第二次作业.

1. (1) $D_0=\{A\in M_n(F)\mid \det(A)=0\}$ 证:D_0是$M_n(F)$的子空间 $\Leftrightarrow n=1$.

Pf: 当$n=1$时, det是恒同映射. 当$n>1$时,

$$\mathrm{diag}_n(1,0,\cdots,0)+\mathrm{diag}_n(0,1,\cdots,1)=E_n.$$

于是D_0加法不封闭, 不是子空间.

(2) $T_0=\{A\in M_n(F)\mid tr(A)=0\}$. 证: T_0是$M_n(F)$的子空间.

Pf: 设$A=(a_{ij})$, $B=(b_{ij})$是T_0中矩阵, $\alpha,\beta\in F$.

(直接法) $tr(\alpha A+\beta B)=\alpha tr(A)+\beta tr(B)=0 \Rightarrow \alpha A+\beta B\in T_0$.

(映射法) $T_0=\ker(tr)$且tr是线性的, $\therefore T_0$是子空间.

(解空间法) T_0是关于未知数a_{ij}, $i=1,\cdots,n$, $j=1,\cdots,n$的方程

$$a_{11}+a_{22}+\cdots+a_{nn}=0$$

在$F^{n\times n}$中解空间. $\therefore$是子空间.

2. $\forall A\in M_n(F)$, 设$A=(a_{ij})$.

令 $S=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1}\\ a_{21} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}\in SM_n(F)$ 为对称矩阵. $U=\begin{pmatrix} 0 & a_{12}-a_{21} & \cdots & a_{1n}-a_{n1}\\ & 0 & \cdots & a_{2n}-a_{n2}\\ & & \ddots & \vdots\\ & & & 0\end{pmatrix}\in UM_n(F)$

$\therefore A=S+U. \Rightarrow M_n(F)=SM_n(F)+UM_n(F)$

$\forall A\in SM_n(F)\cap UM_n(F)$. $\because A=(a_{ij})\in UM_n(F)$ $\therefore a_{ij}=0\ \forall i>j$

又$\because A\in SM_n(F)$. $\therefore a_{ij}=a_{ji}$. $\forall i,j$成立. $\therefore a_{ij}=0$. $\forall i,j$.

$\therefore A=0$. 即 $SM_n(F)\cap UM_n(F)=\{0\}$

$$\Rightarrow M_n(F)=SM_n(F)\oplus UM_n(F)$$

3. (1) $U\cap V \subset U\cap(V+W)$, $U\cap W \subset U\cap(V+W)$ $\therefore$ $(U\cap V)+(U\cap W) \subset U\cap(V+W)$

但反过来包含关系不成立.

例: $\mathbb{R}^2$. $U=\langle\binom{1}{1}\rangle$. $V=\langle\binom{1}{0}\rangle$. $W=\langle\binom{0}{1}\rangle$.

$U\cap V = U\cap W=\{\vec{0}\}$ $\therefore$ $(U\cap V)+(U\cap W)=\{\vec{0}\}$.

而 $U\cap(V+W)=U\cap\mathbb{R}^2=U=\langle\binom{1}{1}\rangle$

(2) 取上述例子即可. $U\oplus V=U\oplus W=\mathbb{R}^2$, 但 $V\neq W$.

4. (1) $\sin x, \sin 2x, \dots, \sin nx$ 在 $\mathbb{R}$ 上线无

法一: (归纳) $n=1$. ✓

设 $n-1$ 时成立, $n>1$.

设 $\beta_1\sin x+\cdots+\beta_{n-1}\sin(n-1)x+\beta_n\sin nx=0$. $\beta_1,\dots,\beta_{n-1},\beta_n\in\mathbb{R}$ 不全为0

由归纳假设 $\beta_n\neq 0$, 故可设 $\beta_n=1$.

$\Rightarrow$ $\sin nx=-\beta_1\sin x-\cdots-\beta_{n-1}\sin(n-1)x$

求两次导. $-n^2\sin nx=\beta_1\sin x+\cdots+(n-1)^2\beta_{n-1}\sin(n-1)x$

$$\sin nx=-\frac{\beta_1}{n^2}\sin x-\cdots-\frac{(n-1)^2}{n^2}\beta_{n-1}\sin(n-1)x$$

再由归纳假设 $\frac{\beta_1}{n^2}=\beta_1$, $\frac{2^2\beta_2}{n^2}=\beta_2$, $\dots$, $\frac{(n-1)^2}{n^2}\beta_{n-1}=\beta_{n-1}$

不妨设 $\beta_i\neq 0$ $(1\le i\le n-1)$. $\frac{i^2}{n^2}\beta_i=\beta_i \Rightarrow \frac{i^2}{n^2}=1$. $\longrightarrow\!\leftarrow$

法二: (定积分).

$$\int_0^{2\pi}\sin jt\sin kt\,dt=\begin{cases}\pi, & j=k\\ 0, & j\neq k\end{cases}$$

设 $\beta_1\sin x+\beta_2\sin 2x+\cdots+\beta_n\sin nx=0$, $\beta_i\in\mathbb{R}$.

左右两边同乘 $\sin jx$.

$\beta_1\sin x\sin jx+\cdots+\beta_n\sin nx\sin jx=0$

取 $j=1,2,\dots,n$, 然后在 $[0,2\pi]$ 做定积分, 则

$\beta_1\pi=\beta_2\pi=\cdots=\beta_n\pi=0 \Rightarrow \beta_1=\cdots=\beta_n=0$

(2) 若 $a_1, \cdots, a_n$ 中有两个相同，则 $x^{a_1}, \cdots, x^{a_n}$ 线性相关.

下证：$a_1, \cdots, a_n$ 两两不同时，$x^{a_1}, \cdots, x^{a_n}$ 线性无关.

法一：设 $\beta_1 x^{a_1} + \beta_2 x^{a_2} + \cdots + \beta_n x^{a_n} = 0$. $\beta_i \in \mathbb{R}$.

取 $x = 2^i$, $i = 0, 1, 2, \cdots, n-1$.

$$\beta_1 (2^{a_1})^i + \beta_2 (2^{a_2})^i + \cdots + \beta_n (2^{a_n})^i = 0.$$

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2^{a_1} & 2^{a_2} & \cdots & 2^{a_n} \\ \vdots & \vdots & & \vdots \\ (2^{a_1})^{n-1} & (2^{a_2})^{n-1} & \cdots & (2^{a_n})^{n-1} \end{pmatrix}}_{\substack{\parallel \\ A}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

由 Vandermonde 行列式性质可知 $\det(A) \neq 0$. $\therefore$ $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

法二：不妨设 $a_1 > a_2 > \cdots > a_n$.

设 $\beta_1 x^{a_1} + \beta_2 x^{a_2} + \cdots + \beta_n x^{a_n} = 0$, $\beta_i \in \mathbb{R}$.

则 $\beta_1 + \beta_2 x^{a_2 - a_1} + \cdots + \beta_n x^{a_n - a_1} = 0$. $x \to \infty$. $\Rightarrow$ $\cancel{\alpha_1}\beta_1 = 0$.

依次类推可知

$$\beta_2 = 0, \cdots, \beta_{n-1} = 0 \Rightarrow \beta_n x^{a_n - a_1} = 0 \Rightarrow \beta_n = 0.$$

5. (1) $P_n = \{ f(t) \in K[t] \mid \deg(f) \le n-1 \}$. $\varphi: P_n \longrightarrow P_n$

$f(t) \longmapsto t f'(t) - f(t)$

$\forall \alpha, \beta \in K$. $\forall f(t), g(t) \in P_n$.

$$\begin{aligned}
\varphi(\alpha f(t) + \beta g(t)) &= t(\alpha f(t) + \beta g(t))' - (\alpha f(t) + \beta g(t)) \\
&= t(\alpha f'(t) + \beta g'(t)) - (\alpha f(t) + \beta g(t)) \\
&= \alpha(t f'(t) - f(t)) + \beta(t g'(t) - g(t)) \\
&= \alpha \varphi(f(t)) + \beta \varphi(g(t))
\end{aligned}$$

$\therefore \varphi$ 是线性映射.

(2)

法一: 设 $f(t) = a_{n-1} t^{n-1} + a_{n-2} t^{n-2} + \cdots + a_1 t + a_0$, $a_i \in K$.

$f'(t) = (n-1) a_{n-1} t^{n-2} + (n-2) a_{n-2} t^{n-3} + \cdots + a_1$

若 $t f'(t) - f(t) = (n-1) a_{n-1} t^{n-1} + (n-2) a_{n-2} t^{n-2} + \cdots + a_1 t - (a_{n-1} t^{n-1} + \cdots + a_1 t + a_0)$

$= (n-2) a_{n-1} t^{n-1} + (n-3) a_{n-2} t^{n-2} + \cdots + a_2 t - a_0 = 0$

$\Rightarrow a_0 = 0$. $(i-1) a_i = 0$, $i = 2, \cdots, n-1$.

分类: ① $\mathrm{char}\, K = 0$. 则 $a_2 = a_3 = \cdots = a_{n-1} = 0$.

$\ker \varphi = \{ a_1 t \mid a_1 \in K \} = \langle t \rangle$. 由 t 生成的子空间.

② $\mathrm{char}\, K = p$. 则显然 $(i-1) a_i = 0$. $\forall p \mid i-1$.

$\therefore (i-1) a_i = 0$. $p \nmid i-1 \Rightarrow a_i = 0$: $p \nmid i-1$.

$\ker \varphi = \langle \{ t^{kp+1} \mid k = 0, 1, \cdots, kp+1 \le n-1 \} \rangle$.

法二: $\varphi: P_n \longrightarrow P_n$

$f(t) \longmapsto t f'(t) - f(t) = t^2 \cdot \left(\frac{f(t)}{t} \right)'$

注: 设 $\mathrm{char}\, K = p \ge 0$

$\ker \varphi = \{ f(t) \mid \left(\frac{f(t)}{t} \right)' = 0 \}$. 其中. $\left(\frac{f(t)}{t} \right)' = 0 \Rightarrow \frac{f(t)}{t} \in \langle \{ t^{kp} \mid k = 0, 1 \cdots \} \rangle$

$\Rightarrow f(t) \in \langle \{ t^{kp+1} \mid k = 0, 1, \cdots \} \rangle$.

$\therefore \ker \varphi = \langle \{ t^{kp+1} \mid k = 0, 1 \cdots kp+1 \le n-1 \} \rangle$.