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1. (i) $(\vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3})=(\vec{e_1},\vec{e_2},\vec{e_3})\begin{pmatrix}1&2&1\\0&1&1\\-1&1&1\end{pmatrix}$

$(\vec{\beta_1},\vec{\beta_2},\vec{\beta_3})=(\vec{e_1},\vec{e_2},\vec{e_3})\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}$

$\therefore (\vec{\beta_1},\vec{\beta_2},\vec{\beta_3})=(\vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3})\begin{pmatrix}1&2&1\\0&1&1\\-1&1&1\end{pmatrix}^{-1}\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}$

$$\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\-1&1&1&0&0&1\end{pmatrix}\to\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\0&3&2&1&0&1\end{pmatrix}\to\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\0&0&1&-1&3&-1\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&1&1&0&0\\0&1&0&1&-2&1\\0&0&1&-1&3&-1\end{pmatrix}\to\begin{pmatrix}1&0&0&0&1&-1\\0&1&0&1&-2&1\\0&0&1&-1&3&-1\end{pmatrix}$$

$\therefore (\vec{\beta_1},\vec{\beta_2},\vec{\beta_3})=(\vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3})\begin{pmatrix}0&1&-1\\1&-2&1\\-1&3&-1\end{pmatrix}\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}=(\vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3})\begin{pmatrix}0&1&1\\-1&-3&-2\\2&4&4\end{pmatrix}$

$\therefore$ 过渡矩阵为 $\begin{pmatrix}0&1&1\\-1&-3&-2\\2&4&4\end{pmatrix}$

(ii) 在$(\alpha_1,\alpha_2,\alpha_3)$下坐标:

$$(x_1,x_2,x_3)^t=\begin{pmatrix}1&2&1\\0&1&1\\-1&1&1\end{pmatrix}^{-1}\begin{pmatrix}2\\5\\3\end{pmatrix}=\begin{pmatrix}0&1&-1\\1&-2&1\\-1&3&-1\end{pmatrix}\begin{pmatrix}2\\5\\3\end{pmatrix}=\begin{pmatrix}2\\-5\\10\end{pmatrix}$$

在$(\beta_1,\beta_2,\beta_3)$下坐标

$$(y_1,y_2,y_3)^t=\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}^{-1}\begin{pmatrix}2\\5\\3\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}&-\frac{1}{2}&\frac{3}{2}\\-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\begin{pmatrix}2\\5\\3\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$$

2. (i) $\forall f_1, f_2\in R[x]^{(n)}$

$\cancel{\varphi(\alpha f+\beta g)}$

$\varphi(\alpha f_1+\beta f_2)=(\alpha f_1+\beta f_2)''=\alpha f_1''+\beta f_2''=\alpha\varphi(f_1)+\beta\varphi(f_2)$

$\therefore \varphi$ 是线性映射

$\varphi(1)=0 \quad \varphi(x)=0 \quad \varphi(x^2)=2 \cdots \varphi(x^{n-1})=(n-1)(n-2)x^{n-3}$

$$\therefore (\varphi(1),\varphi(x),\cdots \varphi(x^{n-1}))=(1,x,x^2,\cdots x^{n-1})\begin{pmatrix} 0 & 0 & 2 & & \\ 0 & 0 & & 6 & \\ \vdots & \vdots & & & \ddots \\ & & & & (n-1)(n-2) \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(iii) ~~不是直和~~

~~$\because R[x]^{(n)}\{f\in R[x] \mid \deg(f)<n\}$~~

当 $n=2$ 时 $\quad im(\varphi)=\{0\}$

$\quad ker(\varphi)=R[x]^{(2)}$

$\therefore ker(\varphi)\cap im(\varphi)=\{0\}$

$\therefore ker(\varphi)+im(\varphi)$ 是直和

当 $n\geq 3$ 时: $\quad im(\varphi)=R[x]^{(n-2)}$

$\quad ker(\varphi)=R[x]^{(2)}$

$\therefore ker(\varphi)\cap im(\varphi)=R$ 或 $ker(\varphi)\cap im(\varphi)=R[x]^{(2)}$

$\therefore ker(\varphi)+im(\varphi)$ 不是直和

3. "$\Leftarrow$" $\because \exists \lambda\in F$ s.t $f=\lambda g$

$\therefore \forall \vec{x}\in ker f \quad f(\vec{x})=\vec{0}$

$\therefore (\lambda g)(\vec{x})=\vec{0}$

$\therefore \lambda g(\vec{x})=\vec{0} \quad \therefore g(\vec{x})=\vec{0} \quad \therefore \vec{x}\in ker g \quad \therefore ker f\subset ker g$

$\forall \vec{y}\in ker g. \quad g(\vec{y})=\vec{0}$

$\therefore \lambda g(\vec{y})=\vec{0}$

$\therefore (\lambda g)(\vec{y})=\vec{0} \Rightarrow f(\vec{y})=\vec{0} \quad \therefore \vec{y}\in ker f \quad \therefore ker g\subset ker f$

$\therefore ker f=ker g.$

"$\Rightarrow$" $\because f,g\in V^*\setminus\{0^*\}$

$\therefore$ ~~dim im~~

$\dim im f)=\dim im g)=1$

~~$\therefore \dim ker f$~~

$\therefore f, g$ 都是满射.

$\therefore \exists \vec{v_0}\in V$ s.t $f(\vec{v_0})=1.\in F.$ ~~和方合 $g(\vec{v_0})=$~~

设 $\vec{v}\in V$.

若 $\vec{v}\in\ker f$ 则 $\vec{v}\in\ker g$

$\because f(\vec{v})=\vec{0}$

$\therefore g(\vec{v})=\vec{0}$

$\therefore f(\vec{v})=\lambda g(\vec{v})=\vec{0}$

若 $\vec{v}\notin\ker f$. 则 $f(\vec{v}-f(\vec{v})\cdot\vec{v_0})=f(\vec{v})-f(\vec{v})f(\vec{v_0})=\vec{0}$

$\therefore \vec{v}-f(\vec{v})\vec{v_0}\in\ker f$

$\therefore \vec{v}-f(\vec{v})\vec{v_0}\in \ker g$

$\therefore g(\vec{v}-f(\vec{v})\vec{v_0})=g(\vec{v})-g(\vec{v_0})\cdot f(\vec{v})=\vec{0}$

$\therefore g(\vec{v})=g(\vec{v_0})f(\vec{v})=\vec{0}$

$\because g(\vec{v_0})\in F$

$\therefore f(\vec{v})=g(\vec{v_0})^{-1}g(\vec{v})$ 令 $g(\vec{v_0})^{-1}=\lambda$ 则 $f(\vec{v})=\lambda g(\vec{v})$

$\therefore \exists\lambda\in F$ s.t. $f=\lambda g$ □

4. ~~$\vec{x}=(\vec{e_1},\vec{e_2},\vec{e_3}$~~

$\because \vec{x}=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}$, $\vec{y}=$ ~~$(\vec{e_1}\vec{e_2}$~~ $y_1\vec{e_1}+y_2\vec{e_2}+y_3\vec{e_3}$

$\therefore f(\vec{x},\vec{y})=(x_1,x_2,x_3)A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=5x_1y_2-3x_3y_1+x_3y_3$

$$\therefore A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}5y_2\\0\\-3y_1+y_3\end{pmatrix}$$

$$\therefore A=\begin{pmatrix}0&5&0\\0&0&0\\-3&0&1\end{pmatrix}$$ $\therefore f$ 在标准基下的矩阵为 $\begin{pmatrix}0&5&0\\0&0&0\\-3&0&1\end{pmatrix}$

5. (i) 证明:

由维数公式: $\dim(V_1+V_2)=\dim V_1+\dim V_2-\dim(V_1\cap V_2)$

$=k+k-(k-1)=k+1$

同理 $\dim(V_2+V_3)=\dim(V_1+V_3)=k+1$.

(iii)

$$\dim(V_1+V_2+V_3) = \dim((V_1+V_2)+V_3) = \dim(V_1+V_2)+\dim V_3 - \dim((V_1+V_2)\cap V_3)$$
$$= k+1+k-\dim((V_1+V_2)\cap V_3)$$
$$= 2k+1-\dim((V_1+V_2)\cap V_3)$$

$\because \forall \vec{v} \in V_1\cap V_3+V_2\cap V_3$

$\exists \vec{x} \in V_1\cap V_3,\ \vec{y}\in V_2\cap V_3$ s.t. $\vec{v}=\vec{x}+\vec{y}$

又$\because \vec{x}\in V_1,\ \vec{y}\in V_2$

$\therefore \vec{v}=\vec{x}+\vec{y}\in V_1+V_2$

又$\because \vec{x}\in V_3,\ \vec{y}\in V_3$

$\therefore \vec{v}=\vec{x}+\vec{y}\in V_3$

$\therefore \vec{v}\in (V_1+V_2)\cap V_3$

$\therefore V_1\cap V_3+V_2\cap V_3 \subseteq (V_1+V_2)\cap V_3$

$\therefore \dim(V_1\cap V_3+V_2\cap V_3)\le \dim((V_1+V_2)\cap V_3)$

$\because V_1\subset V_1+V_2 \Rightarrow V_1\cap V_3\subset (V_1+V_2)\cap V_3$

$V_2\subset V_1+V_2 \Rightarrow V_2\cap V_3\subset (V_1+V_2)\cap V_3$

$\therefore V_1\cap V_3+V_2\cap V_3\subset (V_1+V_2)\cap V_3$

$\therefore 2k+1-\dim(V_1+V_2+V_3)\ge \dim(V_1\cap V_3+V_2\cap V_3)$
$$= \dim(V_1\cap V_3)+\dim(V_2\cap V_3)-\dim(V_1\cap V_2\cap V_3)$$
$$= 2k-2-\dim(V_1\cap V_2\cap V_3)$$

$\therefore \dim(V_1+V_2+V_3)\le 3+\dim(V_1\cap V_2\cap V_3)$

$\because V_1+V_2\subseteq V_1+V_2+V_3 \qquad V_1\cap V_2\cap V_3\subseteq V_1\cap V_2$

$\therefore \dim(V_1+V_2)\le \dim(V_1+V_2+V_3)\le 3+\dim(V_1\cap V_2\cap V_3)\le 3+\dim(V_1\cap V_2)$

$\therefore k+1\le \dim(V_1+V_2+V_3)\le 3+\dim(V_1\cap V_2\cap V_3)\le k+2.$

当 $\dim(V_1+V_2+V_3)=k+1$ 时. 结论成立

当 $\dim(V_1+V_2+V_3)=k+2$ 时: $3+\dim(V_1\cap V_2\cap V_3)=k+2$

$\therefore \dim(V_1\cap V_2\cap V_3)=k-1$ 结论成立 □

补充：

(“$\Leftarrow$”易证)

3. $\dim_F(V)<\infty$，$f,g\in V^*\setminus\{0^*\}$. 证：$\ker(f)=\ker(g) \iff \exists\lambda\in F$ s.t. $f=\lambda g$.

Pf: “$\Rightarrow$” $\mathrm{im}(f)$ 是 F 子空间. 且 $\dim_F(F)=1$. $\therefore \dim(\mathrm{im}(f))=0$ or 1.

$\because f\neq 0^*$. $\therefore \dim(\mathrm{im}(f))=1$. 同理 $\dim(\mathrm{im}(g))=1$. i.e. f, g 是满射.

方法一：$\because f$ 是满射. $\therefore \exists\, \vec{v}_0\in V$ s.t. $f(\vec{v}_0)=1$. 令 $\lambda_1=g(\vec{v}_0)$.

$\forall\, \vec{v}\in V$. $f(\vec{v}-f(\vec{v})\vec{v}_0)=f(\vec{v})-f(\vec{v})f(\vec{v}_0)=0$.

$\therefore \vec{v}-f(\vec{v})\vec{v}_0\in\ker(f)=\ker(g)$. $\Rightarrow g(\vec{v}-f(\vec{v})\vec{v}_0)=0$

$\Rightarrow g(\vec{v})=f(\vec{v})g(\vec{v}_0)=\lambda_1 f(\vec{v})$. 取 $\lambda=\lambda_1^{-1}$ 即可.

$\Rightarrow g=\lambda_1 f$. ($\lambda_1\neq 0$. 否则 $g=0$)

方法二：设 $\dim_F(V)=n$. $\vec{e}_1,\dots,\vec{e}_n$ 是 V 一组基. $\vec{e}_1^*,\dots,\vec{e}_n^*$ 是其对偶基.

设 $f=\alpha_1\vec{e}_1^*+\cdots+\alpha_n\vec{e}_n^*$，$g=\beta_1\vec{e}_1^*+\cdots+\beta_n\vec{e}_n^*$. $\alpha_i,\beta_i\in F$.

$\forall\, \vec{x}\in V$，设 $\vec{x}=x_1\vec{e}_1+\cdots+x_n\vec{e}_n=(\vec{e}_1,\dots,\vec{e}_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$.

$\Rightarrow f(\vec{x})=\alpha_1x_1+\cdots+\alpha_nx_n$，$g(\vec{x})=\beta_1x_1+\cdots+\beta_nx_n$. ($\because \vec{e}_i^*(\vec{x})=x_i$)

i.e. $f: V\longrightarrow F$ $\qquad$ $g: V\longrightarrow F$

$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\longmapsto \alpha_1x_1+\cdots+\alpha_nx_n$ $\qquad$ $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\longmapsto \beta_1x_1+\cdots+\beta_nx_n$

记 $K=\ker(f)=\ker(g)$. 由上面分析可知 $\dim(K)=n-1$

$$K=\left\{\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\ \middle|\ \alpha_1x_1+\cdots+\alpha_nx_n=0\right\}=\left\{\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\ \middle|\ \beta_1x_1+\cdots+\beta_nx_n=0\right\}.$$

i.e. K 是齐次方程 $\alpha_1x_1+\cdots+\alpha_nx_n=0$ ~~和~~ 的解空间

也是 $\cdots\cdots$ $\beta_1x_1+\cdots+\beta_nx_n=0$ $\cdots\cdots$

设 K 的一组基为 $\vec{x}_1=\begin{pmatrix}x_{11}\\ x_{12}\\ \vdots\\ x_{1n}\end{pmatrix}$，$\vec{x}_2=\begin{pmatrix}x_{21}\\ x_{22}\\ \vdots\\ x_{2n}\end{pmatrix}$，$\cdots$，$\vec{x}_{n-1}=\begin{pmatrix}x_{n-1,1}\\ x_{n-1,2}\\ \vdots\\ x_{n-1,n}\end{pmatrix}$ $\qquad$ $\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix},\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}\in \mathrm{Sol}(A\vec{y}=0)$

则 $\overbrace{\begin{pmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & & & \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n}\end{pmatrix}}^{A}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}$ $\qquad$ $\mathrm{rank}(A)=n-1 \Rightarrow \dim(\mathrm{Sol}(A\vec{y}=0))=1$

$\therefore \begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}=\lambda\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}$，$\lambda\in F \Rightarrow f=\lambda g$.

5. $\dim(V_i)=k$, $i=1,2,3$, $k>1$. 设 $\dim(V_1\cap V_2)=\dim(V_2\cap V_3)=\dim(V_3\cap V_1)=k-1$.

① 证 $\dim(V_1+V_2)=\dim(V_1+V_3)=\dim(V_2+V_3)=k+1$

② 证 $\dim(V_1\cap V_2\cap V_3)=k-1$ or $\dim(V_1+V_2+V_3)=k+1$.

Pf: ① 维数公式.

$\dim(V_1+V_2)=\dim(V_1)+\dim(V_2)-\dim(V_1\cap V_2)=2k-(k-1)=k+1$.

其它同理可证.

② 法一: $\dim(V_1+V_2+V_3)=\dim((V_1+V_2)+V_3)=\dim(V_1+V_2)+\dim(V_3)-\dim((V_1+V_2)\cap V_3)$

$=k+1+k-\dim((V_1+V_2)\cap V_3)$. (*)

$\because V_1\subset V_1+V_2$, $V_2\subset V_1+V_2$ $\therefore V_1\cap V_3\subset (V_1+V_2)\cap V_3$, $V_2\cap V_3\subset (V_1+V_2)\cap V_3$

$\Rightarrow (V_1\cap V_3)+(V_2\cap V_3)\subset (V_1+V_2)\cap V_3$

$\Rightarrow \dim((V_1+V_2)\cap V_3)\geq \dim((V_1\cap V_3)+(V_2\cap V_3))$

$=\dim(V_1\cap V_3)+\dim(V_2\cap V_3)-\dim(V_1\cap V_2\cap V_3)$

$=2k-2-\dim(V_1\cap V_2\cap V_3)$

$\therefore$ 由 (*) 可知. $\dim(V_1+V_2+V_3)\leq 2k+1-(2k-2-\dim(V_1\cap V_2\cap V_3))$

$\Rightarrow \dim(V_1+V_2+V_3)\leq \dim(V_1\cap V_2\cap V_3)+3$.

$\because V_1+V_2\subset V_1+V_2+V_3$, $V_1\cap V_2\cap V_3\subset V_1\cap V_2$

$\therefore k+1=\dim(V_1+V_2)\leq \dim(V_1+V_2+V_3)\leq \dim(V_1\cap V_2\cap V_3)+3\leq \underbrace{\dim(V_1\cap V_2)+3}_{=k+2}$

i.e. $k+1\leq \dim(V_1+V_2+V_3)\leq k+2$.

分类: ① $\dim(V_1+V_2+V_3)=k+1$ ✓.

② $\dim(V_1+V_2+V_3)=k+2$.

此时有 $k+2\leq \dim(V_1\cap V_2\cap V_3)+3\leq k+2$.

$\Rightarrow \dim(V_1\cap V_2\cap V_3)=k-1$.

法二： $\dim(V_1+V_2+V_3)=\underbrace{\dim(V_1+V_2)}_{k+1}+\underbrace{\dim(V_3)}_{k}-\dim((V_1+V_2)\cap V_3)$

$=2k+1-\dim((V_1+V_2)\cap V_3)$

设 $\dim(V_1+V_2+V_3)\neq k+1$，下证 $\dim(V_1\cap V_2\cap V_3)=k-1$

则 $\dim((V_1+V_2)\cap V_3)\neq k$ . . . . . (*)

由上式知

$\because\ V_1\cap V_3\subset(V_1+V_2)\cap V_3\subset V_3$ . . . . . (**)

$\therefore \underset{k-1}{\underset{\shortparallel}{\dim(V_1\cap V_3)}}\leq\dim((V_1+V_2)\cap V_3)\leq\underset{k}{\underset{\shortparallel}{\dim(V_3)}}$ . . . . . (***)

由(*)和(***)，$\dim((V_1+V_2)\cap V_3)=k-1$.

由(**) $V_1\cap V_3=(V_1+V_2)\cap V_3$

$\Rightarrow V_1\cap V_2\cap V_3=(V_1+V_2)\cap V_2\cap V_3=V_2\cap V_3$.

$\Rightarrow \dim(V_1\cap V_2\cap V_3)=k-1$

例：设 $\mathbb{R}^4$ 中的标准基是 $\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4$.

情形1. $V_1=\langle\vec{e}_1,\vec{e}_2\rangle$，$V_2=\langle\vec{e}_2,\vec{e}_3\rangle$，$V_3=\langle\vec{e}_1,\vec{e}_3\rangle$

此时 $k=2$. $\dim(V_1+V_2+V_3)=3=k+1$，$\dim(V_1\cap V_2\cap V_3)=0\neq k-1$

情形2. $V_1=\langle\vec{e}_1,\vec{e}_2\rangle$，$V_2=\langle\vec{e}_1,\vec{e}_3\rangle$. $V_3=\langle\vec{e}_1,\vec{e}_4\rangle$.

此时 $k=2$，$\dim(V_1+V_2+V_3)=4\neq k+1$，$\dim(V_1\cap V_2\cap V_3)=1=k-1$