}\vec{v}_{k-1})$

由组合系数唯一性可知 $-\frac{\alpha_i}{\alpha_k}=-\frac{\beta_i}{\beta_k}$，$i=1,\cdots,k-1$，与 $\operatorname{rank}\begin{pmatrix}\alpha_1&\cdots&\alpha_k\\\beta_1&\cdots&\beta_k\end{pmatrix}=2$ 矛盾

$\therefore \dim\langle\vec{v}_1,\cdots,\vec{v}_k\rangle\ne k-1$.

方法二：
$$(\vec{v}_1,\cdots,\vec{v}_k)_{n\times k}\begin{pmatrix}\alpha_1&\beta_1\\\vdots&\vdots\\\alpha_k&\beta_k\end{pmatrix}_{k\times2}=(\vec{0}_n\ \vec{0}_n)_{n\times2} \quad (矩阵相乘)$$

Sylvester 不等式：$\operatorname{rank}((\vec{v}_1,\cdots,\vec{v}_k))+\operatorname{rank}\left(\begin{pmatrix}\alpha_1&\beta_1\\\vdots&\vdots\\\alpha_k&\beta_k\end{pmatrix}\right)-k\le\operatorname{rank}((\vec{0}_n,\vec{0}_n))=0$

$\Rightarrow \operatorname{rank}((\vec{v}_1,\cdots,\vec{v}_k))+2-k\le0$

$\Rightarrow \operatorname{rank}((\vec{v}_1,\cdots,\vec{v}_k))\le k-2$. i.e. $\dim\langle\vec{v}_1,\cdots,\vec{v}_k\rangle<k-1$

7. 设 $\vec{u}_1,\cdots,\vec{u}_d$ 是 U 一组基，扩充成 $\mathbb{R}^n$ 的一组基 $\vec{u}_1,\cdots,\vec{u}_d,\vec{u}_{d+1},\cdots,\vec{u}_n$.

(i) $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ s.t. $\varphi(\vec{u}_i)=\begin{cases}\vec{0}, & i=1,\cdots,d\\ \vec{u}_i, & i=d+1,\cdots,n\end{cases}$

$\Rightarrow im(\varphi)=\langle\varphi(\vec{u}_1),\cdots,\varphi(\vec{u}_n)\rangle=\langle\vec{u}_{d+1},\cdots,\vec{u}_n\rangle$ $\therefore \dim(im(\varphi))=n-d$.

由对偶定理可知 $\dim(\ker(\varphi))=d$. 又 $\because U\subset\ker(\varphi)$ $\therefore U=\ker(\varphi)$

(ii) $\psi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ s.t. $\psi(\vec{u}_i)=\begin{cases}\vec{u}_i, & i=1,\cdots,d\\ \vec{0}, & i=d+1,\cdots,n.\end{cases}$

$\Rightarrow im(\psi)=\langle\psi(\vec{u}_1),\cdots,\psi(\vec{u}_n)\rangle=\langle\vec{u}_1,\cdots,\vec{u}_d\rangle=U$.

(iii) 方法一：设 B 是 (i) 中 φ 在标准基下矩阵. $B\in\mathbb{R}^{n\times n}$

$\because \ker(\varphi)=\{\vec{x}\mid B\vec{x}=\vec{0}_n\}$ $\therefore U=\ker(\varphi)=sol(B\vec{x}=\vec{0}_n)$

由对偶定理可知，$rank(B)=n-d$. 设 A 是由 B 中 $(n-d)$ 个线性无关的行向量组成的矩阵. 则 B 中其他行向量都是 A 中行向量的线性组合. 故 $A\vec{x}=\vec{0}_{n-d}$ 与 $B\vec{x}=\vec{0}_n$ 等价. 故 $U=sol(A\vec{x}=\vec{0}_{n-d})$. $A\in\mathbb{R}^{(n-d)\times n}$

方法二：重新构造线性映射：

$\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^{n-d}$ s.t. $\varphi(\vec{u}_i)=\begin{cases}\vec{0}, & i=1,\cdots,d\\ \vec{v}_{i-d}, & i=d+1,\cdots,n.\end{cases}$

其中 $\vec{v}_1,\cdots,\vec{v}_{n-d}$ 是 $\mathbb{R}^{n-d}$ 的一组基.

与由 (i) 类似可证 $\ker(\phi)=U$. 设 $A\in\mathbb{R}^{(n-d)\times n}$ 是 ϕ 在标准基下矩阵.

则 $\ker(\phi)=\{\vec{x}\mid A\vec{x}=\vec{0}_{n-d}\}$ $\therefore U=\ker(\phi)=sol(A\vec{x}=\vec{0}_{n-d})$

8. (ii).

$\mathrm{rank}(A)=\mathrm{rank}(A^2) \Longleftrightarrow \dim(\mathrm{im}(\varphi_A))=\dim(\mathrm{im}(\varphi_A^2))$

$\overset{\text{对偶定理}}{\Longleftrightarrow} \dim(\ker(\varphi_A))=\dim(\ker(\varphi_A^2))$

$\overset{\text{由(i)}}{\Longleftrightarrow} \ker(\varphi_A)=\ker(\varphi_A^2)$

只需证 $\ker(\varphi_A)\cap \mathrm{im}(\varphi_A)=\{\vec{0}\} \Longleftrightarrow \ker(\varphi_A)=\ker(\varphi_A^2)$.

$\Rightarrow$. 已知 $\ker(\varphi_A)\subset\ker(\varphi_A^2)$.

$\forall \vec{x}\in\ker(\varphi_A^2)$. $\vec{0}=\varphi_A^2(\vec{x})=\varphi_A(\varphi_A(\vec{x}))$

$\therefore$ $\varphi_A(\vec{x})\in\ker(\varphi_A)\cap \mathrm{im}(\varphi_A)=\{\vec{0}\}$.

i.e. $\varphi_A(\vec{x})=\vec{0}$ $\therefore$ $\vec{x}\in\ker(\varphi_A)$. 因此 $\ker(\varphi_A^2)\subset\ker(\varphi_A)$.

$\Leftarrow$ $\forall \vec{x}\in\ker(\varphi_A)\cap \mathrm{im}(\varphi_A)$.

$\exists \vec{y}\in\mathbb{R}^n$ s.t. $\varphi_A(\vec{y})=\vec{x}$ 且 $\varphi_A(\vec{x})=\vec{0}$, 则 $\varphi_A(\varphi_A(\vec{y}))=\varphi_A^2(\vec{y})=\vec{0}$

$\therefore$ $\vec{y}\in\ker(\varphi_A^2)=\ker(\varphi_A)$. $\therefore$ $\vec{x}=\varphi_A(\vec{y})=\vec{0}$.