第十二次作业

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P102.8. 证明：记 $M=\begin{pmatrix} A & B \\ B & A \end{pmatrix} \in M_{2n}(\mathbb{R})$ 对 M 做行、列变换：

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \xrightarrow[i=1,\cdots,n]{\vec{M}_i+\vec{M}_{n+i}} \begin{pmatrix} A & B \\ A+B & A+B \end{pmatrix} \xrightarrow[j=1,\cdots,n]{\vec{M}^{(j)}-\vec{M}^{(n+j)}} \begin{pmatrix} A-B & B \\ 0 & A+B \end{pmatrix}$$

则 $\det\begin{pmatrix} A & B \\ B & A \end{pmatrix}=\det\begin{pmatrix} A-B & B \\ 0 & A+B \end{pmatrix}=\det(A+B)\det(A-B)$ □

P103.9. 证明：由题有 $\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}=\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n+XY \end{pmatrix}$

取行列式，则 $\det\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}\cdot\det\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}=\det\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}\cdot\det\begin{pmatrix} E_k & 0 \\ X & E_n+XY \end{pmatrix}$

由 $\det\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}=\det(E_k)\det(E_n)=1$ 非零，

又 $\det\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}=(-1)^{nk}\det\begin{pmatrix} X & E_n \\ E_k+YX & 0 \end{pmatrix}=(-1)^{2nk}\det\begin{pmatrix} E_n & X \\ 0 & E_k+YX \end{pmatrix}$

原式化为 $\det(E_n)\det(E_k+YX)\cdot(-1)^{2nk}=(-1)^{2nk}\det(E_k)\det(E_n+XY)$

即 $\det(E_k+YX)=\det(E_n+XY)$. □

P108.2. 解. 分类讨论如下. 设 $A\in M_n(\mathbb{R})$. 则 $A^\vee\in M_n(\mathbb{R})$

① $\operatorname{rank}(A)=n$，则 $\det(A)\neq 0$，$AA^\vee=|A|E_n$

则 $\operatorname{rank}(AA^\vee)=\operatorname{rank}(|A|E_n)=n\leq\operatorname{rank}(A^\vee)\leq n$

即 $\operatorname{rank}(A^\vee)=n$.

② $\operatorname{rank}(A)=n-1$，则 A 存在 $n-1$ 阶非零子式.

即 $\exists i\in\{1,\cdots,n\}, j\in\{1,\cdots,n\}$, $A_{ij}\neq 0$.

即 $A^\vee$ 中至少有一个非零数 A_{ij}, $\operatorname{rank}(A^\vee)\geq 1$.

又 $\operatorname{rank}(A)=n-1$，$\det(A)=0$，$AA^\vee=|A|E_n=0$

则 $\operatorname{rank}(AA^\vee)=0\geq\operatorname{rank}(A^\vee)+\operatorname{rank}(A)-n$，得 $\operatorname{rank}(A^\vee)\leq 1$

则 $\operatorname{rank}(A^\vee)=1$.

③ $\operatorname{rank}(A)<n-1$. 则 A 的任一 $n-1$ 阶子式均为 0.

即 $\forall i,j\in\{1,\cdots,n\}$, $A_{ij}=0$. 即 $A^\vee=0$

则 $\operatorname{rank}(A^\vee)=0$.

$$\operatorname{rank}(A^\vee)=\begin{cases} n, & \operatorname{rank}(A)=n \\ 1, & \operatorname{rank}(A)=n-1 \\ 0, & \operatorname{rank}(A)<n-1. \end{cases}$$

P111.9 方法一：$\begin{pmatrix} A & C \\ O & B \end{pmatrix}\begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ O & B^{-1} \end{pmatrix} = \begin{pmatrix} E_n & O \\ O & E_m \end{pmatrix} = E_{n+m}$

$\therefore \begin{pmatrix} A & C \\ O & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ O & B^{-1} \end{pmatrix}$

P109.5. 证明：(反证法). 设 $\det(A)=0$，则 $\operatorname{rank}(A)\neq n$，即 $A\vec{x}=\vec{0}$ 有非平凡解.

设该解为 $\vec{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$，其中 $x_1,\cdots,x_n$ 不全为 0，令 $|x_k|=\max\{|x_1|,\cdots,|x_n|\}$.

对第 k 个方程 $(1\leq k\leq n,\ k\in\mathbb{Z}^+)$，$a_{k1}x_1+\cdots+a_{kk}x_k+\cdots+a_{kn}x_n=0$，

则 $|a_{kk}x_k| = |a_{k1}x_1+\cdots+a_{kk-1}x_{k-1}+a_{kk+1}x_{k+1}+\cdots+a_{kn}x_n|$

$\leq |a_{k1}x_1|+\cdots+|a_{kk-1}x_{k-1}|+|a_{kk+1}x_{k+1}|+\cdots+|a_{kn}x_n|$

$\leq |a_{k1}||x_k|+\cdots+|a_{kk-1}||x_k|+|a_{kk+1}||x_k|+\cdots+|a_{kn}||x_k|$

又 $|a_{k1}|<\dfrac{|a_{kk}|}{n-1}$，$\cdots$ $|a_{kn}|<\dfrac{|a_{kk}|}{n-1}$，$|a_{k1}|+\cdots+|a_{kk-1}|+|a_{kk+1}|+\cdots+|a_{kn}|<|a_{kk}|$

则 $|a_{kk}x_k| > |a_{k1}||x_k|+\cdots+|a_{kk-1}||x_k|+|a_{kk+1}||x_k|+\cdots+|a_{kn}||x_k|$.

矛盾，则 $\det(A)\neq 0$. □

P111.9. 证明：由题 $\operatorname{rank}(A)=n$，$\operatorname{rank}(B)=m$，A、B 可逆. 进行乘法运算：

方法二：$\begin{pmatrix} A & C \\ O & B \end{pmatrix}\begin{pmatrix} A^{-1} & O \\ O & E_m \end{pmatrix} = \begin{pmatrix} E_n & C \\ O & B \end{pmatrix}$，$\begin{pmatrix} E_n & C \\ O & B \end{pmatrix}\begin{pmatrix} E_n & -C \\ O & E_m \end{pmatrix} = \begin{pmatrix} E_n & O \\ O & B \end{pmatrix}$

$\begin{pmatrix} E_n & O \\ O & B \end{pmatrix}\begin{pmatrix} E_n & O \\ O & B^{-1} \end{pmatrix} = \begin{pmatrix} E_n & O \\ O & E_m \end{pmatrix} = E_{n+m}$.

即 $\begin{pmatrix} A & C \\ O & B \end{pmatrix}\cdot\begin{pmatrix} A^{-1} & O \\ O & E_m \end{pmatrix}\begin{pmatrix} E_n & -C \\ O & E_m \end{pmatrix}\begin{pmatrix} E_n & O \\ O & B^{-1} \end{pmatrix} = E_{n+m}$

则 $\begin{pmatrix} A & C \\ O & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ O & E_m \end{pmatrix}\begin{pmatrix} E_n & -C \\ O & E_m \end{pmatrix}\begin{pmatrix} E_n & O \\ O & B^{-1} \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ O & B^{-1} \end{pmatrix}$. □

P111.10. 证明：$\det(A)\neq 0$，$\operatorname{rank}(A)=n$，A 可逆. 进行乘法运算.

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E_n & -A^{-1}B \\ O & E_n \end{pmatrix} = \begin{pmatrix} A & O \\ C & D-CA^{-1}B \end{pmatrix}$，两边取行列式，

$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot\det\begin{pmatrix} E_n & -A^{-1}B \\ O & E_n \end{pmatrix} = (-1)^{2n^2}\det\begin{pmatrix} D-CA^{-1}B & C \\ O & A \end{pmatrix}$

$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot\det(E_n)\cdot\det(E_n) = 1\cdot\det(D-CA^{-1}B)\cdot\det(A)$

即 $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD-ACA^{-1}B) = \det(A)\cdot\det(D-CA^{-1}B)$. □

验证. $AC=CA$ 时，$\det(AD-ACA^{-1}B)=\det(AD-CAA^{-1}B)=\det(AD-CB)$

$AB=BA$ 时，$\det(A(D-\cancel{A}CA^{-1}B))=\det[(D-CA^{-1}B)A]$ (交换不变量)

$=\det(DA-CA^{-1}BA)=\det(DA-CA^{-1}AB)$

$=\det(DA-CB)$

作业题解析：

第十二周讲义定理3.1和推论3.3：

设 $A\in M_m(\mathbb{R})$，$B\in M_n(\mathbb{R})$，$C\in\mathbb{R}^{m\times n}$，则

$$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A|\cdot|B|. \qquad \begin{vmatrix} C & A \\ B & 0 \end{vmatrix} = (-1)^{mn}|A|\cdot|B|$$

另：
$$\begin{vmatrix} A & 0 \\ D & B \end{vmatrix} = \begin{vmatrix} A^t & D^t \\ 0 & B^t \end{vmatrix} = |A^t|\cdot|B^t| = |A|\cdot|B|. \qquad D\in\mathbb{R}^{n\times m}$$

$$\begin{vmatrix} 0 & A \\ B & D \end{vmatrix} = (-1)^{mn}\begin{vmatrix} B & C \\ 0 & A \end{vmatrix} = (-1)^{mn}|A|\cdot|B|$$

P_{103}. 9. 方法一：
$$\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n+XY \end{pmatrix}. \qquad X\in\mathbb{R}^{n\times k},\ Y\in\mathbb{R}^{k\times n}$$

$$\begin{vmatrix} E_k+YX & 0 \\ X & E_n \end{vmatrix}\cdot\begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix} = \begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix}\cdot\begin{vmatrix} E_k & 0 \\ X & E_n+XY \end{vmatrix}$$

$$\Rightarrow |E_k+YX|\cdot|E_n|\cdot|E_k|\cdot|E_n| = |E_k|\cdot|E_n|\cdot|E_k|\cdot|E_n+XY|$$

$$\Rightarrow |E_k+YX| = |E_n+XY|$$

方法二：
$$\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & E_n+XY \end{pmatrix} \Rightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_n+XY|$$

$$\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix} = \begin{pmatrix} E_k+YX & 0 \\ 0 & E_n \end{pmatrix} \Rightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_k+YX|$$

$$\Rightarrow |E_n+XY| = |E_k+YX|$$

注：$X\in\mathbb{R}^{n\times k}$，$Y\in\mathbb{R}^{k\times n}$

$\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}$ 可逆 $\Leftrightarrow$ E_n+XY 可逆 $\Leftrightarrow$ E_k+YX 可逆.

例：计算 $\Delta=\begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & & & \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix}$ $a_i, b_i\in\mathbb{R}$

解：$\Delta=\left|E_n+\begin{pmatrix}a_1\\a_2\\ \vdots\\a_n\end{pmatrix}(b_1\ b_2\cdots b_n)\right|=\left|E_1+(b_1\ b_2\cdots b_n)\begin{pmatrix}a_1\\a_2\\ \vdots\\a_n\end{pmatrix}\right|$

$=1+\sum_{i=1}^{n}a_ib_i$

P_{109}.5. 下面是类似问题：(2) 若 $a_{ii}>\sum_{j\neq i}|a_{ij}|$，$i=1,2,\cdots,n$，则 $|A|>0$

设 $A=(a_{ij})\in M_n(\mathbb{R})$ (1) 若 $|a_{ii}|>\sum_{j\neq i}|a_{ij}|$，$i=1,2,\cdots,n$. ~~证明~~ 则：$|A|\neq 0$.

证：(1)(反证法). 假设 $|A|=0$. 则 $A\vec{x}=\vec{0}$ 存在非零解.

设 $\vec{x}=\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix}$ 是 $A\vec{x}=\vec{0}$ 的一个非零解，且 $|x_k|=\max\{|x_1|,\cdots,|x_n|\}>0$ $(1\le k\le n)$

取方程组第 k 行：$a_{kk}x_k+\sum_{j\neq k}a_{kj}x_j=0$

$\therefore |a_{kk}|\cdot|x_k|=\left|\sum_{j\neq k}a_{kj}x_j\right|\le\sum_{j\neq k}|a_{kj}|\cdot|x_j|\le\sum_{j\neq k}|a_{kj}|\cdot|x_k|=|x_k|\cdot\sum_{j\neq k}|a_{kj}|$

$\because |x_k|\neq 0$ $\therefore |a_{kk}|\le\sum_{j\neq k}|a_{kj}|$ 矛盾. $\therefore |A|\neq 0$.

(2). (反证法) 设 $|A|\le 0$，由 (1) 可知 $|A|\neq 0$. 则 $|A|<0$.

令 $f(t)=\begin{vmatrix} a_{11} & a_{12}t & a_{13}t & \cdots & a_{1n}t \\ a_{21}t & a_{22} & a_{23}t & \cdots & a_{2n}t \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}t & a_{n2}t & a_{n3}t & \cdots & a_{nn} \end{vmatrix}$ $0\le t\le 1$.

则 $f(0)=a_{11}a_{22}\cdots a_{nn}>0$. $f(1)=|A|<0$.

由于 $f(t)$ 是 t 在 $\mathbb{R}$ 上的连续函数，则必存在 $t_1\in(0,1)$ s.t.

$f(t_1)=0$. 此时 $a_{ii}>\sum_{j\neq i}|a_{ij}|\ge\sum_{j\neq i}|a_{ij}t_1|$

由 (1) 知 $f(t_1)\neq 0$ 矛盾. 因此假设不成立，则 $|A|>0$.

P_{102}. 5. 证明

$$B_n(s,t)=\begin{vmatrix}\binom{s}{t} & \binom{s}{t+1} & \cdots & \binom{s}{t+n-1}\\ \binom{s+1}{t} & \binom{s+1}{t+1} & \cdots & \binom{s+1}{t+n-1}\\ \vdots & \vdots & & \vdots\\ \binom{s+n-1}{t} & \binom{s+n-1}{t+1} & \cdots & \binom{s+n-1}{t+n-1}\end{vmatrix}=\frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}$$

证：不妨设 $s\geqslant t$，否则 $B_n(s,t)=0$．

第i行第j列元素为

$$\binom{s+i-1}{t+j-1}=\frac{(s+i-1)!}{(t+j-1)!\,(s+i-t-j)!}=\frac{s+i-1}{t+j-1}\,\frac{(s+i-2)!}{(t+j-2)!\,(s+i-t-j)!}$$
$$=\frac{s+i-1}{t+j-1}\binom{s+i-2}{t+j-2}$$

$B_n(s,t)$ 第i行提出公因子 $s+i-1$，$i=1,\cdots,n$

第j列提出公因子 $\frac{1}{t+j-1}$，$j=1,\cdots,n$

$$B_n(s,t)=\frac{s(s+1)\cdots(s+n-1)}{t(t+1)\cdots(t+n-1)}\underbrace{\begin{vmatrix}\binom{s-1}{t-1} & \binom{s-1}{t} & \cdots & \binom{s-1}{t+n-2}\\ \binom{s}{t-1} & \binom{s}{t} & \cdots & \binom{s}{t+n-2}\\ \vdots & \vdots & & \vdots\\ \binom{s+n-2}{t-1} & \binom{s+n-2}{t} & \cdots & \binom{s+n-2}{t+n-2}\end{vmatrix}}_{=B_n(s-1,t-1)}\quad(*)$$

其中
$$\frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}}=\frac{\frac{(n+s-1)!}{n!\,(s-1)!}}{\frac{(n+t-1)!}{n!\,(t-1)!}}=\frac{s(s+1)\cdots(s+n-1)}{t(t+1)\cdots(t+n-1)}$$

∴ 由$(*)$式可得

$$B_n(s,t)=\frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}}B_n(s-1,t-1)=\frac{\binom{n+s-1}{n}\binom{n+s-2}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}}B_n(s-2,t-2)=\cdots$$

$B_n(s-1,t-1)$ 第i行提出 $s+i-2$，$i=1,\cdots,n$

第j列提出 $\frac{1}{t+j-2}$，$j=1,\cdots,n$

不断从各行各列提公因子

直到第1列全为1

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$$= \frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}} B_n(s-t,0) . \quad (s \geq t)$$

只需证 $B_n(s-t,0)=1$.

$$B_n(s-t,0) = \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-1} \\ 1 & \binom{s-t+1}{1} & \binom{s-t+1}{2} & \cdots & \binom{s-t+1}{n-1} \\ \vdots & & & & \vdots \\ 1 & \binom{s-t+n-1}{1} & \binom{s-t+n-1}{2} & \cdots & \binom{s-t+n-1}{n-1} \end{vmatrix} \xlongequal[\substack{\cancel{i=1,\cdots,n-1} \\ i=n-1,\cdots,1}]{-r_i+r_{i+1}} \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-1} \\ 0 & \binom{s-t}{0} & \binom{s-t}{1} & \cdots & \binom{s-t}{n-2} \\ \vdots & & & & \\ 0 & \binom{s-t+n-2}{0} & \binom{s-t+n-2}{1} & \cdots & \binom{s-t+n-2}{n-2} \end{vmatrix}_{n \times n}$$

(由组长定理 $\binom{s+1}{t} - \binom{s}{t} = \binom{s}{t-1}$)

$$\xlongequal{\text{按第1列展开}} \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-2} \\ 1 & \binom{s-t+1}{1} & \binom{s-t+1}{2} & \cdots & \binom{s-t+1}{n-2} \\ \vdots & & & & \\ 1 & \binom{s-t+n-2}{1} & \binom{s-t+n-2}{2} & \cdots & \binom{s-t+n-2}{n-2} \end{vmatrix}_{(n-1)\times(n-1)} = B_{n-1}(s-t,0)$$

$$\xlongequal[\substack{\cancel{i=1,\cdots,n-2} \\ i=n-2,\cdots,1}]{-r_i+r_{i+1}} \begin{vmatrix} 1 & \binom{s-t}{1} & \binom{s-t}{2} & \cdots & \binom{s-t}{n-2} \\ 0 & \binom{s-t}{0} & \binom{s-t}{1} & \cdots & \binom{s-t}{n-3} \\ \vdots & & & & \\ 0 & \binom{s-t+n-3}{0} & \binom{s-t+n-3}{1} & \cdots & \binom{s-t+n-3}{n-3} \end{vmatrix}_{\substack{\cancel{(n-2)\times(n-2)} \\ (n-1)\times(n-1)}} \xlongequal{\text{按第1列展开}} B_{n-2}(s-t,0)$$

$$= \cdots = B_2(s-t,0) = \begin{vmatrix} \binom{s-t}{0} & \binom{s-t}{1} \\ \binom{s-t+1}{0} & \binom{s-t+1}{1} \end{vmatrix} \xlongequal{-r_1+r_2} \begin{vmatrix} 1 & \binom{s-t}{1} \\ 0 & \binom{s-t}{0} \end{vmatrix} = 1$$

$$\therefore B_n(s,t) = \frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}$$

P_{108}. 1. 证明下述公式：$A,B\in M_n(\mathbb{R})$，$\lambda\in\mathbb{R}$

(1) $(A^t)^\vee=(A^\vee)^t$; (2) $(\lambda A)^\vee=\lambda^{n-1}A^\vee$; (3) $(A^\vee)^\vee=|A|^{n-2}A$; (4) $(AB)^\vee=B^\vee A^\vee$.

证：(1) 设 $B=A^t$

$$B^\vee=\begin{pmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ B_{12} & B_{22} & \cdots & B_{n2} \\ \vdots & \vdots & & \vdots \\ B_{1n} & B_{2n} & \cdots & B_{nn}\end{pmatrix}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn}\end{pmatrix}=(A^\vee)^t$$

注意 $B_{ij}=A_{ji}$

$\downarrow$

去掉 $B=A^t$ 第 i 行第 j 列后代数余子式

= 去掉 A 第 i 列第 j 行后代数余子式

(2) 令 $B=\lambda A$

$$B^\vee=\begin{pmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ B_{12} & B_{22} & \cdots & B_{n2} \\ \vdots & \vdots & & \vdots \\ B_{1n} & B_{2n} & \cdots & B_{nn}\end{pmatrix}=\lambda^{n-1}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn}\end{pmatrix}=\lambda^{n-1}A^\vee$$

$B_{ij}=\lambda^{n-1}A_{ij}$

$\Updownarrow$

去掉 $B=\lambda A$ 第 i 行第 j 列后代数余子式

(3) $(A^\vee)^\vee=|A|^{n-2}A$, $n\geq 2$.

$n=2$. 设 $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$，则 $(A^\vee)^\vee=\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}^\vee=\begin{pmatrix} a & b \\ c & d\end{pmatrix}=A$.

$n\geq 3$ ① $|A|\neq 0$. 由 $AA^\vee=|A|E$ 可得 $A^\vee=|A|A^{-1}$.

$$(A^\vee)^\vee=(|A|A^{-1})^\vee=\big||A|A^{-1}\big|(|A|A^{-1})^{-1}=|A|^n|A|^{-1}\cdot|A|^{-1}A=|A|^{n-2}A$$

② $|A|=0$. 要证 $(A^\vee)^\vee=0$

若 $\operatorname{rank}(A)=n-1$，则 $\operatorname{rank}(A^\vee)=1<n-1$. $\therefore$ $\operatorname{rank}((A^\vee)^\vee)=0$ $\Rightarrow$ $(A^\vee)^\vee=0$

若 $\operatorname{rank}(A)<n-1$，则 $\operatorname{rank}(A^\vee)=0$. i.e. $A^\vee=0$ $\Rightarrow$ $(A^\vee)^\vee=0$.

(4) $(AB)^\vee = B^\vee A^\vee$

① 若 A, B 都可逆．则 AB 可逆．

$(AB)\cdot(AB)^\vee = |AB|E$ 且 $AB\cdot B^\vee A^\vee = A(|B|E)A^\vee = |B|\cdot|A|E = |AB|E$

由 AB 逆矩阵的唯一性得 $(AB)^\vee = B^\vee A^\vee$

② 对一般的矩阵 A, B．令 $A(\lambda) = \lambda E + A$，$B(\lambda) = \lambda E + A$．$\lambda \in R$．

则当 λ 充分大时，$A(\lambda), B(\lambda)$ 可逆（$P_{107}.5$）．由 ① 的结论可得

$\therefore$ $(A(\lambda)B(\lambda))^\vee = B(\lambda)^\vee A(\lambda)^\vee$． $(*)$ 对无穷多个 λ 成立．

等式两边的矩阵中的元素都是关于 λ 的多项式，由 $(*)$ 可得其对应位置的多项式相等．

特别地，常数项对应相等，即 $(A(0)B(0))^\vee = B(0)^\vee A(0)^\vee$．

$\Rightarrow$ $(AB)^\vee = B^\vee A^\vee$．

P101. 4. (柯).

$$\Delta_n(k_1,x_1;\cdots;k_m,x_m)=\begin{vmatrix} M^n_{k_1}(x_1)\\ M^n_{k_2}(x_2)\\ \vdots\\ M^n_{k_m}(x_m)\end{vmatrix},$$

其中 $x_1,\cdots,x_m$ 是未知量，$k_1,\cdots,k_m\in\mathbb{N}$，且 $k_1+\cdots+k_m=n$。

$M^n_k(x)$ 是 $k\times n$ 阶矩阵，形如

$$M^n_k(x)=\begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1}\\ 0 & 1 & \binom{2}{1}x & \cdots & \binom{n-1}{1}x^{n-2}\\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}x^{n-3}\\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & \binom{n-1}{k-1}x^{n-k}\end{pmatrix}$$

证明 $\Delta_n(k_1,x_1;\cdots;k_m,x_m)=\prod_{1\le j<i\le m}(x_i-x_j)^{k_ik_j}$

证：先以 $\begin{cases}k_1=2\\k_2=3\end{cases}$ 为例，观察变换过程

$$\Delta_5(2,x_1;3,x_2)=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4\\ 0 & 1 & \binom{2}{1}x_1 & \binom{3}{1}x_1^2 & \binom{4}{1}x_1^3\\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4\\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2 & \binom{4}{1}x_2^3\\ 0 & 0 & 1 & \binom{3}{2}x_2 & \binom{4}{2}x_2^2\end{vmatrix} \overset{-x_1A^i+A^{i+1}}{\underset{i=3,2,1}{=\!=\!=}} \begin{vmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2-x_1 & (x_2-x_1)x_2 & (x_2-x_1)x_2^2 & (x_2-x_1)x_2^3\\ 0 & 1 & \binom{2}{1}x_2-x_1 & \left[\binom{3}{1}x_2-\binom{2}{1}x_1\right]x_2 & \left[\binom{4}{1}x_2-\binom{3}{1}x_1\right]x_2^2\\ 0 & 0 & 1 & \binom{3}{2}x_2-x_1 & \left[\binom{4}{2}x_2-\binom{3}{2}x_1\right]x_2\end{vmatrix}$$

$$=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ x_2-x_1 & (x_2-x_1)x_2 & (x_2-x_1)x_2^2 & (x_2-x_1)x_2^3\\ 1 & \binom{2}{1}x_2-x_1 & \left[\binom{3}{1}x_2-\binom{2}{1}x_1\right]x_2 & \left[\binom{4}{1}x_2-\binom{3}{1}x_1\right]x_2^2\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \left[\binom{4}{2}x_2-\binom{3}{2}x_1\right]x_2\end{vmatrix} =(x_2-x_1)\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 1 & \binom{2}{1}x_2-x_1 & \left[\binom{3}{1}x_2-\binom{2}{1}x_1\right]x_2 & \left[\binom{4}{1}x_2-\binom{3}{1}x_1\right]x_2^2\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \left[\binom{4}{2}x_2-\binom{3}{2}x_1\right]x_2\end{vmatrix}$$

$$\xlongequal{A_3-A_2}(x_2-x_1)\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 0 & x_2-x_1 & (x_2-x_1)\binom{2}{1}x_2 & (x_2-x_1)\binom{3}{1}x_2^2\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \left[\binom{4}{2}x_2-\binom{3}{2}x_1\right]x_2\end{vmatrix} =(x_2-x_1)^2\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \left[\binom{4}{2}x_2-\binom{3}{2}x_1\right]x_2\end{vmatrix}$$

班长定理化简系数. $\binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k-1}$.

$$\xlongequal{A_4-A_3}(x_2-x_1)^2\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2\\ 0 & 0 & x_2-x_1 & (x_2-x_1)\binom{3}{2}x_2\end{vmatrix} =(x_2-x_1)^3\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 0 & 1 & \binom{2}{1}x_2 & \binom{3}{1}x_2^2\\ 0 & 0 & 1 & \binom{3}{2}x_2\end{vmatrix} =(x_2-x_1)^3\begin{vmatrix} M^{5-1}_{2-1}(x_1)\\ M^{5-1}_{3}(x_2)\end{vmatrix}$$

$$= (x_2-x_1)^3 \begin{vmatrix} M_1^4(x_1) \\ M_3^4(x_2) \end{vmatrix} = (x_2-x_1)^3 (x_2-x_1)^3 = (x_2-x_1)^{3\cdot 2}.$$

对一般情况用数学归纳法证:

$n=2$. 则 $k_1=k_2=1$. $\Delta_2(1,x_1;1,x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = (x_2-x_1)^{1\cdot 1} = x_2-x_1$ 成立.

假设，$n-1$ 时，命题成立.

n. 1). 对 $M_{k_1}^n(x_1)$ 作初等列变换: $M_{k_1}^n(x_1) \cdot F_{n-1,n}(-x_1) \cdot F_{n-2,n-1}(-x_1) \cdots F_{1,2}(-x_1)$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x_1 & x_1^2 & \cdots & x_1^{n-3} & x_1^{n-2} \\ 0 & 0 & 1 & \binom{2}{1}x_1 & \cdots & \binom{n-3}{1}x_1^{n-4} & \binom{n-2}{1}x_1^{n-3} \\ & & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{k_1-2}x_1^{n-k_1-1} & \binom{n-2}{k_1-2}x_1^{n-k_1} \end{pmatrix} = \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{k-1,1} & M_{k_1-1}^{n-1}(x_1) \end{pmatrix}$$

$k_1=1$ 时，$M_{k_1-1}^{n-1}(x_1)$ 退化，则 $M_{k_1}^n(x_1) F_{n-1,n}(-x_1) \cdots F_{1,2}(-x_1) = (1, 0, \cdots, 0)$

（这里 $F_{n-1,n}(-x_1), \cdots, F_{1,2}(-x_1)$ 为初等矩阵，$n\times n$ 阶. $F_{ij}(-x_1)$ 表示第 i 列乘 $(-x_1)$ 加到第 j 列）.

2). 对 $M_{k_i}^n(x_i)$ 作对应的初等列变换:

$M_{k_i}^n(x_i) F_{n-1,n}(-x_1) F_{n-2,n-1}(-x_1) \cdots F_{1,2}(-x_1)$.

$$= \begin{pmatrix} 1 & x_i-x_1 & (x_i-x_1)x_i & (x_i-x_1)x_i^2 & \cdots & (x_i-x_1)x_i^{n-3} & (x_i-x_1)x_i^{n-2} \\ 0 & 1 & \binom{2}{1}x_i-x_1 & \left[\binom{3}{1}x_i-\binom{2}{1}x_1\right]x_i & \cdots & \left[\binom{n-2}{1}x_i-\binom{n-3}{1}x_1\right]x_i^{n-4}, & \left[\binom{n-1}{1}x_i-\binom{n-2}{1}x_1\right]x_i^{n-3} \\ 0 & 0 & 1 & \binom{3}{2}x_i-x_1 & \cdots & \left[\binom{n-2}{2}x_i-\binom{n-3}{2}x_1\right]x_i^{n-5}, & \left[\binom{n-1}{2}x_i-\binom{n-2}{2}x_1\right]x_i^{n-4} \\ & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_i-1}x_i-\binom{n-3}{k_i-1}x_1\right]x_i^{n-k_i-2}, & \left[\binom{n-1}{k_i-1}x_i-\binom{n-2}{k_i-1}x_1\right]x_i^{n-k_i-1} \end{pmatrix}$$

$$\triangleq \begin{pmatrix} 1 & (x_i-x_1)(1 \quad x_i \quad x_i^2 \cdots x_i^{n-3} \quad x_i^{n-2}) \\ 0 & N(k_i, x_i, x_1) \end{pmatrix}.$$

其中 $N(k_i, x_i, x_1)$ 为 $(k_i-1)\times(n-1)$ 阶矩阵.

3). $$\Delta_n(k_1,x_1;\cdots,k_m,x_m)=\begin{vmatrix} M^n_{k_1}(x_1)\\ \vdots\\ M^n_{k_m}(x_m)\end{vmatrix}=\left|\begin{pmatrix} M^n_{k_1}(x_1)\\ \vdots\\ M^n_{k_m}(x_m)\end{pmatrix}F_{n-1,n}(-x_1)F_{n-2,n-1}(-x_1)\cdots F_{1,2}(-x_1)\right|$$

$$=\begin{vmatrix} \begin{pmatrix} 1 & 0\\ 0 & M^{n-1}_{k_1-1}(x_1)\end{pmatrix}\\ \begin{pmatrix} 1 & (x_2-x_1)(1,x_2,x_2^2,\cdots,x_2^{n-2})\\ 0 & N(k_2,x_2,x_1)\end{pmatrix}\\ \cdots\\ \begin{pmatrix} 1 & (x_m-x_1)(1,x_m,x_m^2,\cdots,x_m^{n-2})\\ 0 & N(k_m,x_m,x_1)\end{pmatrix}\end{vmatrix}_{n\times n} \overset{\text{第一行展开}}{=\!=\!=} \begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix}(x_2-x_1)(1,x_2,\cdots,x_2^{n-2})\\ N(k_2,x_2,x_1)\end{pmatrix}\\ \cdots\\ \begin{pmatrix}(x_m-x_1)(1,x_m,\cdots,x_m^{n-2})\\ N(k_m,x_m,x_1)\end{pmatrix}\end{vmatrix}_{(n-1)\times(n-1)}.$$

$$=(x_2-x_1)(x_3-x_1)\cdots(x_m-x_1)\begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix}1 & x_2 & x_2^2 & \cdots & x_2^{n-2}\\ & & N(k_2,x_2,x_1) & & \end{pmatrix}\\ \cdots\\ \begin{pmatrix}1 & x_m & x_m^2 & \cdots & x_m^{n-2}\\ & & N(k_m,x_m,x_1) & & \end{pmatrix}\end{vmatrix}_{(n-1)\times(n-1)}. \qquad (*)$$

$$=\left[\prod_{i=2}^{m}(x_i-x_1)\right]\begin{vmatrix} M^{n-1}_{k_1-1}(x_1)\\ \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2}\\ 1 & \binom{2}{1}x_2-x_1 & \left[\binom{3}{1}x_2-\binom{2}{1}x_1\right]x_2 & \cdots & \left[\binom{n-2}{1}x_2-\binom{n-3}{1}x_1\right]x_2^{n-4}, & \left[\binom{n-1}{1}x_2-\binom{n-2}{1}x_1\right]x_2^{n-3}\\ 0 & 1 & \binom{3}{2}x_2-x_1 & \cdots & \left[\binom{n-2}{2}x_2-\binom{n-3}{2}x_1\right]x_2^{n-5}, & \left[\binom{n-1}{2}x_2-\binom{n-2}{2}x_1\right]x_2^{n-4}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1}, & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2}\\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2}, & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1}\end{pmatrix}\\ * \end{vmatrix}$$

利用 班长定理: $\binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k}$ 化简行列式中元素系数.

$r_{k_1+1}-r_{k_1}$ 并将所得第 k_1+1 行的公因子 (x_2-x_1) 提到行列式外

$$= \left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)\begin{array}{c} M_{k_1-1}^{n-1}(x_1) \\ \left|\begin{array}{cccccc} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\ 0 & 1 & \binom{3}{2}x_2-x_1 & \cdots & \left[\binom{n-2}{2}x_2-\binom{n-3}{2}x_1\right]x_2^{n-5} & \left[\binom{n-1}{2}x_2-\binom{n-2}{2}x_1\right]x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1}, & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2}, & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1} \\ & & & * & & \end{array}\right| \end{array}$$

$r_{k_1+2}-r_{k_1+1}$ 并将所得第 k_1+2 行的公因子 (x_2-x_1) 提到行列式外

$$= \left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^2\begin{array}{c} M_{k_1-1}^{n-1}(x_1) \\ \left|\begin{array}{cccccc} 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ 0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2 & \binom{n-2}{1}x_2^{n-3} \\ 0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\ & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-2}x_2-\binom{n-3}{k_2-2}x_1\right]x_2^{n-k_2-1} & \left[\binom{n-1}{k_2-2}x_2-\binom{n-2}{k_2-2}x_1\right]x_2^{n-k_2} \\ 0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2} & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1} \\ & & & * & & \end{array}\right| \end{array}$$

重复上述过程

$= \cdots\cdots$

$r_{k_2+k_1-2} - r_{k_2+k_1-3}$

并将所得第 k_2+k_1-2 行中公因子 (x_2-x_1) 提到行列式之外

$$\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-2}
\left|\begin{array}{c}
M_{k_1-1}^{n-1}(x_1) \\
\begin{pmatrix}
1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\
0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\
0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\
 & \cdots & & & & \\
0 & 0 & 0 & \cdots & \binom{n-3}{k_2-2}x_2^{n-k_2-1} & \binom{n-2}{k_2-2}x_2^{n-k_2} \\
0 & 0 & 0 & \cdots & \left[\binom{n-2}{k_2-1}x_2-\binom{n-3}{k_2-1}x_1\right]x_2^{n-k_2-2} & \left[\binom{n-1}{k_2-1}x_2-\binom{n-2}{k_2-1}x_1\right]x_2^{n-k_2-1}
\end{pmatrix} \\
*
\end{array}\right|$$

$r_{k_2+k_1-1} - r_{k_2+k_1-2}$

并将第 k_2+k_1-1 行公因子 (x_2-x_1) 提到行列式外

$$\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}
\left|\begin{array}{c}
M_{k_1-1}^{n-1}(x_1) \\
\begin{pmatrix}
1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{n-2} \\
0 & 1 & \binom{2}{1}x_2 & \cdots & \binom{n-3}{1}x_2^{n-4} & \binom{n-2}{1}x_2^{n-3} \\
0 & 0 & 1 & \cdots & \binom{n-3}{2}x_2^{n-5} & \binom{n-2}{2}x_2^{n-4} \\
 & \cdots & & & & \\
0 & 0 & 0 & \cdots & \binom{n-3}{k_2-2}x_2^{n-k_2-1} & \binom{n-2}{k_2-2}x_2^{n-k_2} \\
0 & 0 & 0 & \cdots & \binom{n-3}{k_2-1}x_2^{n-k_2-2} & \binom{n-2}{k_2-1}x_2^{n-k_2-1}
\end{pmatrix} \\
*
\end{array}\right|$$

$$=\left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}
\left|\begin{array}{c}
M_{k_1-1}^{n-1}(x_1) \\
M_{k_2}^{n-1}(x_2) \\
\begin{pmatrix} 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ & & N(k_3, x_3, x_1) & & \end{pmatrix} \\
\begin{pmatrix} 1 & x_m & x_m^2 & \cdots & x_m^{n-2} \\ & & N(k_m, x_m, x_1) & & \end{pmatrix}
\end{array}\right|$$

类似做法

对 $N(k_i, x_i, x_1)$ 作行变换

$$\xrightarrow{i=3,\cdots,m} \left[\prod_{i=2}^{m}(x_i-x_1)\right](x_2-x_1)^{k_2-1}\cdots(x_m-x_1)^{k_m-1}
\left|\begin{array}{c}
M_{k_1-1}^{n-1}(x_1) \\
M_{k_2}^{n-1}(x_2) \\
M_{k_3}^{n-1}(x_3) \\
\cdots \\
M_{k_m}^{n-1}(x_m)
\end{array}\right|$$

$$= \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i}\right] \cdot \left[\prod_{2 \le j < i \le m} (x_i - x_j)^{k_i k_j}\right] \cdot \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i(k_1 - 1)}\right]$$

$$= \left[\prod_{i=2}^{m} (x_i - x_1)^{k_i k_1}\right] \left[\prod_{2 \le j < i \le m} (x_i - x_j)^{k_i k_j}\right]$$

$$= \prod_{1 \le j < i \le m} (x_i - x_j)^{k_i k_j} \qquad \square .$$